

# FUNCTIONS AND MATHEMATICAL MODELS

- 1.1 FUNCTIONS DEFINED BY TABLES
- 1.2 FUNCTIONS DEFINED BY GRAPHS
- 1.3 FUNCTIONS DEFINED BY FORMULAS
- 1.4 AVERAGE RATE OF CHANGE

**H**ow hot is it *really*? Anyone who has ever visited Phoenix surely has heard the adage that the dry heat of the desert doesn't feel as hot as the temperature would suggest. Indeed, when it is 100°F in Phoenix with a relative humidity of 10%, it “feels like” it is only 95°F—a full five degrees cooler than the thermometer says!

The “heat index” is a number calculated by meteorologists to measure the effect of humidity on the apparent temperature as felt by the human body. It is designed to report how hot you actually feel at a particular combination of air temperature and relative humidity. The chart in Fig. 1.0.1 shows the heat index for various levels of relative humidity at a fixed air temperature of 85°F.

The chart indicates that when the humidity is 10% the apparent temperature is only 80°F. However, if the humidity is 90%, then a temperature of 85°F seems like 102°F. Thus the humidity has a big effect on how hot we actually feel.

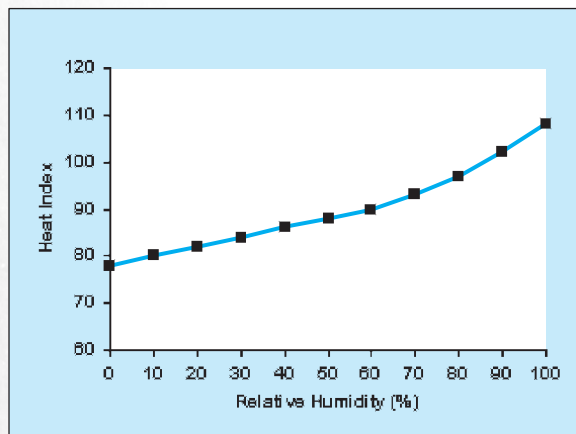


FIGURE 1.0.1

Source: National Weather Service, Buffalo, New York.

We can see from the chart that as the relative humidity increases from 0% to 60%, the apparent temperature increases uniformly from 78°F to 90°F. Notice that this portion of the chart appears to be a straight line. After we reach a relative humidity of 60%, however, the apparent temperature increases more rapidly as the relative humidity increases. That is, this portion of the chart appears to curve upward.

All around us we see quantities, like apparent temperature and relative humidity, that seem to be related to each other in some systematic way. We often describe this dependence of one quantity on another by using the word *function*. Thus, we might say that your weight is (ideally) a function of your height; that your grade on a history test is a function of how long you studied; or that your income is a function of your education level. In each case we are noting a dependence of the first quantity on the second.

In this chapter, we introduce the mathematical concept of a function, we study functions defined by tables, graphs, and rules, and we explore increasing and decreasing functions.

## 1.1 FUNCTIONS DEFINED BY TABLES

The menu at a local Waffle House restaurant gives the price of a breakfast of eggs, toast, jelly, and grits based on the number of eggs the customer orders, as illustrated in the following table of 2005 prices:

NUMBER OF EGGS	PRICE OF BREAKFAST
3	\$2.60
2	\$2.25
1	\$1.80

Most applications of mathematics involve the use of numbers or *variables* to describe real-world quantities. In this situation, suppose we let  $n$  represent the number of eggs ordered and  $p$  the price of the breakfast. Then the table describes a relationship between  $n$  and  $p$ . This relationship is an example of a mathematical *function* because for each number  $n$  of eggs ordered there is a corresponding price  $p$  charged for the breakfast.

The key concept here is that there is only **one price** associated with each number of eggs. The menu, in effect, provides a “rule” for determining price: If you know how many eggs were ordered, you know the price of the breakfast. If you and your friend each ordered a 1-egg breakfast, then you would expect to be charged the same amount. Indeed, if one of you were charged more than \$1.80 for your breakfast, then you surely would complain!

**DEFINITION: Function**

A **function**  $f$  defined on a collection  $D$  of numbers is a rule that assigns to each number  $x$  in  $D$  a specific number  $f(x)$ .

We will refer to the numbers in the set  $D$  as “inputs” and the corresponding  $f(x)$  numbers as “outputs.” In the Waffle House example, we can say that the price of the breakfast is a function of the number of eggs ordered. This means that for each input (number of eggs), there is only one output (price).

We can also look at the table “backwards,” and say that the number of eggs ordered is a function of the price of the breakfast. Then each input (price) has only one output (number of eggs).

**EXAMPLE 1****Is It a Function?**

The following table gives the average price of a 30-second commercial airing during the Super Bowl in the indicated year.

- Is the price of the commercial a function of the year?
- Is the year a function of the price of the commercial?

YEAR	PRICE OF COMMERCIAL (IN MILLIONS)
1998	\$1.3
2000	\$2.1
2001	\$2.3
2003	\$2.1
2005	\$2.4

Source: [www.superbowl-ads.com](http://www.superbowl-ads.com).

**SOLUTION**

- When we ask “Is the price a function of the year?” we are considering the year as the input and the price as the output. Thus, the price of the commercial *is* a function of the year because for each input (year) there is only one output (price).
- When we ask “Is the year a function of the price?” we are considering the price as the input and the year as the output. Thus, the year *is not* a function of the price because for the input \$2.1 million, there are two different outputs (2000 and 2003).

In the definition of function, the number  $f(x)$ —we say “ $f$  of  $x$ ”—is called the *value* of the function  $f$  at the number  $x$ . The “rule” mentioned in the definition can be specified by a table, by a formula, or by a graph, or even by a verbal description that tells how the value  $f(x)$  is found when the number  $x$  is given. While we frequently use  $x$  to denote the variable and  $f$  to denote the function, we can use any other letters that we like or that seem more natural in a particular situation.

When we consider the Waffle House data as defining price as a function of the number of eggs ordered, the set  $D$  is the collection of all possible numbers of eggs—the set of numbers 1, 2, and 3. Since the inputs are numbers of eggs, we will denote them by  $n$  (rather than  $x$ ). Since the outputs are prices, we will denote them by  $p(n)$ .

$n$	$p(n)$
1	\$1.80
2	\$2.25
3	\$2.60

Given a number  $n$  in the first column of the table, we simply look in the second column to find the corresponding price  $p(n)$ . For instance,  $p(3) = 2.60$  because \$2.60 is the price “assigned” to a 3-egg breakfast.

So the mathematical question “Find  $p(n)$ ” is equivalent to the English question “If the input is  $n$ , what is the output?” It is convenient for us to think of this as the “frontward” question “Here is the input, what is the output?”

We can also ask the “backward” question “Here is the output, what is the input?” This is equivalent to the mathematical question “For what value(s) of  $n$  is  $p(n) = 2.25$ ?” From the table, the answer is  $n = 2$ .

Notice that when we use this notation, the *input* value appears *inside* the parentheses. In the “frontward” question “Find  $p(3)$ ,” we are given the input 3 and asked to find the output. In the “backward” question “For what value(s) of  $n$  is  $p(n) = 2.25$ ,” we are looking for the input  $n$ .

## EXAMPLE 2

### Finding Input and Output Values

Rather than keeping a “running total,” the Bureau of the Public Debt uses a daily accounting method to calculate the public debt of the United States. At the end of each day, approximately 50 different agencies (such as Federal Reserve Banks) report certain financial information to the Bureau. At around 11:30 AM EST the next morning, the accounting system produces a figure for the public debt, correct to the nearest penny, for the previous day. Thus, to each date is assigned an official public debt amount, and we therefore can say that the debt depends on the day chosen. In other words, the debt is a function of the date. Here is a portion of a table reporting the public debt in February 2005.

DATE $x$	PUBLIC DEBT $g(x)$
02/18/2005	\$7,689,935,780,269.27
02/17/2005	\$7,689,847,469,266.70
02/16/2005	\$7,671,700,332,790.18
02/15/2005	\$7,674,137,053,033.57
02/14/2005	\$7,630,849,109,540.36

- Find  $g(02/15/2005)$ .
- Find  $x$  if  $g(x) = \$7,689,847,469,266.70$ .

**SOLUTION**

From the table:

- a.  $g(02/15/2005) = \$7,674,137,053,033.57$ .
- b. When  $g(x) = \$7,689,847,469,266.70$ ,  $x = 02/17/2005$ .

## Domain and Range of a Function

When we use functions to describe relationships between variables, it is important that we know what numbers are sensible values to substitute for those variables.

**DEFINITION: Domain**

The collection (or set) of all numbers for which the number  $f(x)$  is defined is called the **domain** (of definition) of the function  $f$ . These numbers are the input values of the function.

We saw in Example 1 that the average price of a 30-second Super Bowl ad is a function of the year it airs. Based on the table, the domain of this function is {1998, 2000, 2001, 2003, 2005}.

**DEFINITION: Range**

The set of all possible values  $y = f(x)$  is called the **range** of the function.

Based on the table in Example 1, the range of the Super Bowl ad function is {1.3, 2.1, 2.3, 2.4}, where the values are in millions of dollars. Notice that it is not necessary to list 2.1 twice, because the range indicates what numbers occur as output values, regardless of how many times each one appears.

### EXAMPLE 3

#### Finding Domain and Range

Consider the following table giving calories from fat and milligrams of cholesterol for several sandwiches from the Subway restaurant:

SUBWAY SANDWICH	CALORIES FROM FAT $x$	CHOLESTEROL (mg) $C(x)$
Honey Mustard Ham	45	25
Cheese Steak	90	35
Buffalo Chicken	130	50
Italian BMT	190	55
Double Meat Cold Cut Combo	250	105

Source: www.subway.com.

This table defines milligrams of cholesterol as a function of calories from fat.

- a. What is the domain of this function?
- b. What is the range of this function?

**SOLUTION**

- a. The domain of the function is  $\{45, 90, 130, 190, 250\}$ .
- b. The range of the function is  $\{25, 35, 50, 55, 105\}$ .

In this section, we have applied the definitions of function, function value, domain, and range to data given in table form. We will continue to apply these same important definitions when we encounter functions defined by graphs or by rules.

## 1.1 Exercises

### Building Your Skills

In Exercises 1–4, determine whether  $B$  is a function of  $A$  and then, if  $B$  is a function of  $A$ , give the domain and the range of the function.

1.

$A$	10	10.5	13	15	16	20
$B$	3	4	9	13	15	23

2.

$A$	10	10.5	13	15	16	20
$B$	3	4	4	9	13	23

3.

$A$	2	4	5	5	8	11
$B$	3	4	9	13	15	23

4.

$A$	2	4	6	8	10	12
$B$	3	3	3	3	3	3

In Exercises 5–8, find (a)  $f(2)$  and (b) the value(s) of  $x$  for which  $f(x) = 2$ .

5.

$x$	0	2	4	6	8	10
$f(x)$	3	4	9	2	7	3

6.

$x$	-1	0	1	2	3	4
$f(x)$	-4	2	2	9	-13	23

7.

$x$	2	4	5	7	8	11
$f(x)$	6	4	7	13	15	23

8.

$x$	-4	-1	0	2	10	17
$f(x)$	2	2	2	2	2	2



## Applying Your Skills

9. The total number of fat grams and calories in various chicken sandwiches are given in the following table:

<i>Fat Grams</i>	20	29	5	33	9	33	26	43
<i>Calories</i>	430	550	320	680	300	750	530	710

Source: Chick-Fil-A Gram Comparison.

- a. Based on this table, are calories a function of fat grams?  
 b. Based on this table, are fat grams a function of calories?
10. In 2005, Kaiser Permanente advertised the following monthly rates for the Personal Advantage health plan for male subscribers in the state of Georgia:

<i>Age (years)</i>	1	4	18	27	35	40	44	45
<i>Cost (\$)</i>	45	45	45	64	68	90	90	125

- a. Based on this table, is cost a function of age?  
 b. Based on this table, is age a function of cost?
11. The College Board reports the average verbal and math SAT scores for college-bound seniors by state. The accompanying table shows averages for the year 2004:

STATE	AVERAGE VERBAL SCORE	AVERAGE MATH SCORE
Arkansas	569	555
Connecticut	515	515
Illinois	585	597
Hawaii	487	514
Texas	493	499

- a. Based on this table, is average math score a function of average verbal score?  
 b. Based on this table, is average verbal score a function of average math score?
12. For a person who is 5 feet, 8 inches tall, the following table gives weight in pounds and corresponding body mass index (BMI). A person whose BMI is less than 18.5 is considered underweight.

<i>Weight (pounds)</i>	121	128	130	133	148	155	166	170	173
<i>BMI</i>	18.4	19.5	19.8	20.2	22.5	23.6	25.2	25.8	26.3

Source: Centers for Disease Control and Prevention.

- a. Explain why, based on this table, BMI is a function of weight in pounds.  
 b. Explain why, based on this table, weight in pounds is a function of BMI.

- c. What is the BMI for a person 5 feet, 8 inches tall who weighs 133 pounds?  
 d. A person whose BMI is 25.0–29.9 is considered overweight. Based on this table, for which weights would a person 5 feet, 8 inches tall be considered overweight?
13. The following table shows the year that the Dow Jones Average first reached certain milestone levels:

<i>Dow Jones Average</i>	500	1000	1500	3000	4000	5000	8000	10,000
<i>Year First Attained</i>	1956	1972	1985	1991	1995	1995	1997	1999

Source: Dow Jones Industrial Average.

- a. Explain why, based on this table, year is a function of Dow Jones Average.  
 b. Explain why, based on this table, the Dow Jones Average is not a function of year.  
 c. When did the Dow Jones average first reach 3000?  
 d. What value(s) did the Dow Jones first attain in 1995?
14. Use a newspaper, a magazine, or the Internet to find
- a. an example of a table of data that represents a function;  
 b. an example of a table of data that does not represent a function.

## 1.2 FUNCTIONS DEFINED BY GRAPHS

In Example 3 of Section 1.1, we saw that the table showing calories from fat and cholesterol for Subway sandwiches represented cholesterol  $c(x)$  as a function of calories  $x$ :

<i>Calories from Fat</i>	45	90	130	190	250
<i>Cholesterol (mg)</i>	25	35	50	55	105

To graphically represent this data, we create a scatter plot, a set of points in the coordinate plane with the input values on the horizontal axis ( $x$ -axis) and the output values on the vertical axis ( $y$ -axis). Figure 1.2.1 shows a calculator scatter plot for these data with calories from fat on the horizontal axis and milligrams of cholesterol on the vertical axis.

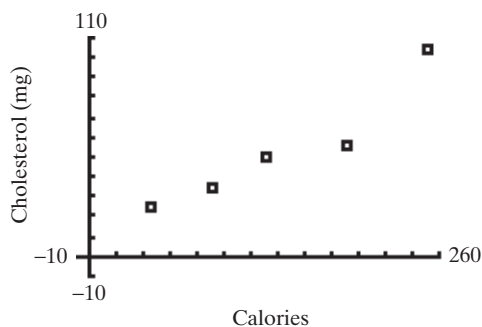


FIGURE 1.2.1 Scatter plot of cholesterol versus calories data.

Recall that our definition of function tells us that each input can have only one output. For the Subway sandwiches, there is only one point in the scatter plot for each of the input values, corresponding to the unique output value for that input.



On the other hand, the table showing fat grams and calories for chicken sandwiches (in Exercise 9 of Section 1.1) did not represent calories as a function of fat grams because the input 33 had two different outputs, 680 and 750:

<i>Fat Grams</i>	20	29	5	33	9	33	26	43
<i>Calories</i>	430	550	320	680	300	750	530	710

Figure 1.2.2 shows a scatter plot for these data with fat grams (input) on the horizontal axis and calories (output) on the vertical axis.

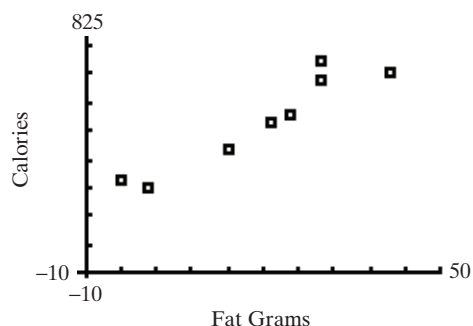


FIGURE 1.2.2 Scatter plot of calories versus fat grams data.

In this scatter plot, we see the two outputs (680 and 750) for the input 33, yielding two different points “stacked” vertically above the input 33.

This understanding that more than one output for a given input shows up on a graph as two points lying on the same vertical line is the source of the “vertical line test” for a function. In order for a graph to represent a function, any vertical line must cross the graph only once.

## EXAMPLE 1

### Is It a Function?

The scatter plot in Fig. 1.2.3 illustrates the population (in thousands) of St. Louis, Missouri, for the census years 1950–2000. Does this scatter plot represent a function?

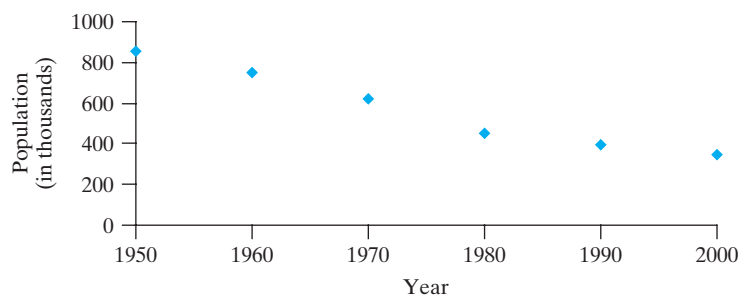


FIGURE 1.2.3 Population of St. Louis.

Source: [www.census.gov](http://www.census.gov).

### SOLUTION

This plot describes a function  $P(x)$  that is defined for each of the years 1950, 1960, 1970, 1980, 1990, and 2000 because for each input (year  $x$ ), the plot shows *exactly one* dot, indicating only one output (population  $P$ ).

Throughout this book we will see how quantities measured over time represent functions with time as the input and the quantity measured (such as population) as the output. Using functions to make predictions of how values will change with time is an important use of mathematics. Planning in business and government for future revenue, expenditure, and resource allocation frequently depends on identifying functions to model the behavior of such quantities.

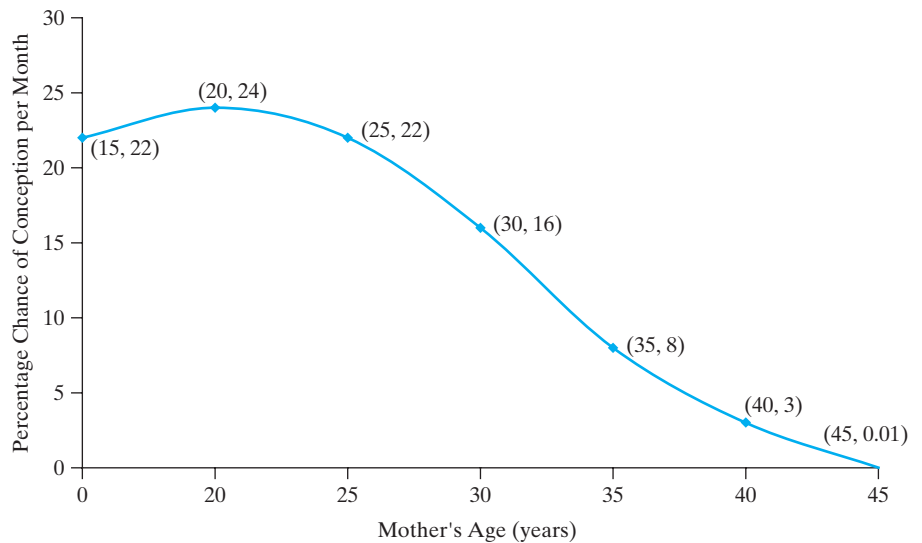
A scatter plot is a representation of a set of individual points. Frequently mathematical relationships are displayed as connected lines or curves. In these situations, we can also apply the vertical line test to determine whether a particular graph represents a function.

## EXAMPLE 2

### Is It a Function?

The graph in Fig. 1.2.4 shows the percentage chance per month that a woman 15–45 years of age will become pregnant. Does this graph represent chance of conception as a function of mother’s age?

**FIGURE 1.2.4** Fertility odds.  
Source: Atlanta Journal-Constitution.



### SOLUTION

We see that any vertical line would cross this graph only once. This indicates that this plot represents a function because each input (age) is associated with only one output (percentage chance of conception).

Just as we did with functions defined by tables, we can find input and output values if functions are given graphically. However, if points on the graph are not labeled or accompanied by a table of values, we can only estimate the values in question.

If we consider the fertility odds graph in Fig. 1.2.4, we can ask the “frontward” question “What is the output when the input is 20?” This question asks us to find the percentage chance of conception when the mother is 20 years old. Since age is represented on the horizontal axis, we find 20 on the horizontal axis and then find the point on the graph directly above it. This appears to be the highest point on the graph, located a bit lower than 25, perhaps 24. So we estimate that the output for an input of 20 is about 24; that is, if a woman is 20 years old, she has about a 24% chance of conceiving per month.

Similarly, suppose we are asked the “backward” question “What is the input when the output is 10?” This question asks us to find the mother’s age when chance of conception is 10%. Since chance of conception is represented on the vertical axis, we find 10 on the vertical axis and find the point (or points) on the graph that lie directly to its right. This point lies a little to the left of 35, so we estimate that the input for an output of 10 is about 34. That is, if the chance of conception is 10%, the mother’s age is about 34 years.

### EXAMPLE 3

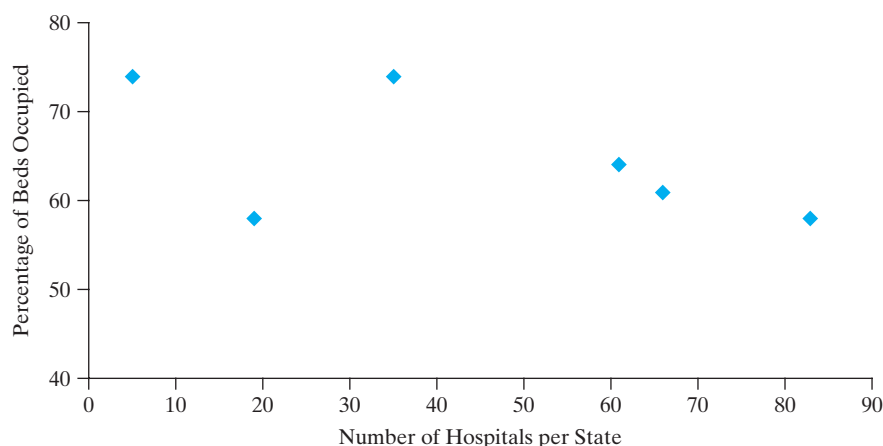
#### Finding Input and Output Values

Figure 1.2.5 shows the number of hospitals and the percentage of beds occupied in the hospitals for several U.S. states.

- Find the percentage of beds occupied if the number of hospitals in the state is 83.
- Find the number of hospitals in the state if 74% of beds are occupied.

**FIGURE 1.2.5** Hospital occupancy.

Source: *The World Almanac and Book of Facts 2005*.



#### SOLUTION

- This question asks us to find the output if the input is 83. If we find 83 on the horizontal axis and look for the point directly above it, we see that the point is located slightly below 60 on the vertical axis. Thus we estimate that about 58% of hospital beds are occupied.
- This question asks us to find the input if the output is 74. There seem to be two points lying at a height of 74. One seems to have a first coordinate of about 5; the second seems to have a first coordinate of about 35. Therefore, we say that the states with 5 or 35 hospitals have 74% of their beds occupied.

If a function is represented graphically, we can find (at least approximately) its domain and range. Recalling that the domain is the set of input values, we look at the hospital occupancy function in Fig. 1.2.5. Estimating these values from the graph, we see that the domain of this function is  $\{5, 19, 35, 61, 66, 83\}$ .

Similarly, since the range is the set of output values, we estimate those values from the graph and find that the range of the function is  $\{58, 61, 64, 74\}$ . Notice that although there are six values in the domain, the range only contains four different

values. This occurs because the inputs 5 and 35 have the same output, 74, and the inputs 19 and 83 both have the output 58.

## EXAMPLE 4

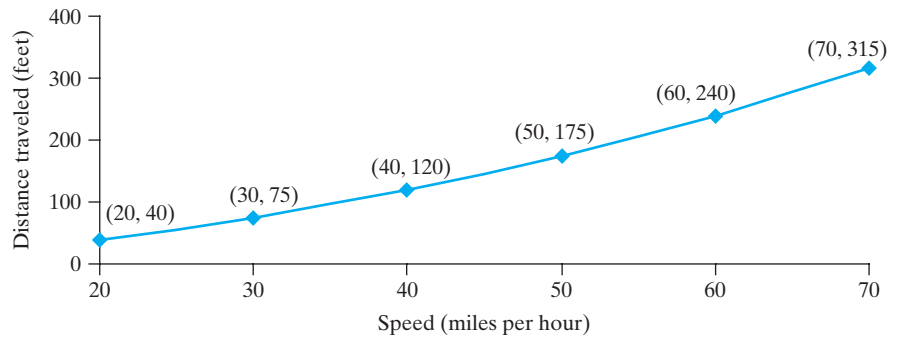
### Finding Domain and Range

Figure 1.2.6 shows the typical distance (in feet) that a car travels after the brakes have been applied for various speeds (in miles per hour).

- Find the domain of this function.
- Find the range of this function.

**FIGURE 1.2.6** Typical stopping distances.

Source: www.highwaycode.gov.uk.



### SOLUTION

- Because the graph is given as a connected curve, the domain consists of all values that lie between the smallest input value and the largest input value, inclusive. That is, the domain contains all real numbers between 20 and 70, including both 20 and 70. Using interval notation, we write the domain as  $[20, 70]$ .
- Similarly, the range of the function consists of all values that lie between the smallest output value and the largest output value, inclusive. So the domain contains all real numbers between 40 and 315, inclusive. In interval notation, the range is  $[40, 315]$ .

## 1.2 Exercises Building Your Skills

In Exercises 1 and 2, make a scatter plot of the data, using  $A$  as input (on the horizontal axis) and  $B$  as output (on the vertical axis). Then explain why the scatter plot does or does not represent  $B$  as a function of  $A$ .

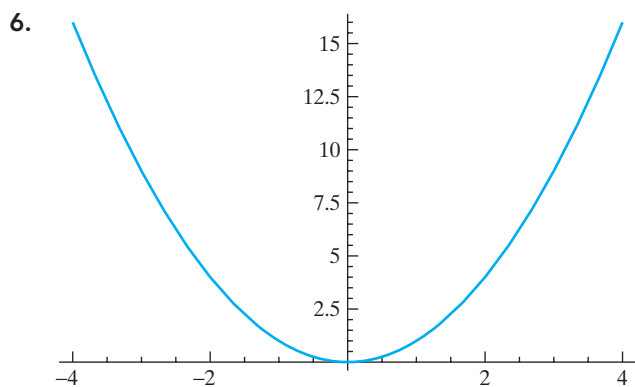
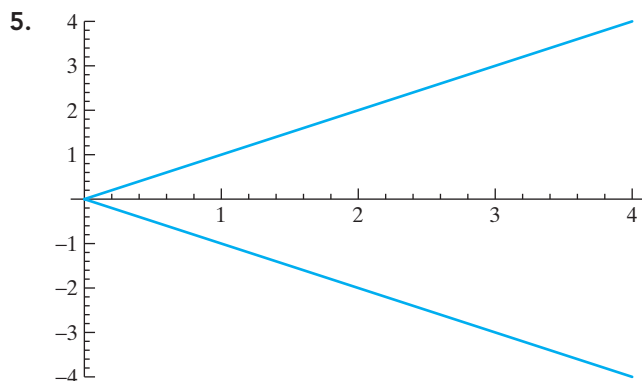
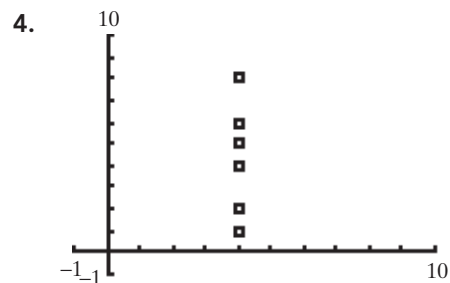
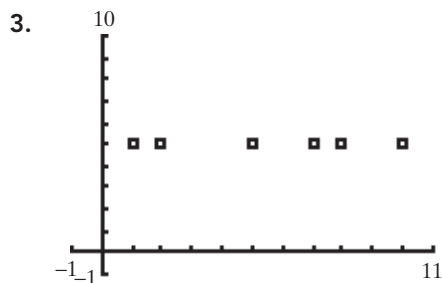
1.

$A$	6	8	9	11	13	14
$B$	3	6	9	6	14	0

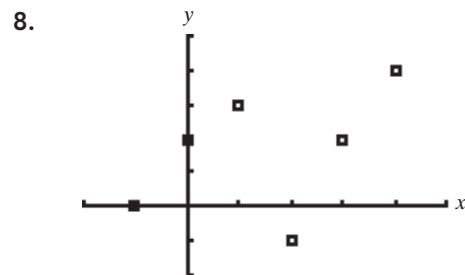
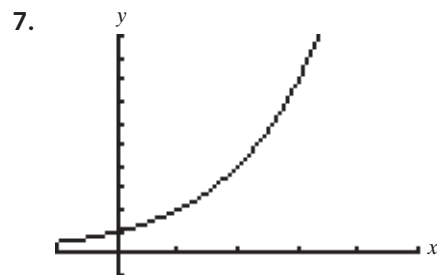
2.

$A$	2	4	5	5	8	11
$B$	2	8	9	13	15	23

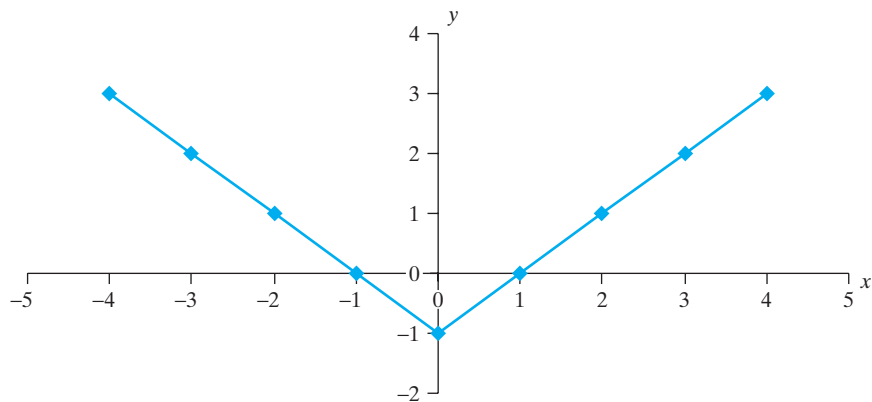
In Exercises 3–6, first determine whether the graph represents a function (with input on the horizontal axis and output on the vertical axis) and then, if the graph represents a function, give its domain and range.



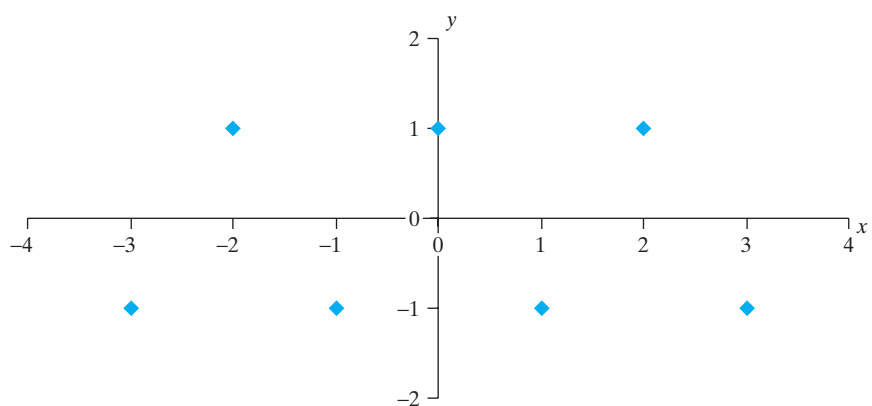
In Exercises 7–10, let  $y = f(x)$ . Find (a)  $f(2)$  and (b) the value(s) of  $x$  for which  $f(x) = 2$ . Each tick mark on the axes represents one unit.



9.

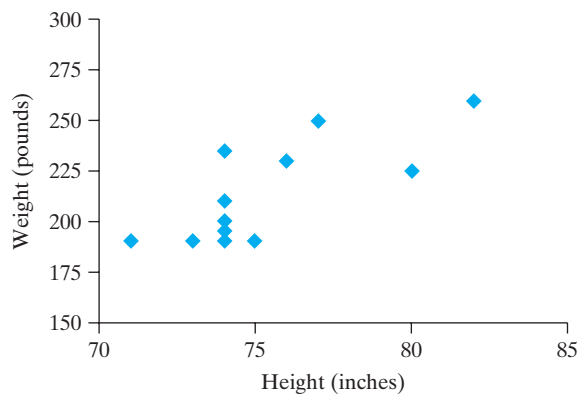


10.



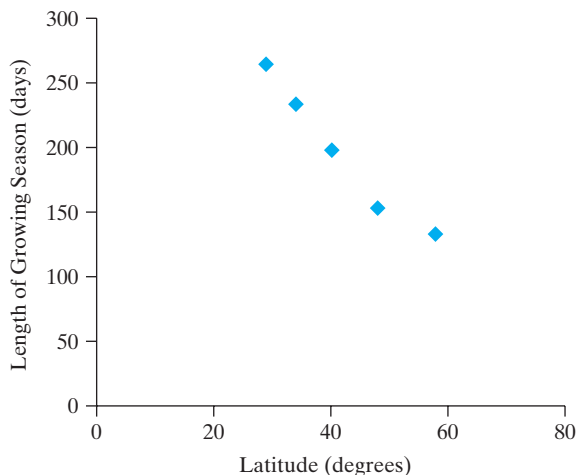
## Applying Your Skills

11. The scatter plot in Fig. 1.2.15 shows the height and weight for 12 pitchers on the Texas Rangers active roster in June 2005. Based on this scatter plot, is weight a function of height? Why or why not?



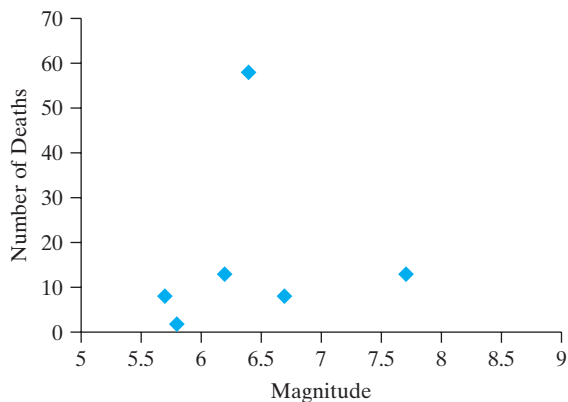
**FIGURE 1.2.15** Texas Rangers pitchers.  
Source: [www.texasrangers.mlb.com](http://www.texasrangers.mlb.com).

12. The scatter plot in Fig. 1.2.16 shows the latitude and length of growing season for several U.S. cities. Based on this scatter plot, is length of growing season a function of latitude? Why or why not?



**FIGURE 1.2.16** Growing season versus latitude.  
 Source: www.census.gov and *The Old Farmer's Almanac*.

13. The scatter plot in Fig. 1.2.17 gives the death toll for California earthquakes of various magnitudes that occurred in the twentieth century.

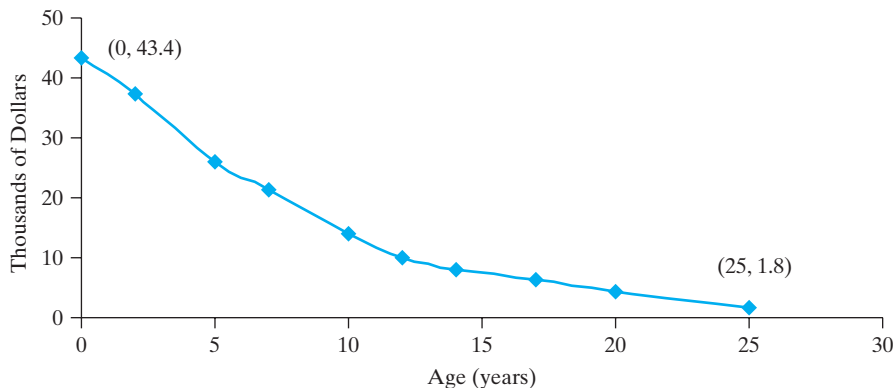


**FIGURE 1.2.17** California earthquakes.  
 Source: National Geophysical Data Center (NGDC).

- Explain why, based on the scatter plot, the number of deaths is a function of the magnitude of the earthquake.
- Give the magnitudes for two earthquakes that have the same number of deaths.
- How many deaths occurred in the earthquake with magnitude 6.4?

14. Fig. 1.2.18 shows data collected by a mathematical modeling student in 2005, giving the price of a used Corvette as a function of its age.

**FIGURE 1.2.18** Price of a used Corvette.

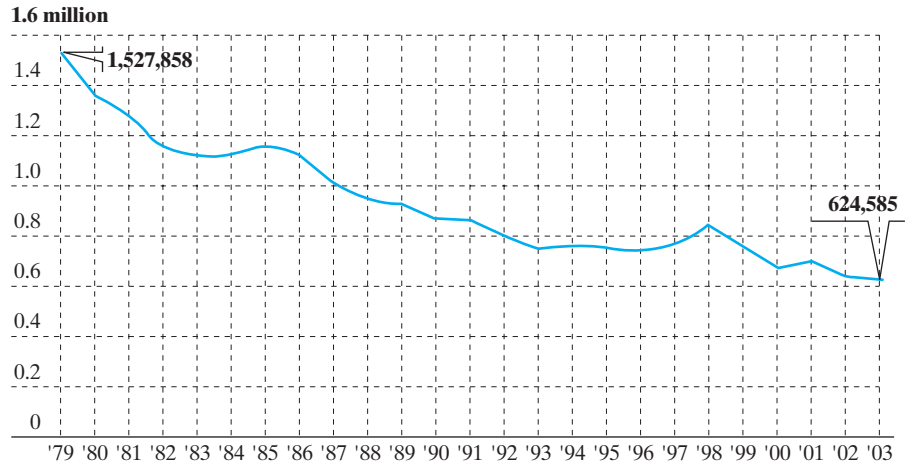




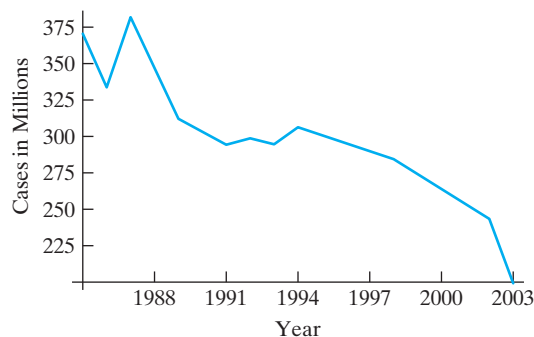
- What is the domain of this function?
- What is the range of this function?
- What is the price of a 10-year-old Corvette?
- How old is a Corvette that costs 10 thousand dollars?

15. In an August 2004 article concerning the political power of the United Auto Workers union, *The Detroit News* included Fig. 1.2.19, which gives the union's membership as a function of the year.

**FIGURE 1.2.19** Membership in the United Auto Workers union.  
Source: *The Detroit News*.



- What is the domain of this function?
  - What is the range of this function?
  - How many UAW members were there in 1987?
  - In what year were there 900,000 UAW members?
16. Fig. 1.2.20 shows the number of cases (in millions) of 7-Up sold for as a function of the year.



**FIGURE 1.2.20** Sales of 7-Up.  
Source: *Atlanta Journal-Constitution*.

- What is the domain of this function?
  - What is the range of this function?
  - In what year were 250 million cases of 7-Up sold?
  - How many cases were sold in 1994?
17. Use a newspaper, a magazine, or the Internet to find:
- An example of a graph that represents a function.
  - An example of a graph that does not represent a function.

## 1.3 FUNCTIONS DEFINED BY FORMULAS

In Sections 1.1 and 1.2, we examined functions that were defined using either a table of values or a graph. In this section, we look at functions defined by formulas, either symbolic or verbal, and consider the relationship among a formula, a table, and a graph.

The Body Mass Index, or BMI, is a measure that assesses the amount of fat in the body. It is commonly used to determine whether individuals are underweight or overweight and is used with both adults and children.

When weight is measured in pounds, the BMI for a child who is 36 inches tall is calculated by the following formula:

$$\text{BMI} = 703 \left( \frac{\text{weight}}{36^2} \right).$$

From this formula, we can see that BMI is a function of weight because for each input value (weight in pounds), we will get exactly one BMI value. For example, if a child's weight is 40 pounds, then his or her BMI is  $703 \left( \frac{40}{36^2} \right) = 21.7$ , rounded to the nearest tenth.

Furthermore, we can use the formula to determine BMI for any given weight. By entering this function rule into our calculator's **Y=** menu as **Y1=703(X/36<sup>2</sup>)**, we can use the table feature to generate a table of function values, as shown in Figs. 1.3.1 and 1.3.2. The values in the **X** column are our inputs

```
TABLE SETUP
TblStart=40
ΔTbl=2
Indent: Auto Ask
Depend: Auto Ask
```

X	Y1
40.0	21.7
42.0	22.8
44.0	23.9
46.0	25.0
48.0	26.0
50.0	27.1
52.0	28.2

X=40

FIGURE 1.3.1 Table setup for BMI function.

FIGURE 1.3.2 Table of BMI function values.

(weights) and the values in the **Y1** column are the outputs (BMI). We see the BMI values displayed correct to one decimal place, and notice that the value obtained for a weight of 40 pounds corresponds to the value we found previously.

To examine this relationship graphically, we can create a scatter plot of BMI versus weight using the data in Fig. 1.3.2 or we can create a graph of the function directly from the function rule. Since the function is already entered into the **Y=** menu of the calculator, all we need do to display the graph is to choose an appropriate viewing window. In order to see all the ordered pairs displayed in Fig. 1.3.2, we must set the window so that our minimum  $x$ -value is smaller than 40, our maximum  $x$ -value is larger than 52, our minimum  $y$ -value is smaller than 21, and our maximum  $y$ -value is larger than 28. Using the settings displayed in Fig. 1.3.3, we obtain the graph in Fig. 1.3.4.

