One of the daily activities for teacher Flo Pearson’s first and second graders is to gather at the classroom calendar and write “Amazing Equations.” These are story problems that have answers equal to the day’s date. On the morning of April 20, for example, children write problems that have an answer of 20.

At the beginning of the lesson, Ms. Pearson has the children sit together on the rug in front of an easel. She asks children to think of and share Amazing Equations that equal 20. As the children state their problems, Ms. Pearson records the problems, modeling the appropriate use of mathematical symbols on a large sheet of blank paper clipped to the easel.

Ms. Pearson wraps up the whole-group part of the lesson by reviewing how to write the equations with symbols and then asking, “Is there more than one way to get to twenty? (Children: “Yes!”) All right, let’s see how many more ways we can find today.”

Working in groups of three, children write story problems with 20 as the answer. Some children use manipulatives to model the story as they write it. Ms. Pearson circulates around the room, asking questions and providing assistance as needed. She encourages students to develop an oral story first and then write down what happened in that story.

At the end of the class time, Ms. Pearson reconvenes the class as a whole group on the rug to share their Amazing Equations. She asks one child from each group to read their story and another to read the corresponding equation.

Erin and Vi’s Amazing Equation:
Erin: “Ten candles on a cake for my little brother’s birthday plus ten more candles equals twenty.”
Vi: “Ten plus ten more equals twenty.”

Colanthia and Cam’s Amazing Equation:
Colanthia: “I am on the tot lot playing...”
Principles and Standards Link 7-1

**Content Strand: Number and Operations**

During the primary grades, students should encounter a variety of meanings for addition and subtraction of whole numbers. Multiplication and division can begin to have meaning for students in prekindergarten through grade 2 as they solve problems that arise in their environment, such as how to share a bag of raisins fairly among four people. In grades 3–5, helping students develop meaning for whole-number multiplication and division should become a central focus. (NCTM, 2000, p. 34)

**Cam:** “Nineteen plus one more is twenty.”

**Matt, Pat, and Rommel’s Amazing Equation:**

**Matt:** “Me and Pat and Rommel and Ms. Pearson we each found five bikes and when we—I didn’t get to finish the rest.”

**Ms. Pearson:** “So we each had five bikes and how many did we end up with? (Twenty). Did you write the equation?”

**Matt:** “Five plus five plus five plus five equals twenty.”

**Nate and Shea’s Amazing Equation:**

**Nate:** “We had nineteen books and Chris gave us one at the library and then we had twenty books.”

**Shea:** “Nineteen plus one equals twenty.”

**Connecting with the Standards**

The National Council of Teachers of Mathematics (NCTM) emphasizes how important it is for children to understand the meaning of operations, stating that “understanding the fundamental operations of addition, subtraction, multiplication, and division is central to knowing mathematics” (NCTM, 1989, p. 41). There are four key aspects of “developing operation sense,” or understanding operations, that include

- recognizing real-world settings for each operation;
- developing an awareness of models and properties of each operation;
- recognizing relationships among the operations; and
- understanding the effects of an operation.

Understanding mathematical operations and their related computational procedures requires that children grasp the concepts of each of the operations, learn the basic facts, and develop computational procedures. This chapter discusses understanding the concepts of addition, subtraction, multiplication, and division. (Learning facts and developing computational procedures are examined in subsequent chapters.)

Carefully examine the story problems written by the children in Ms. Pearson’s class and consider these children’s understanding of operations. How are the problems different? How are they the same? How can teachers help children make sense of operations? One important role elementary teachers play is to help children understand the meaning of the four operations of addition, subtraction, multiplication, and division of whole numbers.

**INTRODUCE OPERATIONS WITH WORD PROBLEMS**

Learning about operations should be based on developing meaning and understanding, which begins with an exploration of real-world settings or story problems. As you read at the beginning of the chapter, even young children are able to write many story problems based on real-life experiences. Children begin to construct meaning for mathematical operations before they enter school through informal, real-life experiences such as sharing cookies or combining collections of cards or action figures. Because these experiences can be translated into word problems and because word problems in familiar contexts are more meaningful to children than symbolic expressions, early instruction on operations should introduce children to addition, subtraction, multiplication, and division by having children solve word problems.

**A Model for Beginning with Word Problems**

Young children develop an understanding of operations by solving a variety of word problems. If those problems come from familiar real-world experiences, children can see more personal relevance, which enables them to more easily analyze the problem and its component parts. Real-world problems can come from everyday classroom opportunities such as routine opening activities like Ms. Pearson’s daily calendar exercise, classroom events, or even examples from children’s literature. In addition, children can help make decisions about sharing and distributing classroom supplies, especially when many-to-one groupings need to be made, which are meaningful contexts for multiplication and division (Kouba & Franklin, 1993). For example, if 10 children get 5 minutes each to use the computer, how much time will be needed? Teachers can take advantage
of these familiar experiences to pose problems and discuss operations. Such rich, familiar contexts in which mathematical operations are introduced naturally not only help children recognize personal relevance and math connections but also help them judge the reasonableness of their answers.

After introducing a real-world problem, a good teacher encourages children to represent or translate it into some model (see Figure 7-1). Initially, this model should be a concrete representation of the setting, for example, using actual objects mentioned in the problem, such as cookies, or blocks or counters to represent the objects in the problem. Later the teacher can introduce pictorial representations. Mental images and children’s natural language skills also play key roles in developing conceptual understanding.

As a teacher, you can make different materials available to children to use in modeling problems. You can also encourage children to be creative in their representations of problem situations, whether concretely or pictorially, and provide ample opportunities for children to discuss and interpret situations presented concretely and pictorially. For example, a daily classroom routine could include children sharing how they modeled a problem. Once children are able to interpret different types of problems and identify and model the operation needed to solve each one, they can write number sentences to represent solution processes. It is important to remember, however, that children must have an extensive number of experiences with meaningful problem solving before they are introduced to symbolic expressions, and symbolic expressions are best introduced as a way to record concrete representations.

Meanings develop over time and, while they are being developed, children should not be pressured to memorize basic facts. In fact, understanding the meaning of the operations is a critical component of mastering the basic facts. Likewise, working with word problems should be the basis of children’s early experiences with operations and must not be delayed until children “know their facts.” The most important consideration is for children to connect their real-life experiences and language with the mathematical language and symbolism associated with each operation (Trafton & Zawojewski, 1990).

It also is important to mention that the “key word” strategy of replacing certain words with others, such as “is means equal” and “of means times,” is not helpful to children. In fact, teaching children to look for “key words” to solve word problems is ineffective and detrimental (Sowder, 1988). Instructional time is better used helping children develop an understanding of the meaning of operations through solving real-world story problems with concrete objects.

