Jeffrey Stockdale starts his fifth-grade math lesson with homework review. He turns to the homework assignment in the teacher’s edition of his textbook and looks out at his class, seated in table groups of four and six. “Your homework was about adding and subtracting fractions. Get out your papers and trade with your table partner so we can check answers.” Students find their homework and make their trades; Jeffrey calls on Emma for the answer to the first problem. “Jon’s paper says $\frac{1}{3}$ of a dozen eggs.” She glances over at her own paper, which Jon is checking. “That’s what I got, too.” Jeffrey confirms that $\frac{1}{3}$ dozen is the correct answer, scanning the room quickly to see if all the students seem to be with him. Students mark their papers and wait patiently for the next problem. He continues through the rest of the homework, calling on different students to supply answers and working through solutions if there are any questions. When they finish, Jeffrey instructs his students to write the number of problems correctly solved at the top of the page. He collects all the papers and moves on to the day’s lesson. After school, he’ll note
Jeffrey Stockdale is sitting on a potential gold mine, but he doesn’t know it—his students’ homework papers are an untapped resource for his math teaching. Jeffrey assigns homework so that students can practice using the math they’ve just studied in class, and he checks papers to make sure that the kids are doing their work. Homework grades also help him keep an eye out for those students who aren’t doing well. Jeffrey clearly recognizes the value of homework as a flag for students’ performance. What he hasn’t yet thought about is homework (and other information from his classroom) as a window into his students’ thinking—a window that could help him to better diagnose what his students do and don’t understand and therefore help to guide his instructional plans.

Jeffrey can open that window by thinking differently about what homework tells him about his students. Class discussion, homework, in-class group work, and tests can all reveal a lot more about how his students are thinking than simply whether they solved a problem correctly. He already recognizes this to some degree since he regularly gives students partial credit for their work; he can further boost his insights into students’ understanding by learning to view his students’ work in new ways. At present, he basically has a single lens that focuses on the correctness of students’ answers. When many students make substantive (rather than “careless”) errors, he knows he has to review or reteach; when most students’ work is correct, he knows he can move on to something new. But Jeffrey can get more mileage out of his students’ work if he brings in other lenses as well. A mathematics lens will help him think about the mathematical purpose of the assignment and can help him assess students’ work more precisely in terms of his mathematical goals. A student thinking lens will help him look for evidence of how his students understand the math and can help him get a clearer picture of where they’re on target and where they’re not. If Jeffrey uses multiple lenses instead of just his one, he gains more information about his students—information that can help guide his instruction. He will know more about the strengths and weaknesses in his students’ mathematical understanding, and he’ll be able to pinpoint more accurately the kinds of upcoming mathematical tasks and conversations that will promote solid learning.

In this book, we aim to support you in taking advantage of your own gold mine by developing lenses that will focus you more sharply on students’ mathematical understanding and reasoning. The better handle you can get on students’ thinking, the better you’ll be positioned to make judicious instructional decisions that will help move their understanding forward. We’ll work on developing those lenses by working with different kinds of artifacts from math classrooms—samples of students’ work, video
and/or transcripts of math discussions in class, and lessons from your textbook—to
develop and refine your ability to analyze student thinking, connect students’ ideas
to important mathematical concepts, and apply these analyses to the daily work of
planning and carrying out lessons.

What Are Artifacts, and Why Analyze Them?

Broadly speaking, artifacts are objects that people make. You probably think about them
in the context of archaeology, where material remains like pottery sherds, systems of
cisterns and water pipes, ruins of buildings, and even the contents of old garbage dumps
help us create accounts of earlier lives and times. But artifacts aren’t only broken or
buried remnants of times gone by. Your own classroom is full of them. Completed math
assignments, cast-off scratch papers, and problem solutions that students share on the
board or overhead are all examples of artifacts from math class. So, too, are the lessons
from your textbook, the work you present on the board or overhead as you introduce
an activity or work through a problem solution, and the comments you write on student
papers. Audio- and videotaped records of lessons are becoming increasingly popular
classroom artifacts. All these examples are potentially rich with information that can
help you be more tuned into your students’ mathematical understanding.