This graph shows BMI not only for the integer weights given in Fig. 1.3.2, but also for any weight between 35 pounds and 55 pounds. For example, to find the



But that is only part of the story. Suppose we were asked the cost if a person used 601 minutes. We can easily see that there is only one “extra” minute, costing 40¢, which must be added to the basic charge of \$39.99. So  $C(601) = \$39.99 + \$0.40 = \$40.39$ . Similarly, if a person used 738 minutes, there would be  $738 - 600 = 138$  additional minutes, each charged at 40¢, making  $C(738) = \$39.99 + 138(\$0.40) = \$95.19$ . If a person used 919 minutes, there would be  $919 - 600 = 319$  additional minutes, each charged at 40¢, making  $C(919) = \$39.99 + 319(\$0.40) = \$167.59$ . In each case, to calculate the cost, we needed to subtract 600 from the minutes used to find the number of additional minutes charged at 40¢ each. In general,  $C(n) = \$39.99 + (n - 600)(\$0.40)$  if  $n > 600$ .

Thus, in order to give a symbolic rule for this function, we must write it in two parts:

$$C(n) = \begin{cases} 39.99 & \text{if } 0 \leq n \leq 600, \\ 39.99 + (n - 600)(0.40) & \text{if } n > 600, \end{cases}$$

where  $n$  represents the number of weekday minutes used and  $C(n)$  is cost in dollars and cents. The graph of this function is shown in Fig. 1.3.6.

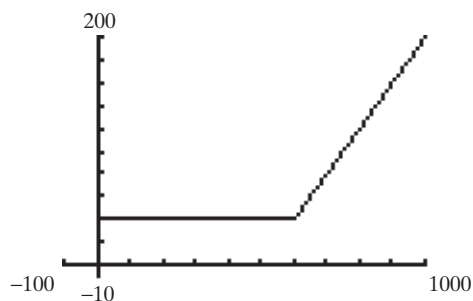


FIGURE 1.3.6 Graph of the “Get More” Plan function.

## EXAMPLE 2

### Finding Input and Output Values

If a certain savings account earns 4% simple interest per year, then the interest  $I$  earned each year is given in terms of the amount  $A$  in the account by the function  $I(A) = 0.04A$ .

- Find the amount in the account if the yearly interest earned is \$18.00.
- Find the interest earned each year if the amount in the account is \$360.00.

### SOLUTION

- This question asks us to find the input if the output is \$18.00. Since  $I(A)$  represents the output for an input of  $A$  dollars, we need to solve the equation

$$18.00 = 0.04A$$

for  $A$ . Thus,  $A = \frac{18.00}{0.04} = 450.00$ , and there is \$450.00 in an account earning \$18.00 interest yearly.

We can also find this input graphically, using the calculator. To the calculator, inputs are  $x$ 's and outputs are  $y$ 's, so we enter  $0.04x$  into **Y1**

and 18.00 into **Y2**. Then we set the window so we can see the graphs of these functions. Since the input  $x$  represents the dollars invested, we use the window **Xmin** = -10, **Xmax** = 800, **Xscl** = 100. The  $y$ 's represent the dollars of interest earned, so we let **Ymin** = -1, **Ymax** = 25, **Xscl** = 5, and then graph the functions. The point where these two graphs intersect has the input value that gives an output of 18.00. Using the **intersect** command in the **CALC** menu of our calculator, we hit **ENTER** to select the “first curve” **Y1**, **ENTER** again to select the “second curve” **Y2**, and **ENTER** once again to select a “guess”. (Sometimes you need to move the cursor to get close to the point of intersection, but usually wherever the cursor “lands” is fine.) We can see from Fig. 1.3.7, that \$450 is the amount invested if the interest earned is \$18.

- b. This question asks us to find the output if the input is \$360.00. Here we need to find  $I(360.00) = 0.04(360.00) = 14.40$ . So, an account of \$360.00 earns \$14.40 each year.

If we use the **value** command in the **CALC** menu, as we did with the BMI function previously, the calculator returns the output value of 14.4, as shown in Fig. 1.3.8.

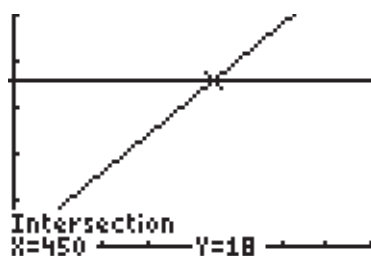


FIGURE 1.3.7 Finding the amount invested for \$18.00 interest earned.

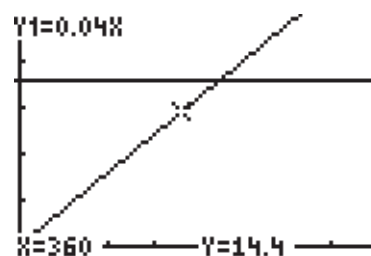


FIGURE 1.3.8 Finding the interest earned for \$360.00 invested.

Since we are dealing with interest earned, we express the answer in dollars and cents as \$14.40.

Examples 1 and 2 illustrate functions that describe relationships between real-world variables. The key to using mathematics to analyze a real-world situation often is recognizing relationships among the variables that describe the situation. The following example illustrates functions defined by formulas that may be familiar from your previous studies in mathematics and science.

### EXAMPLE 3

#### Finding Input and Output Values

- a. The area  $A$  of a circle of radius  $r$  (Fig. 1.3.9) is given by

$$A(r) = \pi r^2 \quad (\text{where } \pi \approx 3.1416),$$

using function notation to indicate that the area depends on the radius. In order to make a circular garden with area 30 square feet, how large (to the nearest hundredth of a foot) should the radius be?

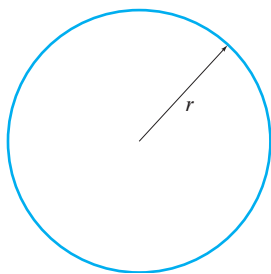


FIGURE 1.3.9 The area  $A$  of a circle is a function of its radius  $r$ .

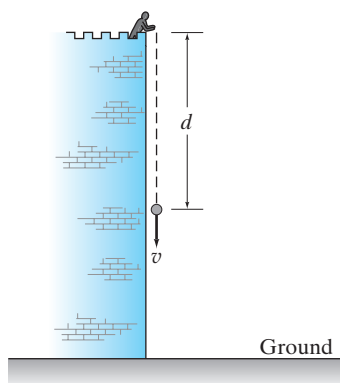


FIGURE 1.3.10 The distance  $d$  the rock has fallen is a function of time  $t$ .

- b. If a rock is dropped from atop a high tower (Fig. 1.3.10) and the acceleration of gravity is  $32 \text{ ft/sec}^2$ , then its velocity  $v$  after  $t$  seconds and the distance  $d$  it has fallen are given by, respectively,

$$v(t) = 32t \quad \text{and} \quad d(t) = 16t^2.$$

How fast is the rock moving after 2.5 seconds, and how far has it fallen?

- c. If the temperature of a 3-gram sample of carbon dioxide is  $27^\circ\text{C}$ , then its volume  $V$  in liters is given by

$$V(p) = \frac{168}{p},$$

where  $p$  is the pressure of the gas in atmospheres. What is the pressure of the gas if its volume is 14 liters?

### SOLUTION

- a. This question asks us to find the input if the output is 30 square feet. Since  $A(r)$  represents the output for an input of radius  $r$ , we need to solve the equation

$$30 = 3.1416r^2$$

for  $r$ . Thus,  $r^2 \approx \frac{30}{3.1416}$ , and  $r \approx \sqrt{\frac{30}{3.1416}} \approx 3.09$  feet (rounded to the nearest hundredth).

- b. This question asks us to find the outputs for both the velocity and distance functions if the input is 2.5 seconds. So we need to find  $v(2.5) = 32(2.5) = 80$  feet per second and  $d(2.5) = 16(2.5^2) = 100$  feet.
- c. Here we are again given output and asked for input. We need to solve the equation

$$14 = \frac{168}{p}.$$

Therefore,  $p = \frac{168}{14} = 12$  atmospheres.

- Of course, we could also use the calculator techniques from Example 2 to find these same input and output values.

## EXAMPLE 4

### Finding Domain and Range

Suppose the total cost  $C$  of manufacturing  $n$  copies of a regional cookbook is \$500 to set up the printing press plus \$6 for each book actually printed. Then  $C$  is given as a function of  $n$  by the formula

$$C(n) = 500 + 6n.$$

Find the domain and the range of this function.

## SOLUTION

- The domain of the cost function  $C$  is the set  $\{1, 2, 3, \dots\}$  of all *positive integers*, because it would be meaningless to speak of printing a negative or fractional number of books. The range of  $C$  is the set of all numbers described by the list  $\{506, 512, 518, \dots\}$  because each book adds an additional \$6 to the total cost of printing.

## EXAMPLE 5

## Finding Domain and Range

The *squaring function* defined by

$$f(x) = x^2$$

assigns to each number  $x$  its square  $x^2$ . What are the domain and the range of  $f$ ?

## SOLUTION

Because every number can be squared, the domain of  $f$  is the set—often denoted by  $\mathbf{R}$ —of all real numbers. Since the output of squaring a real number is never negative, the range of  $f$  is the set of all nonnegative numbers,  $[0, \infty)$ .

Figure 1.3.11 shows a calculator graph of  $y = x^2$ . While the calculator shows only that portion of the graph visible in a standard viewing window (with  $x$  and  $y$  lying between  $-10$  and  $10$ ), we can see that it confirms our belief that no negative numbers are in the range of  $f$ .

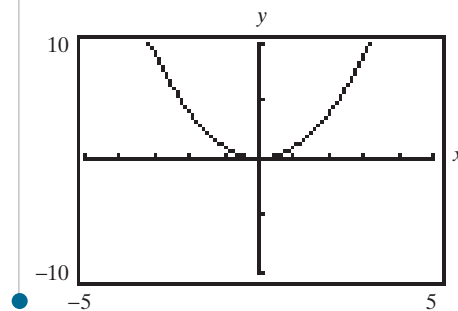


FIGURE 1.3.11 Calculator graph of  $f(x) = x^2$ .

Examples 4 and 5 illustrate the process of finding the range of a function. Frequently, however, we are more concerned with the domain of a function than with its range because it is crucial for us to know what numbers we can use for the input to get “sensible” values for the output. When we are given a function specified by a formula, we will assume that the domain of the function is the set of real numbers for which corresponding output values “make sense.”

When we describe the function  $f$  by writing a formula  $y = f(x)$ , we call  $x$  the *independent variable* and  $y$  the *dependent variable* because the value of  $y$  depends (through the rule or formula of  $f$ ) on the choice of  $x$ . As the independent variable  $x$  changes or varies, then so does the dependent variable  $y$ . For instance, as  $x$  changes from  $-2$  to  $0$  to  $3$ , the value  $y = x^2$  of Example 5 changes from  $4$  to  $0$  to  $9$ .

You may find it useful to visualize the dependence of the value  $y = f(x)$  on  $x$  by thinking of the function  $f$  as a kind of machine that accepts as *input* a number  $x$  and then produces as *output* the number  $f(x)$ , perhaps printed or displayed on a monitor (Fig. 1.3.12).

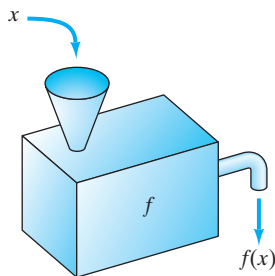


FIGURE 1.3.12 A “function machine” with input  $x$  and output  $f(x)$ .



One such machine is (in effect) the square root key of a simple pocket calculator. When a number  $x$  is entered and this key is pressed, the calculator displays (a decimal approximation to) the number  $\sqrt{x}$ . This square root function  $f(x) = \sqrt{x}$  has domain and range both equal to the set of all nonnegative numbers. A calculator like the TI-84 illustrates its knowledge of the domain by displaying an error message if we ask it to calculate the square root of a negative number.

## 1.3 Exercises Building Your Skills

*In Exercises 1–4 the given formula defines a function. Identify the independent variable and the dependent variable, and then rewrite each formula in function notation.*

1.  $R = 8\sqrt{s}$

2.  $T = 4u - 7$

3.  $C = 2\pi r$

4.  $V = e^3$

*In Exercises 5–10, explain why the formula does or does not represent  $y$  as a function of  $x$ .*

5. The record high temperature  $y$  in Denver for each month  $x$  of the year.

6. The total cost  $y$  of purchasing  $x$  hamburgers at your favorite fast-food restaurant.

7. The price  $y$  of a Christmas tree based on its height  $x$  if

- a 6-foot tree costs between \$20 and \$50;
- an 8-foot tree costs between \$25 and \$75;
- an 11-foot tree costs between \$100 and \$200;
- a 13-foot tree costs \$300 or more.

8.  $y = x^2$

9.  $x = y^2$

10.  $4x - 3y = 8$

*In Exercises 11–14, find and simplify each of the following values:  $f(-1)$ ,  $f(0.5)$ , and  $f(\sqrt{2})$*

11.  $f(x) = 2x + 3$

12.  $f(x) = x^2 + 1$

13.  $f(x) = \frac{1}{2x + 1}$

14.  $f(x) = \sqrt{x^2 + 2}$

*In Exercises 15–17 find all values of  $a$  such that  $g(a) = 13$ .*

15.  $g(x) = 3x + 4$

16.  $g(x) = \sqrt{5x + 4}$

17.  $g(x) = x^2 - 36$

In Exercises 18–20 find the domain and the range of the function  $h$ .

18.  $h(x) = -x + 3$

19.  $h(x) = \sqrt{2x - 3}$

20.  $h(x) = x^2 + 7$

## Applying Your Skills

21. The BMI for a person who weighs 150 pounds is given by

$$\text{BMI} = 703 \left( \frac{150}{h^2} \right),$$

where  $h$  is the person's height in inches.

- Explain why this formula represents BMI as a function of height in inches.
  - How tall (to the nearest inch) is a 150-pound person whose BMI is 24.2?
  - What is the BMI for a 150-pound person who is 72 inches tall?
22. On August 2, 2005, the website XE.com gave the rate for exchanging U.S. dollars and Euros as 1.00 United States dollar = 0.820150 Euro.
- Write a symbolic function  $E(d) = \dots$  giving the number of Euros received when  $d$  dollars are exchanged.
  - How many Euros would you get for \$475?
  - How many dollars did you exchange if you received 492.09 Euros?
  - What are the domain and the range of this function?
23. Straight line depreciation is a method for computing an asset's loss of value over time that assumes that the asset will lose an equal amount of value over each year of its useful life. If you buy a \$3000 computer today for your home business and assume that it will be worth only \$200 at the end of its useful life 5 years from now, it loses \$560 of value each year.
- Write a symbolic function  $V(n) = \dots$  giving the value of the computer as a function of its age in years.
  - How much is the computer worth after 3 years?
  - How old is the computer if it is worth \$1880?
  - What are the domain and the range of  $V$ ?
24. According to the U.S. Postal Service, "First-Class Mail is used for personal and business correspondence. Any mailable item may be sent as First-Class Mail. It includes postcards, letters, large envelopes, and small packages. The maximum weight is 13 ounces." The Postal Service gives its First-Class Mail rates as follows:

<i>First ounce</i>	\$0.37
<i>Each additional ounce</i>	\$0.23

By "each additional ounce," the Postal Service means each additional ounce or fractional part of an ounce, if the letter does not weigh a whole number of ounces. Thus a letter weighing 4.5 ounces costs the same amount as a letter weighing 5 ounces.

- a. Explain why the Postal Service rule describes the postage rate as a function of weight.
  - b. What is the cost to send a letter weighing 6.9 ounces? 7 ounces? 7.1 ounces?
  - c. If a letter costs \$1.06, how much does it weigh?
  - d. Write a multipart rule for the First-Class Mail rate function  $R(w)$ , giving the rate in dollars and cents as a function of the weight in ounces.
  - e. What are the domain and the range of the function  $R$ ?
25. Use a newspaper, a magazine, or the Internet to find
- a. an example of a verbal or symbolic formula that represents a function;
  - b. an example of a verbal or symbolic formula that does not represent a function.

## 1.4 AVERAGE RATE OF CHANGE

According to a July 2005 College Board report, “There’s no escaping the fact that college costs are rising.” This fact probably comes as no surprise to any college student paying for tuition, housing, and books. The following table shows the average cost of tuition at a public four-year college (for an academic year beginning in the fall of each indicated year):

YEAR	TUITION	YEAR	TUITION
1993	\$2535	1999	\$3362
1994	\$2705	2000	\$3487
1995	\$2811	2001	\$3725
1996	\$2975	2002	\$4115
1997	\$3111	2003	\$4694
1998	\$3247		

Based on the table, the cost of tuition is a function of the year. As we examine the table, we see that as the year increases, so does the tuition. We therefore say that the tuition cost is an *increasing* function of the year.

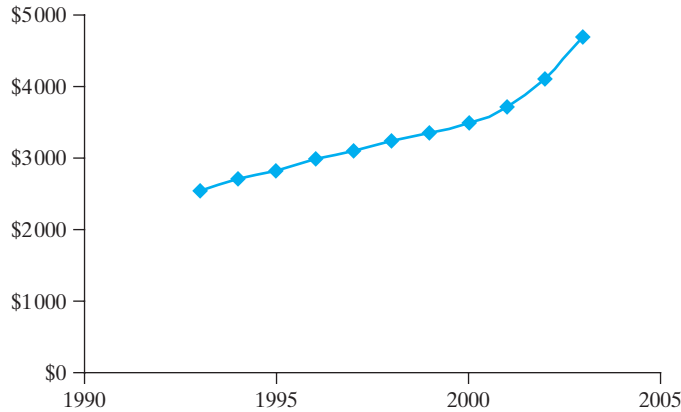
In general, a function  $y = f(x)$  is increasing over an interval of  $x$ -values if, for any two different values  $x_1$  and  $x_2$  in the interval, if  $x_1 < x_2$ , then  $f(x_1) < f(x_2)$ . That is, as the  $x$ -values increase, so do the  $y$ -values.

What does this mean in terms of the graph of the function? If we consider the graph of the increasing tuition cost function shown in Fig. 1.4.1, we see that, as we look from left to right, the graph “goes uphill.” This provides visual confirmation of the fact that as the  $x$ -values increase, so do the  $y$ -values.

Similarly, a function  $y = f(x)$  is *decreasing* over an interval of  $x$ -values if, for any two different values  $x_1$  and  $x_2$  in the interval, if  $x_1 < x_2$ , then  $f(x_1) > f(x_2)$ . That is, as the  $x$ -values increase, the  $y$ -values *decrease*. The graph of a decreasing function “goes downhill” as we look from left to right.

Finally, a function  $y = f(x)$  is constant over an interval of  $x$ -values if, for any two different values  $x_1$  and  $x_2$  in the interval,  $f(x_1) = f(x_2)$ . That is, no matter what the

FIGURE 1.4.1 Graph of an increasing cost function.



$x$ -value is, the  $y$ -value remains the same. Thus, if  $f(x) = c$  for every  $x$ , then the graph of the function is the line  $y = c$ . So the graph of a constant function is a horizontal line.

### EXAMPLE 1

#### Increasing, Decreasing, or Constant?

Recall the cell phone cost function

$$C(n) = \begin{cases} 39.99 & \text{if } 0 \leq n \leq 600, \\ 39.99 + (n - 600)(0.40) & \text{if } n > 600, \end{cases}$$

where  $n$  represents the number of weekday minutes used and  $C(n)$  is cost in dollars and cents. Use the graph of this function, shown in Fig. 1.4.2, to determine the intervals for which  $C$  is increasing, decreasing, or constant.

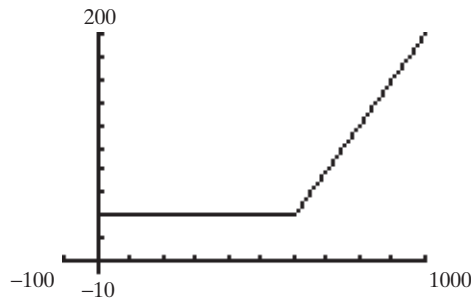


FIGURE 1.4.2 Graph of the “Get More” Plan function.

#### SOLUTION

For all  $n$  between 0 and 600, inclusive,  $C(n)$  is \$39.99, so  $C$  is constant on the interval  $[0, 600]$ . Once  $n$  reaches 600, as  $n$  increases, so does  $C(n)$ , so  $C$  is increasing for “sensible” input values greater than or equal to 600. No matter what month we are in, there are no more than 31 days and there are always at least 8 weekend days. Then the largest number of anytime minutes possible is  $\frac{60 \text{ minutes}}{\text{hour}} \times \frac{24 \text{ hours}}{\text{day}} \times 23 \text{ days} = 33,120$  minutes per month. So  $C$  is increasing on the interval  $[600, 33120]$ . (Most people would probably not argue if you used the interval  $[600, \infty)$  to indicate “sensible” values of  $n$  greater than or equal to 600.)

- Notice that when we report intervals where the function is increasing, decreasing, or constant, we use intervals of *input* values.

## EXAMPLE 2

## Increasing, Decreasing, or Constant?

The accompanying table gives the percentage of the U.S. population that was born outside of the United States as a function of the year. Use the table to determine the intervals where the function is increasing, decreasing, or constant.

YEAR $x$	PERCENTAGE OF POPULATION BORN OUTSIDE U.S. $f(x)$
1940	8.8
1950	6.9
1960	5.4
1970	4.7
1980	6.2
1990	8.0
2000	10.4

Source: *The World Almanac and Book of Facts 2005*.

## SOLUTION

- $f(x)$  is decreasing on  $[1940, 1970]$  and increasing on  $[1970, 2000]$ .

When we discuss how function values change when input values change, we frequently look at the function's average rate of change. For a function  $y = f(x)$ , we define the **average rate of change** of the function over the interval  $[a, b]$  to be

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

Thus the average rate of change is the change in  $y$  divided by the change in  $x$ . For instance, for the function  $f(x)$  in Example 3, the average rate of change over the interval  $[1940, 1950]$  is

$$\frac{\Delta y}{\Delta x} = \frac{f(1950) - f(1940)}{1950 - 1940} = \frac{6.9 - 8.8}{10} = \frac{-1.9}{10} = -0.19.$$

## EXAMPLE 3

## Average Rate of Change

Complete the accompanying table to find the average rate of change of  $f(x)$  over each interval of consecutive  $x$ -values. Note that we are recording the average rate of change between two consecutive data points in the same line of the table as the first of the data points. (The last line of the average rate of change column is blank because average rate of change requires two points.)

YEAR $x$	PERCENTAGE OF POPULATION BORN OUTSIDE U.S. $f(x)$	AVERAGE RATE OF CHANGE $\Delta y/\Delta x$
1940	8.8	-0.19
1950	6.9	
1960	5.4	
1970	4.7	
1980	6.2	
1990	8.0	
2000	10.4	—

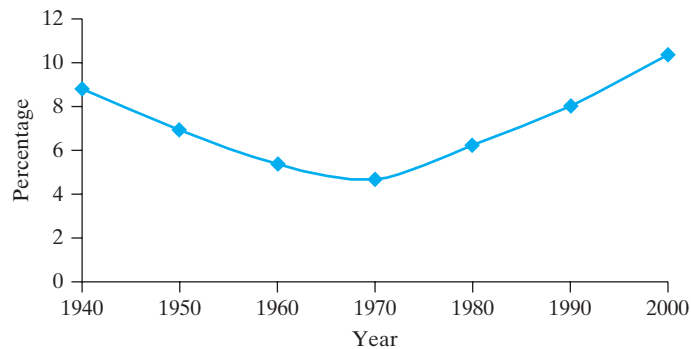
**SOLUTION**

YEAR	PERCENTAGE OF POPULATION BORN OUTSIDE U.S. $f(x)$	AVERAGE RATE OF CHANGE $\Delta y/\Delta x$
1940	8.8	-0.19
1950	6.9	-0.15
1960	5.4	-0.07
1970	4.7	0.15
1980	6.2	0.18
1990	8.0	0.24
2000	10.4	—

Recall that we reported that  $f(x)$  is decreasing on  $[1940, 1970]$  and increasing on  $[1970, 2000]$ . Now we notice that the average rate of change is increasing over the entire interval. So we can say that  $f(x)$  is decreasing at an increasing rate on  $[1940, 1970]$  because the function's output values are getting smaller but average rate of change is getting larger. Similarly,  $f(x)$  is increasing at an increasing rate on  $[1970, 2000]$  because both the function's output values and its average rate of change are getting larger.

If we consider the graph of this function in Fig. 1.4.3, we can see the intervals of increase and decrease. We also observe that the function bends upward like a bowl sitting right side up. This is a property associated with a function with an increasing average rate of change. We say that the graph of the function is *concave upward*.

**FIGURE 1.4.3** Percentage of U.S. population born outside the United States.



**EXAMPLE 4**

## Describing Average Rate of Change

In Section 1.2, we looked at the graph of a woman's percentage chance of conception each month as a function of her age. The following table shows the function values  $f(x)$  for various ages.

- Complete the table to find the average rate of change of  $f(x)$  over each interval of consecutive  $x$ -values.
- Describe the increasing and decreasing behavior of both the function and its average rate of change.

AGE $x$	ODDS OF CONCEPTION $f(x)$	AVERAGE RATE OF CHANGE $\Delta y/\Delta x$
15	22	0.4
20	24	
25	22	
30	16	
35	8	
40	3	
45	0.01	—

**SOLUTION**

- Fortunately, we can use the calculator to complete a table such as this. Store the  $x$ -values in **L1** and the function values in **L2**. Then highlighting **L3**, select **2<sup>nd</sup>**, **STAT**, **OPS**, and choose  **$\Delta$ List**(. Type **2<sup>nd</sup>** **2** for **L2**, then) and  **$\div$** . Again select **2<sup>nd</sup>**, **STAT**, **OPS**, and choose  **$\Delta$ List**(. Now type **2<sup>nd</sup>** **1** for **L1**, then) and **ENTER**. Figure 1.4.4 shows the resulting table.

L1	L2	L3	3
15.000	22.000	.4000	
20.000	24.000	-.4000	
25.000	22.000	-1.200	
30.000	16.000	-1.600	
35.000	8.0000	-1.000	
40.000	3.0000	-.5980	
45.000	.0100	-----	

L3(1) = .4

FIGURE 1.4.4 Finding the average rate of change.

- $f(x)$  is increasing on  $[15, 20]$  and decreasing on  $[20, 45]$ . The average rate of change is decreasing on  $[15, 35]$  and increasing on  $[35, 45]$ .

**EXAMPLE 5**

## Describing Average Rate of Change

The function  $f(x) = \sqrt{x}$  is increasing on its domain  $[0, \infty)$ . Complete the following table to describe its average rate of change on  $[0, 36]$ .



$x$	$f(x)$	AVERAGE RATE OF CHANGE $\Delta y/\Delta x$
0		
1		
4		
9		
16		
25		
36		

**SOLUTION**

The calculator table shown in Fig. 1.4.5 shows that the average rate of change of  $f(x)$  is decreasing on  $[0, 36]$ .

L1	L2	L3	3
0.0000	0.0000	0.0000	
1.0000	1.0000	.3333	
4.0000	2.0000	.2000	
9.0000	3.0000	.1429	
16.0000	4.0000	.1111	
25.0000	5.0000	.0909	
36.0000	6.0000	-----	
L3(1)=1			

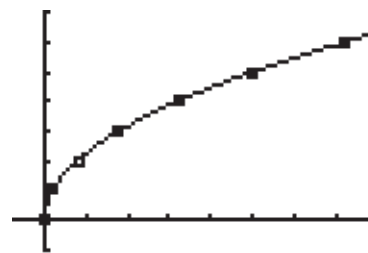


FIGURE 1.4.5 Finding the average rate of change for  $f(x) = \sqrt{x}$ .

FIGURE 1.4.6 Graph of  $f(x) = \sqrt{x}$ .

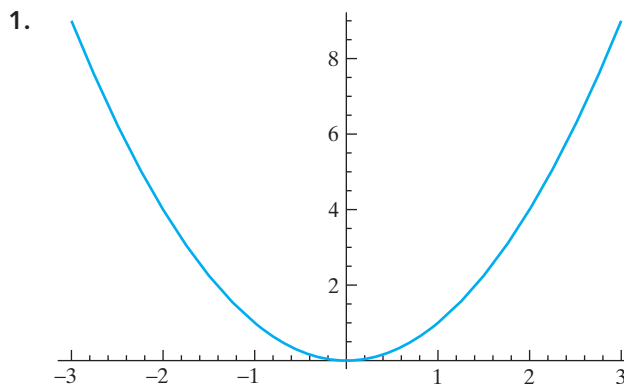
If we examine the graph of the function in Fig. 1.4.6, we see that the graph goes uphill from left to right because the function is increasing. It also curves downward because the average rate of change is decreasing. We say that this graph is *concave downward*.

As we continue through this book we will see the important role that the average rate of change plays in describing a function.

## 1.4 Exercises

### Building Your Skills

In Problems 1–4, determine the intervals over which the indicated function is increasing, decreasing, or constant.

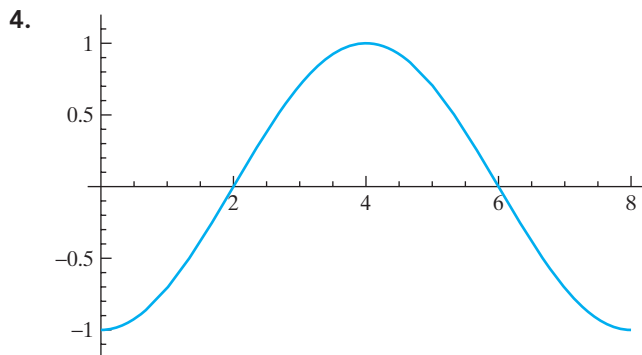


2.

$x$	-6	-3	0	2	5
$f(x)$	-10	-8	1	3	8

3.

$x$	0	2	4	6	8	10
$f(x)$	6	7	8	8	8	7



In Exercises 5–8, find the average rate of change for the indicated function, and determine where the average rate of change is increasing, decreasing, or constant.

5.

$x$	-6	-3	0	2	5
$f(x)$	-128	-62	4	48	114

6.

$x$	0	5	10	15	20
$f(x)$	25	30	34	33	32

7.

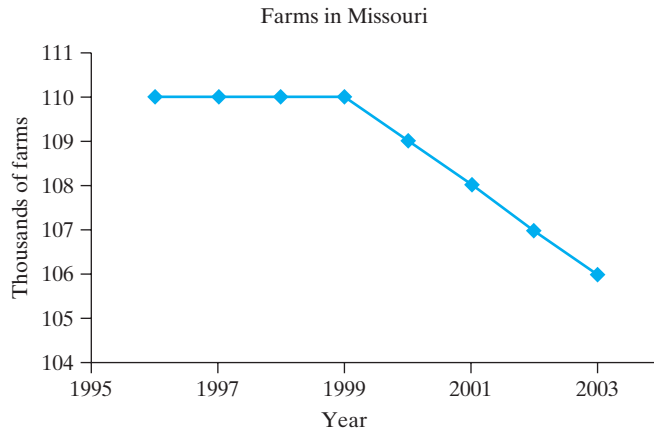
$x$	1	3	5	7	9
$f(x)$	18	6	-3	-9	-12

8.

$x$	-1	0	4	11	23
$f(x)$	18	15	9	0	-12

## Applying Your Skills

9. The accompanying graph shows the number of farms (in thousands) in Missouri as a function of the years 1996 through 2003. Determine the intervals of years for which this function is increasing, decreasing, or constant.



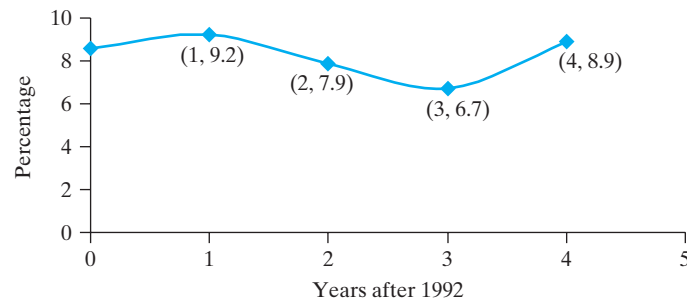
Source: Missouri Agricultural Statistics Service.

10. The following table gives the median age at first marriage for U.S. women as a function of the year. Use the table to determine the years over which the median age is increasing, decreasing, or constant.

Year	1993	1995	1997	1999	2001	2003
Number	24.5	24.5	25.0	25.1	25.1	25.3

Source: World Almanac and Book of Facts 2005.

11. The following graph shows the percentage  $y$  of all music sold that was rap music  $x$  years after 1992. The  $y$ -intercept of 8.6 indicates that 8.6% of all music sold in 1992 was rap music.



Source: World Almanac and Book of Facts 1998.

- Determine the intervals of  $x$ -values for which the function is increasing.
  - Determine the intervals of  $x$ -values for which the function is decreasing.
  - Determine the intervals of  $x$ -values for which the average rate of change of the function is increasing.
  - Determine the intervals of  $x$ -values for which the average rate of change of the function is decreasing.
12. The accompanying table gives the number (in thousands) of new cases of diagnosed diabetes among adults aged 18–79 years in the United States as a function

of the year. Find the average rate of change for this function and use your answer to determine the intervals of years for which,

- the number of new cases of diabetes is increasing at a decreasing rate;
- the number of new cases of diabetes is increasing at an increasing rate.

Year	1997	1998	1999	2000	2001	2002	2003	2004
Number	878	921	979	1104	1213	1304	1349	1356

Source: www.cdc.gov.

## Chapter 1 | Review

In this chapter, you learned about functions and function models. After completing the chapter, you should be able to

- determine whether a relation described numerically, graphically, or symbolically represents a function;
- find the domain and range of a function;
- find the output value of a function for a given input value;
- find the input value(s) of a function for a given output value;
- determine the intervals over which a function is increasing, decreasing, or constant;
- determine the average rate of change of a function over an interval.

## Review Exercises

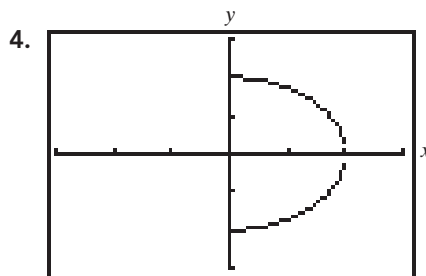
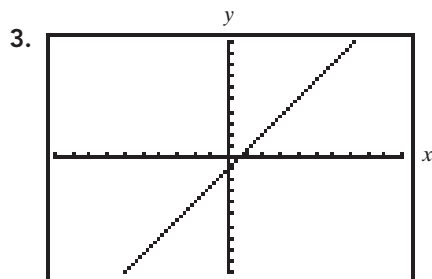
In Exercises 1–6, decide whether, based on the table, graph, or formula,  $y$  is a function of  $x$ .

1.

$x$	2	4	6	6	8	11
$y$	3	4	9	13	15	23

2.

$x$	2	4	5	6	8	11
$y$	3	4	9	9	15	23



5.  $y = x^2 - 4$

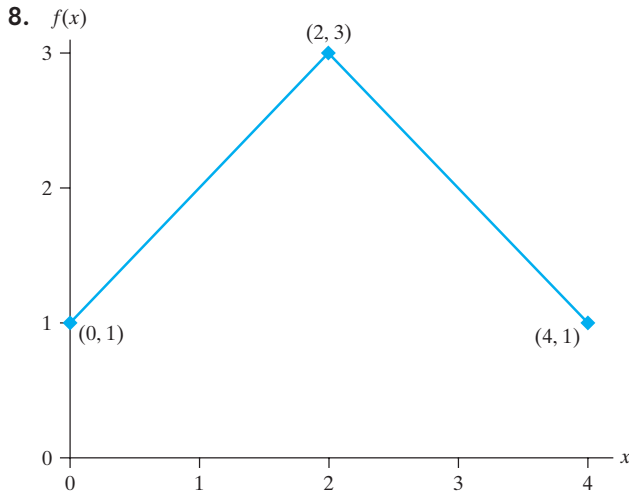
6.  $x = |y|$

In Exercises 7–9, find the domain and range of each function.

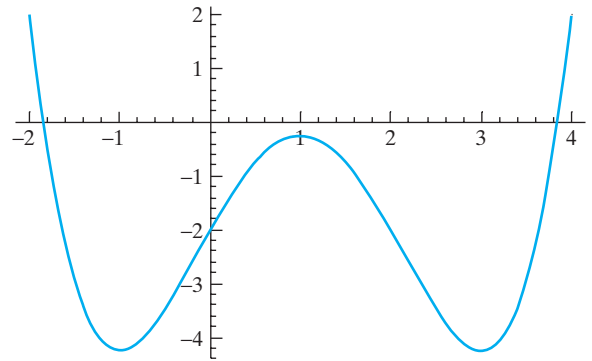
7. The number of fat grams  $F(x)$  in a sandwich based on the number of calories  $x$ , according to the following table.

Calories	430	550	320	680	300	750	530	710
Fat Grams	20	29	5	33	9	33	26	43

Source: Chick-Fil-A Fat Gram Comparison.

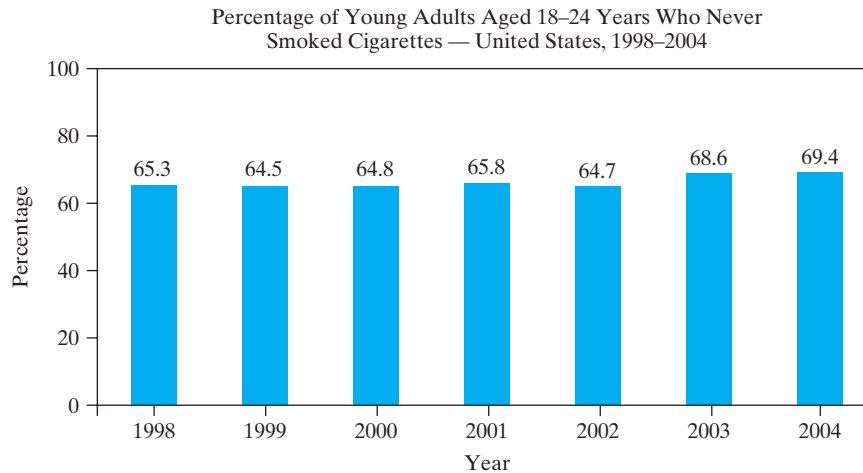


10. The function graphed as follows is defined on the interval  $[-2, 4]$ . Determine the intervals over which the function is increasing or decreasing.



9.  $f(x) = |x| - 3$

11. Often in the media, functions are displayed using bar graphs rather than scatter plots, as in the following graph.

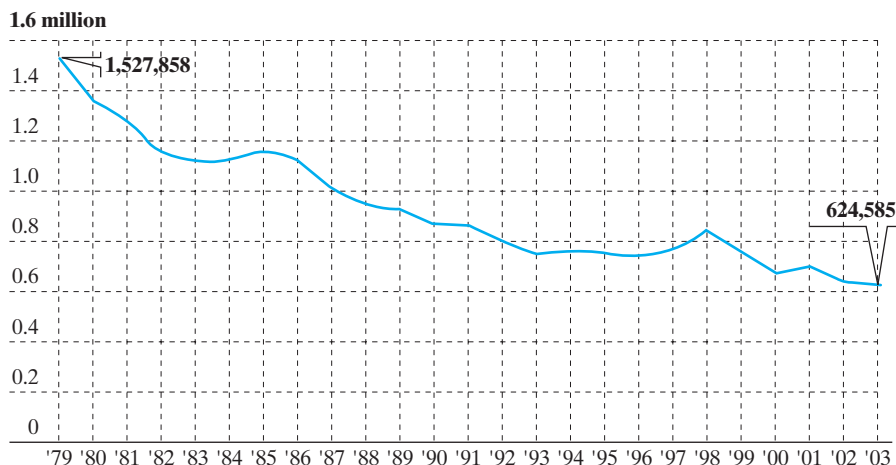


Source: [www.cdc.gov/mmwr](http://www.cdc.gov/mmwr).

- a. Make a table of values for this function, letting  $x$  be years after 1998 and  $P(x)$  the percentage of young adults who never smoked cigarettes.
- b. Use your calculator to make a scatter plot of this function, using  $x$ -window  $[0, 7]$  and  $y$ -window  $[0, 100]$ .
12. For many years, insurance companies published guidelines of ideal weights  $W$  for various heights  $h$ . The most common rule for calculating ideal weight for a woman was 100 pounds for a woman 60 inches tall plus 5 additional pounds for each inch over 60 inches.
- a. Write a function rule  $W(h)$  to determine the ideal weight for a woman who is at least 60 inches tall.
- b. Use your rule to determine the ideal weight for a woman who is 66 inches tall.
13. The monthly cost function for a “light” plan for dial-up Internet access is given by
- $$C(h) = \begin{cases} 6.95 & \text{if } 0 \leq x \leq 5, \\ 6.95 + 2(h - 5) & \text{if } x > 5, \end{cases}$$
- where  $C$  is in dollars and  $h$  is the number of hours used. If a subscriber’s bill was \$23.65, how many hours of Internet time did she use that month?

14. The text on the graph showing UAW membership as a function of year states that there has been a steady decline in membership from 1979 to 2003. However, if you look carefully at the graph, you will see that

there are brief intervals where the function appears to be increasing or constant. Identify the intervals of years over which this function is increasing or constant.



Source: *The Detroit News*.

15. (Chapter Opener Revisited) In the discussion that introduced this chapter, we looked at the graph of the heat index at a fixed air temperature of 85°F as a function of the relative humidity.

- Use the given table of values for that function to determine the average rate of change over consecutive input values.
- Use your answer from part (a) to determine the intervals of input values where the average rate of change is increasing, decreasing, or constant.

- In our discussion, we noted that for relative humidities of 0% to 60%, the graph appeared to be a straight line. Over those input values, was the average rate of change increasing, decreasing, or constant?
- In our discussion, we noted that for relative humidities of 60% to 100%, the graph appeared to curve upward. Over those input values, was the average rate of change increasing, decreasing, or constant?

Relative Humidity (%)	0	10	20	30	40	50	60	70	80	90	100
Apparent Temperature (°F)	78	80	82	84	86	88	90	93	97	102	108

## INVESTIGATION Exploring Rate of Change

In this chapter, you learned about the behavior of functions and their average rates of change. This activity will allow you to investigate these concepts numerically, graphically, and verbally.

To raise money for their sorority’s service project, Kari and Keisha decided to sell cookies outside the student center. After the first week of sales, they examined

their sales record and made a chart that includes the number of cookies sold per day along with the change from the previous day. Complete a chart like the one shown for each of the situations described. Then make a plot that shows the day on the  $x$ -axis and the number of cookies sold on the  $y$ -axis. Describe in words how each graph looks.

DAY	COOKIES SOLD	AVERAGE RATE OF CHANGE (cookies per day)
1	40	N/A
2		
3		
4		
5		
6		
7		

1. Cookie sales remained constant.
2. Cookie sales increased at a constant rate.
3. Cookie sales increased at an increasing rate.
4. Cookie sales increased at a decreasing rate.
5. Cookie sales decreased at a constant rate.
6. Cookie sales decreased at an increasing rate.
7. Cookie sales decreased at a decreasing rate.