**Figure 7-1** A Model for Introducing Operations with Word Problems

<table>
<thead>
<tr>
<th>Real-World Setting or Problem</th>
<th>Models</th>
<th>Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Concrete</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Pictorial</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mental images</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Language</td>
<td></td>
</tr>
</tbody>
</table>

**Encoding and Decoding Word Problems**

Once children understand the meaning of a variety of problem situations, provide them with opportunities to both encode and decode their number sentences. In other...
Chapter 7  Developing Whole-Number Operations: Meaning of Operations

Principles and Standards Link 7-3
Content Strand: Number and Operations

Students can learn to compute accurately and efficiently through regular experience with meaningful procedures. They benefit from instruction that blends procedural fluency and conceptual understanding. (NCTM, 2000, p. 87)

Types of Addition and Subtraction Word Problems

The four addition and subtraction problem types—Join, Separate, Part-Part-Whole, and Compare—are distinguished by the presence or absence of action and the types of relationships involved. The basic structure of each of these problem types is illustrated in Figure 7-2.

Let’s take a look at examples of each problem type. In each of the addition and subtraction problem types, two quantities are given and one is unknown. The examples in Figure 7-2 use the fact family 4, 7, and 11 to illustrate each problem type.

Join problems. In a Join problem, elements are being added or joined to a set. The three quantities involved are the starting amount, the change amount, and the resulting amount.

Separate problems. In a Separate problem, elements are being removed from a set. As in Join problems, the three quantities involved are the starting amount, the change amount, and the resulting amount. In Figure 7-3 (see p. 6) Amy, a second grader, wrote and illustrated a Separate problem and wrote a corresponding number sentence. Notice that she showed that one fish swam away by crossing it out in the picture.

Part-Part-Whole problems. In Part-Part-Whole problems there is no action. Instead, these problems focus on the relationship between a set and its two subsets. The three quantities involved are the two parts and the whole. Unlike Join and Separate problems, there is no change over time. Figure 7-4 shows a sample of a Part-Part-Whole problem (see p. 6). Notice that Amy’s use of two different colors of snails clearly shows the two parts of the problem.

Compare problems. There is no action in Compare problems. Instead, they involve comparisons between two different sets. The three quantities involved are the two wholes and the difference. These four types of addition and subtraction word problem result in 11 different kinds of addition and subtraction problems. Figure 7-2 provides a comprehensive look at these types of problems so you can better examine their similarities and differences.

UNDERSTANDING ADDITION AND SUBTRACTION

Researchers have identified four types of addition and subtraction problems: Join, Separate, Part-Part-Whole, and Compare (Carpenter & Moser, 1982). Join and Separate problems involve action. Part-Part-Whole and Compare problems do not involve action but are identified by the relationships of the quantities in the problems. Research shows that this classification system matches the way children think about these problems (Fennema, Carpenter, Levi, Franke, & Empson, 1997).

words, not only should children translate a real-life setting into a model or mathematics sentence (encoding), but they should also, given a model or mathematics sentence, be able to write a word problem that illustrates that situation. Encourage children to describe and justify their translation. For example, you might give children the following problem to solve:

Tom had 7 cookies and gave 3 to Matt. How many cookies does Tom have left?

You might ask the children to solve the problem using counters or by drawing a picture and to write a number sentence that matches the problem situation. On another occasion, you might give children a number sentence and ask them to write a word problem that illustrates that number sentence. As a follow-up to the “Amazing Equations” lesson you read about at the beginning of the chapter, Ms. Pearson may choose to give children story problems and ask them to model them with manipulatives, or to give children number sentences and ask them to write corresponding story problems.

Different parts of this generalized conceptual model, depicted in Figure 7-1, are emphasized at different stages in the three-part process just described. When developing the concept or meaning of operations, it is most effective for children to focus on the real-world setting, the model for the setting, and the translation of one to the other. Mathematical symbols are not ignored but are used only as a tool.

In summary, research suggests that exposure to a wide range of word problems from the beginning of the school experience significantly improves children’s mathematics performance (Sigler, Fuson, Ham, & Kim, 1986). Engaging children in solving word problems before they learn the basic facts enables them to focus on the nature of the problem and to model it to find an unknown answer (Burns, 1991).
**Figure 7.2** Structure of the Four Types of Addition and Subtraction Problems

<table>
<thead>
<tr>
<th>Problem Type</th>
<th>Example Problem</th>
<th>Model of Problem Type</th>
<th>Number Sentence</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Join</strong></td>
<td><em>Result Unknown:</em> Peter had 4 cookies. Amy gave him 7 more cookies. How many cookies does Peter have now?</td>
<td><img src="image1" alt="Model" /></td>
<td>4 + 7 = ___</td>
</tr>
<tr>
<td></td>
<td><em>Change Unknown:</em> Peter had 4 cookies. Amy gave him some more cookies. Now Peter has 11 cookies. How many cookies did Amy give Peter?</td>
<td><img src="image2" alt="Model" /></td>
<td>4 + ___ = 11</td>
</tr>
<tr>
<td></td>
<td><em>Start Unknown:</em> Peter had some cookies. Amy gave him 7 more cookies. Now Peter has 11 cookies. How many cookies did Peter have to start with?</td>
<td><img src="image3" alt="Model" /></td>
<td>___ + 7 = 11</td>
</tr>
<tr>
<td><strong>Separate</strong></td>
<td><em>Result Unknown:</em> Peter had 11 cookies. He gave 7 cookies to Amy. How many cookies does Peter have now?</td>
<td><img src="image4" alt="Model" /></td>
<td>11 − 7 = ___</td>
</tr>
<tr>
<td></td>
<td><em>Change Unknown:</em> Peter had 11 cookies. He gave some cookies to Amy. Now Peter has 4 cookies. How many cookies did Peter give to Amy?</td>
<td><img src="image5" alt="Model" /></td>
<td>11 − ___ = 4</td>
</tr>
<tr>
<td></td>
<td><em>Start Unknown:</em> Peter had some cookies. He gave 7 cookies to Amy. Now Peter has 4 cookies. How many cookies did Peter have to start with?</td>
<td><img src="image6" alt="Model" /></td>
<td>___ − 7 = 4</td>
</tr>
</tbody>
</table>