So how are artifacts useful for improving your practice? As we noted in the vignette
about Jeffrey Stockdale, artifacts can serve as windows into students’ thinking, and
it’s the development of students’ thinking that is at the core of our work as teachers. If
we are to help shape what students know by building on their current ideas and skills,
we need to sort out what aspects of their thinking are irrelevant or incorrect and what
aspects are absent, nascent, or incomplete. By listening to students and examining their
work, we can gain insights into the nature of their understanding and plan learning
opportunities accordingly.

Artifacts are also useful for learning to zero in on what’s important in math class. As a
teacher, you’re called on to make hundreds of on-the-fly decisions over the course of a day.
Some of these decisions are about where you should be focusing your attention, some are
about what you will attend to, and some are about how you’ll respond to what you’re
noticing (Jacobs, Lamb, & Philipp, 2010). Classroom life itself is so fast paced that you
are often making these decisions without even being aware of how you’re processing
information. As you scan the room, for example, you know that Cecelia and Libby are
goofing off in the far left-hand corner of the room just as sure as you know that Justin and
Jennifer are on task, even though both pairs of students are talking animatedly and laying
out base-10 blocks, and you would be hard pressed to say, right then and there, exactly
how you can tell the difference. Working with artifacts offers the opportunity to explore
classroom events and student work without the real-time pressure of having to register,
interpret, and respond to them as they are actually occurring. Examining artifacts from the classroom but *outside* of the classroom allows for more leisurely analysis, discussion, and reflection on issues related to mathematics, mathematics learning, and mathematics teaching. Investigating classroom artifacts can help you develop and practice new ways of noticing and interpreting students’ thinking, imagining questions you might ask to gather more information about their ideas, and planning lessons and activities.

By learning how to use multiple lenses to analyze classroom artifacts, you'll be developing more tools to help you focus your teaching on your students’ strengths as well as to their learning needs. In this book we’ll focus on developing the following:

- Specific ways to examine and analyze students’ mathematical thinking
- Skills at honing in on specific mathematics that can emerge in student interactions and presentations
- Increased awareness of possible student reasoning that lies beneath both correct and incorrect answers

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**Learning to See through Multiple Lenses**

To work with classroom artifacts productively, we need to know how to attend to information they contain. It turns out that the way we attend to information involves an interesting mix of perceiving and interpreting those perceptions—what we know drives what we can see, and what we see drives what we can know and learn.

We all make thousands of decisions every day about where to pay attention and what to ignore. Virtually every task you undertake requires you to notice some aspects of the environment and to ignore others. When you’re driving down the street, chances are that you don’t notice the colors of the clothes that people on the sidewalk are wearing (you may not even notice the people themselves). If you decide to give yourself a treat and go to a coffee shop to finish the fabulous thriller you’re reading, you probably don’t register the comings and goings of other patrons.

Psychologists Daniel Simons, Chris Chabris, and their colleagues have studied how people filter what they attend to in a provocative series of studies about processing unexpected events. In one study, participants watched a video of two basketball teams. Each team of three players passes a ball among their teammates. Study participants were instructed to count the number of passes made by one of the teams. About halfway through the video, an unexpected event occurred. Some participants saw a woman carrying an umbrella walk right through the middle of the game. Others saw a version of the video where a person in a gorilla costume ambled through the game, stopping briefly in the middle of the frame to look at the camera and perform the prototypical gorilla “chest pound” before completing the stroll through the scene. Overall, nearly half the participants failed to notice the anomalous event at all. The study dramatically
Chapter 1
Turning to the Evidence

illustrates that focusing intently on a task can “blind” you to other, seemingly attention-grabbing aspects of your environment. (Try this basketball task—and others—with an unsuspecting friend. You can read about the study in *The Invisible Gorilla* [Chabris & Simons, 2010] and find videos at http://www.theinvisiblegorilla.com and http://www.youtube.com/user/profsimons.)

Now, we don’t mean to suggest that you would miss Mary Poppins or King Kong ambling through your classroom, even if you were attending closely to something your students were saying or doing. But we do want to emphasize that it’s impossible to monitor everything that’s happening in a room with 20 or 30 (or maybe even more) students in it. In fact, it’s imperative that we tune out some information in order to focus on the events that are really important.