# LINEAR FUNCTIONS AND MODELS

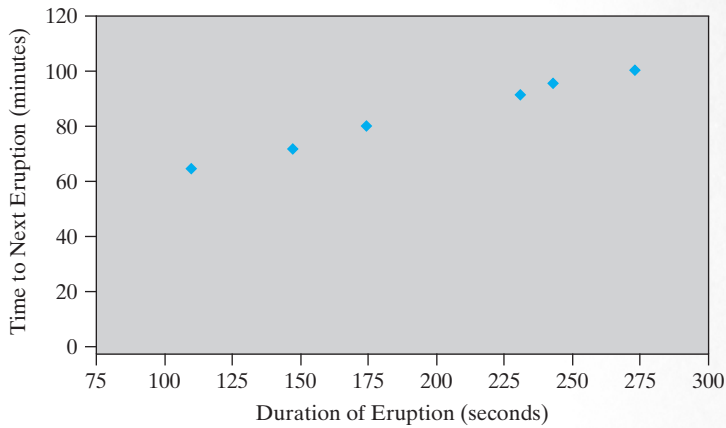
- 2.1 CONSTANT CHANGE AND LINEAR GROWTH
- 2.2 LINEAR FUNCTIONS AND GRAPHS
- 2.3 PIECEWISE-LINEAR FUNCTIONS
- 2.4 FITTING LINEAR MODELS TO DATA

**O**n March 1, 1872, President Ulysses S. Grant signed a law declaring that approximately 2 million acres of land near the headwaters of the Yellowstone River would be “dedicated and set apart as a public park or pleasuring ground for the benefit and enjoyment of the people.” This act created the world’s first national park, Yellowstone National Park.

Seventy-five percent of all the geysers on Earth are in Yellowstone National Park, and the largest concentration of geysers in the world occurs in the Upper Geyser Basin of the park. This is the area that contains the most famous of the geysers, Old Faithful, which was named for its consistent performance by explorers in 1870. According to the National Park Service, “Old Faithful erupts more frequently than any of the other big geysers, although it is not the largest or most regular geyser in the park. Its average interval between eruptions is about 91 minutes, varying from 65–92 minutes.” An eruption lasts from 1.5 to 5 minutes, during which between 3700 and 8400 gallons of boiling water are expelled, with the height of the eruption between 106 and 184 feet.

Visitors to Yellowstone are interested in seeing Old Faithful erupt, and many people mistakenly believe that it erupts every hour. The problem is, as the Park Service indicates, that the geyser doesn’t erupt on a fixed schedule, although there do seem to be patterns in its eruptions. Figure 2.0.1 shows a plot of data from selected Old Faithful eruptions in January 2003 as recorded by park rangers and volunteers in the Old Faithful Visitor Center logbook and transcribed for the Internet by Lynn Stephens.

In this graph, the input variable is the duration of an eruption and the output variable is the time to the next eruption. We can make several observations about this graph. First, the graph represents a function since for each input value, there is



**FIGURE 2.0.1** Old Faithful eruptions.  
 Source: [www.geyser-study.org](http://www.geyser-study.org).

only one output value. Second, this function is increasing; that is, as the duration of the eruption increases, so does the time to the next eruption. Finally, the points in this scatter plot lie approximately on a (straight) line.

Since the time to the next eruption is what visitors want to know, it would be useful to determine a function rule that would allow us to make such a prediction. In this chapter, we study functions whose graphs are lines and learn how to determine rules for such functions. These rules will allow us to predict output values for given input values—exactly what we would want to do for the Old Faithful geyser.

## 2.1 CONSTANT CHANGE AND LINEAR GROWTH

Temperature is commonly measured in the United States in Fahrenheit degrees, but in much of the rest of the world it is measured in Celsius degrees. The following table illustrates the relation between the Celsius temperature  $C$  and the Fahrenheit temperature  $F$ . (This relation is a function because each Celsius temperature corresponds to exactly one Fahrenheit temperature.)

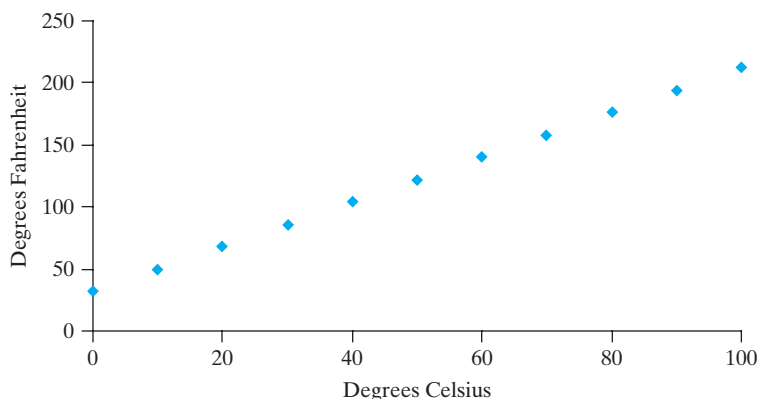
C (DEGREES)	F (DEGREES)
0	32
10	50
20	68
30	86
40	104
50	122
60	140
70	158
80	176
90	194
100	212

Here, successive Celsius temperatures appear in the first column of the table at equal intervals of 10 degrees. Note also that successive Fahrenheit temperatures appear in the second column at equal intervals of 18 degrees. The fact that equal differences in one column correspond to equal differences in the other indicates a special kind of relationship between corresponding Celsius and Fahrenheit temperatures.

Using the techniques from Section 1.4, we see that the average rate of change  $\frac{\Delta F}{\Delta C}$  is *constant*; that is,  $\frac{\Delta F}{\Delta C} = \frac{18}{10} = 1.8$ . This means that for each 10 degrees that the Celsius temperature increases, the Fahrenheit temperature increases by 18 degrees, or alternatively, for each degree that the Celsius temperature increases, the Fahrenheit temperature increases by 1.8 degrees.

Because the average rate of change is constant, there is a special kind of relationship between corresponding Celsius and Fahrenheit temperatures. The scatter plot in Fig. 2.1.1 shows this relationship. We see from this plot that these points all appear to lie on the same (straight) line. So what is special about the data in our original table of temperatures is this: It is described by a function whose graph is a straight line. That is, because the average rate of change of this function is constant, the graph of the function is a line. We call a function whose graph is a line a *linear* function.

FIGURE 2.1.1 Scatter plot of degrees F versus degrees C.



You can verify that if  $C$  and  $F$  are corresponding entries in the table, then if we write  $F$  as a function of  $C$ , we have

$$F(C) = 1.8C + 32. \quad (1)$$

One way to check this is to substitute each value  $C = 0, 10, 20, \dots, 100$  into (1) and verify in each case that the corresponding value of  $F$  shown in the table results.

Another way is to rewrite (1) as

$$y = 1.8x + 32 \quad (2)$$

with  $x$  in place of  $C$  and  $y$  in place of  $F(C)$ . With **Y1 = 1.8X + 32** and using the **TBLSET** menu with **TblStart = 0**, and **ΔTbl = 10**, a calculator yields the table shown in Fig. 2.1.2, the same columns of numerical values as in the Celsius–Fahrenheit table.

If we graph the function in (2) in the calculator window defined by

$$\begin{aligned} X_{\min} &= 0, X_{\max} = 100, X_{\text{scl}} = 10, \\ Y_{\min} &= 0, Y_{\max} = 300, Y_{\text{scl}} = 30, \end{aligned}$$

we get the picture shown in Fig. 2.1.3. Here we see again that the graph of this function is a line. Furthermore, we see **all** points on the graph (in this particular window), not merely the pairs given in the original table.

X	Y1	
0	32	
10	50	
20	68	
30	86	
40	104	
50	122	
60	140	

X=0

FIGURE 2.1.2  $Y1=1.8X+32$  table with  $TblStart=0$  and  $\Delta Tbl=10$ .

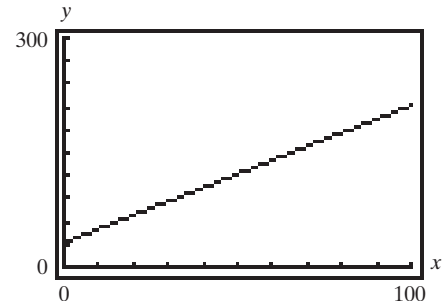


FIGURE 2.1.3 The graph  $y = 1.8x + 32$ .

#### DEFINITION: Linear Function

A **linear function** is a function of the form

$$f(x) = ax + b. \quad (3)$$

Note that the right-hand side in (3) contains both a constant term and an  $x$ -term (but no higher powers). A particular linear function  $f$  is determined when the values of the constant *coefficients*  $a$  and  $b$  are specified.

Since  $f(0) = a \cdot 0 + b = b$ , the constant term  $b$  is the output when the input is 0. Thus, the graph of the linear function  $y = f(x)$  intersects the  $y$ -axis at the point  $(0, b)$ . That is, **the constant  $b$  is the  $y$ -intercept of the line** (Fig. 2.1.4) and therefore measures its vertical location.

**The constant coefficient  $a$  is the average rate of change of the linear function** and measures how the output value of the function changes for each one-unit change in the input value.

The roles of  $a$  and  $b$  in a linear function make it easy for us to determine a function's symbolic rule if we are given its average rate of change and the output for an input value of 0.

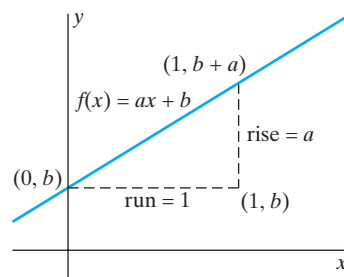


FIGURE 2.1.4 The geometric roles of  $a$  and  $b$ .

Suppose the following table records the growth in the population of a certain hypothetical city during the 1990s:

YEAR	POPULATION (THOUSANDS)
1990	110
1991	116
1992	122
1993	128
1994	134
1995	140
1996	146
1997	152
1998	158
1999	164
2000	170

We see that each number in the second column is found by adding 6 to the previous one. Thus the population of this city grew each year during the 1990s by the same amount of 6 thousand people. Since its average rate of change is constant, the population of the city is a **linear** function of time  $t$ .

In order to find a symbolic rule for this function, let's take  $t = 0$  in 1990. (We will call this technique “resetting the clock,” and will commonly use it when our input variable is calendar years.) The result of starting with a population of  $P(0) = 110$  thousand people and adding 6 thousand people  $t$  times in succession—once each year—can be described by saying that  $t$  years after 1990 the population is given by

$$P(t) = 110 + 6t \quad \text{or} \quad P(t) = 6t + 110 \quad (\text{thousand}) \quad (4)$$

Note that the constant term in (4) is the city's *initial population* and the coefficient of  $t$  is its annual change in population.

More generally, suppose a city has an initial population of

$$P(0) = P_0$$

at time  $t = 0$  and that its constant annual change in population is  $a$ . Then after  $t$  years the population has increased by

$$\Delta P = a \cdot t \quad (5)$$

(using the  $\Delta$ -notation for differences). Therefore, the city's population after  $t$  years is

$$P = \text{initial population} + \text{change} = P_0 + \Delta P = P_0 + at.$$

Thus a population  $P(t)$  with **initial population**  $P_0$  and (constant) **annual change**  $a$  is described by the linear function

$$P(t) = P_0 + at. \quad (6)$$

Not all populations are described by linear functions, but a population model like the one in (6)—with a constant term and a  $t$ -term—is called a *linear population*

*model.* In this section we discuss both linear population models and other linear models for which we know a function's initial value. In Sections 2.2 and 2.3 we discuss linear models for a wider variety of situations.

### EXAMPLE 1

#### Finding a Linear Population Model

On January 1, 1999, the population of Ajax City was 67,255 and had increased by 2935 people during the preceding year. Suppose this rate of increase continues, with 2935 more people added to the population of Ajax City each subsequent year. Find each of the following:

- A linear function  $P(t)$  modeling the population of Ajax City.
- The predicted population of Ajax City on October 1, 2002, based on the model.
- The month and the calendar year in which the population of Ajax City reaches 100 thousand.

#### SOLUTION

- In a given situation, we must decide when to start the clock, and we can count population however we wish—either by the person or by the thousand (for instance). If we reset the clock by letting  $t = 0$  on 1/1/1999 and measure population in thousands, it makes it easier to set an appropriate calculator window. Then  $P(0) = 67.255$  is our initial population and 2.935 is the function's constant average rate of change. Therefore, the linear function model for this population is

$$P(t) = 2.935t + 67.255, \text{ for } P \text{ in thousands and } t = 0 \text{ on } 1/1/1999. \quad (7)$$

- This is once again our standard question “Here's the input, what's the output?” Since we reset the clock, October 1, 2002, is 3 years and 9 months—that is, 3.75 years—after January 1, 1999. So we substitute  $t = 3.75$  in (7) and calculate

$$P(3.75) = 67.255 + 2.935 \times 3.75 = 78.261$$

either by hand or by entering our population function into our calculator as **Y1**, and using the **value** command from the **CALC** menu. Thus our linear model predicts an Ajax city population of 78.261 thousand, or 78,261 people on October 1, 2002.

- In this case, we need to answer the “backward” question “Here's the output, what's the input?” We can solve this algebraically by setting  $P(t) = 100$  and solving for  $t$ :

$$67.255 + 2.935t = 100, \quad (8)$$

so

$$2.935t = 100 - 67.255 = 32.745$$

and

$$t = \frac{32.745}{2.935} = 11.157 \text{ years.}$$



Starting with January 1, 1999, when  $t = 0$ , exactly 11 years later is January 1 of the year  $1999 + 11 = 2010$ . So the population of Ajax City hits 100 thousand

$$0.157 \text{ years} = 0.157 \text{ years} \times \frac{12 \text{ months}}{\text{year}} = 1.88 \text{ months}$$

into the year 2010. Note that 1.88 months means that we have completed all of January and are somewhere toward the end of February. So the population of Ajax City is 100 thousand during February 2010.

Note that for the purpose of “calendar month” problems like this, we will consider the year to be divided into 12 equal months. This is slightly inaccurate, but it is standard practice to take the differing lengths of individual months into account only when a question asks for a specific day of a particular month.

In Example 1, we solved part (c) algebraically. You may prefer to find the answer to such questions either graphically or numerically. Examples 2 and 3 illustrate these methods.

## EXAMPLE 2

### Finding an Input Graphically

Use graphical methods to find the year and month in which the population of Ajax City is 100 thousand.

#### SOLUTION

In order to determine the year and month in which the population of Ajax City is 100 thousand, we need to determine when the output  $P(t)$  has the value 100. Since both  $P(t)$  and 100 represent outputs, we enter them into the **Y=** menu of the calculator, plot their graphs, and see where the two graphs intersect. Because a graphing calculator requires that  $x$  and  $y$  (rather than  $t$  and  $P$ ) be used as the independent and dependent variables, Fig. 2.1.5 shows the **Y=** menu with the appropriate entries

$$y = 67.255 + 2.935x \quad \text{and} \quad y = 100.$$

Figure 2.1.6 shows the resulting plot with viewing window

$$\begin{array}{ll} \mathbf{Xmin} = 0 & \mathbf{Ymin} = 0 \\ \mathbf{Xmax} = 20 & \mathbf{Ymax} = 150 \end{array}$$

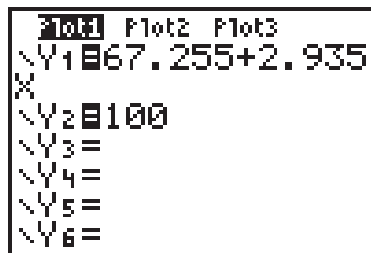


FIGURE 2.1.5 The **Y=** menu for the graphical solution of Example 2.

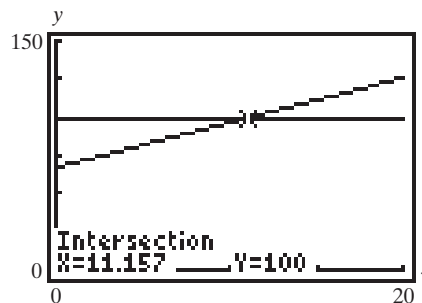


FIGURE 2.1.6 The graphs  $Y1 = 67.255 + 2.935X$  and  $Y2 = 100$  intersect at  $(11.157, 100)$ .

We have used the calculator's **CALC intersect** facility to locate the point of intersection.

- The indicated result (11.157, 100) agrees with our previous algebraic solution  $t = 11.157$  of Equation (8).

### EXAMPLE 3

#### Finding an Input Numerically

Use numerical methods to find the year and month in which the population of Ajax City is 100 thousand.

#### SOLUTION

We now proceed to solve Example 1(c) using the calculator's table-making facility. First we must enter the function we wish to tabulate. This is done in the same way as when we wish to graph a function—using the **Y=** menu. (In this case our function is already stored in **Y1** from Example 2; we can clear **Y2** since we no longer need it.)

To prepare to tabulate values of the function we've defined, we must specify the sort of table we want using the calculator's "table set" menu. We want to start our table at  $x = 0$  and proceed by yearly increments of  $\Delta x = 1$ . Thus our table's *starting point* and its *increment* between successive entries are specified by entering these two values as shown in Fig. 2.1.7.

TABLE SETUP
TblStart=0
ΔTbl=1
Indent: Auto Ask
Depend: Auto Ask

FIGURE 2.1.7 Initial table setting for the Ajax City population function.

Now we're ready to go! When we execute the calculator's **TABLE** command, we get the table shown in Fig. 2.1.8. This table doesn't go quite far enough to reach a population of 100 thousand, so we use the calculator's down arrow to scroll down. The calculator obligingly fills in additional values, with the result shown in Fig. 2.1.9. Again we see the population is closest to 100 thousand when  $x \approx 11$  years (the symbol  $\approx$  indicates "approximate equality" as opposed to  $=$ ,

X	Y1
0.000	67.255
1.000	70.190
2.000	73.125
3.000	76.060
4.000	78.995
5.000	81.930
6.000	84.865

X=0

FIGURE 2.1.8 Initial table for the Ajax City population function.

X	Y1
7.000	87.800
8.000	90.735
9.000	93.670
10.000	96.605
11.000	99.540
12.000	102.48
13.000	105.41

X=13

FIGURE 2.1.9 Additional table values for the Ajax City population function.



which means exact equality). But the population is still a bit short of 100 thousand after precisely 11 years, so the exact population is hit sometime during the twelfth year after January 1, 1997.

To see more closely when during the twelfth year the population reaches 100 thousand, we use the calculator's "table set" menu to specify the new starting value  $x = 11$  and the new increment  $\Delta x = 0.1$  (Fig. 2.1.10). The resulting table (Fig. 2.1.11) indicates that the population is closest to 100 thousand after 11.2 years (rounded off to the nearest *tenth* of a year). This is because 100.13 thousand after 11.2 years is closer to 100 thousand than is 99.834 thousand after 11.1 years.

TABLE SETUP		
TblStart=11		
$\Delta$ Tbl=.1		
Indent:	Auto	Ask
Depend:	Auto	Ask

FIGURE 2.1.10 Second table setting for the Ajax City population function

X	Y <sub>1</sub>	
99.9000	99.540	
11.100	99.834	
11.200	100.13	
11.300	100.42	
11.400	100.71	
11.500	101.01	
11.600	101.30	

X=11

FIGURE 2.1.11 Second table for the Ajax City population function.

To narrow it down still further—in the interval between  $x = 11.1$  years and  $x = 11.2$  years—we go back to the calculator's table set menu and specify starting point  $x = 11.1$  and increment  $\Delta x = 0.01$ . (Note that we typically divide our desired interval of  $x$ -values into 10 "pieces.") Then the **TABLE** command yields the table shown in Fig. 2.1.12. Now we see that the population of Ajax City is closest to 100 thousand after  $x \approx 11.16$  years (rounded off accurate to the nearest *hundredth* of a year).

X	Y <sub>1</sub>	
99.9000	99.834	
11.110	99.863	
11.120	99.892	
11.130	99.922	
11.140	99.951	
11.150	99.980	
11.160	100.01	

X=11.1

FIGURE 2.1.12 Third table for the Ajax City population function.

- To satisfy yourself that we will obtain the same answer here as we did in Example 1(c) and Example 2, you should perform this "table zoom" procedure one more time, with starting point  $x = 11.15$  and increment  $\Delta x = 0.001$ . Do you see that the population is closest to 100 thousand when  $x = 11.157$  years?

Perhaps it seems to you that this "table zooming" is a great deal more work than solving the linear equation  $67.255 + 2.935t = 100$  algebraically or graphically. Most people (including the authors) would agree with you. The method is

presented here to illustrate that you will have a choice of algebraic, graphical, and numerical methods for solving the problems you will encounter. In another situation, you may find using the table preferable.

## Slope and Rate of Change

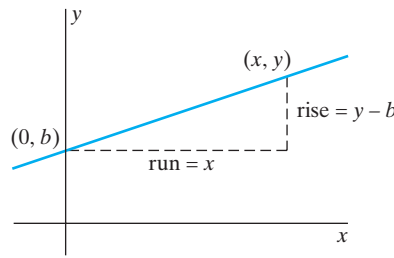
You may recall from previous mathematics courses the **slope-intercept equation**

$$y = mx + b \quad (9)$$

of a straight line in the  $xy$ -plane with coefficients

$$m = \text{slope} \quad \text{and} \quad b = \text{y-intercept} \quad (10)$$

as illustrated in Fig. 2.1.13.



**FIGURE 2.1.13** The slope-intercept equation  $y = mx + b$  of a straight line.

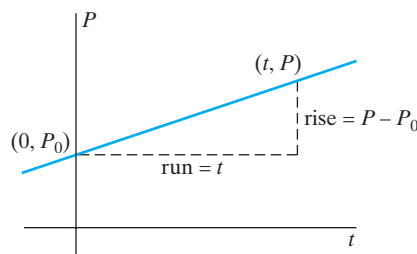
The population function

$$P(t) = P_0 + at \quad [\text{or, equivalently, } P(t) = at + P_0] \quad (11)$$

has the same linear form as the slope-intercept equation (4) but with independent variable  $t$  instead of  $x$  and with dependent variable  $P$  instead of  $y$ . The coefficients (or *parameters*) in the linear function (11) are

$$a = \text{average rate of change} \quad \text{and} \quad P_0 = \text{initial population.} \quad (12)$$

Comparing (9) and (11), we note the correspondence between the **slope** of a straight line and the **rate of change** of a linear function. Indeed, the rate of change  $a$  of the linear function  $P(t) = P_0 + at$  is simply the slope of its straight line graph in the  $tP$ -plane (Fig. 2.1.14).



**FIGURE 2.1.14** The graph of the linear function  $P(t) = P_0 + at$ .

## EXAMPLE 4

## Another Linear Population Model

On January 1, 1992, the population of Yucca City was 46,350 and on July 1, 1994, it was 56,925. Suppose this rate of population increase continues for the foreseeable future.

- Write a linear function  $P(t) = P_0 + at$  giving the population of Yucca City at time  $t$  (where  $t = 0$  on January 1, 1992).
- Find the predicted population of Yucca City on October 1, 2000.
- In what month of what calendar year will the population of Yucca City double?

## SOLUTION

- The statement that “ $t = 0$  on January 1, 1992” means that we are “resetting the clock.” Furthermore, we can measure our population in thousands, making our initial population of  $46,350 = 46.350$  thousand. This procedure makes it easier for us to find a suitable window for viewing the graph of the function.

July 1, 1994, is  $2\frac{1}{2}$  years after January 1, 1992, so the population of Yucca City is

$$P = 46.350 \quad \text{when} \quad t = 0, \quad \text{and}$$

$$P = 56.925 \quad \text{when} \quad t = 2.5.$$

Since we have our initial population (46.350), all we need in order to write our population function is its average rate of change:

$$a = \frac{\Delta P}{\Delta t} = \frac{56.925 - 46.350}{2.5 - 0} = \frac{10.575}{2.5} = 4.230 \quad (\text{thousand people per year}).$$

So our population function for Yucca City is

$$P(t) = 46.350 + 4.230t, \quad (13)$$

where  $t$  is years after 1992 and  $P$  is measured in thousands.

- On October 1, 2000, we have completed 8 years and 9 months since January 1, 1992, so October 1, 2000 represents  $t = 8\frac{9}{12}$  or 8.75. Algebraically, to find the population when  $t = 8.75$ , we use (13) to find

$$P(8.75) = 46.350 + 4.230 \times 8.75 \approx 83.363.$$

Thus our linear model predicts 83,363 people in Yucca City on October 1, 2000.

- In this context, “double” means reaching the population that is two times the initial population. So we need to determine when  $P(t)$  reaches  $2 \times 46.350$  (thousand) by setting  $P(t) = 92.700$  and solving for  $t$ :

$$46.350 + 4.230t = 92.700$$

$$4.230t = 46.350$$

$$t = \frac{46.350}{4.230} \approx 10.957$$

So the function  $P(t)$  reaches the value 92.700 (thousand) at 10.957 years after January 1, 1992. Ten years after January 1, 1992, is January 1, 2002, and 0.957 year is

$$0.957 \text{ year} \times \frac{12 \text{ months}}{\text{year}} = 11.484 \text{ months}$$

into the year 2002. So 11 months of 2004 have been completed, and we are somewhere in the twelfth month. Therefore, the population of Yucca City should double during the twelfth month of the year 2002, that is, during December 2002.

Figures 2.1.15 and 2.1.16 show graphical solutions for parts (b) and (c), respectively. While you may find the algebraic solutions easier while we are working with linear functions, you should be familiar with graphical techniques as well. You may prefer the graphical techniques when we solve similar problems involving functions that are not linear.

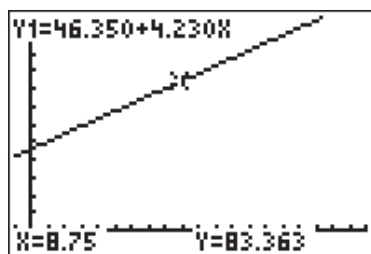


FIGURE 2.1.15 The graphical solution for Example 4(b).

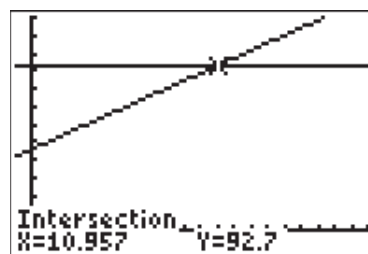


FIGURE 2.1.16 The graphical solution for Example 4(c).

## Other Linear Models

The techniques that we used to create linear population models also apply to other linear models. All we need to know is the function's output for an input of 0 (its initial value) and its average rate of change.

### EXAMPLE 5

#### Finding a Linear Function Model

Suppose that the local garden club wants to raise money for neighborhood beautification and decides to sell a small cookbook containing its members' recipes. A print shop charges \$200 to set up the press and \$2 for each book produced.

- Find a linear function model that gives the total cost  $C$  of the book as a function of the number of books produced  $n$ .
- Find the total cost if 250 books are produced.
- How many books can be produced if the garden club's budget for this project is \$3000?
- If the garden club plans to sell these cookbooks for \$12 each, how many cookbooks must the members sell to break even?

## SOLUTION

- a. In order to write a linear function model, we need to identify both the function's initial value and its constant average rate of change. In this situation, the initial value is \$200 because this is the total cost if we produce 0 books. The constant average rate of change is \$5 because the cost increases by \$5 for each book that is produced. Thus, the total cost function is

$$C(n) = 200 + 5n.$$

- b. To find the total cost if 250 books are produced, we need to find the output when the input is 250. So  $C(250) = 200 + 5 \cdot 250 = 1450$ , and the total cost of the 250 cookbooks is \$1450.
- c. Here we need to find the input that produces an output of \$3000. Let's use the calculator's table feature to answer this question. We begin by storing our function in the **Y=** menu, letting **Y1=200+5x**. We know from part (b) that 250 books cost only \$1450, so the input we are looking for is larger than 250. Let's try setting **TblStart=300** and **ΔTbl=50**. Then the table in Fig. 2.1.17 shows that **Y1** is \$2950 when **X** is 550 and \$3200 when **X** is 600. Using the table once again with **TblStart=550** and **ΔTbl=10**, we see in Fig. 2.1.18 that **Y1** is exactly \$3000 when **X** is 560. Therefore, if the garden club's budget is \$3000, then 560 cookbooks can be produced.

X	Y1
300.00	1700.0
350.00	1950.0
400.00	2200.0
450.00	2450.0
500.00	2700.0
550.00	2950.0
600.00	3200.0

X=300

FIGURE 2.1.17 Table 1 for the numerical solution for Example 5(c).

X	Y1
550.00	2950.0
560.00	3000.0
570.00	3050.0
580.00	3100.0
590.00	3150.0
600.00	3200.0
610.00	3250.0

X=550

FIGURE 2.1.18 Table 2 for the numerical solution for Example 5(c).

- d. Breaking even means that the garden club sells enough cookbooks to cover the cost of producing the books; that is, the revenue generated by selling the cookbooks must equal the cost of producing them. We already have the linear function  $C(n)$  representing the total cost of producing  $n$  cookbooks; we must now find a function to represent the revenue earned by the cookbook sales. Since no revenue is earned if no cookbooks are sold, the initial value of our revenue function is \$0. The average rate of change for this function is \$12 per book, so the revenue is a linear function of the number of books  $n$  that are sold. If we use  $R(n)$  to indicate the revenue function, we then have

$$R(n) = 0 + 12n \quad \text{or} \quad R(n) = 12n.$$

Leaving our original cost function  $C(n)$  stored in **Y1** and putting the revenue function  $R(n)$  in **Y2**, we choose an appropriate window to view

the graphs, and then use the **intersect** command from the **CALC** menu as shown in Fig. 2.1.19. Thus, we see that the cost and revenue functions have the same output (\$342.86) when the input is 28.57. But recall that the input here is number of cookbooks, and the garden club is not going to sell (nor is anyone likely to buy) a fraction of a cookbook. So to cover the cost of making the books sold, the garden club must both produce and sell 29 books.

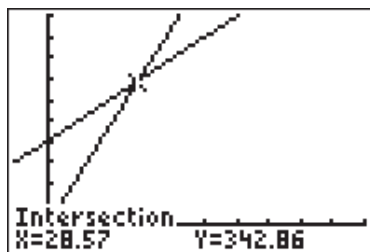


FIGURE 2.1.19 The graphical solution for Example 5(d).

- Do you see that if the garden club produces and sells fewer than 29 books, the total cost will be greater than the revenue generated by sales? Since the garden club wants to make money, they need to sell more than 29 books because then the revenue earned is more than the cost of the books. However, it is important for the club to determine what the demand for the cookbooks is likely to be
- because any unsold books will decrease the overall profit.

Notice that parts (b) and (c) are the same “Here’s the input, what’s the output?” and “Here’s the output, what’s the input?” questions that we answered in Example 1, as well as in Chapter 1, when we looked at functions of all different types. Part (d) is a variation on the “Here’s the output, what’s the input?” question. While we are indeed looking for an input, the output is “hidden” in a sense. The output that we “have” is the output that is the same for our two linear functions. You might wish to practice using algebraic, graphical, and numerical methods to answer such questions so that you can decide which method or methods you prefer.

## 2.1 Exercises Building Your Skills

In Exercises 1–4, first determine whether  $B$  is a function of  $A$  and then, if  $B$  is a function of  $A$ , decide (based on the average rate of change  $\Delta B/\Delta A$ ) whether  $B$  is a linear function of  $A$ .

1.

$A$	10	10.5	13	15	16	20
$B$	3	4	9	13	15	23

2.

A	10	10.5	13	15	16	20
B	3	4	4	9	13	23

3.

A	2	4	5	5	8	11
B	3	4	9	13	15	23

4.

A	2	4	6	8	10	12
B	3	3	3	3	3	3

In Exercises 5–8 a population function  $P(t)$  is given, with  $t$  in years and  $P$  in thousands. Find the initial population and the function's constant rate of change  $a$ .

5.  $P(t) = 123 + 6t$
6.  $P(t) = 387 - 8t$
7.  $P(t) = 487.139 + 20.558t$
8.  $P(t) = 666.333 - 42.789t$

In Exercises 9–12, write a population function  $P(t)$  with the given initial population  $P_0$  and constant rate of change  $a$ .

9.  $P_0 = 42$  and  $a = 5$
10.  $P_0 = 73$  and  $a = -6$
11.  $P_0 = 324.175$  and  $a = 15.383$
12.  $P_0 = 786.917$  and  $a = -21.452$

In Exercises 13–16 use the “resetting the clock” method to find a linear population function  $P(t)$  with the average rate of change  $a$ . Be sure to indicate the calendar year in which  $t = 0$ .

13.  $P(1987) = 375$  and  $a = 12$
14.  $P(1983) = 685$  and  $a = -24$
15.  $P(1991) = 227.625$  and  $a = 17.234$
16.  $P(1993) = 847.719$  and  $a = -60.876$

In Exercises 17–20 use the linear model  $P(t) = P_0 + at$  to represent each city's population. Be sure to indicate the calendar year in which  $t = 0$ .

17. City A had a population of 35,500 on January 1, 1985, and it was growing at the rate of 1700 people per year. Assuming that this annual rate of change in the population of City A continues, find:
  - a. Its population on January 1, 2000.
  - b. The month of the calendar year in which its population reaches 85 thousand.

18. City B had a population of 375 thousand on January 1, 1992, and it was growing at the rate of 9250 people per year. Assuming that this annual rate of change in the population of City B continues, find:
- Its population on January 1, 2000.
  - The month of the calendar year in which its population reaches 600 thousand.
19. City C had a population of 45,325 on January 1, 1985, and a population of 50,785 on January 1, 1990. Assuming that this annual rate of change in the population of City C continues, find:
- Its population on January 1, 2000.
  - The month of the calendar year in which its population reaches 75 thousand.
20. City D had a population of 428 thousand on January 1, 1992, and a population of 422 thousand on January 1, 1997. Assuming that this annual rate of change in the population of City D continues, find:
- Its population on January 1, 2000.
  - The month of the calendar year in which its population is 400 thousand.
21. Find the month of the calendar year during which Cities A and C of Problems 17 and 19 have the same population.
22. Find the month of the calendar year during which Cities B and D of Problems 18 and 20 have the same population.

## Applying Your Skills

23. Consumer credit experts are concerned that Americans' credit card debt continues to increase. *USA Today* reported that in 1995, U.S. bank credit card loans totaled \$358 billion, while in 2004, they totaled \$697 billion.
- Find a linear function model  $C(t) = \dots$  that gives the U.S. bank credit card loans as a function of  $t$ , number of years after 1995. (Round the average rate of change to three decimal places.)
  - According to your function model, in what month and year will U.S. bank credit card loans reach \$1000 billion (1 trillion dollars)?
  - Use your model to predict the U.S. bank credit card loans in 2006.
24. In 1997, Allergan, the maker of Botox, earned \$90 million in sales of Botox, used as injections for both cosmetic and medical purposes. Allergan's revenues from Botox in 2001 were \$310 million.
- Find a linear function model  $R(t) = \dots$  that gives Botox revenue as a function of  $t$ , the number of years after 1997.
  - Use your model to predict Allergan's Botox revenue in 2010.
  - According to your function model, in what month and year will Allergan's Botox revenue reach \$500 million?
25. According to the U. S. Census, the population of St. Louis was 397 thousand in 1990 and 348 thousand in 2000.
- Find a linear function model  $P(t) = \dots$  that gives the population of St. Louis as a function of  $t$ , the number of years after 1990.
  - Use your model to predict the population of St. Louis in 2003.
  - If the actual population of St. Louis in 2003 was 332 thousand, was your answer in (b) a reasonable prediction?



- d. Use your model to predict the month of the calendar year in which the population of St. Louis falls to 300 thousand.
26. According to the U.S. Census, the population of Lexington, Kentucky, was 225 thousand in 1990 and 261 thousand in 2000.
- Find a linear function model  $P(t) = \dots$  that gives the population of Lexington as a function of  $t$ , the number of years after 1990.
  - Use your model to predict the population of Lexington in 2003.
  - If the actual population of Lexington in 2003 was 267 thousand, was your answer in **(b)** a reasonable prediction?
  - Use your model to predict the month of the calendar year in which the population of Lexington grows to 300 thousand.
27. Use your models from Exercises 25 and 26 to predict the month of the calendar year in which the populations of St. Louis and Lexington are the same.
28. One of the services that Statewide Landscape provides is grinding out stumps of trees that have fallen or been otherwise removed. The company charges a \$45 “trip fee” and \$2.00 per inch of diameter of the stump (or stumps) to be ground.
- Find a linear function model  $P(i) = \dots$  that gives the price of stump grinding as a function of the number of inches  $i$  of diameter.
  - If the customer has a budget of \$275 for stump grinding, how many inches of diameter of tree stumps can be ground?
  - If a total diameter of 200 inches of tree stumps needs to be ground, how much will the customer be charged?
29. In 2006, AT&T advertised a “one rate” plan for long distance customers in New Mexico, with a \$3.95 monthly fee and long distance calls charged at 7¢ per minute. A competing company, Qwest, offered a “15¢ Single Rate Plan” for long distance, which featured a 99-cent monthly fee and a charge of 15¢ per minute for calls.
- Find a linear function model  $A(t) = \dots$  that gives the monthly cost of AT&T long distance service as a function of  $t$ , the number of minutes used.
  - How much is the monthly cost of the AT&T long distance if the customer uses 1 hour of long distance time?
  - Find a linear function model  $Q(t) = \dots$  that gives the monthly cost of Qwest long distance service as a function of  $t$ , the number of minutes used.
  - How much is the monthly cost of the Qwest long distance if the customer uses 1 hour of long distance time?
  - When is it cheaper to use the AT&T plan, and when is it cheaper to use the Qwest plan?
30. In 2006, Bell South advertised both a “Dollar Plan” and a “Nickel Plan” to its long distance customers in Charlotte, North Carolina. The Dollar Plan had a service charge of \$1.00 a month, and a charge of 10¢ per long distance minute. The Nickel Plan had a \$5.95 monthly fee, with long distance calls charged at 5¢ per minute.
- Find a linear function model  $D(t) = \dots$  that gives the monthly cost of the Dollar Plan for long distance as a function of  $t$ , the number of minutes used.
  - How much is the monthly cost of the Dollar Plan if the customer uses 1 hour of long distance time?
  - Find a linear function model  $N(t) = \dots$  that gives the monthly cost of the Nickel Plan as a function of  $t$ , the number of minutes used.
  - What is the monthly cost of the Nickel Plan if the customer uses 1 hour of long distance time?

- e. When is it cheaper to use the Dollar Plan, and when is it cheaper to use the Nickel Plan?
31. As a person ages, the number of additional years he or she can expect to live declines. U.S. government statisticians use population data to provide estimates of how much longer a person will live based on gender and current age. A newborn boy (0 years old) in 2001 was expected to live 74.4 years, while a man 30 years old in 2001 was expected to live an additional 46.2 years.
- Find a linear function model  $L(a) = \dots$  that gives the years of life expectancy as a function of  $a$ , a male's age in 2001.
  - Use your model to predict how much longer a male is expected to live if he is 18 in 2001.
  - In 2001 how old is a male who is expected to live 32.1 additional years?
32. The excerpt shown in Fig 2.1.20 is from the Pew Hispanic Center's June 2005 paper "Unauthorized Migrants: Numbers and Characteristics" discusses the

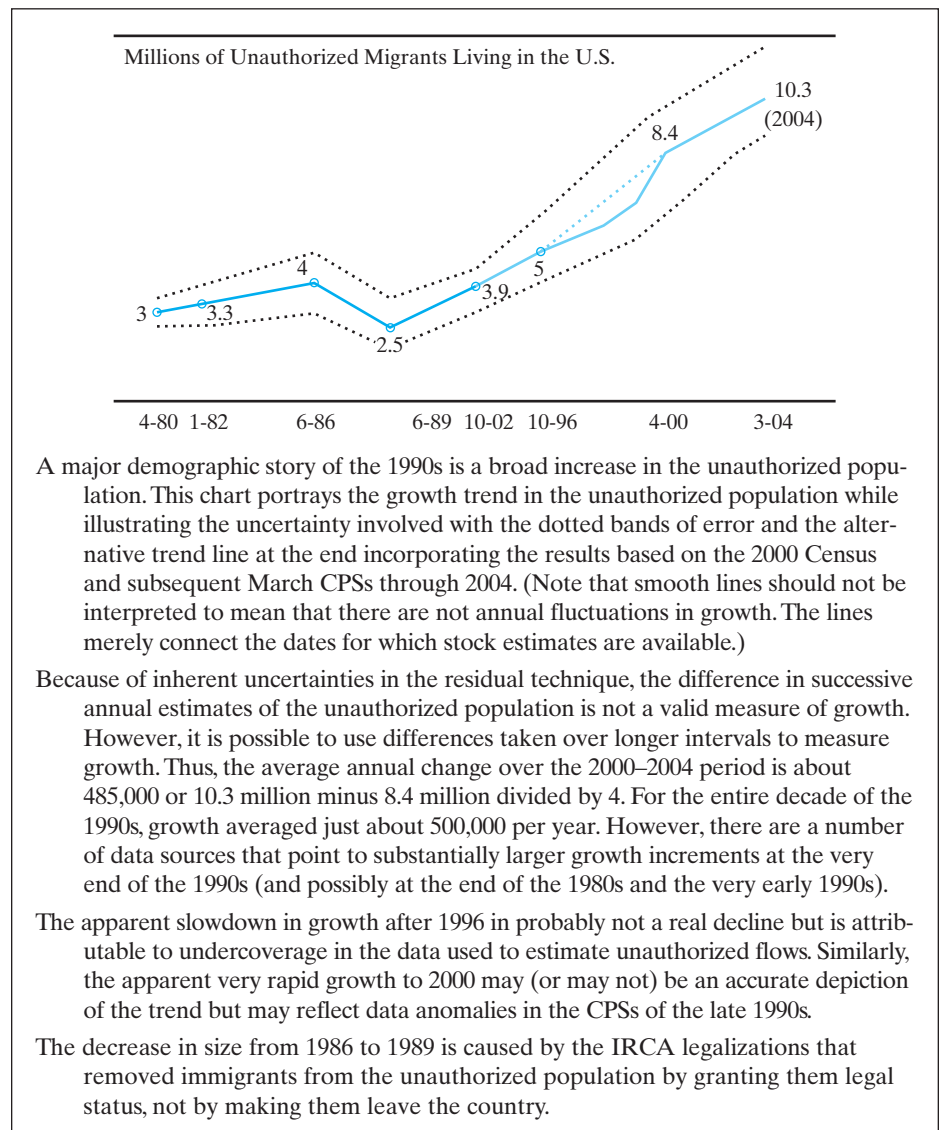


FIGURE 2.1.20 Unauthorized at New High—Details of Trend Uncertain

difficulty of accurately predicting the number of illegal immigrants in the United States.