(continued)
### PROBLEMS WITH NO ACTION

<table>
<thead>
<tr>
<th>Problem Type</th>
<th>Example Problem</th>
<th>Model of Problem Type</th>
<th>Number Sentence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part-Part-Whole</td>
<td><strong>Whole Unknown:</strong> Peter has some cookies. Four are chocolate cookies and 7 are oatmeal raisin cookies. How many cookies does Peter have?</td>
<td><img src="image" alt="Part-Part-Whole Model" /></td>
<td>(4 + 7 = _)</td>
</tr>
<tr>
<td></td>
<td><strong>Part Unknown:</strong> Peter has 11 cookies. Four are chocolate cookies and the rest are oatmeal raisin cookies. How many oatmeal raisin cookies does Peter have?</td>
<td><img src="image" alt="Part-Part-Whole Model" /></td>
<td>(4 + _ = 11)</td>
</tr>
<tr>
<td>Compare</td>
<td><strong>Difference Unknown:</strong> Peter has 11 cookies and Amy has 7 cookies. How many more cookies does Peter have than Amy?</td>
<td><img src="image" alt="Comparison Model" /></td>
<td>(7 + _ = 11)</td>
</tr>
<tr>
<td></td>
<td><strong>Larger Unknown:</strong> Amy has 7 cookies. Peter has 4 more cookies than Amy. How many cookies does Peter have?</td>
<td><img src="image" alt="Comparison Model" /></td>
<td>(7 + 4 = _)</td>
</tr>
<tr>
<td></td>
<td><strong>Smaller Unknown:</strong> Peter has 11 cookies. Peter has 4 more cookies than Amy. How many cookies does Amy have?</td>
<td><img src="image" alt="Comparison Model" /></td>
<td>(4 + _ = 11)</td>
</tr>
</tbody>
</table>

---

**Figure 7-2**  Continued

**Figure 7-3**  A Separate Problem Written by a Second Grader

**Figure 7-4**  A Part-Part-Whole Problem Written by a Second Grader
Examining Children’s Reasoning: Linking to “Amazing Equations”

Now let’s relate these problem types to the story problems the children wrote for their “Amazing Equations” listed at the beginning of this chapter. Use the terminology described in the previous section and in the samples given in Figure 7-5 to identify the problem type for each problem.

What do you notice about the problems the children wrote? Most of these problems are Join Result Unknown, with one Part-Part-Whole, Whole Unknown. No one wrote a Separate problem or a Compare problem. What might account for this? It may be that children are most familiar with Join situations as compared to other problem types.

How might Ms. Pearson use this information? In future lessons, she might decide to expose the children to Separate, Part-Part-Whole, and Compare problems in order to expand their experiences with these problem types. Ms. Pearson might decide to share a problem of her own during an “Amazing Equations” lesson, perhaps writing a Separate, Part-Part-Whole, or Compare problem and asking the children to write a number sentence for it. For example, she might decide to use Nate and Shea’s library book context from Figure 7-5 and write a Separate or Compare problem. Providing children with new types of problems set in familiar contexts and situations helps them make sense of problems and connect new learning with existing understandings.

Examining Children’s Reasoning: Strategies for Solving Story Problems

Children solve story problems in several different ways, including strategies that utilize direct modeling, counting, derived facts, and recall (Carpenter, Fennema, Franke, Levi, & Empson, 1999). Children use direct modeling by representing the quantities in the problem with concrete objects. Children use counting strategies such as counting on, counting back, or skip counting to solve the problem. Children use derived facts to build on or modify facts they have memorized. Children use recall to retrieve facts they have memorized to quickly solve the problem.

To see how children use these strategies, view Clip #1 on the IMAP CD. In this clip three different children are solving the same basic problem:

Nicole (or some other child) had 6 stones (or some other object). How many more stones (or other object) does she/he need to collect to have 13 altogether?

What problem type is this problem? Which problem-solving strategy is each child using? Figure 7-6 summarizes the strategy each child used.

Note that a child who uses the recall strategy has memorized the needed fact and needs only to retrieve it from memory. This is the most sophisticated strategy, and a long-term goal of instruction, but one that is developed over time. (Chapter 8 focuses on helping children memorize the basic facts after they have developed an understanding of the meaning of the operations.)

For more practice in identifying problem types and solution strategies, view the clip on the IMAP CD titled “Clip #2—Dillon,” which shows a second-grade boy solving seven different problems. You also will have an opportunity to view and reflect on this clip in question 1 of the “For Your Journal” section at the end of this chapter.

Figure 7.5 Sample Problems Written by Children in “Amazing Equations” Lesson

<table>
<thead>
<tr>
<th>Children</th>
<th>Amazing Equation</th>
<th>Problem Type</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matt</td>
<td>“I was at a store and bought seventeen pieces of candy. Two more pieces of candy that my uncle gave to me. I need one more and I got it from my big sister Katy.”</td>
<td>Join Result Unknown</td>
<td>Action involved</td>
</tr>
<tr>
<td>Darnell</td>
<td>“There were nineteen roaches in Dee’s house then one more came from Jennifer’s house. Then there was twenty roaches in Dee’s house.”</td>
<td>Join Result Unknown</td>
<td>Action involved</td>
</tr>
<tr>
<td>Erin and Vi</td>
<td>“Ten candles on a cake for my little brother’s birthday plus ten more candles equals twenty.”</td>
<td>Unclear—perhaps Part-Part-Whole, Whole Unknown</td>
<td>Unclear, but the problem does not seem to involve action</td>
</tr>
<tr>
<td>Nate and Shea</td>
<td>“We had nineteen books and Chris gave us one at the library and then we had twenty books.”</td>
<td>Join Result Unknown</td>
<td>Action involved</td>
</tr>
</tbody>
</table>
Chapter 7  Developing Whole-Number Operations: Meaning of Operations

Figure 7-6  Strategies Used by Different Students to Solve the Same Problem

<table>
<thead>
<tr>
<th>Child</th>
<th>Problem</th>
<th>How Solved</th>
<th>Solution Strategy</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nicole</td>
<td>Nicole had 6 stones. How many more stones does she need to collect to have 13 altogether? 6 + ___ = 13</td>
<td>Counts out 6 cubes. Counts more cubes till she has 13 cubes altogether. Separates out original 6 cubes and counts remaining cubes.</td>
<td>Direct Modeling</td>
<td>She models all the quantities in the problem with cubes.</td>
</tr>
<tr>
<td>Michaela</td>
<td>Your sister has 6 marbles. How many more marbles does she need to have 13 marbles altogether? 6 + ___ = 13</td>
<td>Counted on from 6, saying “seven, eight, nine, ten, eleven, twelve, thirteen.” She uses her fingers to keep track of how many times she’s counted.</td>
<td>Counting</td>
<td>She does not need to count 6, but instead starts at 6 and counts on to 13.</td>
</tr>
<tr>
<td>Miguel</td>
<td>Ramona has 6 goldfish. But she wants 13. How many more does she need to buy? 6 + ___ = 13</td>
<td>“Six plus six is twelve, and one plus is seven is thirteen.” Since 6 plus 6 is 12, and 13 is one more than 12, the answer must be one more than 6, or 7.</td>
<td>Derived Facts</td>
<td>He uses a fact he has memorized, 6 + 6, to find the result to a problem he has not memorized.</td>
</tr>
</tbody>
</table>

Using Models to Solve Addition and Subtraction Problems

Direct modeling. Addition and subtraction problems can be modeled with many different types of materials, including real-world objects (such as pencils) and manipulative materials (such as poker chips). The term direct modeling refers to the process in which children use concrete materials to exactly represent the problem as it is written. For example, consider the following Join problem:

Joyce had 3 pencils. Scott gave her 5 more pencils. How many pencils does Joyce have now?