Our attention isn’t influenced only by competing demands—expectations can also narrow what we attend to and therefore the information that we process, even when we’re not preoccupied with other tasks. For example, previous experiences with our students can lead us to anticipate their level of success and the amount of additional support they may need in completing tasks. But our expectations themselves may obscure students’ strengths or weaknesses.

Take the case of Ms. Hamlin and Cindy. We observed Ms. Hamlin’s seventh-grade class as they were beginning a geometry unit. The lesson involved using pentominoes to create congruent figures. It was springtime, and teacher and students knew each other quite well by this time of the year. Ms. Hamlin asked her students to make four or five congruent figures from the pentominoes. Because she felt the activity was a challenging one, she debated whether she should use a simpler, potentially less frustrating activity with her weaker students but decided to give the lesson a try “as is” with the whole group. One of the students she was concerned about was Cindy, a quiet girl on an individualized education plan who struggled with basic math facts and consistently needed extra math support in and out of class. Cindy found 20 congruent figures almost before Ms. Hamlin had finished passing out the materials. Ms. Hamlin was bowled over by Cindy’s ability to think spatially. She was equally surprised by the difficulties she saw among some of her “gifted and talented” students and was puzzled about where they were getting tripped up.

In this instance, Ms. Hamlin decided to override her expectation that the lesson would be too difficult for her less able students. What if she’d gone with her concerns and had presented an alternative, “dumbed-down” lesson for students like Cindy—or if she’d decided not to use the lesson at all but to substitute a much simpler one? Had she made either of these decisions, she wouldn’t have discovered that Cindy had powerful ways of thinking spatially—ways that the more traditionally talented math students seemed to lack. She also wouldn’t have seen the pride and confidence that Cindy displayed during that lesson or the respect shown her by the other students. Hopefully, Ms. Hamlin began to look for other instances of Cindy’s mathematical strengths and to explore ways to use her strong spatial reasoning to help her understand other math ideas.
Our Vision of Mathematics Learning and Teaching

Our vision of mathematics learning and teaching corresponds closely to that articulated by the National Council of Teachers of Mathematics (NCTM, 2000, 2006) and, more recently, the Common Core State Standards (CCSS, 2010). Central to this vision is the notion that we all actively build knowledge and understanding through our efforts to make sense of our experiences. We interpret new experiences—a demonstration in class, a discussion with a friend, or the unexpected results of an event or activity—in terms of what we currently know and we also stretch our existing understanding to accommodate our new encounters (Lave, 1988; Lave & Wenger, 1991; Piaget, 1970; Werner, 1948; Wertsch, 1985). None of our learning begins with a totally blank slate or with simply adopting ideas wholesale from the outside. Instead, learning proceeds by using what we already know to make sense of our new experiences with people, objects, and ideas, gradually shaping our understanding into more complete, complex, and interconnected knowledge.

This view of learning has implications for instruction. For example, effective teaching explicitly engages students in sense making by having students explain and justify their reasoning. It also emphasizes work that’s “authentic”—meaningful, valuable, and representative of the kinds of activities and thinking that engage actual practitioners of the discipline instead of just rote make-work. For math teaching, this means helping students learn to do mathematics—to learn to think and inquire with the same kinds of tools and approaches that practicing mathematicians use. This goal is what’s behind the process principles articulated in NCTM documents and, more recently, in the CCSS Mathematical Practices (see Table 1.1).

