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Fourth Edition

Mathematical Proofs

A Transition to Advanced Mathematics

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To

the memory of my mother and father G.C.

the memory of my brother Russ A.D.P.

my mother and the memory of my father P.Z.

Contents

Preface to the Fourth Edition

As the teaching of calculus in many colleges and universities has become more problem-oriented with added emphasis on the use of calculators and computers, the theoretical gap between the material presented in calculus and the mathematical background expected (or at least hoped for) in advanced calculus and other more advanced courses has widened. In an attempt to narrow this gap and to better prepare students for the more abstract mathematics courses to follow, many colleges and universities have introduced courses that are now commonly called "transition courses." In these courses, students are introduced to problems whose solution involves mathematical reasoning and a knowledge of proof techniques, where writing clear proofs is emphasized. Topics such as relations, functions and cardinalities of sets are encountered throughout theoretical mathematics courses. Lastly, transition courses often include theoretical aspects of number theory, combinatorics, abstract algebra and calculus. This textbook has been written for such a course.

The idea for this textbook originated in the early 1980s, long before transition courses became fashionable, during the supervision of undergraduate mathematics research projects. We came to realize that even advanced undergraduates lack a sound understanding of proof techniques and have difficulty writing correct and clear proofs. At that time, we developed a set of notes for these students. This was followed by the introduction of a transition course, for which a more detailed set of notes was written. The first edition of this book emanated from these notes, which in turn has ultimately led to this fourth edition.

While understanding proofs and proof techniques and writing good proofs are major goals here, these are not things that can be accomplished to any great degree in a single course during a single semester. These must continue to be emphasized and practiced in succeeding mathematics courses.

Our Approach

Since this textbook originated from notes that were written exclusively for undergraduates to help them understand proof techniques and to write good proofs, the tone is student-friendly. Numerous examples of proofs are presented in the text. Following common practice, we indicate the end of a proof with the square symbol ■. Often we precede a proof by a discussion, referred to as a *proof strategy*, where we think through what is needed to present a proof of the result in question. Other times, we find it useful to reflect on a proof we have just presented to point out certain key details. We refer to a discussion of this type as a *proof analysis*. Periodically, we present and solve problems, and we may find it convenient to discuss some features of the solution, which we refer to simply as an *analysis*. For clarity, we indicate the end of a discussion of a proof strategy, proof analysis, analysis, or solution of an example with the diamond symbol ♦.

A major goal of this textbook is to help students learn to construct proofs of their own that are not only mathematically correct but clearly written. More advanced mathematics students should strive to present proofs that are convincing, readable, notationally consistent and grammatically correct. A secondary goal is to have students gain sufficient knowledge of and confidence with proofs so that they will recognize, understand and appreciate a proof that is properly written.

As with the first three editions, the fourth edition of this book is intended to assist the student in making the transition to courses that rely more on mathematical proof and reasoning. We envision that students taking a course based on this book have probably had a year of calculus (and possibly another course such as elementary linear algebra) although no specific prerequisite mathematics courses are required. It is likely that, prior to taking this course, a student's training in mathematics consisted primarily of doing patterned problems; that is, students have been taught methods for solving problems, likely including some explanation as to why these methods worked. Students may very well have had exposure to some proofs in earlier courses but, more than likely, were unaware of the logic involved and of the method of proof being used. There may have even been times when the students were not certain what was being proved.

New to This Edition

The following changes and additions to the third edition have resulted in this fourth edition of the text:

- Presentation slides in PDF and LaTeX formats have been created to accompany every chapter. These presentations provide examples and exposition on key topics. Using short URLs (which are called out in the margin using the icon), these presentations are linked in the text beside the supplemental exercises at the end of each chapter. These slides can be used by instructors in lecture, or by students to learn and review key ideas.
- The new Chapter 7, "Reviewing Proof Techniques," summarizes all the techniques that have been presented. The placement of this chapter allows instructors and students to review all of these techniques before beginning to explore different contexts in which proofs are used. This new chapter includes many new examples and exercises.
- The new Chapter 13, "Proofs in Combinatorics," has been added because of a demand for such a chapter. Here, results and examples are presented in this important area of discrete mathematics. Numerous exercises are also included in this chapter.
- The new online Chapter 18, "Proofs with Real and Complex Numbers," has been added to provide information on these two important classes of numbers. This chapter includes many important classical results on real numbers as well as important results from complex variables. In each case, detailed proofs are given.