- This paper suggests that considering the average rate of change over longer intervals is a reasonable way to measure the growth of the unauthorized migrant population. Use the data from April 1980 (3 million migrants) and from April 2000 (8.4 million migrants) to find a linear model that gives the number of unauthorized migrants as a function of years after April 1980.
- Use your model to predict the number of unauthorized migrants in April 2004. How does your answer compare to the value given in the chart for March 2004?
- In what year does the model predict that the number of unauthorized migrants will grow to 15 million?

## 2.2 LINEAR FUNCTIONS AND GRAPHS

Recall that in the last section we investigated the relationship between temperature measured in Celsius degrees and temperature measured in Fahrenheit degrees. We determined that, because the average rate of change was constant, we could express Fahrenheit temperature as a linear function of Celsius temperature.

The table of values indicated that for a Celsius temperature of 0 degrees, the Fahrenheit temperature was 32 degrees. Furthermore, we found that the average rate of change was 1.8 degrees  $F$  per degree  $C$ . Using the techniques of Section 2.1, we can write this temperature function as  $F(C) = 32 + 1.8C$ .

Suppose that, instead of having the function's initial value (its output when input = 0), we were given two pairs of  $(C, F)$  values, say  $(30, 86)$  and  $(70, 158)$ . As before, we can find the function's average rate of change:

$$\frac{\Delta F}{\Delta C} = \frac{158 - 86}{70 - 30} = \frac{72}{40} = 1.8.$$

But how do we determine a rule for this function? We know that the function has this same average rate of change between any two ordered pairs. So consider  $(C, F(C))$  any pair of Celsius and Fahrenheit temperatures. Then

$$\frac{\Delta F}{\Delta C} = \frac{F(C) - 86}{C - 30} = 1.8, \quad \text{and so} \quad F(C) - 86 = 1.8(C - 30).$$

Therefore

$$F(C) = 86 + 1.8(C - 30). \tag{1}$$

To verify that this is the very same linear function obtained earlier (as well as in Section 2.1), we can simplify (1) to get

$$F(C) = 86 + 1.8C - 54, 50.$$

$$F(C) = 32 + 1.8C.$$

In general, suppose that we know a function's average rate of change  $a$  and a pair of input and output values  $(x_1, f(x_1))$ . Then (as before)

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_1)}{x - x_1} = a.$$

So

$$f(x) - f(x_1) = a(x - x_1),$$

and then

$$f(x) = f(x_1) + a(x - x_1). \quad (2)$$

This is the **point-slope form** of a linear function. In this form, we see visibly displayed both the slope  $a$  of the function's graph and a particular point  $(x_1, f(x_1))$  lying on the graph.

### EXAMPLE 1

#### Finding a Linear Function Model

Use the point-slope form to find a linear function  $f(x)$  such  $f(2) = 1$  and  $f(4) = 15$ .

#### SOLUTION

To find a linear function having these two pairs of input-output values, we first find the average rate of change  $a$ :

$$\frac{\Delta f}{\Delta x} = \frac{f(4) - f(2)}{4 - 2} = \frac{15 - 1}{4 - 2} = \frac{14}{2} = 7.$$

Now we substitute appropriate values in (2), the point-slope form of the linear function:

$$\begin{aligned} f(x) &= f(2) + 7(x - 2), \\ f(x) &= 1 + 7(x - 2), \end{aligned} \quad (3)$$

or

$$\begin{aligned} f(x) &= f(4) + 7(x - 4), \\ f(x) &= 15 + 7(x - 4). \end{aligned} \quad (4)$$

If we prefer to have the slope-intercept form of the function (as in Section 2.1), we merely simplify either (3) or (4).

That is,  $f(x) = 1 + 7x - 14 = 7x - 13$  or  $f(x) = 15 + 7x - 28 = 7x - 13$ . Notice that whichever version of the point-slope form we use, we end up with the same slope-intercept form of the function.

We can use the point-slope form to find linear functions in applied settings as well.

### EXAMPLE 2

#### Finding a Linear Cost Function

AT&T's "one rate" plan for long distance customers in New Mexico charged a monthly fee with long distance calls charged at 7¢ per minute. In June 2006, a customer used 87 long distance minutes, and her bill was \$10.04.

- Use the point-slope form to find a linear function  $C(t)$  giving the monthly cost of long distance service as a function of the number of minutes of long distance calls made.

- b. What was the monthly service charge for the “one rate” plan?  
 c. How many minutes did the customer use in July 2006 if her bill was \$8.92?

**SOLUTION**

- a. Here the average rate of change  $a$  is the per-minute charge (\$0.07), and  $C(87) = 10.04$ . So, the cost of long distance service as a function of minutes is given by

$$C(t) = 10.04 + 0.07(t - 87). \quad (5)$$

- b. The service charge is the fee just for having the calling plan, even if no minutes of long distance time are used. So the service charge is the output for the input of 0, or the function’s initial value. This is once again our standard question, “Here’s the input, what’s the output?” with the given input being 0. So we substitute  $t = 0$  in (5) and calculate

$$C(0) = 10.04 + 0.07(0 - 87).$$

Then

$$C(0) = 10.04 + 0.07(-87),$$

and so

$$C(0) = 3.95.$$

Thus, the monthly service charge for the “one rate” plan is \$3.95.

- c. In this case, we are given an output and are required to find the appropriate input. As always, we have several choices of methods for determining the input that makes the function output 8.82. We might first notice that now that we have found the function’s initial value [in part (b)], we can write an equivalent slope-intercept form for our function:

$$C(t) = 3.95 + 0.07t. \quad (6)$$

We can store our original function from (5) in **Y1** and the slope-intercept form from (6) in **Y2**. Using the table settings **TblStart=0** and **ΔTbl=10**, we see in Figs. 2.2.1 and 2.2.2 that the function values are the same. So the two different forms do describe the same function, and we are free to use whichever one we prefer.

X	Y1	Y2
0.0000	3.950	3.950
10.000	4.650	4.650
20.000	5.350	5.350
30.000	6.050	6.050
40.000	6.750	6.750
50.000	7.450	7.450
60.000	8.150	8.150

X=0

**FIGURE 2.2.1** Table 1 for the numerical solution for Example 2(c).

X	Y1	Y2
70.000	8.850	8.850
80.000	9.550	9.550
90.000	10.250	10.250
100.00	10.950	10.950
110.00	11.650	11.650
120.00	12.350	12.350
130.000	13.050	13.050

X=130

**FIGURE 2.2.2** Table 2 for the numerical solution for Example 2(c).

Furthermore, we see that the cost reaches \$8.92 for some  $t$  between  $t = 70$  and  $t = 80$ . If we reset our table to start at 70 and use increments of 1 (rather than 10), we obtain the table in Fig. 2.2.3. So, it is an input of 71 that generates an output of 8.92. Therefore, if a customer's bill is \$8.92, she used 71 minutes of long distance.

X	Y1	Y2
70.000	8.850	8.850
71.000	8.920	8.920
72.000	8.990	8.990
73.000	9.060	9.060
74.000	9.130	9.130
75.000	9.200	9.200
76.000	9.270	9.270
X=70		

FIGURE 2.2.3 Table 3 for the numerical solution for Example 2(c).

- While we solved this problem numerically, we would have obtained the same result if we had used either an algebraic or a graphical method.

The method used in the previous examples can also be used to find linear population models if we choose not to “reset the clock.”

### EXAMPLE 3

#### Finding a Linear Population Model

According to the Census Bureau, the population of Providence, Rhode Island, was 160,728 in 1990 and 173,618 in 2000.

- Use the point-slope form of a linear function to describe the population of Providence as a function of the year.
- According to your model, in what month and year will the population of Providence be 200,000?
- According to your model, what will Providence's population be in 2008?

#### SOLUTION

- We first need to find the average rate of change  $a$ :

$$a = \frac{\Delta P}{\Delta t} = \frac{173618 - 160728}{2000 - 1990} = \frac{12890}{10} = 1289.$$

Then

$$P(t) = 160728 + 1289(t - 1990).$$

- To find the year in which the population reaches 200,000, we need to find the input for which the function's output is 200,000. That is, we need to solve the equation  $160728 + 1289(t - 1990) = 200000$ . In order to solve this equation graphically, we store our population function in **Y1** and the output value 200,000 in **Y2**. Recalling that the input values for our function are actual calendar years, we set the window as indicated in Fig. 2.2.4. Then Fig. 2.2.5 shows the intersection point found

```

WINDOW
Xmin=1980
Xmax=2050
Xscl=10
Ymin=-10000
Ymax=220000
Yscl=10000
Xres=1

```

FIGURE 2.2.4 The window setting for Example 3(b).

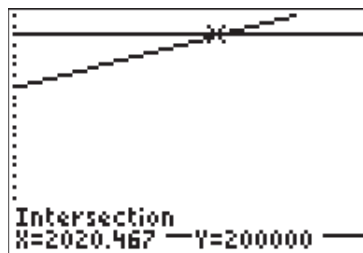


FIGURE 2.2.5 The intersection point for Example 3(b).

by the **intersect** feature of the calculator. Unless we are told otherwise, we assume that our input years begin on January 1, so 2020.467 would be  $0.467 \times 12 = 5.604$  months into the year 2020. Our model predicts that the population of Providence will grow to 200,000 during the sixth month (June) of 2020.

- c. Here we are looking for  $P(2008)$ , which is  $160728 + 1289(2008 - 1990) = 183930$ . Thus, the population of Providence in 2008 is predicted to be 183,930 people.

Notice that in Example 3, we left our function in point-slope form because we were using actual calendar years there. It makes no sense to simplify this function to put it into slope-intercept form since this intercept would indicate the population of Providence in the year 0 (more than 20 centuries ago). There are several problems with this notion. First, there is no year designated “0” in the Western calendar. We have the year 1 BC, followed by the year 1 AD. Second, even if there were such a year way back then, there was no city of Providence. Finally, if we went backward 2000+ years using this model, we would predict a *negative* value for the population, which is certainly not sensible.

This example demonstrates a problem we may encounter when we create functions to model real-world data. Sometimes our meaningful or “sensible” domain does not include 0, making meaningless any value we might find there for the  $y$ -intercept. As we noted in Chapter 1, consideration of the domain of a function is a critical part of the function definition.

## Straight Lines and Linear Graphs

The **graph** of the linear function  $f(x) = ax + b$  is the straight line consisting of all points  $(x, y)$  in the  $xy$ -plane that satisfy the equation

$$y = ax + b. \quad (7)$$

Although we often abbreviate and speak of “the straight line  $y = ax + b$ ,” it is important to understand the differences among

- the linear *function*  $f$ ,
- its defining *formula*  $f(x) = ax + b$ ,
- the *equation*  $y = ax + b$ , and
- the *graph* with equation  $y = ax + b$ .

In particular, the words *function*, *equation*, and *graph* should not be used interchangeably. In mathematics (as elsewhere) it's important to “say what you mean and mean what you say.”

### EXAMPLE 4

#### Distinguishing Rising and Falling Lines

Define  $Y1 = A \cdot X + B$  in the  $Y=$  menu of your graphing calculator. Then enter several different pairs of slope-intercept values as indicated in Fig. 2.2.6, pressing **GRAPH** after each is entered.

1→A: -2→B	-2
-2→A: 5→B	5
3→A: -7→B	-7
-0.5→A: 3→B	

FIGURE 2.2.6 Various slopes  $a$  and  $y$ -intercepts  $b$ .

Describe how changing the parameter  $a$  affects the graph of the line.

#### SOLUTION

As illustrated in Figs. 2.2.7 and 2.2.8, we find that

- the line *rises* (from left to right) if the slope  $a$  is *positive*;
- the *larger* is  $a > 0$ , the more *steeply* the line rises;
- the line *falls* (from left to right) if the slope  $a$  is *negative*;
- the *larger* in absolute value is  $a < 0$ , the more *steeply* the line falls.

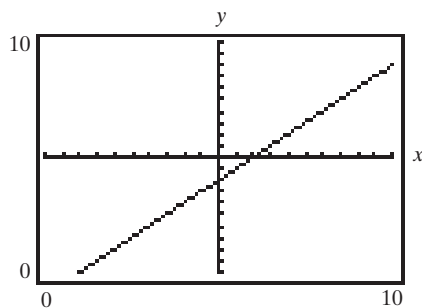


FIGURE 2.2.7 Slope  $a = 1$  and  $y$ -intercept  $b = -2$ , so  $y = x - 2$ .

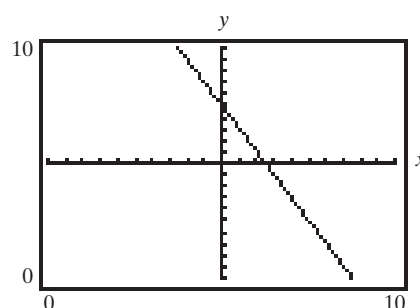


FIGURE 2.2.8 Slope  $a = -2$  and  $y$ -intercept  $b = 5$ , so  $y = -2x + 5$ .

- A straight line is a simple example of a graph of an equation.

#### DEFINITION: Graph of an Equation

The **graph** of an equation involving two variables  $x$  and  $y$  consists of all points in the  $xy$ -plane whose coordinates  $(x, y)$  satisfy the equation.



For instance, there are three types of equations whose graphs are straight lines:

- Equations whose graphs are vertical lines
- Equations whose graphs are horizontal lines
- Equations whose graphs are slanted lines (neither vertical nor horizontal)

### EXAMPLE 5

#### Recognizing a Vertical Line

Sketch and describe a graph of the equation  $x = -3$ .

#### SOLUTION

The graph of the equation  $x = -3$ , illustrated in Fig. 2.2.9, is a vertical line. Only ordered pairs  $(x, y)$  whose first coordinate  $x$  equals  $-3$  satisfy this equation. The second coordinate  $y$  can be any real number. Notice that in this relationship,  $y$  is *not* a function of  $x$  because the single value  $-3$  of  $x$  is paired with infinitely many different values of  $y$ . A graph that *is* a vertical line certainly fails our “vertical line test” for a function.

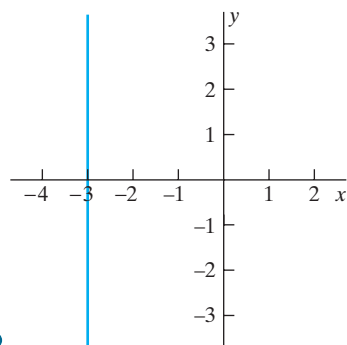


FIGURE 2.2.9 The graph of  $x = -3$ .

### EXAMPLE 6

#### Recognizing a Horizontal Line

Sketch and describe a graph of the equation  $y = 2$ .

#### SOLUTION

The graph of the equation  $y = 2$ , illustrated in Fig. 2.2.10, is a horizontal line. In this case, the first coordinate  $x$  can be any real number, but the second coordinate  $y$  must equal 2. Here  $y$  *is* a function of  $x$  because each  $x$ -value is paired with only one  $y$ -value. What makes this function unusual is that every point on the graph has the same  $y$ -value as every other point.

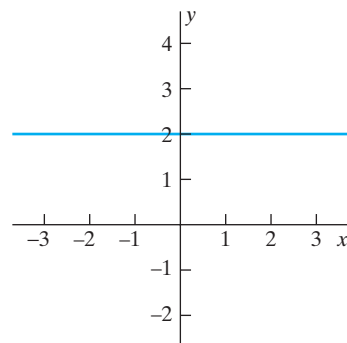


FIGURE 2.2.10 The graph of  $y = 2$ .

## EXAMPLE 7

## Recognizing a More General Line

Sketch and describe a graph of the equation  $y = -2x + 1$ .

## SOLUTION

Figure 2.2.11 shows the graph of the equation  $y = -2x + 1$ . Note that it is a slanted line, falling from left to right, with  $y$ -intercept 1. We see that again  $y$  is a function of  $x$  since each  $x$ -value is paired with only one  $y$ -value. Unlike the previous example, different points on the graph have different  $y$ -values.

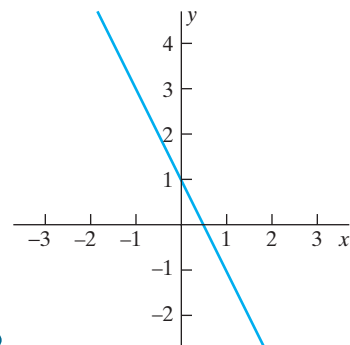


FIGURE 2.2.11 The graph of  $y = -2x + 1$ .

The graph of a function is a special case of the graph of an equation.

## DEFINITION: Graph of a Function

The **graph** of the function  $f$  is the graph of the equation  $y = f(x)$ .

Thus the graph of the function  $f$  consists of all points in the plane whose coordinates have the form  $(x, f(x))$  with  $x$  being in the domain of  $f$ . Observe that the second coordinate of any such point is the value of  $f$  at its first coordinate (Fig. 2.2.12).

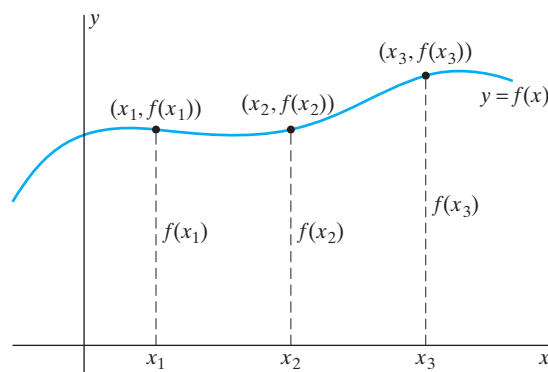


FIGURE 2.2.12 The graph of the function  $f$ .

We have been using this definition somewhat intuitively when graphing a function in our calculator. We have typed our function rule into the **Y=** menu in the calculator, using **X** as our input value. Thus the definition confirms our

understanding that the notation  $f(x)$  is another (somewhat fancy) name for  $y$ . This notation is convenient in that it gives us a shorthand for specifying a particular set of input and output values. Rather than having to say (for example) “when  $x = 2$ ,  $y = 5$ ,” we can say  $f(2) = 5$ . Much of the mathematical notation that may appear confusing at first glance actually makes it easier for us to “talk” about mathematical relationships.

## 2.2 Exercises | Building Your Skills

*In Exercises 1–4 use the point-slope form to find a linear function  $f(x)$  having the given values.*

1.  $f(2) = 7$  and  $f(5) = 13$
2.  $f(3) = 5$  and  $f(7) = 17$
3.  $f(-2) = 19$  and  $f(3) = -16$
4.  $f(1) = 8$  and  $f(3) = -10$

*In Exercises 5–10 write (in slope-intercept form) an equation of the line  $L$  described and sketch its graph.*

5.  $L$  passes through the origin and the point  $(2, 3)$ .
6.  $L$  is vertical and has  $x$ -intercept 7.
7.  $L$  is horizontal and passes through the point  $(3, -5)$ .
8.  $L$  has  $x$ -intercept 2 and  $y$ -intercept  $-3$ .
9.  $L$  passes through  $(-1, -4)$  and has slope  $\frac{1}{2}$ .
10.  $L$  passes through  $(4, 2)$  and rises (left to right) at a 45-degree angle.

*Three points  $A$ ,  $B$ , and  $C$  lie on a single straight line if and only if the slope of  $AB$  equals the slope of  $BC$ . In Exercises 11–14 plot the three given points and then use the slopes  $AB$  and  $BC$  to determine whether all three points lie on a single line.*

11.  $A(-1, -2)$ ,  $B(2, 1)$ ,  $C(4, 3)$
12.  $A(-2, 5)$ ,  $B(2, 3)$ ,  $C(8, 0)$
13.  $A(-1, 6)$ ,  $B(1, 2)$ ,  $C(4, -2)$
14.  $A(-3, 2)$ ,  $B(1, 6)$ ,  $C(8, 14)$

*In Exercises 15–18 find a linear population function  $P(t)$  with the average rate of change  $a$ . Do not “reset the clock.”*

15.  $P(1987) = 375$  and  $a = 12$
16.  $P(1983) = 685$  and  $a = -24$
17.  $P(1991) = 227.625$  and  $a = 17.234$
18.  $P(1993) = 847.719$  and  $a = -60.876$

## Applying Your Skills

19. The Fahrenheit temperature  $F$  and the absolute (Kelvin) temperature  $K$  are linear functions of each other. Moreover,  $K = 273.16$  when  $F = 32$ , and  $K = 373.16$  when  $F = 212$ .
- Find a linear function  $K(F)$  that gives the absolute temperature as a function of the Fahrenheit temperature.
  - Use your function from part (a) to find the Fahrenheit temperature when the Kelvin temperature is 0 (“absolute zero”).
  - What is the Kelvin temperature when the Fahrenheit temperature is 0?
20. The owner of a grocery store finds that she can sell 980 gallons of milk each week at \$2.69 per gallon and 1220 gallons of milk each week at \$2.49 per gallon.
- Find a linear function  $G(p)$  that gives the number of gallons sold as a function of the price of the milk.
  - How many gallons would she then expect to sell at \$2.99 per gallon?
  - According to your function, if the owner sold 800 gallons of milk, what was the price per gallon?
21. The length  $L$  in centimeters of a copper rod is a linear function of its Celsius temperature  $C$ .
- If  $L = 124.942$  when  $C = 20$  and  $L = 125.131$  when  $C = 110$ , find a linear function  $L(C)$  that gives the length of the rod as a function of its Celsius temperature.
  - What is the length of the rod if the temperature is  $50^\circ\text{C}$ ?
  - If the rod is 125.0764 centimeters long, what is its Celsius temperature?
22. Taking a cruise has become an increasingly popular form of vacationing for Americans. In 1995, 4.4 million Americans took a cruise; in 2003 the number had risen to 8.2 million. (Source: *USA Today*)
- Assuming that the number of Americans taking cruises was increasing at a constant rate over this period, find a function  $A(y)$  that gives the number of Americans taking a cruise as a function of the year.
  - In what year can the travel industry expect the number of Americans taking a cruise to reach 12 million?
23. According to the U.S. Department of Agriculture, there were 705 farms in Cheyenne County, Nebraska, in 1987, and 615 farms in 2002.
- Find a linear function  $F(t)$  that gives the number of farms in Cheyenne County as a function of the year.
  - Use your model to find the year in which the number of farms in Cheyenne County will fall to 580.
24. The number of calories that a person burns depends not only on what type of activity is performed, but also on the person’s weight. A person who weighs 130 pounds will burn 590 calories playing racquetball competitively for an hour, while a 155-pound individual will burn 704 calories doing the same activity.
- Find a linear function  $C(w)$  that gives the number of calories burned per hour of competitive racquetball as a function of weight.

- b. According to your model, how many calories will a person weighing 180 pounds burn if he or she plays competitive racquetball for an hour?
25. It is fairly common knowledge that crickets are sensitive to temperature and chirp faster as the temperature rises. There are many different formulas for obtaining the temperature when you know how frequently a cricket chirps. Most of these formulas are simple linear functions.
- a. Suppose that you count the number of times a cricket chirps in 10 seconds and find that it chirped 14 times when the temperature was  $61^{\circ}\text{F}$ . When the temperature was  $88^{\circ}\text{F}$  the cricket chirped 32 times in 10 seconds. Find a linear function  $T(c)$  that gives the temperature in degrees Fahrenheit as a function of the number of cricket chirps in a 10-second interval.
- b. What is the temperature if the cricket chirps 24 times in 10 seconds?
26. According to the U.S. Census, the population of Virginia Beach, Virginia, was 393 thousand in 1990 and 425 thousand in 2000.
- a. Find a linear function  $P(t)$  that gives the population  $P$  as function of the year  $t$ .
- b. Use your model to predict the population of Virginia Beach in 2003.
- c. According to your model, what is the yearly rate of change in the population of Virginia Beach?
- d. Use your model to predict the month of the calendar year in which the population of Virginia Beach grows to 500 thousand people.
27. According to the U.S. Census, the population of Milwaukee, Wisconsin was 628 thousand in 1990 and 597 thousand in 2000.
- a. Find a linear function  $P(t)$  that gives the population  $P$  as function of the year  $t$ .
- b. Use your model to predict the population of Milwaukee in 2003.
- c. According to your model, what is the yearly rate of change in the population of Milwaukee?
- d. Use your model to predict the month of the calendar year in which the population Milwaukee is 400 thousand people.
28. Use your models from Exercises 30 and 31 to predict the month of the calendar year in which the populations of Virginia Beach and Milwaukee are the same.

## 2.3 PIECEWISE-LINEAR FUNCTIONS

Sometimes function relationships cannot be described by a single linear function. There are many situations in which different intervals of input values generate different linear functions. For instance, a population  $P$  might grow at one constant rate on one time interval and at a different constant rate on another time interval. We call such functions piecewise-linear functions because each section (or piece) of the graph is a portion of a line.

### DEFINITION: Piecewise-linear Function

A *piecewise-linear function* is a function that is defined by different linear functions on different intervals.

## EXAMPLE 1

## Finding a Piecewise-Linear Population Model

Suppose that the population of Springfield was 150 thousand in 1970 and from 1970 to 1990 the population grew at the rate of 10 thousand per year. However, due to new industry acquired in 1990, additional people started moving in steadily. As a result, after 1990 the population of Springfield grew at the increased rate of 20 thousand people per year. Find a piecewise-linear function  $P(t)$  that gives the population of Springfield as a function of the year.

## SOLUTION

In 1970 the population was 150 thousand, giving us a  $(t, P)$  ordered pair  $(1970, 150)$  and an average rate of change  $a = 10$ . Using the techniques of Section 2.2, we find that the population  $P$  is described by the function  $P(t) = 150 + 10(t - 1970)$  for  $t$  from 1970 to 1990. Noting that the population of Springfield in 1990 was 350 thousand (why?), and the average rate of change thereafter was  $a = 20$ , we use the same techniques to determine that the city's population is described by  $P(t) = 350 + 20(t - 1990)$  for  $t$  from 1990 onward. Thus, the piecewise-linear function giving Springfield's population is described for  $t \geq 1970$  by

$$P(t) = \begin{cases} 150 + 10(t - 1970) & \text{if } 1970 \leq t \leq 1990, \\ 350 + 20(t - 1990) & \text{if } t > 1990. \end{cases}$$

● Figure 2.3.1 shows the graph of the population function  $P(t)$ .

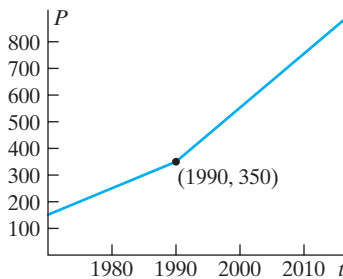


FIGURE 2.3.1 The piecewise-linear population function of Example 1.

In your previous study of mathematics, you've probably encountered the absolute value function,  $f(x) = |x|$ . The absolute value of a number measures how far the number is away from 0, without regard to its direction from 0. Thus, both  $|-2|$  and  $|2|$  are equal to 2 because both numbers are two units away from 0; negative 2 is two units to the left of 0, while positive 2 is 2 units to the right of 0.

Figure 2.3.2 shows the calculator graph **Y1 = abs(X)** of the absolute value function defined mathematically by

$$|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

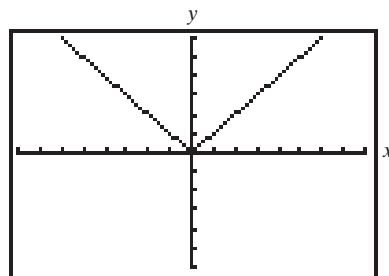


FIGURE 2.3.2 Graph of  $y = |x|$ .

We see that the function is defined differently for  $x < 0$  and for  $x \geq 0$ . The two “pieces” of the function rule correspond to the two straight line “pieces” of the graph. The graph “turns the corner” at the origin because it consists of

- the left half of the line  $y = -x$ , which falls from left to right with slope  $-1$ , and
- the right half of the line  $y = x$ , which rises from left to right with slope  $+1$ .

While such functions definitions may seem odd initially, piecewise functions frequently turn up in applied settings.

## EXAMPLE 2

### Finding a Piecewise-Linear Cost Function

In 1999, Mindspring Enterprises offered a “light” plan for Internet access in which the monthly cost was \$6.95 for up to 5 hours of connection time, with additional hours charged at \$2.00 each. Find a piecewise linear function that gives the cost  $C$  of Internet access as a function of the number of hours  $h$  of connection time.

### SOLUTION

If a customer uses between 0 and 5 hours of connection time, the monthly cost remains the same. That is,  $C(h) = 6.95$  for  $h$  between 0 and 5, inclusive. Then, for each additional hour, the customer is charged \$2 per hour. So for over 5 hours,  $C(h) = 6.95 + 2(h - 5)$ . Thus, our two-part rule for this function is

$$C(h) = \begin{cases} 6.95 & \text{if } 0 \leq h \leq 5 \\ 6.95 + 2(h - 5) & \text{if } h > 5. \end{cases}$$

Figure 2.3.3 shows the graph of this function.

Note that the graph consists of two straight-line pieces—one horizontal and one slanted—that meet at the point  $(5, 6.95)$  and correspond to the two parts of the formula. (You may recognize this symbolic form from Chapter 1, where you answered a “Here’s the output, what’s the input?” question for this function.)

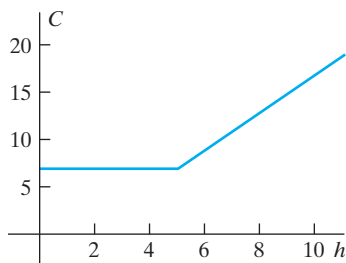


FIGURE 2.3.3 The graph of the cost function  $C(h)$  for the Mindspring “light” plan.

## Graphical, Numerical, and Symbolic Viewpoints

In working examples in Chapter 1 and this chapter we demonstrated graphical, numerical, and symbolic methods of solution, and you have probably used all of these methods at one time or the other as well. Likewise, applications of piecewise functions can involve looking at the same function from different viewpoints.

## EXAMPLE 3

### Describing Distance As a Function of Time

Suppose that a car begins (at time  $t = 0$  hours) in Hartford, Connecticut and travels to Danbury 60 miles away at a constant speed of 60 miles per hour. The car stays in Danbury for exactly 1 hour, and then returns to Hartford, again at a constant speed of 60 miles per hour.

The car's distance from Hartford  $d$  is a function of the time  $t$  in hours.

- Describe the car's distance from Hartford graphically.
- Describe the car's distance from Hartford symbolically.

### SOLUTION

- Here we are asked to create a graph of the function  $d(t)$  from its verbal description. For the first hour, the car is traveling a constant rate of speed away from Hartford, so the distance is increasing at a constant rate. This means that for  $t$  between 0 and 1,  $d$  is an increasing linear function of  $t$ . In order to graph  $d(t)$  on this interval, we only need two points. We know that at time  $t = 0$ ,  $d = 0$ , and at time  $t = 1$ ,  $d = 60$ , so our graph is the line segment connecting  $(0, 0)$  and  $(1, 60)$ .

For the next hour, the car remains in Danbury, so for  $t$  between 1 and 2,  $d$  is a constant function of  $t$ . So on this interval, the graph  $d$  is the line segment connecting  $(1, 60)$  and  $(2, 60)$ .

During the last hour, the car returns to Hartford, again at a constant rate of speed. Thus, the distance is decreasing at a constant rate, and for  $t$  between 2 and 3,  $d$  is a decreasing linear function of  $t$ . At time  $t = 3$ , the car has returned to Hartford and  $d = 0$ . The last “piece” of our graph is the line segment connecting  $(2, 60)$  and  $(3, 0)$ .

The graph of  $d(t)$  is shown in Fig. 2.3.4.

- Now we will build a symbolic function  $d(t)$  based on the verbal description. For the first hour, the constant average rate of change of distance (with respect to time) is 60 miles per hour. The car's initial distance from Hartford is 0. So for  $t$  between 0 and 1,  $d(t) = 60t$ .

During the next hour, the car stays in Danbury, so its distance from Hartford is constant—60 miles. That is, for  $t$  between 1 and 2,  $d(t) = 60$ .

Finally, during the last hour, the car is returning to Hartford, so the constant average rate of change of distance (with respect to time) is  $-60$  miles per hour. Our starting point for the journey back is at  $t = 2$ ,  $d = 60$ . Using the point-slope form, we find that  $d(t) = 60 - 60(t - 2)$  for  $t$  between 2 and 3. [We can simplify this to  $d(t) = 180 - 60t$  if we wish, or merely leave it as it is.]

We have used the word “between” in a vague manner in discussing this symbolic form. We can see that the three “pieces” of graph “match up”—there are no breaks or jumps in the graph. This means that our three function rules also “match up” at their endpoints, so we may include the endpoint  $t$ -values in whichever “piece” of the domain we prefer. For no particular reason, we will include these two values ( $t = 1$  and  $t = 2$ ) with their left-hand rules. Then the distance from Hartford (in miles) as a function of time (in hours) is given by the three-part formula

$$d(t) = \begin{cases} 60t & \text{if } 0 \leq t \leq 1, \\ 60 & \text{if } 1 < t \leq 2, \\ 60 - 60(t - 2) & \text{if } 2 < t \leq 3. \end{cases}$$

The domain of this function is the interval  $0 \leq t \leq 3$ , and we can see from its graph that its range is the interval  $0 \leq d \leq 60$ .

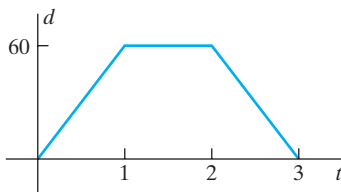


FIGURE 2.3.4 Graph of  $d(t)$ .



In Example 2, we began with a verbal description of the function and then constructed both a graphical description and a symbolic description. In the next example, we begin with a table of values.

### EXAMPLE 3

#### Finding and Graphing a Step Function

Figure 2.3.5 shows a portion of the rate table for UPS Next Day Air delivery from the state of Alaska.

- Any fraction of a pound more than the weight shown in the rate chart requires the next higher rate.

- The Letter rate applies only to document, correspondence and electronic media shipments sent in a UPS Express Envelope. Express Envelopes containing items other than those listed above and Express Envelopes exceeding one pound that are

Zones	22	24
	Alaska Metro	48 Contiguous States, Hawaii Metro and Puerto Rico
Letter	\$15.25	\$17.25
1 lbs.	21.00	27.25
2	23.75	30.00
3	26.25	32.50
4	28.25	35.50
5	30.25	38.00

FIGURE 2.3.5 Rates for package delivery from Alaska.

Source: www.ups.com.

- Find a piecewise-linear function  $R(w)$  that gives the rate to send a package from Alaska to one of the “lower 48” states as a function of its weight  $w$  if the weight is no more than 5 pounds.
- Graph  $R(w)$ .

#### SOLUTION

- In the table, we see the rates for the whole number of pounds from 1 through 5. Furthermore, the note accompanying the table tells us that any fraction of a pound more than the weight shown requires the next higher rate. That means that any package that weighs up to and including 1 pound costs \$27.25; a package weighing more than 1 pound but not more than 2 pounds costs \$30.00, and so on. So our rate function is given by

$$R(w) = \begin{cases} 27.75 & \text{if } 0 < w \leq 1, \\ 30.00 & \text{if } 1 < w \leq 2, \\ 32.50 & \text{if } 2 < w \leq 3, \\ 35.50 & \text{if } 3 < w \leq 4, \\ 38.00 & \text{if } 4 < w \leq 5. \end{cases}$$

Notice that  $R(w)$  is a special kind of piecewise-linear function because each of its linear pieces is also constant.

- The easiest way to graph such a function in the calculator is to store each part of the function rule in a separate location in the function editor. In order to get the proper graph, on each section of the domain we

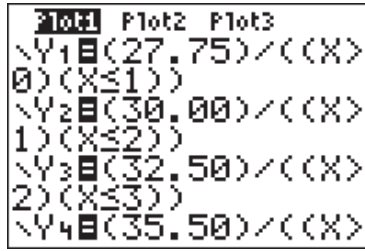


FIGURE 2.3.6 Entering  $R(w)$  into the calculator.

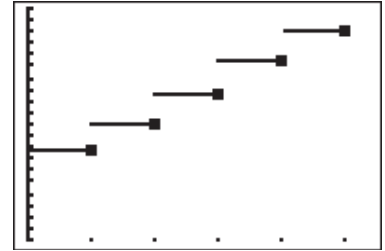


FIGURE 2.3.7 The calculator graph of  $R(w)$ .

put the function rule in parentheses and then use “/” to separate the rule from its domain restriction (also in parentheses). The first three pieces of the rule are shown in Fig. 2.3.6. Then the graph (with a couple of tricks to show the endpoints clearly) is shown in Fig. 2.3.7.

From the graph, you can see why we describe a function like  $R(w)$  as a “step” function.

Most people encounter piecewise-linear functions when they pay their income tax on or before April 15 of each year. While the IRS gives us its “rules” in table form, we can use the point-slope form to convert those rules into linear functions.

**EXAMPLE 4**

**Income Tax As a Piecewise-Linear Function**

Schedule X of the 2005 Form 1040 instructions specifies the tax paid by a single taxpayer as shown in Fig. 2.3.8.

If your taxable income is:		The tax is:	
<i>Over—</i>	<i>But not over—</i>		<i>of the amount over—</i>
\$0	\$7,300	----- 10%	\$0
7,300	29,700	\$730.00 + 15%	7,300
29,700	71,950	4,090.00 + 25%	29,700
71,950	150,150	14,652.50 + 28%	71,950
150,150	326,450	36,548.50 + 33%	150,150
326,450	-----	94,727.50 + 35%	326,450

FIGURE 2.3.8 Income tax for single taxpayers.

- Find a piecewise-linear function  $T(I)$  that gives the income tax owed for a taxable income of  $I$  dollars.
- Find the income tax owed by a single taxpayer with a taxable income of \$50,000.
- Find the income tax owed by a single taxpayer with a taxable income of \$150,000.

## SOLUTION

- a. The information in the IRS table determines a left-hand endpoint  $(I, T)$  and an average rate of change  $a$  for each interval. (This average rate of change over each interval is called the *marginal tax rate*. You can see that the marginal tax rate increases as the income increases.) The following table summarizes this information.

INTERVAL	LEFT ENDPOINT $(I, T)$	AVERAGE RATE OF CHANGE $a$
$0 < I \leq 7300$	$(0, 0)$	0.10
$7300 < I \leq 29700$	$(7300, 730)$	0.15
$29700 < I \leq 71950$	$(29700, 4090)$	0.25
$71950 < I \leq 150150$	$(71950, 14652.50)$	0.28
$150150 < I \leq 326450$	$(150150, 36548.50)$	0.33
$I > 326450$	$(326450, 94727.50)$	0.35

Using the point-slope form on the first interval, we have  $T(I) = 0 + 0.10(I - 0)$ , or  $T(I) = 0.10I$ . Similarly, on the second interval, we have  $T(I) = 730 + 0.15(I - 7300)$ . If we continue in this fashion for each subsequent interval, we have

$$T(I) = \begin{cases} 0.10I & \text{if } 0 < I \leq 7300, \\ 730 + 0.15(I - 7300) & \text{if } 7300 < I \leq 29700, \\ 4090 + 0.25(I - 29700) & \text{if } 29700 < I \leq 71950, \\ 14652.50 + 0.28(I - 71950) & \text{if } 71950 < I \leq 150150, \\ 36548.50 + 0.33(I - 150150) & \text{if } 150150 < I \leq 326450, \\ 94727.50 + 0.35(I - 326450) & \text{if } I > 326450. \end{cases}$$

- b. Because \$50,000 falls into the third interval, a single taxpayer with a taxable income of \$50,000 owes income tax of

$$T(50000) = 4090 + 0.25(50000 - 29700) = \$9165.$$

- c. \$150,000 falls into the fourth interval, so a single taxpayer with a taxable income of \$150,000 owes

$$T(150000) = 14652.50 + 0.28(I - 71950) = \$36,506.20.$$

This taxpayer has three times the taxable income of the taxpayer in (b) but owes almost four times as much tax. This illustrates the fact that the federal income tax is *progressive*—people with larger incomes pay larger percentages of their income in tax.

Most states also have income tax structures that are progressive and thus are represented by piecewise-linear functions. Several states have a “flat” tax rate (the same percentage regardless of income); such tax structures are simple linear functions. A few states tax only dividends and interest, using a flat tax rate.

## 2.3 Exercises Building Your Skills

Sketch the graphs of each function in Exercises 1–4. Label any points where the function rule changes.

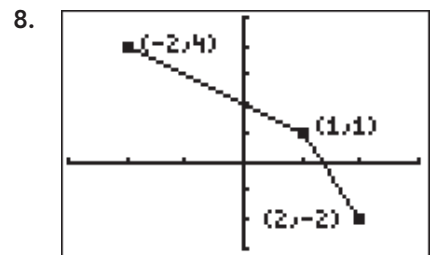
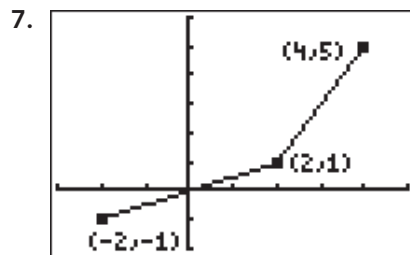
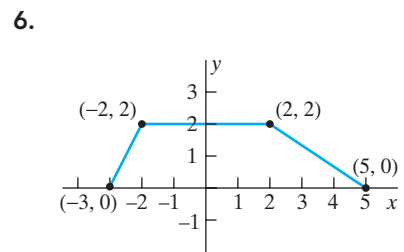
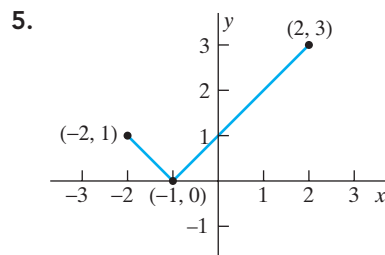
1.  $f(x) = |x - 1|$

2.  $f(x) = |x| - 1$

3.  $f(x) = \begin{cases} 2 & \text{if } x < 0 \\ 3 & \text{if } x \geq 0 \end{cases}$

4.  $f(x) = \begin{cases} 5 - 2x & \text{if } -1 \leq x < 2 \\ 1 + x & \text{if } 2 < x \leq 6 \end{cases}$

In Exercises 5–8 write a symbolic description of the function whose graph is pictured.



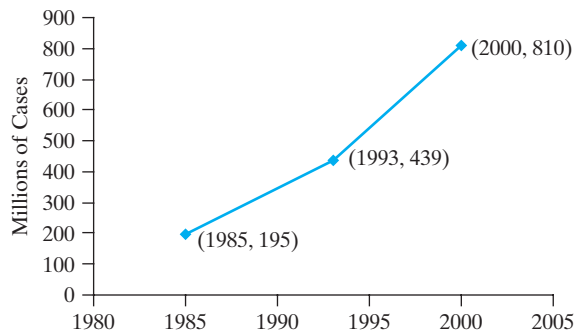
Each of the Exercises 9–12 describes a trip you made along a straight road connecting two cities 120 miles apart. Sketch the graph of the distance  $d$  from your starting point (in miles) as a function of the time  $t$  elapsed (in hours). Also describe the function  $d(t)$  symbolically.