This problem can be solved by having two children (representing Joyce and Scott) directly model or act out the problem. Have one child display three pencils and another child give her five more pencils. The children can then count the number of pencils in the joined sets of pencils now held by Joyce.

It usually is easier for children to solve a problem by modeling it with the actual objects referred to in the problem, as done in the previous example. Ask the children to solve the problem and discuss its solution. Then, after solving a number of problems using the actual objects in the problem for direct modeling, begin to use counters. You should be aware, however, that using a manipulative material such as poker chips to model a problem about pencils is somewhat more abstract and will be a bit more difficult for young children to successfully solve.

Problems involving action, such as Join and Separate, are easiest for children to solve by direct modeling. Part-Part-Whole and Compare problems are more difficult to model. Delay introducing these types of problems until children can successfully use direct modeling to solve Join and Separate problems.

Note that diagrams such as that shown in Figure 7-7 are sometimes confusing. For example, when asked how many trucks are in set C, a young child may respond that there are none. Some young children do not understand class inclusion, that is, that set A and set B are both part of set C. These children may say that all the trucks are in sets A and B, so there are none in set C. This is one reason why concrete objects should be used first in developing problem-solving skills. When children physically join...
two sets, the initial sets are no longer apparent but the children can remember that they started with two sets, and the difficulty sometimes associated with a diagram like the one that appears in Figure 7-7 does not occur.

**Modeling Separate problems.** Separate problems are fairly easy for children to model. To do so, remove an amount from the starting amount and observe the difference. Consider the following problem:

*Megan had 6 cookies (the starting amount). She gave 2 to her brother. How many cookies does Megan have left?*

Using counters, children could set out six, physically remove two, then count the four remaining counters. Using a pictorial model, children might place an X through the two cookies given away.

It is usually more difficult for children to use pictorial models for subtraction, and these models can be confusing at times. A typical textbook picture to illustrate 3 – 1 would show three objects, with one of the objects crossed out. Some children write 2 – 1 because they see groups of two and one in the picture.

A better approach would be to show or have children draw a “before” and “after” picture as in Figure 7-8. The pictures show that there were three flowers at the beginning, that one flower was “taken away” (as shown in the “after” picture with one flower crossed out), and there are two flowers left, which is the difference.

Another way to show an amount being taken away is to circle that set and use an arrow to show it being removed. Children can easily learn to identify the starting amount, the amount being taken away, and the difference in such diagrams.

Page (1994) developed a set of sequenced lessons intended to develop an understanding of subtraction as separating or taking away. An important feature of these lessons is a chart similar to the one in Figure 7-9 in which children record elements of their subtraction stories.

**Modeling Part-Part-Whole and Compare problems.**

Modeling Part-Part-Whole and Compare problems involves matching two sets by using one-to-one correspondence and then observing the number of objects in one set that are unmatched. The real-life setting for this approach to subtraction usually focuses on “how many more” or “how many fewer” there are in one group than another. Figure 7-10 uses a pictorial model to represent the following problem:

*Manuel had 6 candies and Keisha had 4 candies. How many more candies did Manuel have than Keisha?*

Children could solve this problem by creating one group of six candies or counters to represent Manuel’s candies and another group of four candies or counters to represent Keisha’s candies. To find how many more candies Manuel had, children can arrange the candies to show a one-to-one correspondence between Manuel’s and Keisha’s candies, placing one candy from Manuel’s set beside or on top of another from the other set and observing
the number of candies that do not have a match, showing how many more candies Manuel had than Keisha.

Using measurement models to model problems. Another way to model addition and subtraction problems is by using measurement models. In this approach lengths, rather than discrete, countable objects, are used to represent the quantities in the problem. Towers of Unifix cubes, trains of Cuisenaire rods, and a number line are appropriate and effective ways to represent different quantities.

Unifix cubes can be broken apart and used as individual cubes, but because they interlock, they also can be used as “towers” or “trains” to indicate length. To represent addition, two towers can be constructed separately and then joined. Figure 7-11 illustrates using towers of Unifix cubes to solve $3 + 6 = 9$.

Cuisenaire rods are color-coded wooden rods. The unit rod is white, and each of rods 2 through 10 is a different color. The red rod (2) and the light green rod (3) are joined in the illustration below to solve the problem $2 + 3 = 5$. The combined length is five unit rods, which is equivalent to the yellow rod. A measurement model such as this one is useful for representing a real-world problem.

Addition and subtraction also can be represented on a number line, which is a semiconcrete model. Children can use toy grasshoppers or kangaroos to hop along the number line, making it more concrete. The addition sentence $2 + 7 = 9$ may be modeled by starting at 0, then a hop of 2 units followed by a hop of 7 units along the number line, as shown in Figure 7-12.

Similarly, separate problems also can be solved using the number line model. For example, to solve the problem $6 - 2 = ____$, a child could begin at 6, take a hop backward of 2 units, and observe the resulting position on the line, which will be at the number 4.

The number line is often a difficult model for children to understand and should not be the first model used to represent an operation. For example, children sometimes want to start at “1” instead of “0” and confuse spaces with points on the line. However, after children’s experiences with other models, introduce the number line, which is handy and concise and a common model for representing integers and the operations with integers, concepts that help develop algebraic thinking. These concepts will be discussed in Chapter 17. Also, you may want to use a more abstract model such as a number line to assess children’s ability to transfer from a concrete model to a more symbolic representation.

Writing Number Sentences for Addition and Subtraction

After children have had many experiences modeling and talking about real-life problems, encourage them to use mathematical symbols to represent problems. Figure 7-13 illustrates the three-step process of modeling a problem situation with a concrete model, with a semiastract model, and with symbols. Activity 7-1 presents a way for the process to go in the opposite direction, that is, from pictorial models to real-world problem situations. This activity can be modified to provide children with additional opportunities to write story problems that match situations. The key here is to provide children with meaningful real-world situations in which mathematics problems occur.

Although it is important that children explore addition and subtraction word problems of each type, there is often more than one correct way to write a number sentence for a problem. For example, consider the problem below:

Steve wants to buy a book that costs $11. He has $7. How much more money does he need to buy the book?

One child might represent this problem as $7 + ____ = 11$, whereas another child might represent it as $11 - 7 = ____$. Either way is acceptable. What is most important is encouraging children to make sense of problem situations and to represent those situations with symbols in meaningful ways.
UNDERSTANDING MULTIPLICATION AND DIVISION

Multiplication and division problems are fundamentally different from addition and subtraction problems because of the different types of quantities represented in multiplication and division problems. For example, consider the differences between these problems:

**Problem 1:** Peter has 2 cookies. Amy gave him 3 more cookies. How many cookies does Peter have now?
**Problem 2:** Peter has 2 bags with 3 cookies in each bag. How many cookies does Peter have?