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Chapters 16 (Proofs in Ring Theory), 17 (Proofs in Linear Algebra), and 19 (Proofs in Topology) continue to exist online to allow faculty to tailor the course to meet their specific needs.
- More than 250 exercises have been added. Many of the new exercises fall into the moderate difficulty level and require more thought to solve.
- Section exercises for each chapter have been moved from the end of the chapter to the end of each section within a chapter. (Exceptions: for the review chapter [7] and the online chapters [16–19], all exercises remain at the end of the chapter.)
- In the previous edition, there were exercises at the end of each chapter called Additional Exercises. These summative exercises served to pull together the ideas from the various sections of the chapter. These exercises remain at the end of the chapters in this edition, but they have been renamed Supplemental Exercises.

Contents and Structure

Outline of Contents

Each of the Chapters 1–6 and 8–15 is divided into sections, and exercises for each section occur at the end of that section. There is also a final supplemental section of exercises for the entire chapter appearing at the end of that chapter.

Since writing good proofs requires a certain degree of competence in writing, we have devoted **Chapter 0** to writing mathematics. The emphasis of this chapter is on effective and clear exposition, correct usage of symbols, writing and displaying mathematical expressions, and using key words and phrases. Although every instructor will emphasize writing in his or her own way, we feel that it is useful to read Chapter 0 periodically throughout the course. It will mean more as the student progresses through the course.

Chapter 1 contains a gentle introduction to sets, so that everyone has the same background and is using the same notation as we prepare for what lies ahead. No proofs involving sets occur until Chapter 4. Much of Chapter 1 may very well be a review for many.

Chapter 2 deals exclusively with logic. The goal here is to present what is needed to get into proofs as quickly as possible. Much of the emphasis in Chapter 2 is on statements, implications and quantified statements, including a discussion of mixed quantifiers. Sets are introduced before logic so that the student's first encounter with mathematics here is a familiar one and because sets are needed to discuss quantified statements properly in Chapter 2.

The two proof techniques of direct proof and proof by contrapositive are introduced in **Chapter 3** in the familiar setting of even and odd integers. Proof by cases is discussed in this chapter as well as proofs of "if and only if" statements. **Chapter 4** continues this discussion in other settings, namely divisibility of integers, congruence, real numbers and sets.

The technique of proof by contradiction is introduced in **Chapter 5**. Since existence proofs and counterexamples have a connection with proof by contradiction, these also occur in Chapter 5. The topic of uniqueness (of an element with specified properties) is also addressed in Chapter 5.

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Proof by mathematical induction occurs in **Chapter 6**. In addition to the Principle of Mathematical Induction and the Strong Principle of Mathematical Induction, this chapter includes proof by minimum counterexample.

Chapter 7 reviews all proof techniques (direct proof, proof by contrapositive, proof by contradiction and induction) introduced in Chapters 3–6. This chapter provides many examples to solidify the understanding of these techniques, emphasizing both how and when to use the techniques. Exercises in this chapter are distributed randomly with respect to the method of proof used.

The main goal of **Chapter 8** (Prove or Disprove) concerns the testing of statements of unknown truth value, where it is to be determined, with justification, whether a given statement is true or false. In addition to the challenge of determining whether a statement is true or false, such problems provide added practice with counterexamples and the various proof techniques. Testing statements is preceded in this chapter by a historical discussion of conjectures in mathematics and a review of quantifiers.

Chapter 9 deals with relations, especially equivalence relations. Many examples involving congruence are presented and the set of integers modulo n is described.

Chapter 10 involves functions, with emphasis on the properties of one-to-one (injective) and onto (subjective) functions. This gives rise to a discussion of bijective functions and inverses of functions. The well-defined property of functions is discussed in more detail in this chapter. In addition, there is a discussion of images and inverse images of sets with regard to functions as well as operations on functions, especially composition.

Chapter 11 deals with infinite sets and a discussion of cardinalities of sets. This chapter includes a historical discussion of infinite sets, beginning with Cantor and his fascination and difficulties with the Schröder–Bernstein Theorem, then proceeding to Zermelo and the Axiom of Choice, and ending with a proof of the Schröder–Bernstein Theorem.

All of the proof techniques are employed in **Chapter 12**, where numerous results in the area of number theory are introduced and proved.

Chapter 13 deals with proofs in the area of discrete mathematics called combinatorics. The primary goal of this chapter is to introduce the basic principles of counting such as multiplication, addition, pigeonhole and inclusion-exclusion. The concepts of permutations and combinations described here give rise to a wide variety of counting problems. In addition, Pascal triangles and the related binomial theorem are discussed. This chapter describes many proofs that occur in this area including many examples of how this subject can be used to solve a variety of problems.