<table>
<thead>
<tr>
<th>TABLE 1.1 Common Core State Standards for Mathematical Practice</th>
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<tr>
<td><strong>MATHEMATICAL PRACTICES</strong></td>
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<td>1. Make sense of problems and persevere in solving them.</td>
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<tr>
<td>2. Reason abstractly and quantitatively.</td>
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<td>3. Construct viable arguments and critique the reasoning of others.</td>
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<td>4. Model with mathematics.</td>
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<td>5. Use appropriate tools strategically.</td>
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<td>6. Attend to precision.</td>
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<tr>
<td>7. Look for and make use of structure.</td>
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<td>8. Look for and express regularity in repeated reasoning.</td>
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Source: Copyright © 2010. National Governors Association Center for Best Practices and Council of Chief State School Officers. All rights reserved.
This view of learning as an active, constructive process and the emphasis on sense making means that each student will understand the work you do in class in his or her own way. This view is sometimes challenging to keep in mind with respect to students’ mathematical work. It’s fairly common for people to assume that others approach math problems pretty much the same way they do; for those of us who were taught that the way to solve problems was the way the teacher showed us, this tendency might be even stronger. But even when math was about memorizing facts and procedures, we weren’t necessarily all thinking exactly the way the teacher taught us. We know a man in his late 50s who remembers being taught that the “right” way to add long columns of numbers was to start at the top of the column and muscle your way straight down to the bottom. Although he knew his math facts and could have worked from top to bottom, when faced with long columns of numbers to add, he would try to simplify his task by secretly finding pairs of digits in the column that summed to 10. He also firmly believed that he was cheating by looking for the easy combinations. Today, he would be applauded for his number sense. At the time, he believed that he would have been chided for his lack of raw computational power if his teacher had known how he was solving the problems.

It's therefore important to keep in mind that we can’t always assume we know how students are thinking from just having the final products of their work. Sometimes students can get correct answers despite tangled thinking. Conversely, students who produce incorrect answers can still have used promising and productive approaches. One of the big challenges we face as teachers is figuring out whether students’ reasoning is sound, not just whether their answers are right or wrong.

This student’s partner used four cubes to build a train representing a value of 20; the teacher asks, “in your train, how much is a cube worth?”
After more than 20 years of mathematics education reform, most teachers are open to the idea that learning mathematics involves developing conceptual understanding as well as gaining computational skill and accuracy. They’re also open to the idea that not all students think about the mathematics they are learning in exactly the same way. This view fits much better with psychological theory and research about learning, but it also places a bigger burden on you, as the teacher, to better understand your students’ thinking and to remember that you can’t assume that students will easily grasp ideas that seem to you to be simple and perhaps even obvious.

Take as an example the nearly ubiquitous use of base-10 blocks to teach about place value. Because seeing the “ten-ness” in these materials is quite straightforward to us, it’s difficult to imagine that this wouldn’t be the case for children as well. But remember, we already understand our number system, so the unit cubes, rods, flats, and large (1000s) cubes offer us a straightforward way to represent an already well-developed set of ideas about powers of 10 and ways that we write numbers to represent the values of different “places.”

But consider Zach, a second grader we observed during a lesson in which students were using base-10 blocks to solve addition problems using regrouping. He seemed to be having a lot of trouble remembering to trade 10 unit cubes for a rod when representing his answer. As he worked, we realized that his difficulty was not simply a matter of forgetting to make the trades. Despite his teacher’s repeated reminders that he needed to trade 10 unit cubes for a rod, he didn’t really understand that the rod represented a group of 10 units.

On three separate occasions, his teacher Betsy came over to him to work one-on-one, and each time Betsy asked him how many unit cubes were “in” the rod. This was not a trivial or obvious question to him—each time she asked, he had to check. The first time, Zach put the rod on the floor and lined up unit cubes next to it. Because the nubby carpet and his own motor limitations made this task difficult, he couldn’t align them without leaving some space between the cubes, and he could fit only eight in a row alongside of the rod (see Figure 1.1). He looked up at her and answered, somewhat hesitantly, “eight?”

Betsy responded by suggesting that he put the unit cubes on top of the rod instead of beside it. Doing it this way, Zach aligned the 10 unit cubes along the rod, counted, and

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Figure 1.1 Zach’s base-10 block arrangement

![Zach's base-10 block arrangement](image)
said that there were 10 units in the rod. To confirm his work, she took off the unit blocks, held up the rod, and pointed to each segment as he counted each of the 10 segments. Three or four minutes later, she came back over and asked him again how many blocks were in the rod. Again he had to count. This time he miscounted and answered, “nine.” He seemed unperturbed when Betsy reminded him that they’d counted the rod and gotten 10 blocks a few minutes before. They reestablished that there were 10 blocks in the rod, but a few minutes later, Zach was once again unsure.