9. You traveled for 1 hour at 45 miles per hour, then realized you were going to be late, and therefore traveled at 75 miles per hour for the next hour.
10. You traveled for 1 hour at 60 miles per hour, were suddenly engulfed in dense fog, and therefore drove back home at 30 miles per hour.
11. You traveled for 1 hour at 60 miles per hour, stopped for  $\frac{1}{2}$  hour while a herd of bison crossed the road, and then drove on toward your destination for the next hour at 60 miles per hour.
12. You traveled for  $\frac{1}{2}$  hour at 60 miles per hour, suddenly remembered that you'd left your wallet at home, drove back at 60 miles per hour to get it, and finally drove 2 hours at 60 miles per hour to reach your destination.

## Applying Your Skills

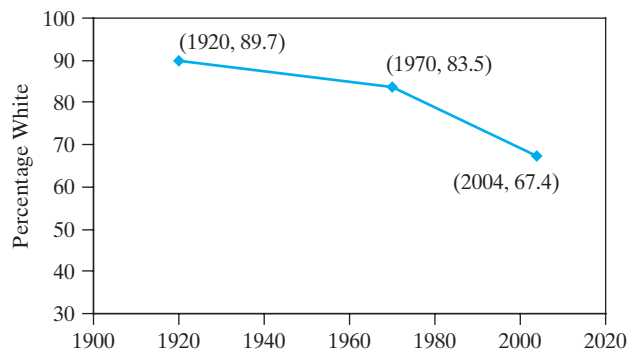
13. According to the U.S. Census Bureau, the population of Tucson, Arizona, was 45 thousand in 1950, and from 1950 to 1970 it grew at the average rate of 10.9 thousand people per year. After 1970, the population of Tucson grew at the average rate of 7.5 thousand people per year.
  - a. Find a piecewise-linear function  $P(t)$  that gives the population  $P$  of Tucson as a function of the year  $t$ .
  - b. According to your model, in what year did the population of Tucson reach 100 thousand?
  - c. Use your model to predict the population of Tucson in 2010.
14. According to the U.S. Census Bureau, the population of Charlotte, North Carolina, was 18 thousand in 1900, and from 1900 to 1950 it grew at the average rate of 2.32 thousand people per year. After 1950, the population of Charlotte grew at the average rate of 6.26 thousand people per year.
  - a. Find a piecewise-linear function  $P(t)$  that gives the population of Charlotte as a function of the year.
  - b. Use your model to determine the year in which the population of Charlotte grew to 500 thousand.
  - c. Based on your model, what was the population of Charlotte in 1980?
15. According to the U.S. Census Bureau, the population of Rochester, New York, was 163 thousand in 1900, and from 1900 to 1950 it grew at the average rate of 3.38 thousand people per year. After 1950, the population of Rochester declined at the average rate of 2.24 thousand people per year.
  - a. Find a piecewise-linear function  $P(t)$  that gives the population of Rochester as a function of the year.
  - b. Based on your model, what was the population of Rochester in 1932?
  - c. Use your model to predict the year in which the population of Rochester returns to its 1900 level.
16. According to the U.S. Census Bureau, the population of San Francisco was 775 thousand in 1950, and from 1950 to 1980 it declined at the average rate of 3.2 thousand people per year. After 1980, the population of San Francisco grew at the average rate of 4.9 thousand people per year.
  - a. Find a piecewise-linear function  $P(t)$  that gives the population of San Francisco as a function of the year.
  - b. Based on your model, what was the population of San Francisco in 1975?
  - c. Use your model to determine the year in which the population of San Francisco returned to its 1950 level.
17. For an express delivery letter weighing up to 8 ounces sent to a certain destination, the charge  $C$  is \$8.00 for the first 8 ounces plus 80 cents for each additional ounce or fraction thereof. Sketch the graph of the step function  $C(x)$  that gives the charge as a function of the total number  $x$  of ounces.
18. In a certain city, the charge  $C$  for a taxi trip of at most 10 miles is \$5.00 for the first 2 miles (or fraction thereof) plus 75 cents for each half-mile (or fraction thereof) up to a total of 10 miles. Sketch the graph of the step function  $C(x)$  that gives the charge as a function of the total number  $x$  of miles.
19. In 2006, the first-class mail rate in the United States was 39 cents for the first ounce and 24 cents for each additional ounce or fraction thereof (up to 13 ounces).

- a. Find a piecewise linear function  $P(x)$  that gives the first-class mail rate for letters not weighing more than 5 ounces.
- b. Sketch the graph of  $P(x)$ .
20. In 2006, the fee for insurance coverage on parcels sent from the United States to Canada was \$1.30 for values not over \$50, \$2.20 for values not over \$100, and \$1.00 for each additional \$100 of value (or fraction thereof), up to a value of \$700.
- a. Find a piecewise linear function  $F(v)$  that gives the fee for insurance coverage for parcels with value up to \$700.
- b. Sketch the graph of  $F(v)$ .
21. While other soft drinks struggled to retain their share of the market, Mountain Dew saw a dramatic increase in sales beginning in 1993, as indicated in Fig 2.3.13.



**FIGURE 2.3.13** Growth in Mountain Dew sales.  
Source: Atlanta Journal Constitution.

- a. Find a piecewise linear function  $C(t)$  that gives the number of cases sold (in millions) as a function of the year.
- b. Use your model to determine when the number of cases of Mountain Dew sold was 300 million.
- c. Use your model to determine when the number of cases of Mountain Dew sold was 500 million.
- d. According to your model, how many cases of Mountain Dew were sold in 2005?
22. The U.S. Census Bureau maintains a “population clock” that keeps track of the population of the United States. In 1967, as the population clock reached 200,000,000, a Chinese-American baby was born in Atlanta and designated (by *Life Magazine*) as the 200,000,000th American. Since that time, the population has continued to grow (reaching 300,000,000 by the time you read this), and has become increasingly diverse. The percentages of the population that are African-American, Asian, and Hispanic have increased, while the percentage of the population that is white has decreased, as indicated in Fig. 2.3.14.



**FIGURE 2.3.14** Change in race/ethnicity of U.S. population.  
Source: Atlanta Journal-Constitution.

- a. Find a piecewise linear function  $P(t)$  that gives the percentage of the population that is white as a function of the year.
- b. According to your model, what percentage of the population was white in 1949?
- c. According to your model, what percentage of the population will be white in 2010?
- d. Use your model to determine when the percentage of the population that is white was 75%.
23. The following table gives the 2004 Connecticut state tax rates for persons who are head of household. (Source for tax information in Exercises 23–27 is *The World Almanac and Book of Facts 2005*.)

IF YOUR TAXABLE INCOME IS OVER	BUT NOT OVER	THE TAX YOU OWE IS	OF THE AMOUNT OVER
\$0	\$16,000	3%	\$0
\$16,000		\$480 + 5%	\$16,000

- a. Find the tax owed on a taxable income of \$5700.
- b. Find the tax owed on a taxable income of \$16,000.
- c. Find the tax owed on a taxable income of \$43,000.
- d. Write a formula defining a piecewise linear function  $T(I) = \dots$  that gives tax owed in Connecticut as a function of taxable income.
24. The following table gives the 2004 Mississippi state tax rates for single persons.

IF YOUR TAXABLE INCOME IS OVER	BUT NOT OVER	THE TAX YOU OWE IS	OF THE AMOUNT OVER
\$0	\$5000	3%	\$0
\$5000	\$10,000	\$150 + 4%	\$5000
\$10,000		\$350 + 5%	\$10,000

- a. Find the tax owed on a taxable income of \$3350.
- b. Find the tax owed on a taxable income of \$6000.
- c. Find the tax owed on a taxable income of \$38,000.
- d. Write a formula defining a piecewise linear function  $T(I) = \dots$  that gives tax owed in Mississippi as a function of taxable income.
25. The following table gives the 2004 Louisiana state tax rates for married persons filing jointly.

IF YOUR TAXABLE INCOME IS OVER	BUT NOT OVER	THE TAX YOU OWE IS	OF THE AMOUNT OVER
\$0	\$12,500	2%	\$0
\$12,500	\$25,000	\$250 + 4%	\$12,500
\$25,000		\$550 + 6%	\$25,000

- a. Find the tax owed on a taxable income of \$4400.
- b. Find the tax owed on a taxable income of \$17,000.
- c. Find the tax owed on a taxable income of \$56,000.
- d. Write a formula defining a piecewise linear function  $T(I) = \dots$  that gives tax owed in Louisiana as a function of taxable income.

26. The following table gives the 2004 District of Columbia tax rates for married persons filing separately.

IF YOUR TAXABLE INCOME IS OVER	BUT NOT OVER	THE TAX YOU OWE IS	OF THE AMOUNT OVER
\$0	\$10,000	5%	\$0
\$10,000	\$30,000	\$500 + 7.5%	\$10,000
\$30,000		\$2000 + 9.3%	\$30,000

- Find the tax owed on a taxable income of \$4400.
  - Find the tax owed on a taxable income of \$17,000.
  - Find the tax owed on a taxable income of \$36,000.
  - Write a formula defining a piecewise linear function  $T(I) = \dots$  that gives tax owed in the District of Columbia as a function of taxable income.
27. The state of Michigan had a 2004 flat tax rate of 3.9% of taxable income.
- Find the tax owed on a taxable income of \$5700. Compare this value to the tax owed on the same amount in the state of Connecticut (found in Exercise 23).
  - Find the tax owed on a taxable income of \$43,000. Compare this value to the tax owed on the same amount in the state of Connecticut (found in Exercise 23).
  - Write a formula defining a linear function  $T(I) = \dots$  that gives tax owed in Michigan as a function of taxable income.
  - For what incomes would a taxpayer in Michigan pay more tax than a comparable taxpayer in Connecticut? For what incomes would the Michigan taxpayer pay less?

## 2.4 FITTING LINEAR MODELS TO DATA

In Section 2.1 we saw that a population whose growth is modeled by a linear function grows with a constant rate of change—that is, with the same change in population each year. In this section we discuss the modeling of data of a sort that might be said to display an “almost constant” rate of change—with the annual changes from year to year being approximately but not exactly equal. As an example of data that might therefore be described as “almost but not quite linear,” the following table shows the population of Charlotte, North Carolina, as recorded in the decade census years of 1950–1990.

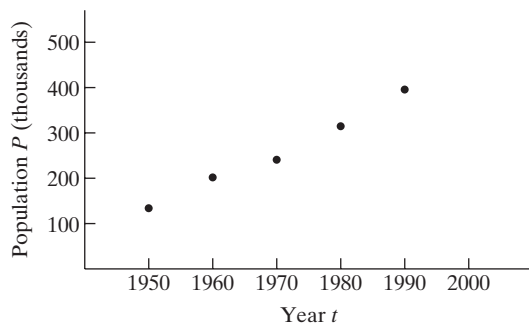
YEAR	POPULATION (THOUSANDS)	CHANGE (THOUSANDS)
1950	134	
1960	202	68
1970	241	39
1980	315	74
1990	396	81

Source: U.S. Census Bureau.



The third column of this table shows (for each decade year) the change in the population during the preceding decade. We see that the population of Charlotte increased by roughly 70 to 80 thousand people during the 1950s, the 1970s, and the 1980s. It increased by somewhat less during the 1960s, but still with a change measured in roughly comparable tens of thousands. We might wonder whether this qualifies as “almost linear” population growth. The way to answer such a question is to plot the data and take a look.

The data points corresponding to this table are plotted in Fig. 2.4.1. Surely most people would agree that these points appear to lie on or near some straight line.



**FIGURE 2.4.1** Scatter plot of the Charlotte 1950–1990 population data.

But how can we find a straight line that passes through or near each data point in the figure? One way is simply to pass a straight line through the first and last data points—those for 1950 and 1990.

## EXAMPLE 1

### Finding a Linear Population Model

Use the populations from 1950 and 1990 to find a linear function that describes (at least approximately) the population of Charlotte over this time period.

#### SOLUTION

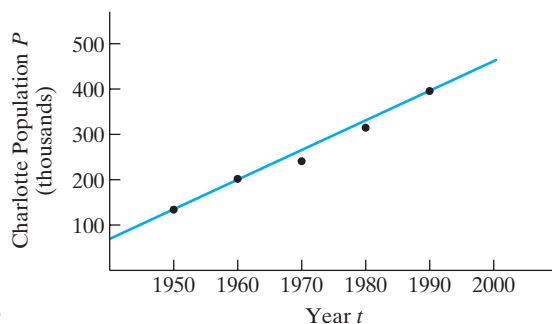
First, we find the average rate of change of this function:

$$a = \frac{\Delta P}{\Delta t} = \frac{396 - 134}{1990 - 1950} = \frac{262}{40} = 6.55 \text{ thousand/year.}$$

Then, using the point-slope form with  $t_1 = 1950$  and  $P_1 = 134$  gives the linear population model

$$P(t) = 134 + 6.55(t - 1950). \quad (1)$$

Figure 2.4.2 shows the graph of  $P(t)$  along with our original census population data points for Charlotte.



**FIGURE 2.4.2** The graph of  $P(t) = 134 + 6.55(t - 1950)$  approximating the 1950–1990 growth of Charlotte.

The linear function we found looks like a pretty “good fit” to the data—the 1950 and 1990 data points automatically lie on the line (why?), while the 1960 point seems to lie on the line and the 1970 and 1980 points lie just below the line. The following table shows the *discrepancies* between the actual populations and those “predicted” by the linear function in (3) for each of the 1950–1990 census years. We see that the actual 1960 population is just 2.5 thousand (only about 1%) larger than the 1960 population predicted by the linear model in (3), whereas the actual 1970 and 1980 populations are 24 thousand and 15.5 thousand, respectively, less than the corresponding linear predictions. We generally refer to these discrepancies as “errors,” and will do so from here on.

$t$	$P$ (ACTUAL) (THOUSANDS)	$P(t)$ (PREDICTED) (THOUSANDS)	ERROR $P - P(t)$ (THOUSANDS)
1950	134	134	0
1960	202	199.5	2.5
1970	241	265	-24
1980	315	330.5	-15.5
1990	396	396	0

Note that the figures in each (horizontal) row of the figure satisfy the formula

$$P_{\text{actual}} = P(t) + \text{error}.$$

## Smaller Errors and Fitting Data Better

Now we discuss the concept of a linear model that **best fits** given population (or other) data such as those in the following table:

$t$ (year)	$t_1$	$t_2$	$t_3$	$t_4$
$P$ (thousands)	$P_1$	$P_2$	$P_3$	$P_4$

In the case of the Charlotte population, there were five given data points. However, here we assume for simplicity in the general discussion that four data points are given. The final procedure will be analogous whatever the number of data points.

The table gives the actual populations  $P_1, P_2, P_3,$  and  $P_4$  specified at  $n = 4$  different times  $t_1, t_2, t_3,$  and  $t_4$ . We may ask what linear model of the form

$$P(t) = b + at \quad (2)$$

“best fits” the given data. That is, what should the numerical values of the coefficients  $a$  and  $b$  be in order that the model best fits the data? But the real question is: What does this mean? What does it mean for the model to “fit” the data well?

Figure 2.4.3 shows the errors that correspond to the discrepancies between the given data points in the  $tP$ -plane and the straight-line graph of (2). The  $i$ th error  $E_i$  is the vertical distance between the actual data point  $(t_i, P_i)$  and the corresponding point  $(t_i, P(t_i))$  on the line that is “predicted” by the linear model. It

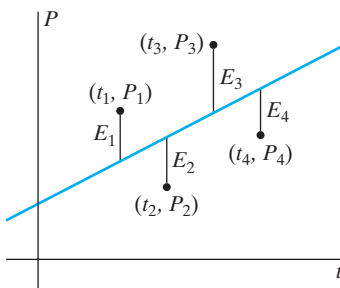


FIGURE 2.4.3 The errors in the linear model  $P(t) = at + b$ .

is worth emphasizing that  $P_i$  is the *actual* observed value of the population at time  $t_i$ , while  $P(t_i)$  is the value *predicted* by the linear model, so

$$\text{error} = \text{actual} - \text{predicted}. \quad (3)$$

One might suspect that the linear model fits the data well if the *sum of the errors* is small. However, observe that each error  $E_i$  defined is *signed*. It is positive if the  $i$ th data point lies above the graph of  $P(t) = at + b$  but is negative if the data point lies below the line. Consequently, it is possible for large positive errors to cancel out large (in absolute value) negative errors in the sum of all the errors. For instance, Fig. 2.4.3 indicates two large positive errors (data points above the line) and two large negative errors (data points below the line). Although each of these four errors is numerically large, their sum may be quite small, or even 0, just as

$$(+72) + (-54) + (+29) + (-47) = 0.$$

Thus the fact that the sum of the errors is small does *not* guarantee that all the individual errors are small.

It is therefore customary to use the sum of the squares of the errors as a measure of the overall discrepancy between the given data points and a proposed linear model.

**DEFINITION: Sum of Squares of Errors**

The phrase “**Sum of Squares of Errors**” is so common in data modeling that it is abbreviated to **SSE**. Thus the **SSE** associated with a data model based on  $n$  data points is defined by

$$\text{SSE} = E_1^2 + E_2^2 + E_3^2 + \cdots + E_n^2. \quad (4)$$

Note that, however many data points are given, **the SSE is the sum of the squares of all their errors**. In plain words, if we write the actual populations in one column and the predicted populations in a second column, then the SSE is the sum of the squares of the differences between corresponding entries in the two columns:

$t_i$	ACTUAL $P_i$	LINEAR $P(t_i)$	ERROR $E_i$	$E_i^2$
1950	134	134	0	0
1960	202	199.5	2.5	6.25
1970	241	265	-24	576
1980	315	330.5	-15.5	240.25
1990	396	396	0	0

**EXAMPLE 2**

Finding the SSE for a Linear Model

Find the SSE for the linear population model in Example 1.

**SOLUTION**

Recall that the linear model for Charlotte’s population in Example 1 is

$$P(t) = 134 + 6.55(t - 1950). \quad (5)$$

The table showing the Charlotte census data now includes a final column showing the squares of the errors. The SSE is the sum of the numbers in this final column; that is,

$$\text{SSE} = 0 + 6.25 + 576 + 240.25 + 05 = 822.5.$$

- So the SSE for our Charlotte population model is 822.5.

The linear function  $P(t) = 134 + 6.55(t - 1950)$  can be simplified as follows:

$$P(t) = 134 + 6.55t - 12772.5$$

$$P(t) = 6.55t - 12638.5. \quad (6)$$

So  $P(t) = 6.55t - 12,638.5$  in (6) corresponds to choosing the numerical values  $a = 6.55$  and  $b = -12638.5$  in the slope-intercept linear model  $P(t) = b + at$ . We wonder if it is possible to find different numerical values of  $a$  and  $b$  that yield a smaller SSE than the value 822.5 found in Example 2.

### EXAMPLE 3

#### Finding a Smaller SSE

Find a linear function with a smaller SSE for the Charlotte population data.

#### SOLUTION

Looking at Fig. 2.4.2 as well as at the table for Example 2, we see that the graph of  $P(t) = 6.55t - 12,638.5$  passes right through the first and last data points, but passes 24 units above the third data point. Let's think of a line that splits the difference and passes 12 units below the first and last data point as well as 12 units above the third data point. Can you see that we can make this change by subtracting 12 from the value  $b = -12638.5$  in (8), so that the new line lies 12 units lower in the  $tP$ -plane? Our new linear model is

$$P(t) = 6.55t - 12650.5 \quad (7)$$

with  $m = 6.55$  as in (6) but now  $b = -12650.5$  (rather than  $-12,638.5$ ). The following table shows the computation of the SSE for this new altered linear model. (We started with the previous table, then recalculated the numbers in the third, fourth, and fifth columns in turn. You should verify the results for yourself.)

$t_i$	ACTUAL $P_i$	LINEAR $P(t_i)$	ERROR $E_i$	$E_i^2$
1950	134	122	12	144
1960	202	187.5	14.5	210.25
1970	241	253	-12	144
1980	315	318.5	-3.5	12.25
1990	396	384	12	144

Now the sum of the numbers in the final column is

$$\text{SSE} = 144 + 210.25 + 144 + 12.25 + 144 = 654.5$$

- (as compared with 822.5 for the linear model of Example 2). Thus we have succeeded in reducing somewhat the SSE.

So although either of the functions

$$P(t) = 6.55t - 12638.5 \quad \text{or} \quad P(t) = 6.55t - 12650.5 \quad (8)$$

can be used as a linear model for the growth of Charlotte during the 1950–1990 period, the latter one fits the observed census data at least slightly better because it has a smaller sum of squares of errors.

But exactly *what* does the SSE really mean? Well, the SSE for either of the models in (8) is a sum of five “squared errors,”

$$\text{SSE} = E_1^2 + E_2^2 + E_3^2 + E_4^2 + E_5^2.$$

We should therefore divide by 5 to get the average of these squared errors,

$$\text{average squared error} = \frac{E_1^2 + E_2^2 + E_3^2 + E_4^2 + E_5^2}{5} = \frac{\text{SSE}}{5}.$$

But then we should take the *square root* of the average *squared* error to get the average error itself,

$$\text{average error} = \sqrt{\frac{\text{SSE}}{5}}.$$

This is for Charlotte with  $n = 5$  data points. To define the average error for a model fitting  $n$  given data points we need only divide the SSE instead by  $n$ .

**DEFINITION: Average Error**

The **average error** in a linear model fitting  $n$  given data points is defined in terms of its SSE by

$$\text{average error} = \sqrt{\frac{\text{SSE}}{n}}. \quad (9)$$

This formula says simply that **the average error is the square root of the average of the squares of the individual errors** (or discrepancies between predicted and actual data values).

**EXAMPLE 4**

Finding Average Error

Find the average error for each of the population models in (8).

**SOLUTION**

For the first linear model  $P(t) = 6.55t - 12638.5$  in (8), we calculated its  $\text{SSE} = 822.5$ , so with  $n = 5$  in (9) we find that its average error is

$$\text{average error} = \sqrt{\frac{822.5}{5}} \approx 12.826.$$

But the SSE for the second linear model  $P(t) = 6.55t - 12650.5$  in (8) is only 654.5, so *its* average error is

$$\text{average error} = \sqrt{\frac{654.5}{5}} \approx 11.441.$$

These average errors—like the populations themselves—are measured in thousands. We may therefore say that the populations “predicted” by our first model for the five census years 1950, 1960, 1970, 1980, and 1990 err by an average of 12,826 people, whereas the predictions of our second model err by an average of 11,441 people. This is a tangible statement of the extent to which the second model fits the actual census data better than does the first model.

## The Best Fit—The Least Possible Average Error

The question is this: What choice of the numerical coefficients  $a$  and  $b$  in the linear model  $P(t) = b + at$  will minimize the average error—that is, will result in the least possible average error? This optimal linear model will be the one that we say “best fits” the given data.

Students in past generations often plotted data points on a piece of graph paper, then carefully maneuvered a ruler so that it appeared visually to come as close as possible to these points “on the average.” But the modern graphing calculator comes equipped with the facility to solve “best-fitting” problems more precisely. Here we describe how this is done. You should use your graphing calculator to carry out the steps we describe.

Figure 2.4.4 shows how to enter the Charlotte population data in a calculator. The list of census years from 1950 to 1990 is stored as a list **L1**, and the corresponding list of recorded populations is stored as a list **L2**. (Note that calculator “lists” are enclosed in curly braces.) Figure 2.4.5 the resulting **STAT EDIT** menu displaying the data in table form. Here the individual items—either a year or the corresponding population—can be changed or “edited” one at a time (for instance, if an error was made in entering either list originally).

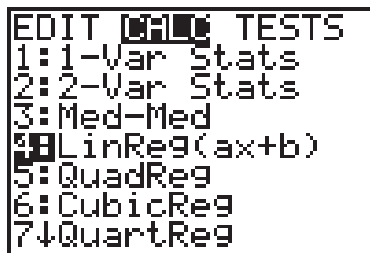
```
{1950, 1960, 1970,
1980, 1990} → L1
{1950 1960 1970...
{134, 202, 241, 315
, 396} → L2
{134 202 241 31...
```

FIGURE 2.4.4 Storing the 1950–1990 population census data for Charlotte.

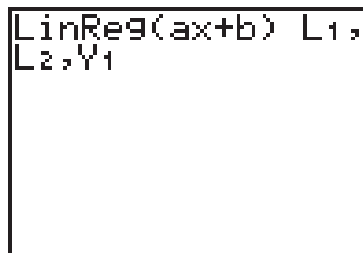
L1	L2	-----	1
1950	134		
1960	202		
1970	241		
1980	315		
1990	396		
-----	-----		
L1(1) = 1950			

FIGURE 2.4.5 The TI-84 **STAT EDIT** menu.

Figure 2.4.6 shows the calculator's **STAT CALC** menu. Item **4: LinReg(ax+b)** on this menu is the calculator's so-called “linear regression” facility for finding the linear function  $y = ax + b$  that best fits the selected data. (Curve-fitting is called “regression” in statistics and calculator lingo.) When we select item **4** on this menu, the **LinReg(ax+b)** function is entered on the home calculating screen. As shown in Fig. 2.4.7, we must then enter the names of our list **L1** of input values, our list **L2** of output values, and the name **Y1** of the **Y=** menu variable where we want the resulting linear formula saved.



**FIGURE 2.4.6** The TI-84 **STAT CALC** menu.

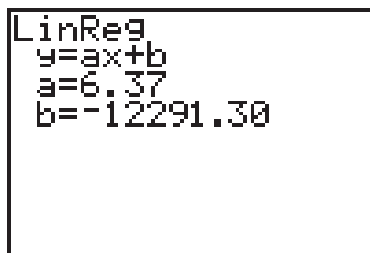


**FIGURE 2.4.7** Fitting the **X**-data in list **L1** and the **Y**-data in list **L2**.

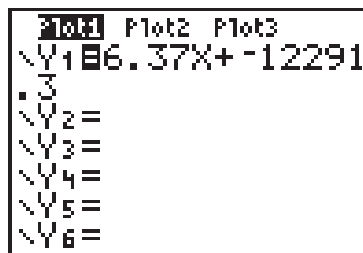
Figure 2.4.8 shows the display that results when this command is entered, showing that the linear function that best fits our data is

$$y = 6.37x - 12291.3. \quad (12)$$

The **LinReg(ax+b) L1, L2, Y1** command automatically enters Equation (12) in the **Y=** menu, ready for plotting as shown in Fig. 2.4.9. With **Plot 1** turned **On** in the **STAT PLOT** menu (Fig. 2.4.10), **GRAPH** then gives the plot of the best fitting linear function shown in Fig. 2.4.11, where the original census data points are shown as small squares.



**FIGURE 2.4.8** The best straight-line fit to our data points.



**FIGURE 2.4.9** The best-fitting linear function ready for graphing.

The calculator automatically uses **X** as the input variable and **Y** as the output variable. For our calendar year–population situation, we use  $t$  as the input vari-

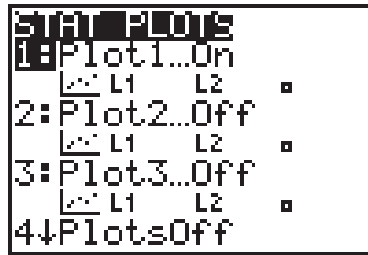


FIGURE 2.4.10 Preparing to plot the data with the best-fit straight line.

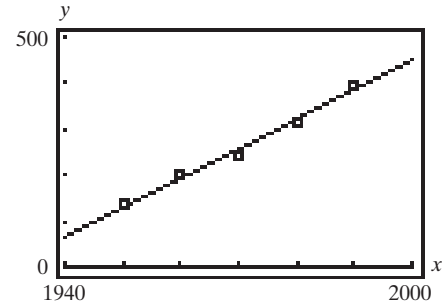


FIGURE 2.4.11 The line of best fit.

able and  $P$  as the output variable, and Equation (12) says that the linear model that best fits the 1950–1990 Charlotte census data is given by

$$P(t) = 6.37t - 12291.3 \quad (13)$$

This is the model that “best fits” the data in the sense that it has the smallest sum of the squares of the errors (SSE). While we cannot show that no model has a smaller SSE (this would involve mathematics beyond the scope of this text), we can demonstrate that the SSE for this model is smaller than either of the two we have already found. Should you want further examples, you can construct some linear models of your own and find their (larger) SSEs.

## EXAMPLE 5

### Finding the Average Error for the Best-Fitting Model

Find the SSE and the average error for the linear model that best fits the 1950–1990 Charlotte census data.

#### SOLUTION

The following table gives the actual population values and the values predicted by our best-fitting linear model, along with the errors and their squares. (You should verify these table entries.)

$t$ (CALENDAR YEAR)	$P$ (ACTUAL POPULATION IN THOUSANDS)	$P(t)$ (PREDICTED POPULATION IN THOUSANDS)	ERROR (ACTUAL – PREDICTED) (THOUSANDS)	ERROR SQUARED (THOUSANDS)
1950	134	130.2	3.8	14.44
1960	202	193.9	8.1	65.61
1970	241	257.6	–16.6	275.56
1980	315	321.3	–6.3	39.69
1990	396	385	11.0	121.00



Hence the SSE associated with the optimal linear model in (13) is

$$14.44 + 65.61 + 275.56 + 39.69 + 121.00 = 516.30.$$

Therefore the average error is given by

$$\text{average error} = \sqrt{\frac{516.30}{5}} \approx 10.162.$$

Thus the linear model  $P(t) = 6.37t - 12291.3$  predicts 1950–1990 census year populations that differ (on the average) by 10,162 people from those actually recorded for Charlotte.

- By comparison, our first linear model  $P(t) = 6.55t - 12638.5$  had an SSE of 822.5 and an average error of 12.826, while our second model  $P(t) = 6.55t - 12650.5$  had an SSE of 654.5 and an average error of 11.441.

Once we have a linear model that predicts the 1950–1990 census year populations for Charlotte, we can use it to predict the population of Charlotte in years for which we have no data. This is really the point of mathematical modeling since it would be foolish to use an approximate value for the population in 1970 when we have the exact value.

This issue raises several points of which you should be aware. First, your calculator will always find you a “best” linear model, whether your data “looks” linear or not. Before constructing such a model, you should consider whether it is appropriate. You can do this by plotting your data or finding the average rates of change between consecutive data points to verify that the average rate of change is more or less constant. (A set of data that increases and decreases at noticeably different rates is not a good candidate for a linear model.)

Second, you should pay attention to the type of prediction you plan to make. If you are making a prediction about a year between two data points, say 1962 (between census years 1960 and 1970), this is called **interpolation**. It is generally safe to make such a prediction because, barring some unusual circumstance, the population is not likely to fluctuate wildly over this 10-year period. (Of course, such unusual circumstances do sometimes occur. Consider the population of New Orleans on January 1, 2005, before Hurricane Katrina, compared to its population on January 1, 2006, approximately 4 months after the hurricane.)

If you make a prediction about a year earlier or later than all of your data, this is called **extrapolation**. You must be careful about predicting too far beyond (or before) the information you have because circumstances that affect the actual population may be very different from those existing when the data are collected.

## Applications of Linear Modeling

Thus far, we have discussed only the modeling of linear population growth. But the world is full of other apparently linear data waiting to be modeled.

## EXAMPLE 6

## Finding an Appropriate Linear Model

The per capita consumption of cigarettes in 1930 and the lung cancer death rate (deaths per million males) for 1950 in the four Scandinavian countries were as follows:

COUNTRY	CIGARETTE CONSUMPTION IN 1930 $c$	DEATH RATE IN 1950 $D$
Norway	250	95
Sweden	300	120
Denmark	350	165
Finland	1100	350

It is hard to ignore the fact that higher cigarette consumption appears to be correlated with higher lung cancer death rates 20 years later.

- Verify that this relationship is approximately linear.
- Find the linear model that best fits this data.

## SOLUTION

- We do **MEM ClrAllLists** to purge the Charlotte population data from our calculator, then enter the  $c$ - and  $D$ -data in the lists **L1** and **L2** (Fig. 2.4.12). Then we use the command **L3=ΔList(L2)/ΔList(L1)** (as we did in Section 1.4) to determine whether the average rate of change is approximately constant. The result is shown in Fig. 2.4.13.

```
{250, 300, 350, 1100}
0} → L1
{250 300 350 1100}
{95, 120, 165, 350}
→ L2
{95 120 165 350}
```

FIGURE 2.4.12 The cigarette consumption and lung cancer death rate of Example 6.

L1	L2	L3	3
250.00	95.00	.30	
300.00	120.00	.90	
350.00	165.00	.25	
1100.0	350.00	-----	
-----	-----		
L3(1) = .5			

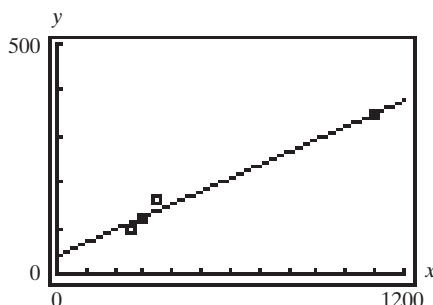
FIGURE 2.4.13 Checking the average rate of change.

At first glance, the average rates of change stored in **L3** might appear very different. But notice that each one is positive and between 0 and 1. So we conclude that the average rate of change for this table of values is approximately constant, and so we can model these data with a linear function.

- Now we use the **STAT CALC** command **LinReg(ax+b) L1, L2, Y1**, which produces the linear function

$$y = 0.28x + 40.75, \quad (14)$$

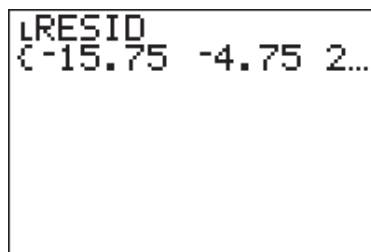
which best fits our data (with our calculator set in two-decimal-place mode). With an appropriate window and the same **STAT PLOT** settings as before, we get the plot of this linear function and the table data points that is shown in Fig. 2.4.14.



**FIGURE 2.4.14** The line best fitting the cigarette consumption versus lung cancer deaths data.

Thus, using function notation and our  $c$  and  $D$  variables, we find that the linear model that best fits the given data is  $D(c) = 0.28c + 40.75$ .

The calculator also calculates automatically the discrepancies between the data and the predictions of the linear model in (14). These errors are stored in the **LIST** menu under the name **RESID**. If you scroll through the list indicated in Fig. 2.4.15, you see that these errors or “residuals” are  $-15.75$ ,  $-4.74$ ,  $26.25$ , and  $1.25$ .



**FIGURE 2.4.15** The errors or “residuals” in Example 6.

Hence the average error of our linear model is

$$\text{average error} = \sqrt{\frac{(-15.75)^2 + (-4.75)^2 + (26.25)^2 + (1.25)^2}{4}} = 15.50.$$

Thus the average error in the linear model’s predictions (for the four Scandinavian countries) is 15.5 lung cancer deaths per million males.

For a comparison, the 1930 per capita cigarette consumption in Australia was 470, so the linear model predicts

$$D(470) = (0.28)(470) + 40.75 = 172.35 \approx 172$$

lung cancer deaths per million males in Australia. The actual number of such deaths in Australia in 1950 was 170 per million. (Given an average error of 15.50, this agreement is better than we have any right to expect, especially ignoring any

differences there may be between Australian and Scandinavian health and lifestyles.)

Finally, let us note what the linear model  $D(c) = 0.28c + 40.75$  means. If we substitute  $c = 0$  (no cigarette consumption), we get  $D(0) = 40.75$ . Thus we should expect 40.75 lung cancer deaths per million males even if no cigarettes at all are smoked. Then the average rate of change (or slope)  $a = 0.28$  implies an additional 0.28 death per unit increase in per capita cigarette consumption. That is, since  $4 \times 0.28 = 1.12$ , there should be approximately one additional death for each four-unit increase in per capita cigarette consumption. This sort of *interpretation* of a linear model frequently is more important than any specific numerical predictions of the model.

Before you begin working on the exercises, we need to discuss decimal-place settings on the calculator. When using real-world data, the values of  $a$  and  $b$  in the best-fitting linear model seldom come out “nicely.” This means that if your calculator is in “float” mode, these values could be reported with as many as 10 decimal places. Because we frequently give our population values in thousands, it is convenient for interpreting the model if  $a$  and  $b$  are given as three-place decimals. Thus, the answers given in the back of the book reflect a calculator setting of three decimal places. If you use a different setting on your calculator, your answers may vary a bit from ours.

## 2.4 Exercises | Building Your Skills

In Exercises 1–4 the population  $P$  (in thousands) of a city in three different years is given, thereby providing three known  $(t, P)$  data points. Find the average error for the linear function  $P(t) = at + b$  whose graph contains:

- The first and third of these points.
- The second and third of these points.

(Note that in some exercises we have “reset the clock,” while in others we use actual calendar years.)

1.

$t$ (years)	0	10	20
$P$ (thousands)	200	270	320

2.

$t$ (years)	0	10	20
$P$ (thousands)	460	390	300

3.

$t$ (years)	1980	1990	2000
$P$ (thousands)	715	605	435

4.

$t$ (years)	1980	1990	2000
$P$ (thousands)	615	805	1155

In Exercises 5–8 the population  $P$  (in thousands) of a city in four different years is given, thereby providing four known  $(t, P)$  data points. For each set of data, find:

- The linear model that best fits the data.
- The SSE and average error for this optimal linear model.

5.

$t$ (years)	0	10	20	30
$P$ (thousands)	360	300	300	240

6.

$t$ (years)	0	10	20	30
$P$ (thousands)	240	320	360	360

7.

$t$ (years)	1960	1970	1980	1990
$P$ (thousands)	600	800	950	1050

8.

$t$ (years)	1960	1970	1980	1990
$P$ (thousands)	565	485	385	265

## Applying Your Skills

Exercises 9–12 give the 1950–1990 U.S. census data for a city whose average rate of growth is approximately constant.

- Find the linear model  $P(t) = at + b$  that best fits this census data. (You may “reset the clock” or not, as you prefer.)
- Construct a table showing the actual populations, predicted populations, and errors for the decade years 1950–1990.
- Use your linear model to predict the city’s population in the year 2000.

9. San Diego, California:

$t$ (years)	1950	1960	1970	1980	1990
$P$ (thousands)	334	573	697	876	1111

10. Buffalo, New York:

$t$ (years)	1950	1960	1970	1980	1990
$P$ (thousands)	580	533	463	358	328

11. Newark, New Jersey:

$t$ (years)	1950	1960	1970	1980	1990
$P$ (thousands)	439	405	382	329	275

12. Garland, Texas:

$t$ (years)	1950	1960	1970	1980	1990
$P$ (thousands)	11	39	81	139	181

13. The following table shows the percentage of a three-person American family's weekly income spent on gasoline for the indicated year.

YEAR $t$	1966	1976	1986	1996	2006
Percentage Spent on Gasoline $P$	5.5	4.4	3.0	2.7	4.2

Source: St. Louis Post-Dispatch.

- Use a plot of the data to determine whether a linear function is a suitable model for these data.
  - If a linear model is suitable, find the linear function that best fits the data.
14. The following table shows the revenue in billions of dollars for the state of Georgia for the years 1996 through 2000.

YEAR $t$	1996	1997	1998	1999	2000
Revenue $R$ (billions of dollars)	9.9	10.5	11.1	12.1	13

Source: Atlanta Journal-Constitution.

- Use a plot of the data to determine whether a linear function is a suitable model for these data.
  - If a linear model is suitable, find the linear function that best fits the data.
15. The following table below shows the U.S. life expectancy (estimated number of years of life remaining) as a function of current age.

AGE $A$	0	10	20	35	50	70
Years of Life Remaining $L$	77.2	67.9	58.1	43.9	30.3	14.6

Source: The World Almanac and Book of Facts 2005.

- Use a plot of the data to determine whether a linear function is a suitable model for these data.
  - If a linear model is suitable, find the linear function that best fits the data.
16. The following table shows the total number of marine species as a function of the total number of gallons of tanks for various U.S. aquariums.

Gallons $g$	1	1.3	2.5	2.5	4.75	5
Marine Species $M$	692	536	536	625	1600	500

Source: Atlanta Journal-Constitution.

- a. Use a plot of the data to determine whether a linear function is a suitable model for these data.
- b. If a linear model is suitable, find the linear function that best fits the data.
17. The following table gives the number of CDs (in millions) sold in the United States for the even-numbered years 1988 through 1996. A plot of these data indicates that CD sales were increasing at a reasonably constant rate over these years.

T (YEAR)	1988	1990	1992	1994	1996
<i>S</i> (millions of CDs)	149.7	286.5	407.5	662.1	778.9

Source: *The World Almanac and Book of Facts 1998* and *The World Almanac and Book of Facts 2005*.

- a. Find the linear model  $S(t) = at + b$  that best fits these data.
- b. Compare the model's prediction for the year 1995 with the actual 1995 CD sales of 722.9 million.
- c. Compare the model's prediction for the year 2002 with the actual 2002 CD sales of 803.3 million.
- d. What do you think may have accounted for the discrepancy between the 2002 predicted and actual values?
18. The following table gives the number of cassette tapes (in millions) sold in the United States for various years between 1994 and 2001.

T (YEAR)	1994	1995	1997	2000	2001
<i>S</i> (millions of cassettes)	345.4	272.6	172.6	76.0	45.0

Source: *The World Almanac and Book of Facts 2005*.

- a. Find the linear model  $S(t) = at + b$  that best fits these data.
- b. Explain the meaning of the average rate of change for this linear model in terms of the situation.
- c. According to this model, in what year will no more cassette tapes be sold? Is this answer reasonable?
19. The following table gives the number of athletes participating in the Winter Olympics for various years between 1924 and 2002 as a function of the number of countries represented.

Number of countries <i>C</i>	16	25	37	49	64	77
Number of Athletes <i>A</i>	258	464	1072	1272	1801	2399

Source: *National Council of Teachers of Mathematics*.

- a. Find the linear model that best fits these data.
- b. What does the average rate of change of the model mean in terms of the situation?
- c. Use your model to predict the number of athletes participating if 80 countries are represented.

20. The following table gives the winning times (in seconds) for the men's 400-meter hurdles in the Summer Olympics as a function of the year.

$t$ (years after 1920)	0	8	16	36	48	56
$W$ (seconds)	54.0	53.4	52.4	50.1	48.12	47.64

Source: *The World Almanac and Book of Facts 2005*.

- Find the linear model  $W(t)$  that best fits these data.
  - Find the average error for the linear model, and explain what it means in terms of the situation.
  - Given that the Summer Olympics are held only every 4 years, during which Summer Olympics does the model predict that the winning time will fall below 45 seconds?
21. The following table gives the 2004 military monthly pay scale for a Warrant Officer (W-4) based on years of service.