How are these problems the same, and how are they different? Both problems contain the numbers 2 and 3 and are about cookies. In fact, the questions in each problem are almost identical (“How many cookies does Peter have [now]?”). But look more closely: What do the “2” and the “3” mean in each problem? In problem 1, both numbers and the answer represent a certain number of cookies. But in problem 2, each number means something different: The “3” represents the number of cookies in each bag, whereas the “2” stands for the number of bags. The answer, 6, represents the total number of cookies in both bags. This is an example of why multiplication and division are more complex and often harder for children to understand than addition and subtraction problems: Children have more factors to pay attention to when solving multiplication and division problems.

Another difference between multiplication and division as compared to addition and subtraction is the type of counting children are asked to do. In addition and subtraction, children use counting by ones to find the result. But in multiplication and division, children are counting by groups, sometimes called *skip counting*, to find the result. The transition to counting by groups is significant and often more difficult for children.

**Making the Transition from Adding to Multiplying**

How can teachers help children understand multiplication and division? The basics are the same as for addition
and subtraction: Good instruction helps children connect multiplication and division to representations that make sense to them (Kouba & Franklin, 1993). Just as you have children use their own words to describe relationships and problem situations, you help children link their own less formal language with the more formal mathematical language.

Language is very important in helping children understand multiplication. Many of the difficulties children have with multiplication relate to the clarity and familiarity of the language used. For example, a child might understand “give each child four cookies” but not understand “give four cookies per child.” The following are important tips to help children understand the language associated with multiplication.

Encourage children to use the phrase “groups of” to indicate creating a number of equal groups. Children can learn to use language such as “I have three groups of five—that’s fifteen.” Note, however, that the meanings of other expressions such as “three fives” and “three times five” should be developed before the symbolic expression “3 × 5 = 15” is expected to be used.

Help children understand the meaning of each quantity. Notice that the multiplication problem 2 × 3 = 6 can be interpreted as “2 groups of 3 objects” or “3 × 3 = 6.” This is different from 3 × 2 = 6, which means “3 groups of 2 objects” or “2 + 2 + 2 = 6.” Even though the total number of objects is the same in both problems, the problems have different meanings. Until children understand the commutative property, saying that “2 × 3 is the same as 3 × 2” will confuse many children. For young children, three groups of two objects is fundamentally different from two groups of three objects. Children think in concrete terms: Two children who each get three pieces of candy are luckier than three children who each get two pieces of candy (Anghileri & Johnson, 1992). The fact that the total amount of candy is the same may not be important to the child who is thinking about the lucky children who each got three pieces of candy!

The following sections discuss different types of multiplication and division problems and describe ways to support children in developing understanding of these operations.

**Types of Multiplication Word Problems**

Researchers have identified several different types of multiplication and division problems (Greer, 1992). These include Equal Groups, Area and Array, Multiplicative Comparison, and Combination problems. Examples of each of the four types of multiplication problems are shown in Figure 7-14.

**Equal Groups problems.** Equal Groups problems are based on making a certain number of equal-sized groups. The three numbers in the problem represent the number of groups, the size of the groups, and the total number of objects. Equal Groups problems generally are the most familiar type of multiplication problems. In Equal Groups problems, the multiplication sign can be interpreted as “groups of.” For example, the problem 2 × 3 means “2 groups of 3” or “2 copies of 3” using the equal groups interpretation of multiplication.

**Area and Array problems.** Area and Array problems involve finding the area of a rectangular region or finding the total number of objects in a rectangular array (or arrangement). The area of a rectangle can be found by covering the region with unit squares and counting them or by multiplying the length by the width. In contrast, arrays are rectangular arrangements of discrete, countable objects, such as desks arranged in rows in a classroom.

**Multiplicative Comparison problems.** Multiplicative Comparison problems involve comparing two quantities multiplicatively. In other words, these problems describe *how many times as much* one quantity is compared to another quantity. These situations also can be thought of as stretching the original quantity by a certain factor. These problems are more difficult to model with concrete objects.

**Combination problems.** Combination problems, also known as Cartesian products, involve different combinations that can be made from sets of objects, such as the number of outfits that can be made from two blouses and three pairs of slacks. This type of problem is the most difficult type of multiplication and division problem to model.
### Figure 7.14  Structure of the Four Types of Multiplication and Division Problems

<table>
<thead>
<tr>
<th>Problem Type</th>
<th>Example Problem</th>
<th>Model of Problem Type</th>
<th>Number Sentence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equal Groups</td>
<td><strong>Multiplication:</strong> Maria has 2 bags of oranges. There are 3 oranges in each bag. How many oranges does Maria have altogether?</td>
<td><img src="image1" alt="Equal Groups Model" /></td>
<td>$2 \times 3 = 6$</td>
</tr>
<tr>
<td></td>
<td><strong>Partitive Division:</strong> Maria has 6 oranges. She put the oranges into 2 bags with the same number of oranges in each bag. How many oranges are in each bag?</td>
<td><img src="image2" alt="Partitive Division Model" /></td>
<td>$6 \div 2 = 3$</td>
</tr>
<tr>
<td></td>
<td><strong>Measurement Division:</strong> Maria has 6 oranges. She put 2 oranges into each bag. How many bags of oranges did she make?</td>
<td><img src="image3" alt="Measurement Division Model" /></td>
<td>$6 \div 2 = 3$</td>
</tr>
<tr>
<td>Area and Array</td>
<td><strong>Multiplication:</strong> Maria’s parents have some orange trees planted behind their house. There are 2 rows of orange trees with 3 trees in each row. How many orange trees are planted behind Maria’s house?</td>
<td><img src="image4" alt="Area and Array Model" /></td>
<td>$2 \times 3 = 6$</td>
</tr>
<tr>
<td></td>
<td><strong>Division:</strong> Maria’s parents are going to plant 6 orange trees behind their house. They want to plant the trees in 2 equal rows. How many orange trees should they plant in each row?</td>
<td><img src="image5" alt="Division Model" /></td>
<td>$6 \div 2 = 3$</td>
</tr>
<tr>
<td>Multiplicative Comparison</td>
<td><strong>Multiplication:</strong> Maria has 3 oranges. Tony has 2 times as many oranges as Maria does. How many oranges does Tony have?</td>
<td><img src="image6" alt="Multiplicative Comparison Model" /></td>
<td>$2 \times 3 = 6$</td>
</tr>
<tr>
<td></td>
<td><strong>Division:</strong> Tony has 6 oranges, which is 2 times as many oranges as Maria has. How many oranges does Maria have?</td>
<td><img src="image7" alt="Division Model" /></td>
<td>$6 \div 2 = 3$</td>
</tr>
<tr>
<td>Combination</td>
<td><strong>Multiplication:</strong> How many different outfits can be made with 2 blouses and 3 pairs of slacks?</td>
<td><img src="image8" alt="Combination Model" /></td>
<td>$2 \times 3 = 6$</td>
</tr>
<tr>
<td></td>
<td><strong>Division:</strong> How many pairs of slacks are needed to make 6 different outfits by using 2 blouses?</td>
<td><img src="image9" alt="Division Model" /></td>
<td>$6 \div 2 = 3$</td>
</tr>
</tbody>
</table>
Introducing Children to Division

As in multiplication, the clarity and familiarity of the language used in division problems is very important. Children experience division situations throughout their everyday lives—but they may not recognize that when they’re sharing food with other family members, they’re doing division.