Adults know in their bones that the rod represents 10 unit cubes and that this quantity is stable. Many second graders see the situation similarly, but some are still developing this understanding. Zach seemed to still be in this “under development” phase. His lack of certainty about the number of unit cubes that made up a rod and his comfort with assigning a different value to the rod each time he counted its segments suggests that he had yet to firmly establish a one-to-one correspondence between the number of segments in the rod and the count of those segments. Until he understands this relationship, he will see something different in the base-10 blocks than his teacher Betsy does. And until he does, the rod will not help him understand the idea of bundling units into groups of 10 during addition or subtraction.

It’s likely that at least some of your students, too, will wrestle with ideas in math class that you have assumed would be simple and straightforward to them. Sometimes figuring out what, exactly, is the source of the problem can be challenging and time consuming. Sometimes, the challenge is to decide whether a student’s different way of thinking is sensible and valid. Other times, your challenge involves thinking about how to address students’ errors in a way that will help them better understand the math.

For example, it’s not uncommon for teachers to chalk up multidigit addition errors to students’ carelessness in forgetting to “carry” when writing their answers. Here are two examples of incorrect problem solutions that might be interpreted this way (see Figure 1.2). But let’s stop and consider these errors a moment. How would you interpret them? Do they seem similar to you? Why (or why not)?

To us, they appear quite different. We could imagine that Ethan’s answer, though, looks like an oversight or a careless error to us. Instead, we think there’s something either about the structure of our number system itself or about the way that we write numbers using place value that she’s still trying to figure out. While Ethan’s teacher may need do no more than ask him whether he’s satisfied with his answer or to tell him he’s made a computational error he should look for, Asheran’s teacher will likely want to spend some time with her trying to understand what it is that she does and doesn’t yet understand.

Our point here is that recent theory and research emphasize that we can’t assume that others think about mathematics exactly the same way we do—particularly when those others are children. Our vision of mathematics teaching includes cultivating the
disposition to attend to the details of students’ thinking, recognizing novel as well as common approaches to problems, assessing the mathematical validity of students’ work, and helping students draw connections between their thinking and important mathematical ideas and skills.

A Framework for Using Artifacts Skillfully

So how can you cultivate these approaches to teaching? We think that a promising approach is to tap into that potential gold mine that classroom artifacts offer to inquire into students’ mathematical thinking. But tapping the mother lode involves knowing how to release the gold; prospecting for information about students’ thinking with artifacts is surely more effective when you know where and what to look for. We developed the Skillful Use of Artifacts framework, described briefly in this book’s preface, to help you focus your efforts where the payoff is likely to be worthwhile.

The framework highlights the importance of attending to both the mathematical thinking and the mathematical content embodied in the artifact. It also articulates specific strategies for engaging with artifacts in the service of these two main foci of attention. You can think about these strategies as tools for polishing those new lenses you’re using to understand the mathematical work you and your students undertake in class.

For example, the entries in the “Attention to Thinking” column of Table 1.1 offer strategies to help you get a clearer idea of how students make sense of the mathematics they are working on. Instead of assuming that you understand what students are thinking, these strategies help you set your own perspective aside and use classroom artifacts as data to examine for evidence of how—and possibly why—students approached problems.
as they did. The entries in the “Attention to Content” column help keep instruction focused on developing students’ conceptual understanding and mathematical habits of mind. (This doesn’t mean that you should ignore students’ acquisition of facts and skills; rather, you should see them as important tools for supporting conceptual understanding and problem solving. And, of course, we should recognize that one student’s “simple fact” may, for another student, be a developing idea.)

The exercises in each of the chapters that follow provide the opportunity to explore the two main components of the framework, using a variety of different artifacts. We hope that as you work through this book, you will begin to internalize the ideas in the framework and develop the disposition to approach your teaching with new lenses for considering artifacts from your own practice.
In this section, we consider the left-hand column of the Skillful Use of Artifacts framework, focusing on classroom artifacts as data about students’ thinking. In some ways, it’s artificial to try to distinguish between examining artifacts in terms of thinking and content because in the end we’re interested in both. The purpose of analyzing artifacts is ultimately to get a fuller picture of students’ thinking about particular content. So, while organizing the work into a section on attention to thinking and another on attention to content is a bit artificial, we need to disentangle them at least temporarily in order to study these two important components in more detail. Since we have to start somewhere, we decided to begin with attention to thinking because, in the end, teaching is fundamentally about helping students to think more powerfully.