$Y$ (years of service)	2	4	8	12	16	20
$P$ (pay in dollars)	3356	3547	3872	4194	4617	4944

Source: *The World Almanac and Book of Facts 2005*.

- Find the linear model that best fits these data.
  - According to your model, what is the pay for a Warrant Officer (W-4) with 15 years of service?
  - Use a graph or a table of values for this model to find the years in which the model underestimates a warrant officer's pay.
22. The Consumer Price Index (CPI) is a measure of the cost to consumers of various goods and services in comparison to the cost of those items in previous years. The following table gives the CPI for all urban consumers for housing for selected years from 1987 to 1996. The years 1982–1984 serve as the base years for comparison, with the cost of housing in those years equal to 100.

$T$ (years after 1987)	0	3	4	7	9
$CPI$	115.6	130.5	135.0	145.4	154.0

- Find the linear model that best fits these data.
- Use this linear model to predict the CPI for housing in 1989. Is your answer a reasonable approximation to 124.9, the actual 1989 CPI for housing? Explain your answer in terms of the plot of the data or the average rates of change between consecutive table entries.
- Use the linear model to predict the year in which the CPI for housing will reach 165. What assumption are you making when you make this prediction?
- Use the linear model to predict when the CPI for housing will reach 250. Why is it *not* a good idea to make such a prediction?



23. Thus far we have constructed linear models for data that represent a function of some independent variable. Frequently in the “real world,” we are confronted with a set of data that does not actually describe a function, but that suggests an underlying relationship that might be modeled by a function. An example of such data is the relationship between height and weight for the seven infielders on the Los Angeles Dodgers roster on July 12, 1997.

<i>Height in inches <math>h</math></i>	70	74	71	71	76	70	73
<i>Weight in pounds <math>W</math></i>	163	170	180	145	222	185	200

- Find the linear model that best fits these data.
- Use the linear model to predict the weight of a major league infielder who is 6 feet tall.
- Should you use this model to predict the weight of any American male who is 6 feet tall? Why or why not?

## Chapter 2 | Review

In this chapter, you learned about linear and piecewise-linear function models. After completing the chapter, you should be able to:

- Determine whether a function described numerically, graphically or symbolically represents a linear function or a piecewise-linear function.
- Determine the average rate of change of a linear function.
- Find a linear function that models given linear data.
- Find a piecewise-linear function that models given piecewise-linear data.
- Find the best-fitting linear model for data that is “approximately” linear.
- Interpret the slope and  $y$ -intercept of a linear model in terms of the situation modeled.

## Review Exercises

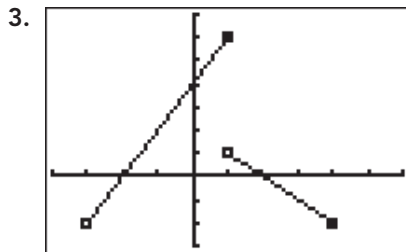
In Exercises 1–6, decide whether, based on the table, graph, or formula,  $y$  is a linear or a piecewise-linear function of  $x$ .

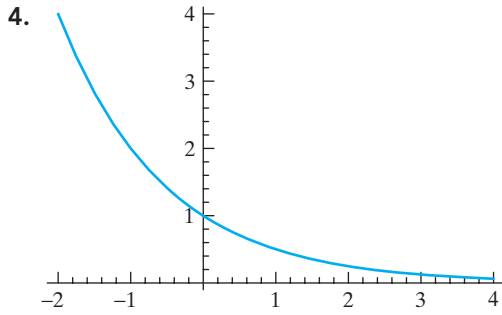
1.

$x$	2	4	5	6	8	11
$y$	3	1.5	0	-1.5	-3	-4.5

2.

$x$	2	4	6	8	10	12
$y$	3	1.5	0	-1.5	-3	-4.5





5.  $3x + 4y = 2$

6.  $y = x^2 + 3x$

7. Find the average rate of change for the linear function given by the following table.

$x$	-3	0	1.5	4	6
$f(x)$	-7.2	-3	-0.9	2.6	5.4

8. The population of Libertyville was 227 thousand in 2000 and was decreasing at an average rate of 2.3 thousand people per year.

- Find a linear population model  $P(t)$  that gives the population of Libertyville as a function of the number of years after 2000.
- According to your model, what was the population of Libertyville in 2000?
- Use your model to predict the year in which the population of Libertyville falls to 200 thousand.

9. In 1989, 51% of students at Ohio State University graduated within 6 years of entering the university. By 1999, the percentage graduating within 6 years had risen to 68%.

- Find a linear model that gives the percentage of students graduating within 6 years as a function of the year.
- Explain what the average rate of change of your model means in terms of the situation.
- Use your model to predict the percentage of students graduating within 6 years in 2006.

10. Economists frequently use linear functions to construct models for the relationship between price and demand for a small business. Generally, we consider the selling **price**  $p$  of the item to be a linear function of the **demand**  $x$  (the number of units consumers will purchase). Pomelia has begun a home business making decorative lawn sprinklers out of copper tubing. She has found that she can sell 20 sprinklers per month if

they are priced at \$60 each but only 12 sprinklers per month if they are priced at \$70.

- Find a linear function  $p(x)$  giving the price of the sprinklers as a function of the number sold.
- How many sprinklers will she sell if the sprinklers are priced at \$55?
- If Pomelia wants to sell at least 30 sprinklers per month, what price should she charge?
- Explain what the intercepts of the line mean in this situation.

11. According to Knight-Ridder Newspapers, homeowners who hire a tree service to grind and remove tree stumps from their yards can expect to pay about \$95 for the first 10 inches of tree stump diameter and \$1 for each additional inch thereafter.

- Find a piecewise-linear function  $C(d)$  that gives the charge for stump grinding and removal as a function of the diameter of the stump in inches.
- Find  $C(16)$  and explain what it means in terms of the situation.
- If a homeowner paid \$108 for stump grinding and removal, what was the diameter of the tree stump?

12. Nambe Mills in Santa Fe, New Mexico, manufactures tableware made of a metal alloy. The pieces are sand casted and then shaped, ground, buffed, and polished. The table following gives the total grinding and polishing times (in minutes) for various pieces of Nambeware and their corresponding prices (in dollars).

<i>Time</i>	109.38	16.41	23.77	13.25	44.25	64.30	34.16
<i>Price</i>	260	39	49.50	31	89	165	75

Source: Nambe Mills.

- Find the best-fitting linear model for price as a function of grinding and polishing time.
  - According to your model, what should be the price of a piece of Nambeware if its grinding and polishing time is 53.18 minutes?
  - Interpret the slope of your model in terms of the situation.
13. (Chapter Opener Revisited) In the discussion that introduced this chapter, we looked at a plot that displayed the relationship between the duration of an Old Faithful eruption and the time to its next eruption. The plot was based on data from January, 2003, which is displayed in the following table.

<i>Duration <math>d</math> (seconds)</i>	110	147	174	231	243	273
<i>Times to Next Eruption <math>T</math> (minutes)</i>	66	73	81	92	96	101

- a. It is known that the time to the next Old Faithful eruption can be predicted using the duration of an eruption and an appropriate regression model. Use the table of values to find the best-fitting linear model  $T(d)$  that gives the time (in minutes) to the next eruption based on the duration (in seconds) of an eruption.
- b. Use your model to predict the time to the next eruption if an eruption lasted 90 seconds.
- c. Use your model to predict the time to the next eruption if an eruption lasts 260 seconds.
- d. Complete the following table by finding the times to next eruption predicted by your linear model in part (a) and the corresponding errors. (Round your predicted values to the nearest whole minute.)

<i>Duration (seconds)</i>	110	147	174	231	243	273
<i>Times to Next Eruption (minutes)</i>	66	73	81	92	96	101
<i>Predicted Value <math>T(d)</math> (minutes)</i>						
<i>Error (Actual–Predicted)</i>						

- e. According to Ralph Taylor, who has logged and analyzed much of the Old Faithful data, in recent years there have been more eruptions lasting longer than 90 seconds and fewer eruptions of shorter duration. Therefore, most predictions are made using two different regression models—one for longer durations and one for shorter durations. (Adjustments are made for the very few durations that fall in between.) The following table gives the time to next eruption predicted in the Old Faithful Visitor Center (OFVC) Logbook for each of our data points. Complete the table to find the errors for the Logbook predictions.

<i>Duration (seconds)</i>	110	147	174	231	243	273
<i>Times to Next Eruption (minutes)</i>	66	73	81	92	96	101
<i>Time Predicted in OFVC Logbook (minutes)</i>	65	65	84	94	94	94
<i>Error (Actual–Predicted)</i>						

- f. Find the SSEs for your model and for the Logbook predictions. Based on the SSEs, which predictions are better? (Note that our model is based on a very small set of very linear data.)

## INVESTIGATION Modeling Used Car Prices

In this chapter, you have learned that a linear function is one whose rate of change is constant. As a car ages, its market value decreases, frequently at a rate that is approximately constant. This activity will allow you to create a linear model to describe the price of a used car as a function of its age.

Select a particular make and model of a car or truck that has been in production for at least 10 years (such as a Ford Mustang, a Chevy Silverado, or a Toyota Camry). Then collect prices from a newspaper or the Internet for 10 different vehicles from at least five different model years. Try to choose vehicles that are similar in terms of

features; for example, if you choose a Camry, select only SE models or only LE models. Be sure to save a copy of your data from the original source, and attach it to your report.

Once you have your prices:

1. Make a table showing the age of the vehicle in years as input and the price in dollars as output. (A vehicle of the current year is 0 years old, last year's model is 1 year old, etc.)
2. Find the best-fitting linear model  $P(t)$  that gives the price of a used vehicle of your model as a function of its age.
3. Find the SSE and the average error for your model.
4. Make a graph showing the scatter plot of your data, along with the regression line. Based on your graph, is a linear function an appropriate model for your data?
5. Explain what the slope and  $y$ -intercept of the line mean in terms of the age and price of your vehicle.
6. Based on your model, how much should a 6-year-old vehicle of your make and model cost?
7. Based on your model, how old is a car of your make and model when it becomes "free"?

# NATURAL GROWTH MODELS

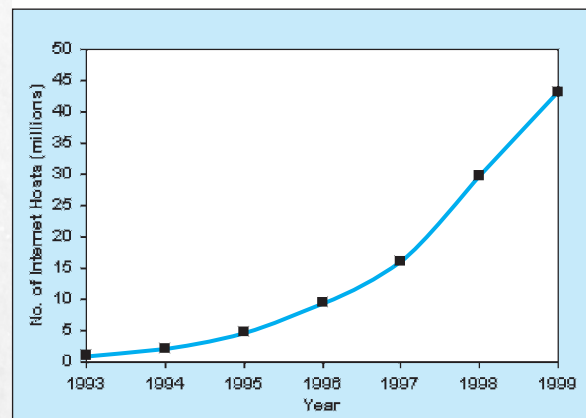
- 3.1 PERCENTAGE GROWTH AND INTEREST
- 3.2 PERCENTAGE DECREASE AND HALF-LIFE
- 3.3 NATURAL GROWTH AND DECLINE IN THE WORLD
- 3.4 FITTING NATURAL GROWTH MODELS TO DATA

Some time during the last day, you probably “visited” the Internet. You may have checked your e-mail, participated in a chat room, or downloaded a new ring tone for your cell phone, activities that were unavailable just a few years ago. During the 1990s the use of the Internet exploded, so that today millions of people around the world use it each day for school, work, and entertainment.

The chart in Fig. 3.0.1 shows a plot of data portraying the growth of the Internet from 1993 to 1999. The Internet Software Consortium considered a “host” to be a computer system connected directly to the Internet. We can see

**FIGURE 3.0.1** Internet growth from 1993 to 1999.

*Source: Internet Software Consortium Internet Domain Survey.*



from the increasing steepness of the graph that the growth of the Internet was rather slow at first, but then began to increase more rapidly. Because this function is not increasing at a constant rate, it is not a linear function. To a mathematician, a graph with this shape suggests a quite different kind of function, one that grows at the same *percentage* rate each year.

In this chapter, we study functions that model such growth and illustrate their applicability to a wide variety of real-world situations.

### 3.1 PERCENTAGE GROWTH AND INTEREST

Suppose that \$1000 is invested on July 1, 2000, in a savings account that pays 10% annual interest. This means that, on each subsequent July 1, the amount in the account is increased by 10%—that is, “interest” equal to  $1/10$  of the current amount is added to the account. Thus,  $(1/10)(\$1000) = \$100$  in interest is added to the account on July 1, 2001, so the new amount then is  $\$1000 + \$100 = \$1100$ . Next,  $(1/10)(\$1100) = \$110$  in interest is added to the account on July 1, 2002, so the new amount then is  $\$1100 + \$110 = \$1210$ . The following table shows the resulting amount on each July 1 (after the year’s interest has been added) for the first 10 years. The third column of the table shows the change in the amount from the previous year. You should use your calculator to verify all the entries in this table. That is, add  $1/10$  of each year’s amount to that amount to calculate the subsequent year’s amount.

DATE	AMOUNT	CHANGE
July 1, 2000	\$1,000.00	
July 1, 2001	\$1,100.00	\$ 100.00
July 1, 2002	\$1,210.00	\$ 110.00
July 1, 2003	\$1,331.00	\$ 121.00
July 1, 2004	\$1,464.10	\$ 133.10
July 1, 2005	\$1,610.51	\$ 146.41
July 1, 2006	\$1,771.56	\$ 161.05
July 1, 2007	\$1,948.72	\$ 177.16
July 1, 2008	\$2,143.59	\$ 194.87
July 1, 2009	\$2,357.95	\$ 214.36
July 1, 2010	\$2,593.74	\$ 235.79

Observe that that the annual change increases from each year to the next. Indeed, the change in each of the last 2 years is over twice the change in the first year. Hence it is not the amount of the change that is constant from year to year, but rather the percentage of change that remains constant. Thus growth at a constant *percentage* rate—10% annually in this case—is quite different from a constant rate of change.



Many changing quantities, ranging from bank accounts to animal populations, are very different in appearance but very similar in the way they change. In equal units of time, such as during any 1-year period, from one year to the next, they grow by the same *percentage*. In this section we review the language that is needed to measure and analyze the growth of such quantities.

## Percentage Increase

For a brief review of percentages, recall first that 1 **cent** is  $1/100$  of a dollar. This helps us to remember that the word **percent** means “one hundredth.” Thus **1 percent of something is  $1/100$  of it**. Then  $p$  percent of it is  $p$  times 1 percent of it, which is the same as  $p/100$  of it. So, if we’re talking about a quantity denoted by  $A$ , then

$$p \text{ percent of } A = \frac{p}{100} \times A, \quad (1)$$

We often use the abbreviation % for the word “percent.” Thus **the symbol % simply stands for the number  $1/100$** . That is,

$$\% = \frac{1}{100} = 0.01. \quad (2)$$

So, whenever we see the symbol % we can replace it with the number 0.01 if we wish. For instance,

$$7\% = 7 \times \% = 7 \times 0.01 = 0.07.$$

This fact is all we need to clear up the air of mystery that surrounds percentages among some people.

The phrase “percent of” simply means “% times.” Hence

$$p\% \text{ of } A = p \times \frac{1}{100} \times A = p \times 0.01 \times A. \quad (3)$$

For those who remember (3), there really is no mystery to percentages. It may help to regard the symbol % as a combination of the division symbol / and the two zeroes in the quantity  $1/100$  in (2).

### EXAMPLE 1

#### Using Percentages

Find each of the following:

- 6% of 100
- 15% of 270
- 7.6% of 385

#### SOLUTION

$$\text{a. } 6\% \text{ of } 100 = 6 \times \frac{1}{100} \times 100 = 0.06 \times 100 = 6$$

$$\text{b. } 15\% \text{ of } 270 = 15 \times \frac{1}{100} \times 270 = 0.15 \times 270 = 40.5$$

$$\text{c. } 7.6\% \text{ of } 385 = 7.6 \times \frac{1}{100} \times 385 = 0.076 \times 385 = 29.26$$

- (Whenever you see such calculations here, you should use your calculator to check them to make sure you know what's going on.)

We can simply replace the word **percent** with the symbol  $\% = 0.01$  when “translating” an appropriate sentence into a mathematical equation. The following table includes some additional pairs of corresponding words and symbols that should be familiar:

WORD	SYMBOL
percent	%
of (“times”)	$\times$
and (“plus”)	$+$
is (“equals”)	$=$

Often we need to find the result of increasing some quantity by a given percentage. **To increase  $A$  by  $p\%$  means to increase  $A$  by adding  $p\%$  of  $A$  to  $A$  itself.**

## EXAMPLE 2

### Percentage Increase

Suppose a shirt is priced at \$26.50. If this price is increased by 6%, find the new price of the shirt.

#### SOLUTION

$$26.50 + (0.06)(26.50) = 26.50 + 1.59 = 28.09,$$

- so the new price of the shirt is \$28.09.

## Interest and Constant Percentage Growth

We now begin our study of quantities that grow by equal percentages during equal time intervals. A typical such quantity is the amount  $A$  invested in a savings account that draws interest that is compounded annually. If the **annual interest rate** is  $p\%$ , this means that at the end of each year,  $p\%$  of the amount at the beginning of the year is added to the account as “interest.” **That is, the amount in the account at the beginning of the year is increased by  $p\%$  at the end of the year.** Let's consider a specific example as a way of developing an appropriate function rule for the amount in the account after  $t$  years.

Suppose that \$500 is invested in an account paying 6% interest, compounded annually. We would like to determine the amount  $A$  in the account after any given number of years  $t$ , that is,  $A(t)$ . For the sake of having a specific number to



talk about, let's suppose we are interested in  $A(20)$ , the amount in the account after 20 years.

We can proceed as we did in Example 2, and we find the following:

- After 1 year, we have

$$A(1) = \$500 + 0.06(\$500) = \$500 + \$30 = \$530. \quad (4)$$

- After 2 years, we have

$$A(2) = \$530 + 0.06(\$530) = \$530 + \$31.80 = \$561.80. \quad (5)$$

- After 3 years we have (rounded to the nearest cent)

$$A(3) = \$561.80 + 0.06(\$561.80) = \$561.80 + \$33.71 = \$595.51. \quad (6)$$

We could continue on in this manner all the way to 20 years, but this is just not very efficient. Let's look carefully at what we have done.

In (4), we calculated that the amount  $A$  in the account in dollars after 1 year is  $A(1) = \$500 + 0.06(\$500)$ . If we factor that expression, we see that

$$A(1) = \$500(1 + 0.06) = \$500(1.06).$$

Similarly, factoring (5), we obtain  $A(2) = \$530(1 + 0.06) = \$530(1.06)$ . But \$530 is  $A(1)$ , so we can replace \$530 with  $\$500(1.06)$  to get

$$A(2) = [\$500(1.06)](1.06) = \$500(1.06)^2.$$

Again, factoring (6), we find  $A(3) = \$561.80(1 + 0.06) = \$561.80(1.06)$ . Since \$561.80 is  $A(2)$ ,

$$A(3) = [\$500(1.06)^2](1.06) = \$500(1.06)^3.$$

Do you see the pattern that is emerging?

- After **1** year, the amount in the account is  $A(1) = 500(1.06) = 500(1.06)^1$  dollars.
- After **2** years, the amount in the account is  $A(2) = 500(1.06)^2$  dollars.
- After **3** years, the amount in the account is  $A(3) = [500(1.06)](1.06) = 500(1.06)^3$  dollars.

You can probably now guess that the answer to our original question, "How much is in this account after 20 years?" is found by calculating  $A(20) = 500(1.06)^{20}$ . That is, the exponent on the multiplier 1.06 matches the number of years the money has been in the account. We see in Fig. 3.1.1 the calculator computation of this value, \$1603.57.

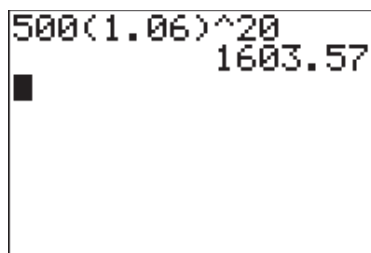


FIGURE 3.1.1 Calculating  $500(1.06)^{20}$ .

Therefore, after  $t$  years, the amount in this account will be  $A(t) = 500(1.06)^t$  dollars. In particular, we see that the amount  $A$  is a function of the number of years  $t$  that the money remains in the account.

Because there is nothing special about the amount we start with or the rate of interest we are earning, we can find a similar function to represent any such annual compound interest situation.

If  $A_0$  denotes the amount originally deposited in the account and the interest rate, written in decimal form, is  $r$ , then the amount in the account after the first year is given by

$$A(1) = A_0 + A_0r = A_0(1 + r).$$

During the second year the amount  $A(1)$  is itself multiplied by  $(1 + r)$ , so at the end of 2 years the amount in the account is

$$A(2) = A(1)(1 + r) = [A_0(1 + r)](1 + r) = A_0(1 + r)^2.$$

During the third year the amount  $A(2)$  is multiplied by  $(1 + r)$ , so at the end of 3 years the amount in the account is

$$A(3) = A(2)(1 + r) = [A_0(1 + r)^2](1 + r) = A_0(1 + r)^3.$$

Again we see the same pattern as before. In each case, the **final exponent** on the right equals the **input value** (time in years) on the left.

Therefore, when  $A_0$  dollars are invested in an account with an annual compound interest rate whose decimal equivalent is  $r$ , then the amount in the account at the end of  $t$  years is

$$A(t) = A_0(1 + r)^t \text{ dollars.} \quad (7)$$

This function rule is frequently written as  $A(t) = P(1 + r)^t$  because the initial amount  $A_0$  is referred to as the *principal* in investment situations.

### EXAMPLE 3

#### Annual Compound Interest

Suppose you deposit \$1000 in a savings account that draws 5% interest compounded annually. How long will you have to wait until you have \$1300 in the account?

#### SOLUTION

If we substitute  $P = 1000$  and  $r = 0.05$  in (7), we get the function  $A(t) = 1000(1.05)^t$ , which gives the amount in the account after  $t$  years. In order to answer this “Here’s the output, what’s the input?” question, we can calculate this amount with the successive values  $n = 1, 2, 3, \dots$  until our money reaches \$1300. An easy way to do this is to store our function in **Y1** and then look at the table of function values for **TblStart = 0** and **ΔTbl = 1**. The calculator screen in Fig. 3.1.2 shows the resulting table.

We see that we had “almost” reached \$1300 after 5 years, but it required 6 years to actually exceed our goal of \$1300. So the answer to the question asked is 6 years. (But if we had asked after what year is the amount in the account *closest* to \$1300, the answer would have been 5 years.)

X	Y1
0.00	1000.0
1.00	1050.0
2.00	1102.5
3.00	1157.6
4.00	1215.5
5.00	1276.3
6.00	1340.1

X=0

FIGURE 3.1.2 Table of function values for  $A(t) = 1000(1.05)^t$ .

## Natural Growth Models

The U.S. Constitution as adopted in 1783 decrees that a census (or count) of the people shall be conducted every 10 years. The third column of the following table records the results from the first six census counts in the decade years 1790, 1800, 1810, 1820, 1830, and 1840. The fourth column shows the results of an annual growth rate of 3%, starting with the initial 1790 population  $P_0 = 3.9$  million. Note the close correspondence, in these early decades of our nation's history, between the actual population figures and those predicted by the “3% mathematical model”  $P(t) = 3.9 \times (1.03)^t$ .

$t$ (YEARS)	YEAR	CENSUS POPULATION (MILLIONS)	$P(t) = 3.9 \times (1.03)^t$ (ROUNDED, MILLIONS)
0	1790	3.9	3.9
10	1800	5.3	5.2
20	1810	7.2	7.0
30	1820	9.6	9.5
40	1830	12.9	12.7
50	1840	17.1	17.1

The world is full of quantities that, like savings accounts and the early U.S. population, appear to grow at a constant percentage rate per unit of time. For instance, this is so common for many other populations—of people, animals, insects, bacteria—that such growth is called **natural growth**. If a population starts at time  $t = 0$  with initial population numbering  $P_0$  and thereafter grows “naturally” at an annual rate of  $r$ , then the number of individuals in the population after  $t$  years is given by

$$P(t) = P_0(1 + r)^t. \quad (8)$$

Observe that this formula is the same as Equation (7), except that here we write  $P$  for population instead of  $A$  for amount. Regardless of the letters we use as variables, a function in this form denotes a quantity growing at a constant percentage rate. A graphing calculator typically requires that **X** (rather than **T**) be used for the independent variable in its **Y=** functions menu. But you should realize that the formula  $P(x) = P_0(1 + r)^x$  defines exactly the same function as (8) because it says to do precisely the same thing with  $x$  or  $t$  (whichever letter we use to denote the independent variable). The independent variable by any other name is (like a rose?) still the independent variable.

We will call the percentage form of  $r$  in either (7) or (8) the **annual growth rate** because this value indicates the percentage by which the quantity is growing each year. We will call the value  $1 + r$  the **annual growth factor** because this is the value by which our amount is multiplied each year.

## EXAMPLE 4

## Using a Natural Growth Model

Suppose the U.S. population, starting at 3.9 million in 1790, had continued indefinitely to grow at a constant 3% annual rate.

- Use your model to predict the U.S. population in 1890.
- According to your model, when would the country's population have reached 100 million?

## SOLUTION

- Using the model  $P(t) = 3.9 \times (1.03)^t$  with  $t = 0$  in 1790, the year 1890 is  $t = 100$ , so  $P(100) = 3.9 \times (1.03)^{100} = 75$  million people (rounded to the nearest million).
- Here we want to find the value of  $t$  such that  $P(t) = 3.9 \times (1.03)^t = 100$  million. In Fig. 3.1.3 we plotted the graphs  $Y1 = 3.9 \cdot 1.03^X$  and  $Y2 = 100$  in the viewing window  $0 < x, y < 150$  and then used the calculator's intersection-finding facility to determine the indicated coordinates  $(109.8, 100)$  of the point of intersection. Thus it would take about 109.8 years (starting in 1790) for the U.S. population to reach 100 million.

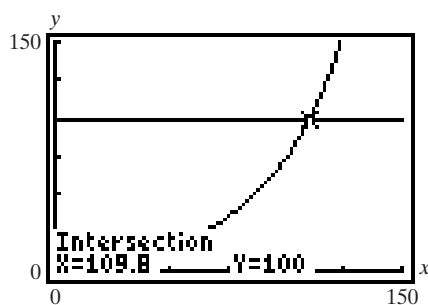


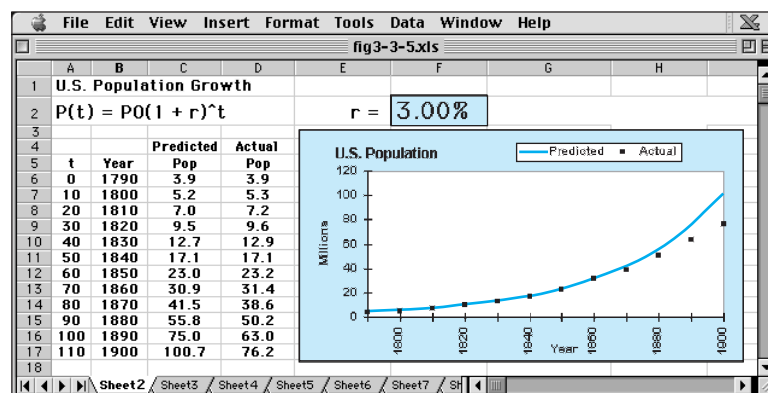
FIGURE 3.1.3  $P = 100$  when  $t = 109.8$

Let's assume for the sake of discussion that each calculated population occurs at the midpoint (July 1) of the corresponding year. The 109 years after July 1, 1790, would be July 1, 1899 (because  $1790 + 109 = 1899$ ). The population of 100 million would then occur about 0.8 year later, early in the year 1900.

A prediction based on an assumed formula is one thing, and actual reality may be another. The results in Example 4 could have been calculated (perhaps using a slide rule instead of a modern calculator) early in the nineteenth century, as soon as enough census data were available to suggest a 3% annual growth rate. But how could one then have been sure that the U.S. population would continue to grow at this same 3% constant annual rate? Did it? One should never merely accept mathematical predictions without wondering about the possibility of a reality check. We cannot be sure of the validity of a mathematical model until we have checked its predictions against real-world facts.

In this case, sitting here more than a century later, we *know* how it turned out. Figure 3.1.4 exhibits a spreadsheet (available at this book's web site as **USpop.xls**) showing both the population predicted by the “3% mathematical model” and the actual population recorded in U.S. census data at 10-year intervals through the year 1900. In the graph, predicted populations are plotted as a smooth curve and the actual census populations are plotted as square dots.

**FIGURE 3.1.4** Predicted and actual U.S. populations for 1790–1900.



We observe that the assumed 3% rate of growth appears to have been maintained initially—perhaps through the first half of the nineteenth century—but the actual U.S. population growth seems to have slowed appreciably during the second half of the century. Indeed, in the year 1900 the actual population of the United States was about 76 million rather than the 100 million that is predicted by the model  $P(t) = 3.9 \times (1.03)^t$ .

## EXAMPLE 5

### Interpreting a Natural Growth Model

The revenue for the state of Georgia from 1990 to 2001 can be modeled approximately by the natural growth function  $R(t) = 6.8(1.068)^t$ , where  $t$  is years after 1990 and  $R(t)$  is measured in billions of dollars.

(Source for data: *Atlanta Journal-Constitution*.)

- According to this model, what was the State of Georgia's revenue in 1990?
- At what annual rate was Georgia's revenue growing over this time period?
- Use the model to predict the year in which Georgia's revenue reaches \$10 billion.
- Use the model to predict Georgia's revenue in the year 2010.

### SOLUTION

- In a natural growth model  $A(t) = A_0(1 + r)^t$ ,  $A_0$  represents the initial amount. So our initial amount, 6.8 (billion dollars) represents the revenue for Georgia in 1990.
- Similarly,  $(1 + r) = 1.068$  is our growth *factor*. Therefore, the growth rate  $r = 1.068 - 1 = 0.068$ . Changing this decimal form to percent form gives us an annual growth rate of 6.8%.

- c. With  $A(t)$  stored in **Y1**, 10 stored in **Y2**, and a viewing window of  $-1 < x, y < 20$ , the **intersect** command on the calculator gives the intersection point shown in Fig. 3.1.5.

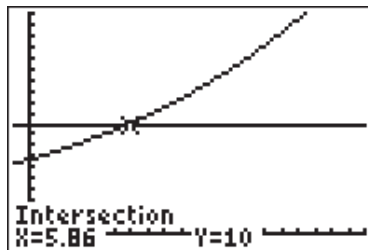


FIGURE 3.1.5  $R = 10$  when  $t = 5.86$ .

Assuming, as we usually do, that the years referred to begin on January 1, 5.86 years takes us to the latter part of 1995. So, the revenue for the state of Georgia is predicted to have reached \$10 billion in 1995.

- d. Since  $A(t)$  is already stored in **Y1** and our calculator window is (just) large enough for the input we are interested in (year 2010,  $t = 20$ ), we can use the **value** command to evaluate  $A(20)$ .

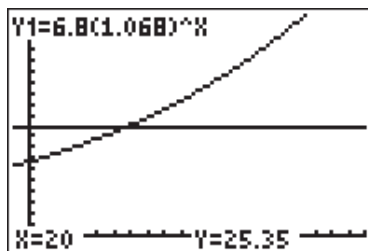


FIGURE 3.1.6 When  $t = 20$ ,  $R = 25.35$ .

As Fig. 3.1.6 indicates, our model predicts a 2010 Georgia revenue of \$23.35 billion.

## Iteration

Modern calculators and computers provide especially simple ways to carry out repetitive calculations like the one in Example 3 (in which \$1000 is deposited in a savings account that draws 5% interest compounded annually). Indeed, this is exactly what modern computers were invented (in the 1940s and 1950s) to do. Suppose we enter

$$1000 \rightarrow A$$

in our calculator to record the initial deposit  $A$ , and then enter the command

$$A*(1 + 0.05) \rightarrow A$$

to calculate and store (as the value of **A**) the amount in the account after 1 year if 5% interest is compounded annually.

Now comes the punch line—if we simply press the **ENTER** key again, the last command line is executed again! That is, the value 1050.00 currently stored as **A**

```

A*(1+.05)→A
1050.00
1102.50
1157.63
1215.51
1276.28
1340.10

```

FIGURE 3.1.7 Example 3 by iteration.

is itself multiplied by the growth factor 1.05, and the result stored as the new value of **A**. Having started the process, each press of the **ENTER** key carries out the computation for another year and stores the result. So, in order to find out how much is in the account after 6 years, we need only type the line **A\*(1+0.05)→A** *once*, and then press the **ENTER** key five times in succession (keeping careful count of how many times we've entered the command). Figure 3.1.7 shows the result, and provides the same conclusion as in Example 3.

This is our first example of *iteration*—doing the same thing again and again. Specifically, the successive results  $A_0, A_1, A_2, A_3, \dots$  are calculated *iteratively* when the same formula is used repeatedly to calculate each new result from the preceding one. In Example 3, this *iterative formula* is the formula

$$A_{n+1} = A_n(1 + 0.05), \quad (9)$$

which says that each amount is multiplied by  $(1 + 0.05)$  to get the next amount. Doing this over and over is called *iterating* the formula. So Fig. 3.1.7 shows the result of iterating (9) six times, starting with  $A_0 = 1000$ .

In this situation iteration provides an alternate strategy for calculating yearly amounts in annual compound interest problems. Iteration has a long history in mathematics; iterative methods were sometimes the only ones available to solve a problem and were used long before calculators and computers were available.

When we repeat the iteration of Fig. 3.1.7, we “plow” the output back into the right-hand side of the equation  $A_{new} = A_{old}(1 + 0.05)$ . Indeed, the word *iterate* apparently stems from the Latin verb *iterare* with the meaning “to plow again.” The concept of iteration is as old as the word. Two thousand years ago the Babylonians introduced the iteration

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{A}{x_n} \right), \quad (10)$$

which can be used to calculate more and more accurate approximations to the **square root**  $\sqrt{A}$  of a given positive number  $A$ . One starts with an initial guess  $x_0$  and then uses (10) to calculate the successive iterates  $x_1, x_2, x_3, \dots$ . It happens that the initial guess  $x_0$  need not be especially accurate—any nonzero guess will do for a start. For instance,  $x_0 = A/2$  is a convenient starting point, and then the **Babylonian square algorithm** can be implemented with the calculator commands

**2 → A**

**A/2 → X**

**(1/2)(X + A/X) → X**

to approximate the square root of the number  $A = 2$ . The first two commands here initialize the variables **A** and **X**, and the last one is the iterative command.

```

1.000000
(1/2)(X+A/X)+X
1.500000
1.416667
1.414216
1.414214
1.414214
1.414214

```

FIGURE 3.1.8 Approximating the square root of 2.

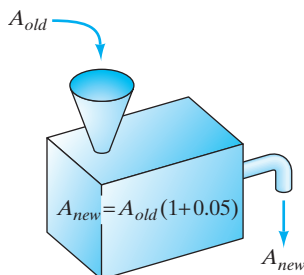


FIGURE 3.1.9 Iteration (9) as an input-output process.

The results shown in Fig. 3.1.8 indicate that the six-place value  $\sqrt{2} \approx 1.414214$  is reached quickly. With any positive number  $A$  and any initial guess  $x_0$  the successive approximations generated by the iteration in (10) eventually “stabilize” in this way, agreeing to the number of decimal places displayed. In Exercises 28 and 29, you can explore this historic method for calculating square roots.

## Input-Output Processes

The iterative formula in (9) or (10) can be visualized as an **input-output process** as illustrated in Fig. 3.1.9. We think of a “black box” with a hidden mechanism inside that performs the computation described by the iterative formula. Specifically, when an **input**  $A = A_{old}$  is fed into the box, the iteration is performed and the **output**  $A_{new}$  is produced. We often find it useful to interpret a complicated iteration as a “simple” input-output process—simple in that for many purposes we need not think explicitly of what is happening inside the box. It may only be important that, whatever the input, the same process always produces the corresponding output.

## 3.1 Exercises

### Building Your Skills

In Exercises 1–4, find each number.

- 5% of 200
- 3.5% of 50
- 2.25% of 18
- 4.19% of 58.35

In Exercises 5–8, calculate the new price (rounded off to the nearest cent) if

- The old price of \$110 is increased by 13%.
- The old price of \$1720 is increased by 4.2%.
- The old price of \$69.50 is increased by 3.9%.
- The old price of \$250 is increased by 8.25%.
- Explain why the following method works to calculate the amount to be paid in a restaurant if you wish to add a 15% tip to the original bill amount  $A$ . First you write down the amount  $A$ . Then you move the decimal point in  $A$  one unit to the left and write this amount under  $A$ . Finally divide this second amount by 2 and write this amount down also. The total amount (including tip) to be paid is then the sum of the three amounts you have written down.



10. The town of Bridgeport had 20 thousand people in the year 2000 and was growing at a rate of 4% per year. Find a natural growth function  $P(t)$  that gives the population of Bridgeport as a function of  $t$ , years after 2000.
11. The town of Jackson had 130 thousand people in the year 2000 and was growing at a rate of 6.3% per year. Find a natural growth function  $P(t)$  that gives the population of Jackson as a function of  $t$ , years after 2000.
12. A small company currently has total sales of \$80,000 and projects that its sales will grow at a rate of 2% per year. What are the company's projected sales in 10 years?
13. A self-employed tax preparer currently has an advertising budget of \$500 and plans to increase this budget by 1.5% per year. In how many years will her advertising budget rise to \$600?

## Applying Your Skills

14. A 1996 presidential primary candidate proposed a “flat tax” under which a family of four would pay as its federal income tax each year an amount equal to 17% of the portion of its taxable income in excess of \$36,000. Suppose that a family's 1996 taxable income was \$50,000.
  - a. Under this proposal, how much would be owed as income tax?
  - b. Suppose that this family's actual federal income tax bill for 1996 was \$6015 plus 28% of the amount of their taxable income over \$40,100. How much income tax did the family owe?
  - c. Under which method of assessing taxes would the family pay less tax?
15. Repeat Exercise 14 for a family whose taxable income was \$200,000.
16. Suppose that \$400 is deposited in an account paying 6% interest compounded annually.
  - a. Find a natural growth function  $A(t)$  that gives the amount in the account after  $t$  years.
  - b. How much is in the account after 5 years?
  - c. How long will it take until the account has \$1000? Give the year after which the amount is **at least** \$1000.
17. Suppose that \$2000 is deposited in an account paying 3.45% interest compounded annually.
  - a. Find a natural growth function  $A(t)$  that gives the amount in the account after  $t$  years.
  - b. How much is in the account after 8 years?
  - c. How long will it take until the account has \$3200? Give the year after which the amount is **at least** \$3200.
18. How long (rounded off accurate to the nearest year) does it take to triple an initial deposit of \$1000 if the annual interest rate is 8%?
19. Suppose that \$1000 is deposited in an account.
  - a. Complete column B to show how long it will take until the initial deposit is doubled for various interest rates. (Give the year after which the amount in the account is **closest to** \$2000.)
  - b. Complete column C by finding the product of the values in column A and column B for each row.

COLUMN A % INTEREST RATE	COLUMN B NUMBER OF YEARS TO DOUBLE	COLUMN C COLUMN A $\times$ COLUMN B
3		
4		
6		
8		
9		

- c. Based on your results, what might the **rule of 72** say?
20. If you invest \$1000 at 12% annual interest, show that you will have \$1973.82 after 6 years and \$2210.68 after 7 years. What initial deposit (accurate to the nearest cent) would lead to precisely \$2000 (accurate to the nearest dollar) after 6 years? To answer this question, you will need to try several different initial values, finding successively better estimates of the needed initial deposit; obviously you need to start with a bit more than \$1000.
21. The population of Jacksonville, Florida, was 635 thousand in 1990 and grew at an average annual rate of 1.49% throughout the 1990s. Assuming that this rate of growth continues,
- Find a natural growth function  $P(t)$  that gives the population of Jacksonville as a function of  $t$ , years after 1990.
  - Use your function model to predict the year in which the population of Jacksonville will grow to 800 thousand.
  - Use your function model to predict the population of Jacksonville in 2003. How does your prediction compare to the actual 2003 population of 733,781?
22. The population of Lincoln, Nebraska, was 192 thousand in 1990 and grew at an average annual rate of 1.64% throughout the 1990s. Assuming that this rate of growth continues,
- Find a natural growth function  $P(t)$  that gives the population of Jacksonville as a function of  $t$ , years after 1990.
  - Use your function model to predict the population of Lincoln in 2003. How does your prediction compare to the actual 2003 population of 235,594?
  - Use your function model to predict the year in which the population of Lincoln will grow to 350,000.
23. The population of Corpus Christi, Texas, was 257 thousand in 1990 and grew at an average annual rate of 0.75 % throughout the 1990s. Assuming that this rate of growth continues,
- Find a natural growth function  $P(t)$  that gives the population of Corpus Christi as a function of  $t$ , years after 1990.
  - Use your function model to predict the population of Corpus Christi in 2003. How does your prediction compare to the actual 2003 population of 279,208?
  - Use your function model to predict the year in which the population of Corpus Christi will grow to 400 thousand.

24. Data from the *St. Louis Post-Dispatch* indicate that the cost in dollars of a gallon of gasoline from 1966 to 2006 can be modeled approximately by the natural growth function  $G(t) = 0.32(1.055)^t$ , where  $t$  is years after 1966.
- According to this model, what was the price of a gallon of gas in 1966?
  - At what annual percentage rate was the price of a gallon of gas growing over this time period?
  - Use the model to predict the price of a gallon of gas in 2016.
  - Use the model to predict the year in which the price of a gallon rises to \$5.
25. The number of people (in millions) enrolled in health maintenance organizations (HMOs) for the 20 years following 1976 can be given approximately by the natural growth model  $P(t) = 6.0(1.12)^t$ , where  $t$  is years after 1976. (Model based on data from *World Almanac and Book of Facts 2005*.)
- According to this model, how many people were enrolled in HMOs in 1976?
  - At what annual percentage rate was the number of people enrolled in HMOs growing over this time period?
  - Use the model to predict the number of people enrolled in HMOs in 1999. How does your prediction compare to 81.3 million, the actual number of people enrolled in HMOs in 1999?
  - Use the model to predict the number of people enrolled in HMOs in 2000. How does your prediction compare to 80.9 million, the actual number of people enrolled in HMOs in 2000?
  - How do you explain the discrepancy in part (d)?
26. According to the *Atlanta Journal-Constitution*, the market research firm NPD Group “reported a dramatic 53% increase in sales of suits, dress pants, sport coats, and jackets to young men” in 2005 over 2004 sales of \$645 million dollars.
- Assuming that this trend continues, find a natural growth function  $S(t)$  that gives the sales as a function of  $t$ , years after 2004.
  - Based on your model, what were the 2005 sales of suits, dress pants, sport coats, and jackets to young men?
  - Use your model to predict the year in which sales will reach \$1 billion? (\$1 billion is \$1000 million.)
27. Data from *USA Today* indicate that in 1994, Brazil’s exports were valued at \$46.4 billion and increased at an average annual rate of 8.51% over the next 10 years. (“Exports” refers to goods and services, and are measured in constant 2000 U.S. dollars.)
- Assuming that this trend continues, find a natural growth function  $E(t)$  that gives the value of Brazil’s exports as a function of  $t$ , years after 1994.
  - Based on your model, what was the value of Brazil’s exports in 2004?
  - Use your model to predict the year in which Brazil’s exports will double.
28. Apply the Babylonian square root algorithm to approximate the (principal) square root of 19. Try different initial guesses and discuss how the accuracy of the initial guess affects the number of iterations required to get six-place accuracy.
29. What happens with the Babylonian algorithm if a negative rather than a positive initial guess is used?
-

## 3.2 PERCENTAGE DECREASE AND HALF-LIFE

In Section 3.1, we considered natural growth functions that describe quantities that increase at a constant annual percentage rate. Natural growth functions can also be used to describe quantities that *decrease* at a constant percentage rate. (Oddly enough, we refer to this entire class of functions as natural “growth” functions, regardless of whether the quantity involved is growing or declining.)