One way to help children make everyday connections to division problems is to use familiar language when first modeling division situations, and then build on these understandings when introducing more formal language. The terms “shared by” and “equal groups” make sense to children and should be used in early experiences. Similarly, the phrase “divided by” is more formal and it should not be used until children are successful at solving problems using less formal language. Then you can help children understand the connections between this informal and formal terminology.

Division with remainders. If children are solving division problems set in meaningful real-world settings, they will encounter some problems that don’t have a whole number as the solution. For example, when sharing 5 cookies among 2 people, each person will get 2 cookies and there will be one left over. Take advantage of this opportunity for a rich discussion about what to do with the cookie that’s left over. Frequently, even young children will suggest that they split the remaining cookie into smaller parts.

This type of problem is an excellent introduction to other types of division problems (those whose solution is not a whole number) and to a discussion about what to do with “leftovers” when dividing. Such discussions about realistic problems can be extended to include situations in which it may be appropriate to continue to divide what’s left into fractional parts, when the leftovers might just be set aside, and when it may be appropriate to make unequal groups. (For example, when 9 children need to ride in 2 cars, it would not be appropriate to cut a child in half or to leave one home, but rather to put 4 children in one car and 5 in the other.) This topic will be discussed in more detail in Chapter 9. Begin introducing division problems to children by helping them understand two basic types, Equal Groups and Area and Array problems. As children mature cognitively they will need to be exposed to the other two types of division problems, Multiplicative Comparison problems and Combination problems. All four types are described in the following section.
Types of Division Word Problems

Each of the four types of multiplication problems can be expressed as a division problem. In addition, there are two types of Equal Groups division problems. The following sections discuss each interpretation and describe ways to support children in developing an understanding of these operations.

We now examine examples of each division problem type. The four types of division problems are Equal Groups (Partitive and Measurement division), Area and Array, Multiplicative Comparison, and Combination problems, which were shown earlier in Figure 7-14.

Equal Groups problems. Equal Groups problems involve splitting a larger group into several smaller groups. The three numbers in the problem represent the number of groups, the size of the groups, and the total number of objects. There are two different types of Equal Groups division problems: Partitive and Measurement division.

In Partitive division problems, the total number of objects is partitioned into a specified number of groups. In the Partitive division example of Figure 7-14, 6 oranges are split, or partitioned, into 2 equal groups. Partitive division is also referred to as “fair sharing.”

In contrast, in Measurement division problems, the total number of objects is measured out into groups of a certain size. In the Measurement division example of Figure 7-14, 6 oranges are split into groups of 2 oranges. Measurement division is also referred to as “repeated subtraction.”

Here’s another way to look at the distinction between these two types of division: When you know the number of groups (or parts) to make, the problem is known as a Partitive division problem. And when you know the size of the groups to be measured out, the problem is known as a Measurement division problem. These distinctions should help you remember the names.

Area and Array problems. Area and Array problems for division involve finding one of the dimensions of a rectangular region or of a rectangular array (or arrangement) when the total area or total number of objects in the arrangement is given.

The following interpretations of division appear much less often, but are included here for the sake of completeness. These problems rarely occur in elementary school mathematics curricula.

Multiplicative Comparison problems. Multiplicative Comparison problems for division involve comparing two quantities multiplicatively, in other words, expressing one quantity in terms of how many times it is when compared to another, such as “Eric ate twice as many cookies as Patrick did.” But in the case of division, the multiplicative relationship, that is, how many times as much one quantity is compared to another quantity, is known, as is one of the quantities being compared. The task is to find the second quantity being compared. For example, consider this problem: “Eric ate twice as many cookies as Patrick did. If Eric ate eighteen cookies, how many cookies did Patrick eat?” These situations also can be thought of as shrinking the original quantity by a certain factor. Figure 7-14 provides an example problem.

Combination problems. Combination problems, also known as Cartesian products, involve different combinations that can be made from sets of objects, such as the number of outfits that can be made from 2 blouses and 3 pairs of slacks. In the case of Combination division problems, the total number of combinations is known, as is the number of one of the elements being combined. The task is to find the number of the second element being combined. Figure 7-14 provides an example problem.

Avoiding Misconceptions and Dead Ends

There are two common misconceptions about multiplication and division: that “multiplication makes bigger” and “division makes smaller.” For example, students will sometimes incorrectly generalize that when you multiply, the answer is bigger than the two factors, whereas when you divide, the answer is smaller than the starting amount (the dividend). These generalizations are correct for whole-number multiplication and division. However, they are incorrect when considering multiplication and division of fractions and decimals. It is important for you to be aware of these misconceptions in order to address them and not to reinforce or accept them when they come up in class discussions.

In addition, there is an instructional “dead end” to avoid: that division means only fair shares (or partitive division). You will make many instructional decisions. These decisions sometimes are even harder to make when faced with the large amount of content to be covered in what never seems to be enough class time. It may be tempting for you to consider omitting some of the interpretations of operations discussed in this chapter. In particular, you may sometimes want to teach only one interpretation of division, partitive division, which often is the meaning of division that is most familiar to children. But if you do not help your students understand both meanings of equal-groups division (partitive and measurement division) you will be leading your students into an instructional dead end. This dead end may not show up until students are introduced to division of fractions, but students who see division only as fair sharing will have more difficulty understanding division of fractions. Consider the following problems:

Problem 1: Peter has 6 cookies. He wants to give 2 cookies to each of his friends. How many friends will get cookies? (Measurement division problem—whole-number divisor)
Problem 2: Peter has 6 cookies. He wants to give \( \frac{1}{2} \) of a cookie to each of his friends. How many friends will get cookies? (Measurement division problem—fraction divisor)

Problem 3: Peter has 6 cookies. He wants to put the cookies into 2 bags with the same number of cookies in each bag. How many cookies will be in each bag? (Partitive division problem—whole-number divisor)

Problem 4: Peter has 6 cookies. He wants to put the cookies into \( \frac{1}{2} \) of a bag with the same number of cookies in each bag. How many cookies will be in each bag? (Partitive division problem—fraction divisor)

Problems 1 and 2 are both measurement division—and both are reasonable problems, as is problem 3, which is partitive division. But what about problem 4? It presents a situation that is harder to visualize.