Chapter 14 deals with proofs that occur in calculus. Because these proofs are quite different from those previously encountered but are often more predictable in nature, many illustrations are given that involve limits of sequences and functions and their connections with infinite series, continuity and differentiability.

The final **Chapter 15** deals with modern algebra, beginning with binary operations and moving into proofs that are encountered in the area of group theory.

It is our experience that many students have benefited by reading and solving problems in these later chapters that deal with courses they are currently taking or are about to take. The same is true for the following online chapters.

The study of proofs in modern algebra continues in the first online **Chapter 16** (`goo.gl/zKdQor`), where the major topic is ring theory. Proofs concerning integral domains and fields are presented here as well.

In **Chapter 17** (`goo.gl/Tmh2ZB`), we discuss proofs in linear algebra, where the concepts of vector spaces and linear transformations are emphasized.

Even though real numbers (and, to a lesser degree, complex numbers) occur throughout mathematics, there are many properties of these two classes of numbers of which students may be unaware. This is the topic for **Chapter 18** (`goo.gl/ymv5kJ`).

The final online chapter is **Chapter 19** (`goo.gl/bWFRMu`), where we discuss proofs in topology. Included in this chapter are proofs involving metric spaces and topological spaces. A major topic here is open and closed sets.

Exercises

There are over 1000 exercises in Chapters 1–19. The degree of difficulty of the exercises ranges from routine to medium difficulty to moderately challenging. As mentioned earlier, the fourth edition contains more exercises in the moderately difficult category. Types of exercises include:

- Exercises that present students with statements, asking students to decide whether they are true or false (with justification).
- Proposed proofs of statements, asking if the argument is valid.
- Proofs without a statement given, which ask students to supply a statement of what has been proved.
- Exercises that call upon students to make conjectures of their own and possibly to provide proofs of these conjectures.

Chapter 3 is the first chapter in which students will be called upon to write proofs. At such an early stage, we feel that students need to (1) concentrate on constructing a valid proof and not be distracted by unfamiliarity with the mathematics, (2) develop some self-confidence with this process and (3) learn how to write a proof properly. With this in mind, many of the exercises in Chapter 3 have been intentionally structured so as to be similar to the examples in that chapter.

Exercises for each section in Chapters 1–6 and 8–15 occur at the end of a section (section exercises) and additional exercises for the entire chapter (supplemental exercises) appear at the end of the chapter as do chapter exercises for Chapter 7. Answers or hints to the odd-numbered section exercises appear at the end of the text as do odd-numbered chapter exercises for Chapter 7. One should also keep in mind, however, that proofs of results are not unique in general.

Teaching a Course from This Text

Although a course using this textbook could be designed in many ways, here are our views on such a course. As we noted earlier, we think it is useful for students to reread (at least portions of) Chapter 0 throughout the course, as we feel that with each reading, the chapter becomes more meaningful. The first part of Chapter 1 (Sets) will likely be familiar to most students, although the last part may not. Chapters 2–6 will probably be part of any such course, although certain topics could receive varying degrees of emphasis (with perhaps proof by minimum counterexample in Chapter 6 possibly even omitted). Chapter 7 reviews all proof techniques introduced in Chapters 3–6. If the instructor believes that students have obtained a strong understanding of these techniques, this chapter could be omitted. Nevertheless, we feel it is good for students to read this chapter and try solving the exercises. Instructors who choose to omit Chapter 7 might find it useful to assign exercises from this chapter, asking students to determine which proof technique is to be used.

One could spend much or little time on Chapter 8, depending on how much time is used to discuss the large number of "prove or disprove" exercises. We think that most of Chapters 9 and 10 would be covered in such a course. It would be useful to cover some of the fundamental ideas addressed in Chapter 11 (Cardinalities of Sets). As time permits, portions of the later chapters could be covered, especially those of interest to the instructor, including the possibility of going to the web site for even more variety.

Supplements and Technology

Online Chapters

Four additional chapters, Chapters 16–19 (dealing with proofs in ring theory, linear algebra, real and complex numbers, and topology), can be found by going to: `goo.gl/bf2Nb3`.

Instructor's Solutions Manual (downloadable)

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The Instructor's Solutions Manual, written by the authors, provides worked-out solutions for all exercises in the text. It is available for download to qualified instructors from the Pearson Instructor Resource Center https://www.pearson.com/us/sign-in.html.

Chapter Presentations

This icon , found beside the supplemental exercises for each chapter, indicates a chapter presentation. The short URLs in the margin of the text provide students with direct access to the presentations in PDF form. The URL `goo.gl/bf2Nb3` provides access to the complete library of presentations in both PDF and editable LaTeX (Beamer) formats.

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