### Percentage Decrease

Some department store “clearance centers” operate on the principle that the price of an article is reduced by an additional 10% each week it remains in stock. Suppose that such a clearance center has a pair of shoes initially priced at \$150. What function describes the price of the shoes in terms of the number of weeks it remains in stock?

Recall that to increase  $A$  by  $p\%$  means to increase  $A$  by adding  $p\%$  of  $A$  to  $A$  itself. Similarly, **to decrease  $A$  by  $p\%$  means to decrease  $A$  by subtracting  $p\%$  of  $A$  from  $A$  itself.**

So for our clearance shoes:

- After 1 week, the price  $P$  would be

$$P(1) = \$150 - 0.10(\$150) = \$135. \quad (1)$$

- After 2 weeks, the price  $P$  would be

$$P(2) = \$135 - 0.10(\$135) = \$121.50. \quad (2)$$

- After 3 weeks, the price  $P$  would be

$$P(3) = \$121.50 - 0.10(\$121.50) = \$109.35. \quad (3)$$

We could continue on in this manner, finding the price of the shoes as long as they remain in stock, or we could use an iterative formula as we did in Section 3.1. But our goal here is to determine a function that describes the price of the shoes.

In (1), we calculated that the price  $P$  of the shoes after 1 week is  $P(1) = \$150 - 0.10(\$150)$ . If we factor that expression, we see that

$$P(1) = \$150(1 - 0.10) = \$150(0.90).$$

Similarly, factoring (2), we get  $P(2) = \$135(1 - 0.10) = \$135(0.90)$ . But \$135 is  $P(1)$ , so we can replace \$135 with  $\$150(0.90)$  to get

$$P(2) = [\$150(0.90)](0.90) = \$150(0.90)^2.$$

And again, factoring (3), we find  $P(3) = \$121.50(1 - 0.10) = \$121.50(0.90)$ . Since \$121.50 is  $P(2)$ ,

$$P(3) = [\$150(0.90)^2](0.90) = \$150(0.90)^3.$$

We see the same kind of pattern here that we observed in increasing natural growth functions. That is,

- after **1** week, the price of the shoes is  $P(1) = 150(0.90) = 500(0.90)^1$  dollars;

- after 2 weeks, the price of the shoes is  $P(2) = 150(0.90)^2$  dollars;
- after 3 weeks, the price of the shoes is  $P(3) = 150(0.90)^3$  dollars.

Therefore we conclude that, after  $t$  weeks, the price of the shoes will be  $P(t) = 150(0.90)^t$  dollars.

## EXAMPLE 1

### Finding an Input Value

If you are willing to pay only \$80 for the shoes from the clearance center, how long must you wait to buy them?

#### SOLUTION

Since the price of the shoes is reduced only once a week, a table displaying function values at the end of each week will indicate when the price of the shoes is \$80 or less. Entering  $P(t)$  into **Y1** and 80 into **Y2**, we set **TblStart=0** and **ΔTbl=1**. Figure 3.2.1 shows the resulting table.

X	Y1
0.000	150.00
1.000	135.00
2.000	121.50
3.000	109.35
4.000	98.415
5.000	88.574
6.000	79.716

X=0

FIGURE 3.2.1 Finding when  $P(t) \leq 80$ .

We see that if the shoes are still in stock after 6 weeks, the price will be \$79.72 (to the nearest cent). Since the price is below \$80, you can now happily purchase the shoes. (Of course, someone else may have been willing to pay more, and may have bought them earlier!)

## Declining Natural Growth Models

In general, suppose that  $A$  is a quantity that is decreasing at a constant percentage rate  $r$  (expressed as a decimal) during each unit of time  $t$ . If  $A_0$  denotes the initial value of the quantity, then the amount after the first unit of time (hour, day, week, year, whatever) is given by

$$A(1) = A_0(1 - r).$$

During the second unit of time, the amount  $A(1)$  is itself multiplied by  $(1 - r)$ , so that

$$A(2) = A(1)(1 - r) = [A_0(1 - r)](1 - r) = A_0(1 - r)^2.$$

During the third unit of time, the amount  $A(2)$  is multiplied by  $(1 - r)$ , so that

$$A(3) = A(2)(1 - r) = [A_0(1 - r)^2](1 - r) = A_0(1 - r)^3.$$

Again we see the same pattern as before. In each case, the **final exponent** on the right equals the **input value** (time) on the left.

Therefore, when a quantity whose initial value is  $A_0$  is decreasing during each unit of time at a rate whose decimal equivalent is  $r$ , then the amount of the quantity after  $t$  units of time is

$$A(t) = A_0(1 - r)^t. \quad (4)$$

## EXAMPLE 2

### Finding a Declining Natural Growth Function

Suppose you invested \$1200 a dot-com stock in 1990, and rather than growing (as you expected), the value of that investment decreased at an annual percentage rate of 4%.

- Find a natural growth function that describes the value of the investment as a function of  $t$ , the number of years after 1990.
- What was the value of your investment in 1995?
- If you have decided to sell the stock when its value declines to \$500, in what month and year will you sell the stock?

### SOLUTION

- If we substitute  $A_0 = \$1200$  and  $r = 0.04$  in (4), we get the natural growth function  $A(t) = \$1200(0.96)^t$  giving the value of the investment—that is, the amount in the account after  $t$  years—as a function of  $t$ .
- In order to answer this “Here’s the input, what’s the output?” question, we need to evaluate our function when  $t = 5$ . Thus  $A(5) = \$1200(0.96)^5 = \$978.45$  (correct to the nearest cent).
- In this case, we are given the output (\$500) and asked to find the appropriate input (time). Letting  $Y1=1200(0.96)^X$  and  $Y2=500$ , and using the window indicated in Fig. 3.2.2, we find that the **intersect** command gives intersection point shown in Fig. 3.2.3.

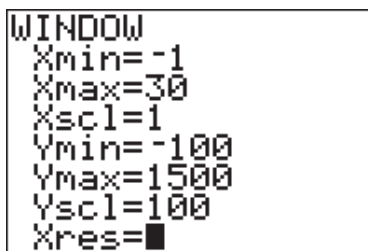


FIGURE 3.2.2 The window for solving  $1200(0.96)^X = 500$ .

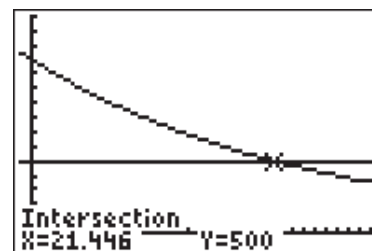


FIGURE 3.2.3  $A = 500$  when  $t = 21.446$ .

Thus, you need to sell the stock after 21.446 years. Since  $(0.446 \text{ year})(12 \text{ months/year}) = 5.352$  months, we have completed 5 months and are in the sixth; you need to sell the stock in June 2011.

If you prefer to learn only one formula for natural growth functions, you should remember the one from Section 3.1,  $A(t) = A_0(1 + r)^t$ . When you need to create a decreasing natural growth function, you can think of its growth rate as negative (because adding the opposite of a number is equivalent to subtracting the number itself).

We saw in Chapter 2 that once we have “built” one or more functions, we can answer many different types of input-output questions, such as,

- What is the value of a function for a given input?
- For what input does a function have a given output?
- For what input is the function’s value double its initial value?
- For what input do two functions have the same output?

We can also answer such questions for the natural growth functions we created in this chapter (and have done so in Examples 1 and 2). Indeed, we will be doing this for all the different types of functions we create throughout this book. This is the key to understanding the “big idea” of mathematical modeling: the questions do not change—it is only the functions that model the situation that are different.

### EXAMPLE 3

#### When Are Two Populations Equal?

In 2000, the population of Baltimore, Maryland, was 651 thousand and was declining at an average annual rate of 1.14%. At the same time, the population of Fort Worth, Texas, was 541 thousand and was increasing at an average annual rate of 2.64%. In what month and year will the populations of Baltimore and Fort Worth be equal?

#### SOLUTION

Our first task is to find a natural growth model for the population of each city. Since Baltimore’s initial population is 651 thousand and its annual rate of decline is 0.0114, the population is given by

$$B(t) = 651(1 - .0114)^t = 651(0.9886)^t,$$

where  $t = 0$  in 2000 and  $B(t)$  is given in thousands.

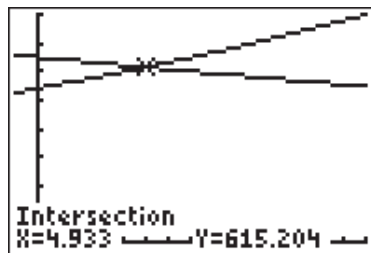
Fort Worth’s initial population is 541 thousand and its annual rate of increase is 0.0264; its population is given by

$$F(t) = 541(1 + .0264)^t = 541(1.0264)^t,$$

where  $t = 0$  in 2000 and  $B(t)$  is given in thousands.

The question we need to answer then is “What value of  $t$  makes  $B(t) = F(t)$ ?” With  $B(t)$  stored in **Y1** and  $F(t)$  stored in **Y2**, and using the window  $-1 < x < 15$ ,  $-10 < y < 800$ , we use the intersect feature of the calculator. Figure 3.2.4 shows the result.

We see (as we expect) one increasing function (the population of Fort Worth) and one decreasing function (the population of Baltimore). You should always verify that the graph your calculator displays reflects what you know about the function or functions you are graphing. It is easy to make an error in

FIGURE 3.2.4  $B(t) = F(t)$  when  $t = 4.933$ .

entering a function into the calculator—if something looks wrong to you, check it before reporting an answer.

In this case, however, all is fine, and the intersection point (4.933, 615.204) tells us that 4.933 years after January 1, 2000, the populations of Baltimore and Fort Worth are equal. Since  $0.933 \times 12$  is 11.196, the populations are equal in December 2004.

The intersection point also tells us the answer to a question we didn't ask. That is, what are the populations of the cities when they are equal? The second coordinate gives us these equal populations. So our models predict that on some day in December 2004 Baltimore and Fort Worth will each have a population of 615.204 thousand, or 615,204 people.

## Half-Life

When discussing an increasing natural growth function, it makes sense to ask when the initial amount doubles. With a decreasing natural growth function, the quantity is getting smaller, so the amount will never be double the initial amount. However, we are frequently interested in answering a similar question—when does the quantity decline to *half* the initial amount?

For a quantity described by a decreasing natural growth function, the length of time it takes for the quantity to decrease to half its initial value is called its **half-life**. While we do not typically talk about the half-life of an investment gone bad or a city's population, this idea is very common in discussing the decay of radioactive elements or the metabolism of drugs in the human body.

### EXAMPLE 4

#### Finding Half-Life

A laboratory has a 50-gram sample of bismuth-210, a radioactive element that decays at a daily rate of approximately 12.94%.

- What is the half-life of bismuth-210?
- How long will it take until only 2 grams of the sample remain?

#### SOLUTION

- The natural growth function describing the number of grams in the sample is  $B(t) = 50(0.8706)^t$ . To determine the half-life, we need to figure out when  $B(t) = 25$ . We will again use the intersect command from the calculator's **CALC** menu, with  $B(t)$  stored in **Y1** and 25 stored in **Y2**. In an appropriate window, the calculator displays the intersection



point shown in Fig. 3.2.5. (For many students, the most difficult part of these problems is finding an appropriate window. We leave you to discover a “good” window here on your own, as practice for future situations. Just keep trying windows until you find one where the intersection of the two graphs shows clearly.)

The intersection point indicates that the half-life of bismuth-10 is (correct to the nearest day) 5 days.

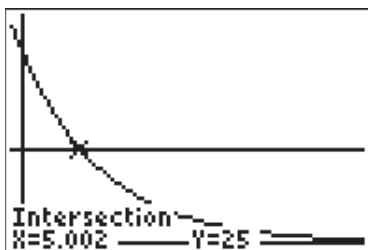


FIGURE 3.2.5  $A = 25$  when  $t = 5.002$ .

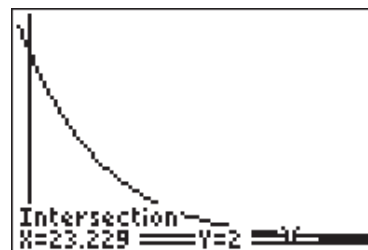


FIGURE 3.2.6  $B = 2$  when  $t = 23.229$ .

- b. Although the wording is much different, this question is the same kind of “Here’s the output, what’s the input?” question that we answered in part (a). (We really are asking you the same questions over and over again!) So if we replace 25 with 2 in **Y2** and use **intersect** once again, the result is shown in Fig. 3.2.6. Thus, it takes a bit over 23 days for only 2 grams of the sample to remain.

There are two important points to note about this example. First, although we gave you a specific number of grams in the sample, this value is not required for the calculation of the half-life. The half-life of a substance does not depend on the initial amount given, but rather on properties inherent in the substance itself. If a particular amount had not been given in Example 4, we could have chosen any amount we like (our favorite number perhaps?) or the arbitrary amount  $A_0$ . In this case, we would need to solve the equation

$$0.5 A_0 = A_0(0.8706)^t.$$

If we divide both sides of this equation by  $A_0$ , we obtain the equivalent equation

$$0.5 = (0.8706)^t.$$

We can solve this equation graphically, as we did before, with  $0.8706^t$  in **Y1**, and 0.5 in **Y2**. (You should try this to see that we get exactly the same half-life.)

Second, we notice that while it took only 5 days for the first 25 grams of the sample to decay, the next 23 grams (so that only 2 grams remain) took approximately 18 days to decay. Why is this? This sample is not decaying by the same *amount* every 5 days—that would make it a linear function. It is decaying by the same *percentage* (50% of what is there) every 5 days—that’s what makes it a natural growth function. Because half of our sample “disappears” every 5 days, we keep halving and halving and halving indefinitely. Thus, at least in theory, all of the sample never goes away. Of course, at some point, whatever scale we are

using to measure the sample is not sensitive enough to recognize that there is any sample left. At this point, we can say that the weight of the sample is “essentially” zero. What we mean by this is that neither we nor the scale we are using can distinguish between the weight of the sample and the real number 0.

So for a declining natural growth function, as our input values increase, our output values get closer and closer to 0 but never actually “get there.” Thus the graph of such a function gets closer and closer to the  $x$ -axis but never actually touches it. (Our calculator has the same difficulty as the scale weighing the sample—if you go out “far enough,” it appears that the output values are all 0. But in reality they are not; it is just the physical limitations of the calculator that prevent us from seeing the difference.)

You probably have had no occasion to think about the half-life of bismuth-210 in your daily life, but the same mathematical principles apply to how your body eliminates certain chemicals from your blood. Metabolism of prescription medications, over-the-counter remedies, and even caffeine behaves like a decreasing natural growth function. (Surprisingly, alcohol—when measured by the drink—is eliminated from the body in a linear fashion, typically at the rate of about 1 drink per hour.)

## EXAMPLE 5

### The Half-Life of Caffeine

In a healthy adult, the amount of caffeine in the bloodstream decreases at an hourly rate of approximately 11.5%.

- What is the half-life of caffeine in a healthy adult?
- A can of Red Bull energy drink contains about the same amount of caffeine as a cup of regular coffee, 100 mg. If you drink a Red Bull or a cup of coffee, how much caffeine will remain in your bloodstream after 2 hours?

Sources: *American Journal of Clinical Pathology* and [www.redbullusa.com](http://www.redbullusa.com).

### SOLUTION

- Since a specific initial value is not given here, the natural growth function describing the amount of caffeine in the bloodstream is  $A(t) = A_0(0.885)^t$ . We need to determine when the amount of caffeine falls to half the initial amount, that is, to  $0.5A_0$ . Thus we need to solve the equation  $0.5A_0 = A_0(0.885)^t$ . As before, we can find an equivalent equation,  $0.5 = 0.885^t$ , and solve this equation for  $t$ . Figure 3.2.7 shows the graphical solution of this equation. Once again, you should verify this result in your calculator.

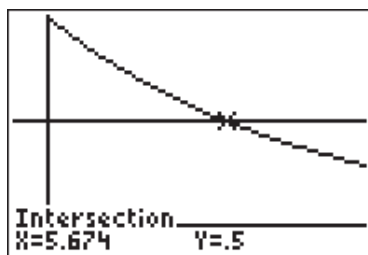


FIGURE 3.2.7  $A = 0.5$  when  $t = 5.674$ .

So the half-life of caffeine in a healthy adult is about 5.7 hours.

- b. Here we do have a specific initial amount, so our natural growth function is  $A(t) = 100(0.885)^t$ . We are asked to determine how much caffeine is in the bloodstream after 2 hours—that is, to find the output for an input value of 2. We evaluate  $A(2) = 100(0.885)^2 = 78.4335$ ; to the nearest milligram, 78 mg of caffeine remain after 2 hours.

What we have presented here is a simplified view of how a body metabolizes caffeine. There are many variables involved—a person’s gender, weight, and medical condition and how quickly he or she consumes the caffeine. But it is worth noting that natural growth models, whether increasing or decreasing, occur in many applications that at first glance seem quite different. What unites all of these situations is that they involve quantities that either increase or decrease at a constant percentage rate.

## 3.2 Exercises | Building Your Skills

*In Exercises 1–6, calculate the new price (rounded off to the nearest cent) if*

- the old price of \$82 is decreased by 7%.
- the old price of \$640 is decreased by 2%.
- the old price of \$69.50 is decreased by 1.5%.
- the old price of \$1080 is decreased by 6.3%.
- the old price of \$535 is decreased by 0.4%.
- the old price of \$192 is decreased by 0.25%.
- Suppose that a quantity whose initial value is 120 is reduced by 25%, and then the result is increased by 25%. Is the final value 120, more than 120, or less than 120? Why?
- Suppose that a quantity whose initial value is 200 is increased by 15%, and then the result is decreased by 15%. Is the final value 200, more than 200, or less than 200? Why?
- The town of Bilston had 35 thousand people in the year 2000 and was declining at a rate of 3% per year. Find a natural growth function  $P(t)$  that gives the population of Bilston as a function of  $t$ , years after 2000.
- The town of Independence had 210 thousand people in the year 2000 and was declining at a rate of 4.1% per year. Find a natural growth function  $P(t)$  that gives the population of Independence as a function of  $t$ , years after 2000.
- A manufacturing company currently loses \$20,000 a year on defective items. It is implementing a plan by which it hopes to reduce this amount by 1% each year for the next 10 years. If the company is successful, how much will it lose on defective items 5 years from now?
- A small accounting firm is spending \$60,000 a year on clerical staff. The president of the firm believes that they can reduce this amount by 5% per year over the next 5 years by hiring temporary workers. Assuming the president is correct, how much would the firm spend on clerical staff in 3 years?

### Applying Your Skills

- Suppose that you invest \$1000 in shares of a new stock you heard about from a friend. Instead of gaining value (as you hoped), the stock loses 2% of its value each month.

- a. Find a natural growth function  $V(t)$  that gives the value of the stock after  $t$  months.
  - b. What is the stock worth after 6 months?
  - c. If you are unwilling to lose more than \$200 on this venture, in how many months must you sell the stock (assuming that it continues to lose value at this rate)?
14. The population of Birmingham, Alabama, was 266 thousand in 1990 and was decreasing at an average annual rate of approximately 1% throughout the 1990s. Assuming that this rate of decline continues,
- a. find a natural growth function  $P(t)$  that gives the population of Birmingham as a function of  $t$ , years after 1990.
  - b. use your function model to predict the year in which the population of Birmingham will fall to 220 thousand.
  - c. Use your function model to predict the population of Birmingham in 2003. How does your prediction compare to the actual 2003 population of 236,620?
15. The population of Toledo, Ohio, was 333 thousand in 1990 and was decreasing at an average annual rate of 0.59% throughout the 1990s. Assuming that this rate of decline continues,
- a. find a natural growth function  $P(t)$  that gives the population of Toledo as a function of  $t$ , years after 1990.
  - b. use your function model to predict the population of Toledo in 2003. How does your prediction compare to the actual 2003 population of 308,973?
  - c. use your function model to predict the year in which the population of Toledo will fall to 280 thousand.
16. The population of Cincinnati, Ohio, was 331 thousand in 2000 and decreased at an average annual rate of 1.53% over the next 3 years. At the same time, the population of Plano, Texas, was 222 thousand and grew at an average annual rate of 2.92%. Assuming that these trends continue, in what month and year will the populations of Cincinnati and Plano be the same?
17. The population of Boston, Massachusetts, was 589 thousand in 2000 and decreased at an average annual rate of 0.4% over the next 3 years. At the same time, the population of Sacramento, California, was 407 thousand and grew at an average annual rate of 3.02%. Assuming that these trends continue, in what month and year will the populations of Boston and Sacramento be the same?
18. Canada's Office of Consumer Affairs reported that in 1982 the average household spending for food constituted 15.1% of household expenditures. From 1950 to 2003 this amount decreased at an average annual rate of 1.53%.
- a. Assuming that this trend continues, find a natural growth function  $F(t)$  that gives percentage expenditure on food as a function of  $t$ , years after 1994.
  - b. Use your model to predict the year in which percentage expenditure on food decreases to 10%.
  - c. Based on your model, what percentage of household expenditures did the average household spend on food in 1996?
19. According to the American Cancer Society, death rates from diseases other than cancer were falling from 1950 to 2003, while cancer death rates remained virtually the same. Measured per hundred thousand, the death rate from heart

disease in 1950 was 586.8. This amount decreased at an average annual rate of 1.74% over the next 53 years.

- a. Assuming that this trend continues, find a natural growth function  $D(t)$  that gives heart disease deaths per hundred thousand as a function of  $t$ , years after 1950.
  - b. Based on your model, how many heart disease deaths per hundred thousand were there in 1967?
  - c. Use your model to predict the year in which heart disease deaths per hundred thousand fall to 300.
20. As indicated in Example 5, the amount of caffeine in the bloodstream decreases at an hourly rate of approximately 11.5%. A double shot of espresso contains 80 mg of caffeine.
- a. Find a natural growth model  $C(t)$  that gives the amount of caffeine in the bloodstream as a function of time  $t$  in hours.
  - b. How much caffeine is in the bloodstream after 3 hours?
  - c. How long does it take for the caffeine level in the blood to fall to 10 mg?
21. A laboratory has a 30-gram sample of radon-222, which decays at a rate of 16.6% per day.
- a. Find a natural growth model  $R(t)$  that gives the amount of radon-222 as a function of  $t$ , time in days.
  - b. How long does it take for the sample to decay to 5 grams?
  - c. How much radon-222 remains after 2 weeks?
22. A laboratory has a 800-mg sample of cesium-137, which decays at an annual rate of 2.28%.
- a. Find a natural growth model  $C(t)$  that gives the amount of cesium-137 as a function of  $t$ , time in years.
  - b. How much of the sample remains after 5 years?
  - c. What is the half-life of cesium-137, correct to the nearest year?
23. Many people use the over-the-counter medication ibuprofen for relief from muscle strains and joint pain. The amount of ibuprofen in an adult's bloodstream decreases at an hourly rate of approximately 29%, and a normal adult dose is 400 mg.
- a. Find a natural growth model  $I(t)$  that gives the amount of ibuprofen in the bloodstream as a function of time  $t$  in hours.
  - b. What is the half-life of ibuprofen?
  - c. How much ibuprofen is in a person's bloodstream after 3 hours?

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### 3.3 NATURAL GROWTH AND DECLINE IN THE WORLD

In Sections 3.1 and 3.2, we discussed natural growth models for which we knew the annual growth rate. However, in many situations we suspect (or assume) constant-percentage growth but know the actual population at only a couple of different times. The following example illustrates several different methods that can then be used to determine the population's growth rate  $r$ .

## EXAMPLE 1

Finding the Growth Rate  $r$ 

The city of Bethel had a population of 25 thousand in 1990 and 40 thousand in 2000. What was the city's percentage rate of growth (rounded off accurate to one decimal place in percentage points) during this decade?

## SOLUTION

We take  $t = 0$  in 1990 to get started. Using the natural growth function  $P(t) = P_0(1 + r)^t$  with  $P_0 = 25$  (thousand), we have

$$P(10) = 25(1 + r)^{10} = 40 \quad (1)$$

since the population 10 years later in 2000 was 40 thousand. We need to solve this equation for  $r$ . Various methods are available.

**Graphical Method** We graph the functions  $Y1 = 25(1+X)^{10}$  and  $Y2 = 40$ , with  $x$  instead of  $r$  denoting the unknown in Equation (1), in the viewing window  $0 < x < 0.10$ ,  $0 < y < 60$ . (For sake of investigation, we estimate initially that the unknown rate  $r = x$  is less than  $10\% = 0.10$ .) Figure 3.3.1 shows this plot; the rising curve is the curve  $y = 25(1 + x)^{10}$ . We used the calculator's intersection-finding facility to solve automatically for the intersection point and obtained the point  $(0.048, 40)$ .

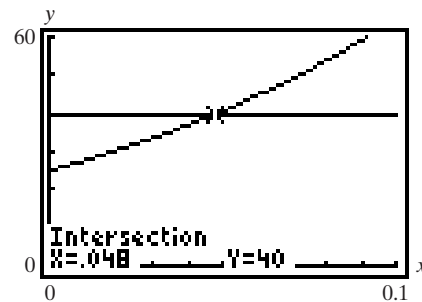


FIGURE 3.3.1 The graphs  $y = 25(1 + x)^{10}$  and  $y = 40$ .

Thus  $x = 0.048$ , and the growth rate  $r = 4.8\%$  describes the growth of Bethel during this decade. Therefore, the mathematical model

$$P(t) = 25 \times 1.048^t \quad (2)$$

gives the population (in thousands) of Bethel as a function of years after 1990.

**Numerical Method** Another approach is to solve Equation (1) by the method of tabulation. We use our calculator to tabulate values of the function  $Y1 = 25*(1 + X)^{10}$  (which we previously stored). Figure 3.3.2 shows the result when we construct a table with **TblStart = 0** and **ΔTbl = 0.001** and then scroll down in this table until the population figures (in the **Y1** column) approach the target of 40 (thousand).

X	Y1
.045	38.824
.046	39.197
.047	39.574
.048	39.953
.049	40.336
.050	40.722
.051	41.112

X = .048

FIGURE 3.3.2  $P$  is closest to 40 when  $r = 0.048$ .

The second column entry that is closest to 40 corresponds to the first column entry of  $x = 0.048 = 4.8\% = r$  for the annual percentage growth rate. This is the same growth rate we found graphically and leads to the same mathematical model shown in Equation (2).

**Symbolic Method** Finally, we illustrate an entirely algebraic approach. In order to solve the equation

$$25(1 + r)^{10} = 40,$$

we divide each side by 25 and obtain the equation

$$(1 + r)^{10} = \frac{8}{5}$$

Taking the tenth root, or the  $1/10$  power, of each side yields

$$1 + r = \left(\frac{8}{5}\right)^{1/10} = 1.048.$$

What we have found here is the growth *factor*—the value by which we multiply each “old” population of Bethel to get each “new” population of Bethel. The question asks us to find the growth *rate*  $r$ . So,

$$r = \left(\frac{8}{5}\right)^{1/10} - 1.$$

The calculator then gives  $(8/5)^{(1/10)} - 1 = 0.048$  (approximately), so once again we see that  $r \approx 4.8\%$ . Pay careful attention to the parentheses in this expression. If you forget either set of parentheses, you will not get the correct growth rate!

As we have seen, it is “the rule rather than the exception” when a choice of different methods is available to solve a given problem. The preceding example illustrates the **rule of three**, which advocates the consideration of **graphical** and **numerical** methods as well as the **symbolic** methods of ordinary algebra. Of course, plotting the graph of a function is **graphical**, calculating a table of values is **numerical**, and solving an equation algebraically (as we did here) is **symbolic**. It is important that you develop, through experience and practice, some judgment as to which of the available methods of approach is likely to work best (or seem easiest to you) in a given situation.

In this situation many students prefer to find the correct natural growth function symbolically, particularly when the growth rate is not specifically requested. Suppose that we are given both the initial value  $A_0$  (when  $t = 0$ ) and a “new value”  $V$  after a specified “elapsed time” of  $N$  years (or other appropriate time unit). Then an easy way to “build” the natural growth function  $A(t)$  is to write

$$A(t) = (\text{initial value}) \left( \frac{\text{new value}}{\text{initial value}} \right)^{t/\text{elapsed time}}, \quad (3)$$

that is,

$$A(t) = A_0 \left( \frac{V}{A_0} \right)^{t/N}. \quad (4)$$

Here's the reasoning behind this formula. We are given that the amount  $A$  increased from  $A_0$  to  $V$  in time  $N$ . Thus we multiply  $A_0$  by the factor  $V/A_0$  to get the new value  $A_0(V/A_0) = V$  after passage of the time period  $N$ . We think of  $t/N$  such time periods having passed in time  $t$ . If we started with  $A_0$  and multiplied by the factor  $V/A_0$  once for each of these elapsed time periods—that is, a total of  $t/N$  times—then we would get the result shown in (4). This is so because multiplication  $t/N$  times by  $V/A_0$  amounts to multiplication by the “aggregate factor”  $(V/A_0)^{t/N}$  that we see in (4).

## EXAMPLE 2

### Finding a Natural Growth Model

The city of Greendale had a population of 46 thousand in 1996 and 41 thousand in 2000.

- Assuming natural growth, find a model that gives the population of Greendale as a function of years after 1996.
- What was the city's population in 2005?
- Use your model to predict the year in which the population of Greendale's falls to 30 thousand.

### SOLUTION

- With  $t = 0$  in 1996, we have  $A_0 = 46$ ,  $V = 41$ , and  $N = 2000 - 1996 = 4$ . Substitution of these values in (4) yields the natural growth model

$$P(t) = 46 \left( \frac{41}{46} \right)^{t/4} \quad (\text{thousands})$$

which gives the population of Greendale  $t$  years after the year 1996.

- In parts (b) and (c), we are once again back to our usual input-output questions. Here we are given input ( $t = 9$  years) and asked for output (population). Since part (c) is the “backward” question, for which we will want to use the **intersect** command, we will go ahead and store  $P(t)$  in **Y1**. This allows us to use the **value** command to evaluate  $P(9)$ . Figure 3.3.3 shows our function, with the required parentheses; Fig. 3.3.4 shows  $P(9)$ .

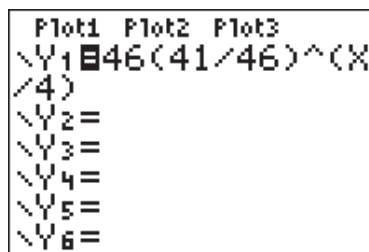


FIGURE 3.3.3 The function modeling Greendale's population.

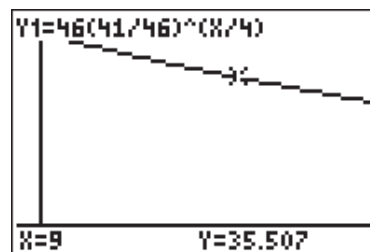


FIGURE 3.3.4 Finding the population when  $t = 9$ .

According to our model, the 2005 population of Greenwood was 35.507 thousand, or 35,507 people.



- c. The window we used in part (b),  $-1 < x < 15$ ,  $-5 < y < 50$ , is not quite large enough in the  $x$ -direction to show clearly the intersection of the graph of **Y1** with the line **Y2 = 30**. So in a new window, with **Xmax=20**, the **intersect** command gives the result shown in Fig. 3.3.5.

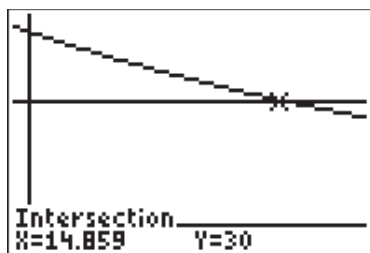


FIGURE 3.3.5 Finding when the population is 30 (thousand).

Our model predicts that by 2010 the population of Greenwood will have decreased to 30 thousand.

Notice that in Example 2, the calculator performs perfectly well without “knowing” explicitly either the annual growth rate  $r$  or the annual growth factor  $1 + r$ . If, for whatever reason, we wanted to determine  $1 + r$ , we could apply the law of exponents

$$a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m.$$

In this case, then,

$$P(t) = 46 \left( \frac{41}{46} \right)^{t/4} = 46 \left[ \left( \frac{41}{46} \right)^{1/4} \right]^t.$$

Since  $\left(\frac{41}{46}\right)^{1/4} = 0.9716$  (correct to four decimal places), the growth factor for the population of Greendale is approximately 0.9716.

Because the growth factor is less than 1, we know that the population is declining rather than growing. (Of course, we knew this from the data we were given.) If we subtract 1 from the growth factor to get the growth rate, we find that  $r = -0.0284 = -2.84\%$ , with the negative sign indicating that the population was decreasing at the rate of 2.84% per year.

## Exponential Models

We have used the function

$$A(t) = A_0(1 + r)^t \tag{5}$$

as a mathematical model for the *natural growth* of a quantity that starts (at time  $t = 0$ ) with initial amount  $A_0$  and thereafter grows at an annual percentage rate whose decimal equivalent is  $r$ .

There are situations (like Example 2) in which the growth rate  $r$  itself is not of specific interest, and it may then be easier to simply think of the “base constant”

$$b = 1 + r \quad (6)$$

that appears (raised to the  $t$ th power) in (5). Then (5) takes the simpler-looking form

$$A(t) = A_0 \cdot b^t. \quad (7)$$

#### DEFINITION: Exponential Function

An exponential function is a function of the form

$$f(x) = a \cdot b^x \quad (8)$$

with **base**  $b$  and **exponent**  $x$  (its independent variable).

Note that  $b$  is a *constant* raised to a *variable power*. By contrast, in an ordinary power function like  $7x^3$ ,  $x$  is a *variable* raised to a *constant power*.

Equation (8) then says that **a natural growth function is an exponential function**. Frequently, the verbal description of a natural growth function gives the base constant  $b$  rather than the growth rate  $r$ ; in this situation, we can write the function directly if we use the exponential form  $A(t) = A_0 \cdot b^t$ .

### EXAMPLE 3

#### Using the Base Constant

Suppose that a population of rabbits initially has 10 rabbits and is doubling every year.

- Find an exponential function that models the growth of this population.
- How many rabbits will there be in 5 years?
- What is the annual rate of growth of this population?

#### SOLUTION

- When we say that a quantity is doubled, we mean that it is multiplied by 2. Since the base constant  $b$  is the factor by which the population is multiplied each year,  $b$  is 2 for this population. With an initial population of 10, our function model is then

$$P(t) = 10 \cdot 2^t.$$

- Because the population is doubling, we can easily find its value when  $t = 5$ . We don't even need the calculator for this computation. We can use inputs from 0 to 5, and then double the population as we go from one year to the next, as shown in the following table. (Of course, we wouldn't want to do this if we were asked about, say, 20 years, and we would be unable to do it “in our head” if we were asked about a fractional number of years.)

$t$	$P(t)$
0	10
1	20
2	40
3	80
4	160
5	320

- c. The base constant  $b$  is just another “name” for the growth factor  $1 + r$ . For this population  $b = 2$ , so we have  $1 + r = 2$ , and thus  $r = 1$ . To give  $r$  as a percent, we write  $1 = 1.00 = 100\%$ . Therefore, a population that is doubling every year is growing at an annual rate of 100%.

A population that *triples* every year is described by the function  $P(t) = P_0 \cdot 3^t$ ; one that *quadruples* every year is described by  $P(t) = P_0 \cdot 4^t$ ; and one that *quintuples* annually is described by  $P(t) = P_0 \cdot 5^t$ . Obviously the larger the base constant  $b$ , the faster the exponential function  $b^t$  grows in value as  $t$  increases.

There’s no reason that it—whatever *it* is—has to happen in a single year. Our rabbit population of Example 3 could double every 3 years, rather than every year. In that case, we can modify the table we constructed to reflect this new situation:

$t$	$P(t)$
0	10
3	20
6	40
9	80
12	160
15	320

You can see that we multiply the initial population of 10 by

- $2^1$  when  $t = 3$
- $2^2$  when  $t = 6$
- $2^3$  when  $t = 9$

and so on. Notice that each exponent is determined by dividing  $t$  by 3. That is, instead of multiplying by  $2^t$ , we are multiplying by  $2^{t/3}$ . So the function describing this population is  $P(t) = 10 \cdot 2^{t/3}$ .

**Natural Growth Model: Multiplication by  $b$  Every  $N$  years**

If a quantity with initial value  $A_0$  grows naturally and is multiplied by the number  $b$  every  $N$  years (or other appropriate unit of time), then it is described by the function

$$A(t) = A_0 \cdot b^{t/N} \quad (9)$$

This formula is equivalent to the earlier formula in (4). To see this, suppose the value of the quantity after the initial period of  $N$  years is denoted by  $V$ . Then the initial amount  $A_0$  has been multiplied by the factor  $b = V/A_0$  to get  $V$ . But substitution of  $b = V/A_0$  in (9) gives the same function  $A(t) = A_0(V/A_0)^{t/N}$  shown in (4).

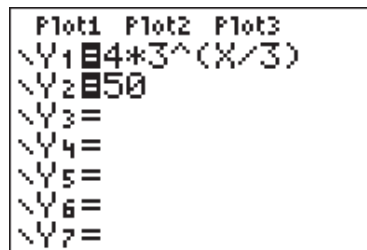
**EXAMPLE 4****A Population That Triples**

Suppose that Quasimodo has four bats in his belfry, and the bats are tripling every 5 years.

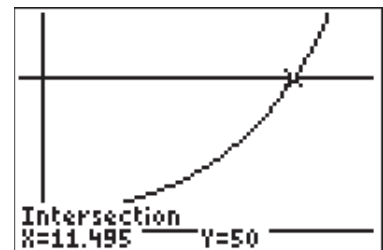
- Find an function  $B(t)$  that gives the number of bats as a function of the time in years.
- How long will it take for the population of bats to grow to 50 bats?
- How many bats will there be in Quasimodo's belfry in 7 years?

**SOLUTION**

- With an initial population of four bats and the population tripling every 5 years, the function model is  $B(t) = 4 \cdot 3^{t/5}$ .
- Storing  $B(t)$  in **Y1** and 50 in **Y2** (Fig. 3.3.6), we use the **intersect** command to obtain the screen shown in Fig. 3.3.7.



**FIGURE 3.3.6** Storing the functions to solve  $B(t) = 50$ .



**FIGURE 3.3.7** The intersection point  $(11.495, 50)$ .

So after approximately  $11\frac{1}{2}$  years the population of bats will grow to 50.

- Here we can use the **value** command, or we can type the function evaluation directly on the home screen of the calculator, as shown in Fig. 3.3.8.

A calculator screen showing the calculation  $4 \cdot 3^{(7/5)}$  resulting in 18.622.

FIGURE 3.3.8 Evaluating  $B(7)$ .

After 7 years, there will be 18 or 19 bats in the belfry (depending on whether we ignore the “fractional” bat or round to the nearest whole bat).

**Warning** We emphasize once again that the parentheses enclosing the exponent  $7/5$  in Example 4(c) are vital (as are the ones in our function definition as well)! If we omitted them we would get

$$4 \cdot 3^{7/5}$$

$$1749.600$$

instead. You should recognize immediately that this cannot be the correct value. Since  $t = 7$ , the population has tripled once (to 12 bats at  $t = 5$ , but has not tripled again (to 36 bats at  $t = 10$ ).

The calculator, of course, finds the correct value for the calculation you requested. The calculator “understands” the ordinary order of operations you learned in elementary algebra. That is, *unless parentheses indicate otherwise*, raising to a power comes before multiplying or dividing (which then occur in order from left to right). Thus the command  $4 \cdot 3^{7/5}$  means to first raise 3 to the seventh power, then multiply this value by 4, and finally divide the result by 5. Without parentheses this calculation is  $4 \cdot (3^7)/5$  instead of  $4 \cdot 3^{(7/5)}$ . Most people dislike superfluous parentheses because they complicate typing (and also make the eyes glaze over) and therefore try not to use them unless actually necessary to tell the calculator precisely what to do. But a good practice is “When in doubt, use parentheses” to make sure the calculator does what you intend.

## EXAMPLE 5

### Radioactive Decay

A nuclear reactor accident at the state’s engineering school has left its campus contaminated with three times the maximal amount  $S$  of radiation that is safe for human habitation. Two and one half months after the accident the campus radiation level has declined to 75% of its original level. Assuming natural decline of this radiation level,

- how long must students and faculty members wait before it is safe for them to return to campus;
- what is the half-life of this substance?

### SOLUTION

- Let us write  $A(t)$  for the amount of radiation still present after  $t$  months. If we measure  $A$  as a multiple of the maximal safe amount  $S$ , then we

won't need to know exactly what  $S$  is (who knows this sort of thing offhand?). For in terms of these “safe units” (SU) we're given that  $A(0) = 3$  SU and we want to find when  $A(t) = 1$  SU, so it's safe for folks to come back to campus.

In addition to the initial amount  $A_0 = 3$ , we're given that the amount of radiation is multiplied by the factor  $b = 75\% = 0.75$  every  $N = 2.5$  months. Therefore Equation (8) gives

$$A(t) = 3 \times 0.75^{t/2.5}.$$

Notice that here time is measured in months rather than in years. To find when  $A = 1$ , we define **Y1 =  $3 \cdot 0.75^{(X/2.5)}$**  and **Y2 = 1** with the idea of seeing where the amount graph crosses the horizontal line  $y = 1$ . Figure 3.3.9 shows a plot in the viewing window defined by  $0 < x < 12$ ,  $0 < y < 4$ . Automatic intersection-finding yields the intersection point  $(9.5471, 1)$ . Thus we see that  $A$  falls to 1 SU in just over  $9\frac{1}{2}$  months. With human lives at stake, we probably ought to add a margin for safety and wait at least 10 months, maybe a full year, before reoccupying the campus.