Problem 4 illustrates the dead end—partitive division situations with fraction divisors are harder to understand; measurement division situations with fractions make more sense to most children. But what if you have decided not to cover the measurement division interpretation? Your students will be at a disadvantage when learning division of fractions. Division of fractions will be discussed in more detail in Chapter 11; it is mentioned here only to encourage you to make sure that students understand both partitive and measurement division in order to avoid a dead end in future learning.

Examining Children's Reasoning: Strategies for Solving Story Problems

The strategies for solving addition and subtraction story problems discussed earlier in this chapter also are used to solve multiplication and division problems. To see how one child uses these strategies, view Clip #23 on the IMAP CD. In this clip Nicole, a second-grade girl, solves the following problem:

At a party there were 18 M&M's to be shared fairly among 3 children. How many M&M's would each child get?

How did Nicole solve the problem? She directly modeled the problem with cubes, separating 18 cubes into 3 equal piles (after a slight adjustment to correct the number of M&M's that were shared).

How might a child use a counting strategy to solve this problem? This is not an easy problem for which to use skip counting, but a child might skip count as follows: “six, twelve, eighteen, so the answer is three, since I counted by six three times.” Skip counting may be best used in this problem to check an estimate. For example, a child might think “three groups of five is fifteen, but I have eighteen M&M’s, so the answer might be six” and then proceed to check that estimate by skip counting by sixes.

How might a child use a derived facts strategy to solve this problem? She would build on a fact she already knows, such as \( 15 \div 3 = 5 \). She might think “I know fifteen divided by three equals five, but I have eighteen M&M’s, which is three more, which means each child will get one more M&M, so they each get six M&M’s.”

How might a child using the recall strategy solve this problem? He would say that the answer is 6, because \( 18 \div 3 = 6 \). Children using the recall strategy have memorized the needed fact and need only to retrieve it from memory. More will be said about helping children memorize basic facts in Chapter 8.

More about Nicole: Think back to Clip #23 that you just viewed, or view it again. Nicole initially got the wrong answer to the problem, saying the answer was 5. What did she do wrong? What does this clip show about Nicole’s understanding of the concept of division?

Nicole seemed to be solving a different problem: \( 15 \div 3 \). When the interviewer reread the problem, what did Nicole do? She quickly took 3 more cubes and put 1 in each pile, saying then that each child gets 6 M&M’s. Nicole initially got the problem wrong not because she did not understand division, but because she was solving a different problem.

This video shows that sometimes children who understand a concept get an incorrect answer for a trivial reason, such as solving a different problem or making a counting mistake. It is important when assessing children’s understanding to carefully examine the way children solve problems to identify those who may understand the concept but got an incorrect answer because of a small or careless error. You can use these observations to guide further instruction and discussions with children.

Using Models to Solve Multiplication and Division Problems

As in addition and subtraction, many different models can be used to illustrate relationships posed in multiplication and division problems. The following section describes some of these models.

Modeling Equal Groups and Multiplicative Comparison problems. Equal Groups problems, such as in the following example, can be modeled using a set model.

Maisha and Peter decided to sell cookies in packages of 3 at their school’s bake sale. Mrs. Walsh bought 6 packages. How many cookies did Mrs. Walsh buy?

A set model is illustrated in Figure 7-15, in which six groups of three are assembled. Activities could include the children placing an equal number of cookies or counters on a specified number of plates or putting an equal number of marbles in bags. The mathematics sentence for the representation is \( 6 \times 3 = 18 \).
Similarly, Measurement division and Partitive division problems can be modeled using a set model. Generally, children are introduced to division using a subtractive, or measurement, setting. They are to find the number of groups. The repeated subtraction process for the following problem is symbolically represented in Figure 7-16.

**Figures 7-15** and **7-16**

Janice has 12 cookies. She wants to put 3 cookies into each bag. How many bags does she need?

In a partitioning setting, the total number and the number of equal groups are known. The child is to determine the number in each group. For example:

**Three children want to share 12 cookies equally. How many cookies will each child get?**

Children can find the solution by acting out a sharing or “dealing out” process (Figure 7-17). At its most basic level, a child might say “one for me, one for Tom, one for Margaret, one for me, one for Tom, one for Margaret . . . .” Later, children will realize that they can share two or more at a time, thereby making the process more efficient.

**Modeling Area and Array problems.** Area and Array problems can be modeled by making a rectangle with the given number of rows of a certain length. Consider the following problem:

Maisha and Peter decided to sell cookies at their school’s bake sale. They arranged the cookies on trays. They put 6 rows of cookies with 3 cookies in each row on one tray. How many cookies were on the tray?

**Modeling Combination problems.** Although Combination problems are the least used of the four approaches, they can be very helpful in building the concept of multiplication—particularly multiplication with zero. Consider the following example:

Lindsay has a choice of 6 flavors of ice cream and 3 different toppings. How many different kinds of ice cream sundaes could Lindsay have?
Some children have difficulty matching each topping to one flavor and then repeating that for each of the flavors. The use of a 6-by-3 chart (similar to an array) facilitates understanding this approach (Kouba & Franklin, 1993).

Consider how the preceding problem about ice cream sundaes would change if there were zero flavors of ice cream and 3 toppings. In this situation, no sundaes could be made, illustrating that \(0 \times 3 = 0\).

**An instructional sequence for modeling multiplication and division.** Kouba and Franklin (1993) use the problem “If 8 plates hold 4 cookies each, how many cookies are on all the plates?” to illustrate a sequential development in understanding multiplication:

- **Developmental Level 1:** A child sets out 8 plates, puts 4 cookies (or objects) on each plate, and counts the total number.
- **Developmental Level 2:** A child makes 8 groups of 4 without using separate objects for plates.
- **Developmental Level 3:** A child makes one group of 4 and recounts it 8 times, keeping track of how many groups have been counted by using fingers or another memory device.

“More advanced levels of representation include counting by fours; counting on when they cannot recall the next multiple, for instance, 4, 8, \ldots, 9, 10, 11, 12, and so on; adding fours; and using such derived facts as ‘Four groups of 4 are 16 and 16 plus 16 is 32’” (Kouba & Franklin, 1993, pp. 575–576). Understanding the developmental sequence of children’s understanding of multiplication helps you assess and plan instruction.

**Another Word About Notation and Children’s Language**

Children have a natural way of talking about the action involved in the operations. Capturing this natural language and using it in the classroom can help children better understand whole-number operations. Do not rush into using symbolic notation. Instead, make a slow, gradual transition from natural language to symbolic language.

Listen to the language children use as they talk about problems. They will use phrases such as “ran away” and “joined in” as they describe real-life actions. You also will hear language such as “and three more,” “start with seven and cross out four,” “three bags with two each,” and “twelve to be shared by four.” Encourage children to write statements about the problems they are solving and to write number sentences using words rather than mathematics symbols.