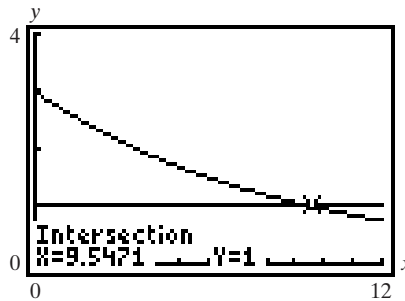


FIGURE 3.3.9 Solving the equation  $3 \times 0.75^{t/2.5} = 1$ .

- b. Recalling that the half-life of a substance is the amount of time for only half it to remain, we replace **Y2 = 1** with **Y2 = 1.5** (half of our initial 3 SU). For the same window as before, **intersect** gives the intersection point  $(6.024, 1.5)$ . Therefore, the half-life of the radioactive substance is approximately 6 months.

## SUMMARY

We say that a quantity **grows** or **declines naturally** provided that

$$A(t) = A_0 \cdot b^t$$

for constants  $A_0$  and  $b$  both greater than zero. Recalling the relation  $b = 1 + r$  between the growth rate  $r$  and the (positive) base constant  $b$ , we see that as  $t$  increases,  $A(t)$

- **increases** if  $r > 0$  (and hence  $b > 1$ ),
- **decreases** if  $r < 0$  (and hence  $b < 1$ ).

This means that, as we scan the graph of  $A(t)$  from left to right, the curve

- **rises** if  $b > 1$  (natural growth),
- **falls** if  $0 < b < 1$  (natural decline).

Letting  $A_0 = 1$ , we can see typical “growth curves” for values of  $b > 1$  in Fig. 3.3.10. Figure 3.3.11 shows some typical “decay curves” for several values of  $b < 1$ .

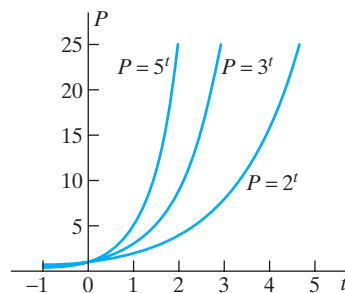


FIGURE 3.3.10 Natural growth curves.

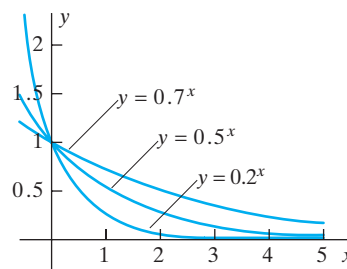


FIGURE 3.3.11 Natural decay curves.

Many quantities grow or decline naturally—that is, are multiplied by equal factors in equal times—and are therefore described by exponential functions.

## 3.3 Exercises Building Your Skills

Each of the tables in Exercises 1–4 gives data from a function that is exponential. Determine the factor by which  $y$  is multiplied each year, and find a function of the form  $y = a \cdot b^x$  that fits these points exactly.

1.

$x$	0	1	2	3	4
$y$	10	20	40	80	160

2.

$x$	0	1	2	3	4
$y$	5	15	45	135	405

3.

$x$	0	1	2	3	4
$y$	100	20	4	0.8	0.16

4.

$x$	0	1	2	3	4
$y$	80	40	20	10	5

In Exercises 5–12, write a natural growth or decay function model.

- $P(0) = 100$  and  $P(1) = 175$ .
- $P(0) = 400$  and  $P(1) = 155$ .
- $P(0) = 120$  and  $P(4) = 40$ .
- $P(0) = 120$  and  $P(3) = 180$ .

9.  $P(0) = 75$  and  $P$  doubles every year.
10.  $P(0) = 125$  and  $P$  triples every year.
11.  $P(0) = 100$  and  $P$  doubles every  $2\frac{1}{2}$  years.
12.  $P(0) = 50$  and  $P$  triples every  $3\frac{1}{4}$  years.
13. Find the half-life for the function model in Exercise 6.
14. Find the half-life for the function model in Exercise 7.
15. Find the annual growth or decay rate for the function model in Exercise 7.
16. Find the annual growth or decay rate for the function model in Exercise 8.

## Applying Your Skills

17. The population of Amsterdam, New York, was 20.7 thousand in 2000 and 18 thousand in 2003. Assuming natural growth,
  - a. find its annual percentage decrease;
  - b. find its predicted population in the year 2010;
  - c. find the year in which the population falls to half the initial amount.
18. The population of Crestwood, Missouri, was 11.2 thousand in 1990 and 11.9 thousand in 2000. Assuming natural growth,
  - a. find its annual percentage increase;
  - b. find the predicted population in the year 2010;
  - c. find the year in which the population doubles.
19. The population of Nevada was 1.2 million in 1990 and 2 million in 2000. Assuming natural growth,
  - a. find its annual percentage increase;
  - b. find the predicted population in the year 2010;
  - c. find the year in which the population reaches 3 million.
20. The population of North Dakota was 642.2 thousand in 2000 and 636.7 thousand in 2003. Assume natural growth;
  - a. find its annual percentage decrease;
  - b. find the predicted population in the year 2010;
  - c. find the year in which the population falls to 500 thousand.
21. The population of the world's more developed regions was 1.002 billion in 1965 and 1.176 billion in 1985, while the population of the world's less developed regions was 2.356 billion in 1965 and 3.706 billion in 1985.
  - a. Using a natural growth model, in what year do you expect the population of the world's more developed regions to double?
  - b. Using a natural growth model, in what year do you expect the population of the world's less developed regions to double?
22. The Giving USA Foundation reports that individual Americans gave 11.19 billion dollars to charity in 1964 and 92.52 billion dollars in 1994.
  - a. Assuming natural growth, find the annual percentage rate of growth in individual American's charitable giving over this time period.



- b. Assuming this trend continued, predict the amount of money given to charity by individual Americans in 2004.
  - c. Compare your prediction in part (b) with the actual value, \$187.92 billion.
23. According to *USA Today*, spending on medical equipment is rising rapidly because of the aging of the baby boomer population and the improving quality of health care in countries like India and China. In 2001, U.S. medical device revenue was \$47.5 billion and was projected to be \$80.2 billion in 2006.
- a. What annual percentage increase in revenue does this represent?
  - b. Use a natural growth model to predict the U.S. medical device revenue in 2010.
  - c. In what year does your model predict U.S. medical device revenue of \$100 billion?
24. In 1979, the typical microcomputer contained 29 thousand transistors, while in 1993, a typical microcomputer contained 3.1 million transistors.
- a. Assuming natural growth, find a function giving the number  $N(t)$  of transistors in a typical microcomputer  $t$  years after 1979.
  - b. Find the annual growth rate  $r$  in part (a), expressed as a percentage.
  - c. At this rate, how many months does it take to double the number of transistors in a typical microcomputer?
  - d. Assuming this annual rate of increase continues, how many transistors (rounded off accurate to the nearest million) did the typical microcomputer contain in the year 2001?
25. A naturally growing bacteria population  $P(t)$  numbers 49 at 12 noon.
- a. Write a formula giving  $P(t)$  after  $t$  hours if there are 294 bacteria at 1 PM.
  - b. How many bacteria are there at 1:45 PM?
  - c. At what clock time (to the nearest minute) that afternoon are there 20 thousand bacteria?
26. How long does it take a naturally growing bacteria population to triple if it doubles in 1.5 hours?
27. The number of bacteria in a culture increased sixfold in 10 hours. Assuming natural growth, how long did it take their number to double?
28. The English language evolves naturally in such a way that 77% of all words disappear (or are replaced) every 1000 years.
- a. Of a basic list of words used by Chaucer in the year 1400, what percentage would have been in use in the year 2000?
  - b. How long will it take until only 1% of Chaucer's words are still in use?
29. A survey by the market research firm NPD Group, as reported in the *San Antonio Express News*, found that over the 20-year period from 1985 to 2005, the percentage of Americans who said they find overweight people less attractive dropped from 55% to 24%.
- a. Find a natural growth model that gives the percentage of Americans who find overweight people less attractive as a function of years after 1985.
  - b. Use your model to predict the percentage of Americans who find overweight people less attractive in 2008.
  - c. In what year will only 5% of Americans find overweight people less attractive?

29. An accident at a nuclear power plant has left the surrounding area polluted with radioactive material that decays naturally. The initial amount of radioactive material present is 15 SU (safe units), and 5 months later it is still 10 SU.
- Write a formula giving the amount  $A(t)$  of radioactive material (in SU) remaining after  $t$  months.
  - What amount of radioactive material will remain after 8 months?
  - How long (in total number of months or fraction thereof) will it be until  $A = 1$  SU so it is safe for people to return to the area?
30. On April 26, 1986, the worst accident in the history of nuclear power occurred at the Chernobyl plant in Ukraine. An estimate of the amount of radioactive cesium-137 released from the plant is 2.7 million curies. If cesium-137 has a half-life of 30 years, how much of the amount released remained on April 26, 2006, the twentieth anniversary of the accident?
31. The National Center for Health Statistics reported that 557,271 Americans died of cancer in 2002, and that number declined to 556,902 deaths in 2003.
- Find a natural growth model that gives the number of Americans who died of cancer as a function of years after 2002.
  - What was the annual percentage decrease in the number of Americans dying of cancer?
  - If this trend continues, how many years will it take for American cancer deaths to fall below 500,000?
32. Thousands of years ago ancestors of the American Indians crossed the Bering Strait from Asia and entered the Western Hemisphere. Since then, they have fanned out across North and South America. The single language that the original Indian settlers spoke has since split into many Indian “language families.” Assume that a language family develops into 1.5 language families every 6000 years. There are now 150 Indian language families in the Americas. About when did the first American Indians arrive?
33. On the April 24, 1999, edition of the Car Talk radio show, Tom and Ray presented a puzzler concerning a fellow who visited a nursery for advice concerning a new lawn. The following scenario is an adaptation of that puzzler. Mike wants to have a lush, new lawn for the Fourth of July. Since it is already May 30, it is too late for grass seed, and sod is too expensive for his budget. Matt, the nursery worker, suggests a new product—a small plug of grass that doubles in size every day. After consulting a diagram of the yard, Matt does some calculations, and reports his conclusion. If Mike plants only one plug on June 1, he will have the lawn he desires on June 30. Being a bit nervous, Mike decides to buy two, just to be on the safe side. If Mike plants two grass plugs on June 1, on what day will he have his new lawn?

## 3.4 FITTING NATURAL GROWTH MODELS TO DATA

In Section 3.1 we discussed briefly the data in the following table, which shows the U.S. population as recorded during the first six census counts, in the decade years 1790, 1800, 1810, 1820, 1830, and 1840.

$t$	YEAR	U.S. CENSUS POPULATION (MILLIONS)	CHANGE (MILLIONS)
0	1790	3.9	
10	1800	5.3	1.4
20	1810	7.2	1.9
30	1820	9.6	2.4
40	1830	12.9	3.3
50	1840	17.1	4.2

We saw that the U.S. population was growing at approximately 3% per year over these 50 years, and we modeled the population with the natural growth function

$$P(t) = 3.9 \times (1.03)^t. \quad (1)$$

But now we wonder whether there is a natural growth function that will fit these data even better.

Recall that in Section 2.4, we used the calculator's linear regression feature to find the best-fitting linear model—the one that had the smallest SSE and average error. This is the criterion we would like to continue to use for judging which model best fits a set of data. So let us first consider the SSE and average error for our “3% mathematical model” for the early U.S. population.

The following table compares the actual population figures with those predicted by this 3% natural growth model, and gives both the errors and their squares.

$t$	ACTUAL POPULATION	$P(t) = 3.9 \times (1.03)^t$	ERROR $E$	$E^2$
0	3.9	3.90	0.00	0.0000
10	5.3	5.24	0.06	0.0036
20	7.2	7.04	0.16	0.0256
30	9.6	9.47	0.13	0.0169
40	12.9	12.72	0.18	0.0324
50	17.1	17.10	0.00	0.0000

*Population values and errors in millions.*

The sum of the squares of the errors shown in the table is

$$\text{SSE} = 0.0000 + 0.0036 + 0.0256 + 0.0169 + 0.0324 + 0.0000 = 0.0755.$$

Since there are  $n = 6$  data points, the average error in this natural growth model is given by

$$\text{average error} = \sqrt{\frac{\text{SSE}}{n}} = \sqrt{\frac{0.0755}{6}} = \sqrt{0.0126} \approx 0.1122. \quad (2)$$

Thus the average discrepancy between the actual census population and the population predicted by the natural model in (1) is about 0.112 million, or 112,000 persons.

We could now search around, trying different natural growth models, to see if we can find one with a smaller SSE and average error. We could adjust either the initial amount or the annual growth rate or both, each time checking to see if we get a better model. However, happily for us, the calculator does not only linear regression, but also many other kinds, including exponential regression. For a given set of data, exponential regression will supply a “best-fitting” function of the form

$$y = a \cdot b^x. \quad (3)$$

In a typical data-modeling situation, the numbers  $a$  and  $b$  in (3) are not known in advance. Indeed the question ordinarily is “What numerical values for  $a$  and  $b$  yield the best natural growth model for the given data?” As before, the “optimal” model is the one that best fits the actual data, and the better of two different models is the one giving the lesser average error.

## The Best Fitting Natural Growth Model

Figure 3.4.1 shows 1790–1840 U.S. population data entered in a calculator. The  $t$ -values 0, 10, 20, 30, 40, 50 corresponding to the decade years from 1790 to 1840 are stored as list **L1**, and the corresponding list of census population figures is stored as list **L2**. Figure 3.4.2 shows the resulting **STAT EDIT** menu displaying the data in table form.

Calculator screen showing data entry for lists L1 and L2. The screen displays the following text:

```
{0,10,20,30,40,50} → L1
{0,10,20,30,40,50} → L2
{3.9,5.3,7.2,9.6,12.9,17.1} → L2
{3.9 5.3 7.2 9.6...
```

FIGURE 3.4.1 Storing the 1790–1840 U.S. census data.

TI-83 **STAT EDIT** screen showing the U.S. population data in table form. The screen displays the following data:

L1	L2	----- 2
0.0	3.9	
10.0	5.3	
20.0	7.2	
30.0	9.6	
40.0	12.9	
50.0	17.1	
-----	-----	
L2(6) = 17.1		

FIGURE 3.4.2 TI-83 **STAT EDIT** screen showing the U.S. population data in table form.

In the calculator’s **STAT CALC** menu, item **0: ExpReg** (below item 9) is the calculator’s so-called “exponential regression” facility for finding the natural growth (or “exponential”) curve  $y = a \cdot b^x$  that best fits the selected data. As we did with linear regression, we will tell the calculator where our inputs are stored (**L1**), where our outputs are stored (**L2**), and where to save our natural growth function rule (**Y1**). So the command we enter is **ExpReg L1, L2, Y1**. Figure 3.4.3 shows the display that results when this command is entered.

The displayed results say that the natural growth curve best fitting the given data is

$$y = 3.9396 \times (1.0300)^x. \quad (4)$$

```

ExpReg
y=a*b^x
a=3.9396
b=1.0300

```

FIGURE 3.4.3 The optimal natural growth function.

In terms of time  $t$  and population  $P$  rather than the calculator's variables  $x$  and  $y$ , the natural growth function best fitting the actual 1790–1840 U.S. population growth is given by

$$P(t) = 3.9396 \times (1.0300)^t. \quad (5)$$

Observe that this optimal population function corresponds to a population that grows at a 3.00% annual rate, starting with an initial population of 3.9396 million when  $t = 0$ . Note also that this best-fitting initial population, about 3.94 million, is different from the actual initial population of 3.9 million. So when we attempt to best-fit a given list of population data, we are free to modify both the initial population and the rate of growth in order to fit the data as closely as possible.

The **ExpReg L1, L2, Y1** command automatically enters Equation (4) in the **Y=** menu. With **Plot 1** turned **On** in the **STAT PLOT** menu, **GRAPH** then gives the plot of the best-fitting natural growth curve shown in Fig. 3.4.4, where the original census data points are shown as small squares.

Here we are less interested in the graph than in a table of values that we can use to compute the average error in our optimal fit. Our calculator now provides the table of (rounded off) values of **Y1 = 3.9396\*1.03^X** shown in Fig. 3.4.5.

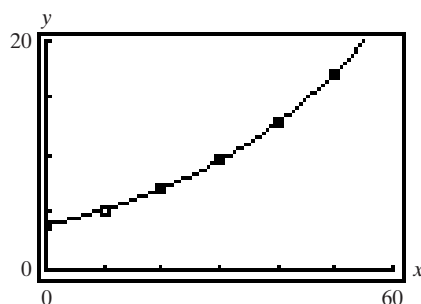


FIGURE 3.4.4 The optimal natural growth fit.

X	Y1	
0.00	3.94	
10.00	5.29	
20.00	7.12	
30.00	9.56	
40.00	12.85	
50.00	17.27	
60.00	23.21	
X=0		

FIGURE 3.4.5 Values of the optimal natural growth function  $P(t) = 3.9396 \times 1.0300^t$ .

When we combine these values with our original U.S. population data, we get the following table. It compares the original census data with the population figures predicted by the optimal natural growth model  $P(t) = 3.9396 \times (1.0300)^t$ .

$t$	ACTUAL POPULATION	$P(t) = 3.94 \times (1.03)^t$	ERROR $E$	$E^2$
0	3.9	3.94	-0.04	0.0016
10	5.3	5.29	0.01	0.0001
20	7.2	7.12	0.08	0.0064
30	9.6	9.56	0.04	0.0016
40	12.9	12.85	0.05	0.0025
50	17.1	17.27	-0.17	0.0289

Population values and errors in millions.

As usual, the final column shows the *squares* of the errors. Hence the SSE associated with the optimal natural growth model in (18) is

$$\text{SSE} = 0.0016 + 0.0001 + 0.0064 + 0.0016 + 0.0025 + 0.0289 = 0.0411.$$

Since there are  $n = 6$  data points, the average error in the optimal model is given by

$$\text{average error} = \sqrt{\frac{\text{SSE}}{n}} = \sqrt{\frac{0.0411}{6}} = \sqrt{0.0069} \approx 0.0828.$$

Thus the average discrepancy between the actual census population and the population predicted by the natural model in (5) is about 0.083 million, or 83 thousand persons.

In this case, the average error of 83 thousand for our “best-fitting” model is less than the average error of 112 thousand in the 3% natural growth model  $P(t) = 3.9 \times (1.03)^t$ . However, contrary to one’s natural expectation, the so-called “best-fitting model” given by a calculator’s exponential-regression facility is not always an exponential model with the least possible average error. The example that follows demonstrates this surprising fact.

## EXAMPLE 1

### Comparing Natural Growth Models

One of the indicators of China’s growing economy is its trade volume with other nations. The table gives this trade volume in billions of dollars for selected years from 1980 to 2004.

YEAR	TRADE VOLUME (BILLIONS OF DOLLARS)
1980	38
1985	70
1990	115
1995	281
2000	474
2004	1155

Source: Atlanta Journal-Constitution.

- Use the first and last data points to find a natural growth model giving trade volume as a function of years after 1980.
- Find the SSE and average error for the model in (a).
- Use exponential regression to find the best-fitting natural growth model giving trade volume as a function of years after 1980.
- Find the SSE and average error for the model in (c).

### SOLUTION

- Using the formula

$$A(t) = A_0 \left( \frac{V}{A_0} \right)^{t/N}$$

from Section 3.3, we obtain the natural growth function

$$A_1(t) = 38 \left( \frac{1155}{38} \right)^{t/24}, \text{ where } A_1 \text{ is measured in billions and } t \text{ is years after 1980.}$$

- As before, we enter our inputs in **L1**, our outputs in **L2**, and our natural growth function  $A(t)$  in **Y1**. We will use our calculator's table to find the squares of the errors as follows:

- In the **STAT EDIT** menu, highlight **L3**. At the bottom of the screen **L3 =** appears. Hit **VARS**, arrow over to **Y-VARS**, hit **ENTER**, select Function, then **ENTER** again to select **Y1**. The bottom line of the **STAT EDIT** menu now reads **L3 = Y1**. Type **(**, then **2ND** and **1** for **L1**, then **)**. The bottom line of the **STAT EDIT** menu now reads **L3 = Y1(L1)**. Hit **ENTER**, and the calculator displays in **L3** (Fig. 3.4.6) the output values predicted by the natural growth model.

L1	L2	L3	Z
0.000	38.0000	38.000	
5.000	70.000	77.393	
10.000	115.00	157.62	
15.000	281.00	321.02	
20.000	474.00	653.81	
24.000	1155.0	1155.0	
-----	-----	-----	
L2(1)=38			

FIGURE 3.4.6 Values predicted by the natural growth model

$$A_1(t) = 38 \left( \frac{1155}{38} \right)^{t/24}.$$

- To find the errors, arrow over and highlight **L4**. At the bottom of the screen **L4 =** appears. Type **2ND** and **2** for **L2**, then **-**, and **2ND** and **3** for **L3**. Hit **ENTER**, and the calculator displays in **L4** the errors (actual - predicted) for the natural growth model.
- Finally, to find the squares of the errors, arrow over and highlight **L5**. At the bottom of the screen **L5 =** appears. Type **2ND** and **4** for **L4**, then the **x<sup>2</sup>** key, and **ENTER**. The calculator displays in **L5** the squares of the errors for this model. Figure 3.4.7 shows the errors in **L4** and their squares in **L5**. (Notice that the errors for the first and last data points are 0 because we built the function to contain those exact points.)

L3	L4	L5	5
38.000	0.000	0.0000	
77.393	-7.393	54.652	
157.62	-42.62	1816.6	
321.02	-40.02	1601.6	
653.81	-179.8	32330	
1155.0	0.000	0.000	
-----	-----	-----	
L5(1)=0			

FIGURE 3.4.7 Predicted values, error, and the squares of the errors for

$$A_1(t) = 38 \left( \frac{1155}{38} \right)^{t/24}.$$

So the SSE for this model is given by

$$\text{SSE} = 0 + 54.652 + 1816.617 + 1601.630 + 32330.207 + 0 = 35803.106,$$

where we have scrolled down through **L5** to obtain three decimal places for each value. Therefore, the average error is  $\sqrt{\frac{35803.106}{6}} = 77.248$ , correct to three decimal places.

- c. We already have our data stored in **L1** and **L2**, so we execute the command **ExpReg L1, L2, Y1**, and obtain the natural growth function **Y1 = 34.218\*1.149^X**. Thus, the best-fitting exponential model for the China trade volume data is  $A_2(t) = 34.218(1.149)^t$ , again with  $A_2$  given in billions and  $t$  as years after 1980.
- d. Following exactly the same steps as in part (b), we obtain Fig. 3.4.8, showing the predicted values, the errors, and their squares.

L3	L4	L5	5
34.218	3.782	14.3004	
68.526	1.474	2.173	
137.23	-22.23	494.26	
274.82	6.176	38.140	
550.37	-76.37	5832.5	
959.26	195.74	38315	
-----	-----	-----	
L5(1)=14.303524			

FIGURE 3.4.8 Predicted values, error, and the squares of the errors for  $A_2(t) = 34.218(1.149)^t$ .

So,

$$\begin{aligned} \text{SSE} &= 14.304 + 2.173 + 494.255 + 38.140 + 5832.463 + 38315.034 \\ &= 44696.369, \text{ and the average error is } \sqrt{\frac{44696.369}{6}} = 86.310. \end{aligned}$$

Note that this average error, for the best-fitting function  $A(t) = 34.218(1.149)^t$ , exceeds the average error of 77.248 for our original exponential model  $A(t) = (38)(1155/38)^{t/24}$ , as found in part (b).

We see that, unlike the linear regression from Section 2.4, exponential regression does not always provide us with the function with the smallest possible SSE. (This is because of the way the calculator does exponential regression, which we can't explain without ideas from the next chapter.)



So if it's not always “the best there is,” why do we bother? First, it is a model that is quite good at fitting our data—it is indeed the “best-fitting” exponential model in a certain technical sense (which we will describe in Chapter 4). Frequently it is better than a “first point-last point” model like the one we created in part (a) of Example 1. Second, we don't have the mathematics that's needed to find the “perfect” model, so we'd rather not spend lots of time and energy trying different models when one that is easy to obtain will do very nicely.

We will continue to refer to the models that we obtain by regression as the models that “best fit” our data (even though that term is sometimes misleading). This gives us the vocabulary by which we will recognize when we are asked to use regression to find a function model.

## Newton's Law of Cooling

Suppose that a hot object with initial temperature  $T_0$  is placed in a relatively cool medium with constant temperature  $A$ . For example, we might put a hot cake (just out of the oven) in a cool room with air temperature  $A$ , or we might put a hot rock in a large water tank with water temperature  $A$ . Then **Newton's law of cooling** says that the difference

$$u(t) = T(t) - A \quad (6)$$

between the temperature of the object at time  $t$  and the temperature  $A$  is a naturally declining quantity. That is,

$$u(t) = a \cdot b^t \quad (7)$$

with appropriate values of the positive parameters  $a$  and  $b$ , where  $b < 1$ . In the case of the hot cake, for instance, we might measure its falling temperature several times with a thermometer and then attempt to fit the resulting data with an exponential function to predict when the cake will be cool enough to serve.

### EXAMPLE 2

#### When Is the Cake Cool?

Suppose a cake is baked in an oven at 350°F. At 1 PM it is taken out of the oven and placed to cool on a table in a room with air temperature 70°F. We plan to slice and serve it as soon as it has cooled to 100°F. The temperature of the cake is measured every 15 minutes for the first hour, with the following results.

Time $t$ (PM)	1:00	1:15	1:30	1:45	2:00
Temperature of cake $T$ (°F)	350	265	214	166	143

When will the cake be cool enough to eat?

#### SOLUTION

To use Newton's law we must replace the cake temperature  $T$  with the difference  $u = T - 70$  that we see in Equation (6), with  $A = 70^\circ\text{F}$  (room temperature). Measuring time  $t$  in hours and subtracting 70 from each temperature entry, we find that the given data take the form

$t$	0	0.25	0.5	0.75	1
$u$	280	195	144	96	73

If we can fit these data with an exponential model  $u(t) = a \cdot b^t$ , then we can find a model for the temperature of the cake and then solve graphically for the time when  $T = 100^\circ\text{F}$  as desired.

We begin by entering the list of  $t$ -values as calculator list **L1** and the list of  $u$  values as list **L2** (Fig. 3.4.9). Then the **STAT CALC** command **ExpReg L1, L2, Y1** produces the result shown in Fig. 3.4.10.

```
{0,0.25,0.5,0.75
,1}→L1
{0.00 .25 .50 ...
{280,195,144,96,
73}→L2
{280 195 144 96...
```

FIGURE 3.4.9 Entering the data of Example 5.

```
ExpReg
y=a*b^x
a=277.534
b=.257
```

FIGURE 3.4.10 The best exponential fit  $u(t) = 277.534(0.257)^t$ .

So the best-fitting exponential function is given by

$$u(t) = 277.534(0.257)^t \quad (8)$$

(on replacing  $X$  with  $t$  and  $Y$  with  $u$ ). Finally, remembering Equation (6), we add  $A = 70^\circ\text{F}$  (the room temperature) to both sides to obtain the cake's temperature function

$$T(t) = 70 + 277.534(0.257)^t \quad (^\circ\text{F}) \quad (9)$$

It's always important to keep your eye on the cake. Here, literally *no one* cares what is the average error in the natural decay function (8). Instead, we all want to know when we can eat our cake. That is, accepting (9) as our mathematical model for the cooling of our cake, for what value of time  $t$  is it true that

$$70 + 277.534(0.257)^t = 100^\circ\text{F}? \quad (10)$$

In Fig. 3.4.11 we have graphed the functions **Y1 = 70+277.534\*(0.257)^X** and **Y2 = 100** and asked for the coordinates of the point of intersection.

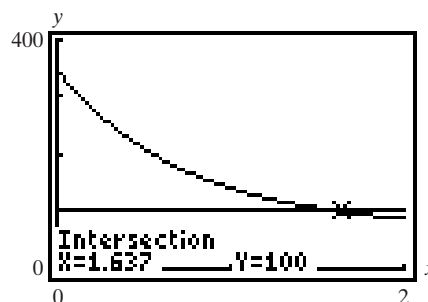


FIGURE 3.4.11 Solving Eq. (10).

- The  $x$ -value of 1.637 tells us that our cake reaches an edible temperature of  $100^\circ\text{F}$  after 1.637 hours (starting at 1 PM). Since  $0.637 \times 60 = 38.220$ , this means we can slice and eat the cake at about 2:38 PM.

In previous sections of this chapter we saw many different quantities whose growth can be modeled with natural growth functions. We can use exponential regression to find the best-fitting model whenever we are given a set of data whose behavior exhibits natural growth.

## 3.4 Exercises Building Your Skills

Each of the tables in Exercises 1–8 gives data from a function that is approximately exponential.

- First use the exponential regression facility of your calculator to find the exponential model of the form  $y = a \cdot b^x$  that best fits these points. Find  $a$  and  $b$  accurate rounded off to three decimal places.
- Calculate the average error in this optimal exponential model.

1.

$x$	0	1	2	3	4
$y$	5	7	10	15	23

2.

$x$	1	2	3	4	5
$y$	35	75	170	385	865

3.

$x$	0	2	4	6	8
$y$	15	30	60	110	220

4.

$x$	1	1.5	2	2.5	3
$y$	22	23	24	25	27

5.

$x$	0	1	2	3	4
$y$	40	27	17	11	7

6.

$x$	1	3	5	7	9
$y$	215	40	7	1	0

7.

$x$	0.5	0.75	1	1.25	1.5
$y$	80	65	55	45	40

8.

$x$	0	5	10	15	20
$y$	1000	775	600	465	360

## Applying Your Skills

In each of Exercises 9–12 the 1960–1990 population census data for a U.S. city is given.

- Find the exponential model  $P = a \cdot b^t$  (with  $t = 0$  in 1960) that best fits these census data.
- What was the city's average annual percentage rate of growth during this 30-year period?
- Use your model to predict the city's population in 2000.

### 9. San Diego

<i>Year</i>	1960	1970	1980	1990
<i>Population (thousands)</i>	573	697	876	1111

### 10. Phoenix

<i>Year</i>	1960	1970	1980	1990
<i>Population (thousands)</i>	439	584	790	1008

### 11. Cleveland

<i>Year</i>	1960	1970	1980	1990
<i>Population (thousands)</i>	876	751	574	506

### 12. Buffalo

<i>Year</i>	1960	1970	1980	1990
<i>Population (thousands)</i>	533	463	358	328

### 13. San Antonio

<i>Year</i>	1960	1970	1980	1990
<i>Population (thousands)</i>	588	654	786	935

### 14. Raleigh

<i>Year</i>	1960	1970	1980	1990
<i>Population (thousands)</i>	94	123	150	212

In Exercises 15–21, use the best-fitting exponential model to determine the answer to each question. (For problems using calendar years, “reset the clock” so that  $t = 0$  in the year for which you first have data.)

- In a certain lake the intensity  $I$  of light beneath the surface is a naturally declining function of the depth  $x$  in feet. The following table gives  $I$  as a percentage of the intensity of light at the surface:

<i>Depth <math>x</math> (feet)</i>	0	5	10	15
<i>Intensity <math>I</math> (%)</i>	100	60	35	20

How deep would you have to dive in this lake so that the light intensity there will be 1% of the surface intensity?

16. The atmospheric pressure  $p$  (in pounds per square inch,  $\text{lb}/\text{in}^2$ ) is a naturally declining function of the altitude  $x$  above sea level. The following table gives  $p$  as a function of  $x$  in thousands of feet:

Altitude $x$ (thousands of feet)	0	4	8	12
Pressure $p$ ( $\text{lb}/\text{in}^2$ )	14.7	12.6	10.9	9.3

- a. Suppose that you (like most people) cannot survive without special conditioning at an air pressure of less than  $7.5 \text{ lb}/\text{in}^2$ . Then how high could you safely fly in an airplane without pressurization?
- b. What is the air pressure atop Mt. Everest (at 29,029 feet)?
17. The amount money earned by winning the Kentucky Derby has grown quite a bit since the Derby's first running in 1875. The following table gives the net amount earned by the winner as a function of the year of the race.

Year $t$	1875	1900	1925	1950	1975
Winnings (\$)	2850	4850	52,950	92,650	209,600

Source: Churchill Downs Incorporated ([www.kentuckyderby.com](http://www.kentuckyderby.com)).

- a. In what year would you expect the prize money to grow to \$1,000,000?
- b. Use your model to predict the prize money in 2007.
18. As cassette tapes and compact discs became more popular, the sales of vinyl singles declined according to the accompanying table.

Year $t$	1988	1989	1992	1994	1996
Millions of units $s$	65.6	36.6	19.8	11.7	10.1

Source: *The World Almanac and Book of Facts 1998*.

Suppose that vinyl singles will be discontinued when their sales fall below 2 million. In what year will this occur?

19. The success of a major league baseball team is frequently tied very closely to the quality of its pitching staff. The relief pitchers (the "bullpen") come in the game to pitch when a pitcher tires or gets in trouble. The table gives the ERA (earned run average) and the number of games saved by the bullpen of the Atlanta Braves over various seasons between 1969 and 2005.

ERA	2.60	3.73	5.11	3.06	4.22
Games saved	57	41	24	49	32

Source: *Atlanta Journal-Constitution*.

If the Braves bullpen had an ERA of 6.50 (a very bad ERA), how many games would you expect them to save in a season?

20. The gross domestic product (GDP) of a country is a very common measure of the vitality of its economy. China's economy has been growing rapidly, as shown by the increase in its GDP in the accompanying table.

<i>Year t</i>	1980	1985	1990	1995	2000	2004
<i>GDP of China (billions of dollars)</i>	266	306	388	706	1072	1647

Source: Atlanta Journal-Constitution.

- a. According to your model, what will the GDP of China be in 2010?
  - b. When will the GDP of China rise to 2000 million (2 trillion) dollars?
21. A BlackBerry is an all-in-one mobile device that includes phone, e-mail, Web browser, and organizer features. The number of BlackBerry subscribers has grown according to the following table.

<i>Year t</i>	2001	2002	2003	2004	2005
<i>Thousands of subscribers S</i>	165	321	534	1070	2510

Source: USA Today.

- a. At what average annual rate did the number of BlackBerry subscribers grow during this period?
  - b. If this trend continues, how many BlackBerry subscribers would you expect there to be in 2010?
22. Suppose a hot cake like the one of Example 2 is taken out of an oven at  $175^{\circ}\text{C}$  and immediately set out on a cool porch where the temperature is  $10^{\circ}\text{C}$ . During the next hour the following temperature readings of the cake are taken:

<i>Time (PM)</i>	5	5:20	5:40	6
<i>Temperature (<math>^{\circ}\text{C}</math>)</i>	175	125	85	60

The cake will be brought in and served when it has cooled to  $35^{\circ}\text{C}$ . When do you expect this to be?

23. A pitcher of buttermilk initially at  $25^{\circ}\text{C}$  is set out on the  $10^{\circ}\text{C}$  porch of Exercise 22. During the next half hour the following temperature readings are taken:

<i>Time (PM)</i>	1	1:10	1:20	1:30
<i>Temperature (<math>^{\circ}\text{C}</math>)</i>	25	20	17	15

The buttermilk will be brought in and served when it has cooled to  $12^{\circ}\text{C}$ . When do you expect this to be?

## Chapter 3 | Review

In this chapter, you learned about exponential functions and models. After completing the chapter, you should be able to

- Determine whether a relation described numerically, graphically, or symbolically represents an exponential function.
- Find the output value of an exponential function for a given input value.
- Find the input value of an exponential function for a given output value.
- Solve an equation or inequality involving an exponential function

- Find an exponential function model that fits given exponential data.
- Find the best-fitting exponential model for data that are approximately exponential.
- Find the annual percentage rate of growth or decay for an exponential model.

## Review Exercises

In Exercises 1–7, the given information (table, graph, or formula) gives  $y$  as a function of  $x$ . Determine whether each function is linear, exponential, or neither.

1. 

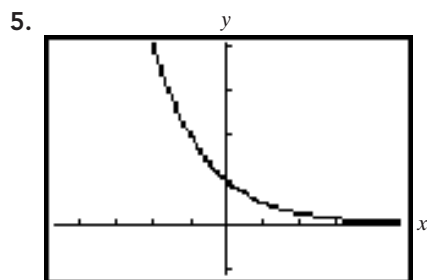
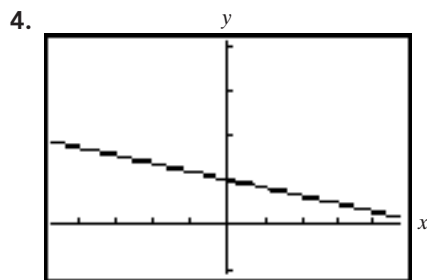
$x$	-3	-2	0	4	7	10
$y$	4.5	2	0	8	24.5	50

2. 

$x$	-2	-1	1	3	4	5
$y$	32	16	4	1	.5	.25

3. 

$x$	-4	-3	-1	2	4	5
$y$	29	26	20	11	5	2



6.  $y = 3x^2$

7.  $y = 3 \cdot 2^x$

8. Suppose that a child who lost her first tooth in 1998 was left \$1.00 by the Tooth Fairy. Let us assume that the Tooth Fairy leaves the same amount for each tooth, if

payments are adjusted for inflation. Then, based on the Consumer Price Index, the Tooth Fairy should have left her mother \$0.19 for a tooth in 1963 and given her grandmother \$0.09 for a tooth in 1938. Find the best-fitting exponential function for these data, and use it to predict how much the Tooth Fairy should leave for the little girl's daughter if she loses a tooth in 2023.

Source for conversion factors: Robert Sahr, Political Science Department, Oregon State University.

9. An accident at a nuclear power plant has left the surrounding area polluted with a radioactive element that decays naturally. The initial radiation level is 10 times the maximum amount  $S$  that is safe, and 100 days later it is still 7 times that amount. How long (to the nearest day after the original accident) will it be before it is safe for people to return to the area?
10. Suppose that, as of January 1, 1999, a state's annual Medicaid expenditures are \$1 billion and increasing at 6% per year, while its annual Medicaid tax income is \$2 billion and is increasing at 3% per year. If these rates continue, during what calendar year will the state's Medicaid budget go into the red (with expenditures exceeding income)?
11. According to the BP Statistical Review of World Energy 2006, the United States produced 7733 thousand barrels of oil a day in 2000 and 6830 thousand barrels daily in 2005. Use an exponential model to determine the average annual rate at which U.S. daily oil production was declining over this time period.
12. The following table gives the gross revenue  $R$  in millions of dollars for the Broadway season that begins in the indicated year  $t$ . Find the best-fitting exponential model for these data and use it to predict the year in which gross revenue will grow to 1000 million (1 trillion) dollars.

$t$ (year)	1984	1989	1992	1996	2001	2005
$R$ (millions)	209	282	328	499	643	862

Source: League of American Theatres and Producers.

13. (Chapter Opener Revisited) In the discussion that introduced this chapter, we looked at a plot that displayed the growth of the Internet over the years from 1993 to 1999. The plot was based on data from the Internet Software Consortium giving the number of Internet hosts, which is displayed in the following table.

<i>Calendar year</i>	1993	1994	1995	1996	1997	1998	1999
<i>t (years after 1993)</i>	0	1	2	3	5	5	6
<i>H (millions)</i>	1.3	2.2	4.9	9.5	16.1	29.7	43.2

- Find a natural growth function  $H_1(t)$  that models the number of Internet hosts as a function of years after 1993 and agrees with the data given for 1993 and 1998.
- At what average annual rate was the number of Internet hosts growing over this time period? What is the sum of squares of errors  $SSE_1$  in the model  $H_1(t)$  considering the given data for the years through 1993 through 1998 but ignoring the data for 1999?
- Use your model to predict the number of Internet hosts in 2004. What was the error in your approximation if the actual value was 317.6 million?
- Your prediction is so much larger than the actual value because the number of Internet hosts did not continue to grow at the rate you found in part (b). The data in the following table give the number of Internet hosts for the years 1998 to 2002.

<i>Calendar year</i>	1998	1999	2000	2001	2002
<i>t (year after 1993)</i>	5	6	7	8	9
<i>H (millions)</i>	29.7	43.2	72.4	109.6	147.3

Source: ISC Internet Domain Survey.

- Find a natural growth function  $H_2(t)$  that models the number of Internet hosts as a function of years after 1993 and agrees with the data given for 1998 and 2002.
- At what average annual rate was the number of Internet hosts growing over this time period? What is the sum of squares of errors  $SSE_2$  in the model  $H_2(t)$  considering the given data for the years through 1998 through 2002?
- Use your model to predict the number of Internet hosts in 2004. Find the error in your approximation, given that the actual value for 2004 was 317.6 million.
- The two exponential functions  $H_1(t)$  and  $H_2(t)$  in parts (a) and (d) agree at the point  $t = 5$  corresponding to 1998. Consider the function  $H(t)$  that equals  $H_1(t)$  for  $t \leq 5$  but equals  $H_2(t)$  for  $t \geq 5$ . Can you see why  $H(t)$  might be called a “piecewise-exponential model” for the number of Internet hosts from 1993 to 2002? What is the SSE of this model considering the data from 1993 to 2002. Can you find SSE quickly—without doing any more real computation—using your answers to parts (b) and (e)?
- What is the average error in the piecewise-exponential model  $H(t)$  of (g) for the data from 1993 to 2002?

## INVESTIGATION Population Projections for U.S. Cities

In this activity, you will create function models based on U.S. Census data for one of the 100 largest cities in the United States. U.S. Census data are available at [www.census.gov](http://www.census.gov).

From the census data for 1990, select the  $xy$ th largest city, where  $x$  and  $y$  are the last two digits of your student identification number. This is your city. (For example, if your student ID number ends in 37, you should find the data for the 37th largest city in the United States. If your ID number ends in 00, choose the 100th largest city.) Record the populations for your city in 1950, 1960, 1970, 1980, and 1990.

- Find the best-fitting linear model for your population data, using years after 1950 as the independent variable.
- Calculate the SSE for your linear model and the average error in the approximation.
- Use your linear model to predict your city’s population in 2000.
- Find the best-fitting exponential model for your population data.
- Calculate the SEE for your exponential model and the average error in the approximation.
- Use your exponential model to predict your city’s population in 2000.
- On the basis of having the smallest average error, which model is the best predictor of your city’s population?
- Use census data to determine your city’s actual population in 2000. Which model came closer to the actual population?