To capture this natural language, write key terms or phrases on cards such as those shown in Figure 7-18. The cards created for a specific class will vary with the language used by the children in the class. Also prepare nu-
meral cards with the numbers encountered in the basic facts. Encourage the children to create sentences for problems and for concrete or pictorial models of the operations by using the cards as in Figure 7-19.

Using number and language cards in the classroom can help make the transition from horizontal to vertical notation more natural. Sentences such as those shown in Figure 7-19 can be arranged in vertical format. Initially, the sentences might be formed as a simple 90-degree rotation of the horizontal sentence (Figure 7-20). Later, the format can be altered to conform more closely to conventional notation. See Figure 7-21.

Over a period of time, introduce the conventional symbols for the different action words. Also introduce the “=” symbol for words such as “makes” and “leaves.” Write these conventional symbols on cards as well and have children use them to generate sentences for problems and models in the same way they did with the natural language cards.

Children should understand that the “=” sign means “is the same number as” or “is another name for.” Having children write several expressions equivalent to a given expression should facilitate this understanding. For example, give children \(5 + 7 = \_\) and ask them to write at least three true expressions (not just one number) in the blank. They may respond with:

\[
\begin{align*}
5 + 7 &= 6 + 6 & 5 + 7 &= 10 + 2 \\
5 + 7 &= 24 + 2 & 5 + 7 &= 15 - 3
\end{align*}
\]

Note that the notation for division is particularly confusing to children. Three symbolic representations are commonly used for division: \(6 ÷ 2\), \(2 | 6\), and \(6/2\) or \(\frac{6}{2}\). The latter format usually is delayed until children are studying fractions. The first usually is read as “six divided by two” and the second as “two goes into six,” although it should also be read as “six divided by two.” The second representation is the most common, but also it is the only symbolic representation that should not be read from left to right, as “two goes into six.” It is important to help children connect the phrase “six divided by two” to each of the common symbolic forms.

**Conclusion**

It is critical that children build a sound understanding of whole-number operations. Therefore, children should be allowed time to manipulate and solidify their ideas. Solving word problems of many different types is necessary for children to develop this “operation sense.” To rush on to “more advanced” work is a mistake that often comes back to haunt children and teachers.
To complete your journal or portfolio entries online, go to the For Your Journal or For Your Portfolio module for this chapter of the Companion Website.

**FOR YOUR JOURNAL**

When you have finished studying this chapter, reflect on the following questions in your math journal:

1. View “Clip #2—Dillon” on the IMAP CD, and then reflect on the following questions:
   a. This 10-minute video clip shows Dillon, a second-grade child, solving seven problems. As you watch the video clip, identify the problem type for each problem he solves and the solution strategy he uses. The problems are listed below:
      1. Ismael has 6 marbles. How many more marbles does Ismael need to buy to have 13 altogether?
      2. Dillon has 14 colored marbles. Eight are blue, and the rest are red. How many red marbles does Dillon have?
      3. Dillon has some toy cars. He goes to the store and buys 4 more toy cars, and then he has 9 toy cars. How many toy cars did Dillon have to start with?
      4. Dillon has 9 marbles and Ismael has 4 marbles. How many more marbles does Dillon have than Ismael?
      5. A pack of gum has 5 pieces. How many pieces of gum would you have altogether if you had 3 packs of gum?
      6. At a party, there were 18 M&Ms. Three children want to share them fairly. How many M&Ms would each child get?
      7. Twenty children are going on a field trip. If each car had seat belts for only 4 children, how many cars would be needed to drive all 20 on the field trip?
   b. How is problem 6 different than problem 7, other than in the contexts and numbers in the problems?
   c. Explain the difference between Dillon’s solution to problem 6 and his solution to problem 7. Pay attention to the three quantities in each problem: the total number, the number of groups, and the size of each group.
   d. Dillon does not need to physically represent each quantity. For example, in problem 1 he does not use cubes to build the set of 6. However, the structure of the problem affects the way he approaches the solution. Choose a problem Dillon solved correctly and a problem he did not solve; discuss what factors you believe made the latter problem more difficult for Dillon.
   e. Select something from this clip that stands out for you. Explain your selection.
   f. Describe factors that make a mathematics problem difficult or easy for children to make sense of, and discuss.

2. View “Clip #24—Richard” on the IMAP CD, and then reflect on the following questions:
   a. This 8-minute video clip shows Richard, a second-grade child, solving four problems. Select something from this clip that stands out for you. Explain your selection.
   b. Explain what you think Richard is thinking during his solution to the second question, “Richard has 14 toy cars. He gives 5 toy cars to Cameron. How many toy cars does Richard have left?” We recommend that you watch the clip of this problem again.
   c. Richard spent a long time solving the problem, “Suppose you had 6 marbles. How many more marbles would you need to buy to have 13 altogether?” Analyze his solution, and discuss how he used the blocks to support his thinking. We recommend that you watch the clip of this problem again.
   d. How does the interviewer use wait time during this interview? Do you think the wait time is appropriate?
   e. Some people have been taught to use key words to decide on the operation to use to solve a story problem. For example, they are told that if the term *sum* appears, they should add. How might the use of key words affect a child solving the third problem, “Suppose you had 6 marbles. How many more marbles would you need to buy to have 13 altogether?”
   f. Imagine that you are working with a child who counts quietly to himself and mumbles under his or her breath. Generate a question or two that you might ask the child to help you understand how the child is thinking.

3. Imagine that you are a first-grade teacher. You want to begin instruction to help your students understand addition and subtraction. What will you do? What problems will you ask students to solve? What materials will you use? What teaching strategies will you employ?

4. Imagine that you are a classroom teacher. A colleague asks why you are spending so much instructional time having children solve word problems rather than drilling them on basic facts. How will you respond?
FOR YOUR PORTFOLIO

When you have finished studying this chapter, complete the following activities to include in your professional portfolio:

1. Interview a child to assess his or her understanding of operations. (You may choose to use problems from the Early Number or Place-Value Interviews on the IMAP CD or problems that you write.) Describe the problems you asked the child to solve and the strategies he or she used to solve problems. Describe what the child seemed to understand and what understanding the child still needs to develop.

2. Write a lesson plan to help introduce children to the Join type of addition problems.

3. Write a lesson plan to help introduce children to the Equal Groups type of multiplication and division problems.

4. Write a lesson plan to help children understand division by zero.

LINKS TO THE INTERNET

ProTeacher: Addition and Subtraction
http://www.proteacher.com/100011.shtml
Contains links to lessons to help children understand and practice addition and subtraction.

ProTeacher: Multiplication and Division
http://www.proteacher.com/100012.shtml
Contains links to lessons to help children understand and practice multiplication and division.

A Sample Core Curriculum for Michigan Schools
http://www.michigan.gov/scope/0,1607,7-155-13476-36378--,00.html
Contains a unit plan for whole-number operations for second grade.

RESOURCES FOR TEACHERS

Books on Whole-Number Operations