Both of the chapters in this section are designed to support greater attention to the specifics of students’ mathematical thinking. Chapter 2 focuses on describing the work captured by artifacts and on using those descriptions as evidence for interpretations of students’ mathematical thinking. The process of gathering and interpreting evidence is core to instruction. Eliciting and interpreting evidence of students’ understanding, for example, lies at the heart of formative assessment practices. Teachers base their instructional decision making and feedback to students largely on the information they gather about students thinking (Black, Harrison, Lee, Marshall, & Wiliam, 2003; Black & Wiliam, 1998; Heritage, 2010).

Chapter 3 concentrates on attending to the strengths as well as the weaknesses in students’ mathematical thinking. Many teachers spend much of their energy figuring out what students don’t yet know or can’t yet do in order to fix erroneous thinking. This approach doesn’t always leave room for teachers to see the potential in their students’ thinking, even if that thinking is incorrect. And since we actually address students’ deficiencies by building out from what they do know, it’s important to become as inclined to recognize and utilize students’ strengths as it is to attend to their weaknesses.
Describing and Interpreting Classroom Artifacts

Fifteen teachers sat around a table in the library at their monthly after-school professional development seminar. Mark, the group’s facilitator, passed out five samples of students’ work on the Crossing the River problem:

Eight adults and two children need to cross a river, and they have one small boat. The boat can hold either one adult or one or two children. Everyone in the group is able to row the boat. How many one-way trips does it take for the eight adults and two children to cross the river? Write a rule for $n$ adults and two children.

The teachers were familiar with the problem, having worked it themselves the month before. Mark stopped by one group just as Lorena picked up Linda’s paper.
Linda hadn’t written an algebraic expression, but she had diagramed an algorithm, using different-colored pencils to indicate trips to and from the opposite side of the river. Lorena glanced at Linda’s work and noted, “This student is learning disabled,” then moved on to the next work sample. Mark stepped into the conversation. “Lorena, your comment sounds like an interpretation of Linda’s work, not a description of it. But you haven’t said what evidence you’re basing your comment on. What did you notice about Linda’s work that led you to your interpretation?”

Lorena hesitated a moment and then explained, “Well, I’ve worked with lots of kids with learning disabilities, and lots of them have trouble with math. For one thing, they need extra help to stay organized enough to even figure out what they’re supposed to be doing. So using colored pencils to keep track of her solution is the kind of thing I’ve seen lots of students with learning disabilities do.”

In this vignette, Lorena’s labeling of Linda as learning disabled is based on scanty evidence. As Mark pointed out to her, it’s a big interpretive leap to go from “Linda used color to keep track of different kinds of trips” to “Linda’s learning disabled”—a leap that seems unwarranted without further, thoughtful consideration of evidence from her work. Yet Lorena’s tendency to categorize and label is something that we all do. In fact, it would be hard to get through the day without making connections between new experiences and old ones. Categorizing our experiences is a cognitive tool that can help keep us from being overwhelmed by the variety of information and situations we encounter daily.

In the classroom, for example, we are continually being called on to quickly size up students’ work and respond to it. Unless we have a cache of experiences to draw on to help us make sense of students’ work, each encounter with a student would require starting the process of sizing up and responding from scratch. Thinking of students in terms of certain categories—mathematically talented, hardworking, easily distracted, or even learning disabled—can help shape our in-the-moment responses to students as well as our broader learning goals and lesson plans.

But in school, as in life, categorizing and labeling has its hazards. If we make assessments on the basis of too little information, we can label students inaccurately (and sometimes unfairly). How likely does it seem to you that Lorena’s characterization of Linda would actually hold water? What other possible interpretations might one reasonably make for Linda’s choice to color code her solution strategy?

Part of the hazard of labeling is that once we’ve placed someone in a particular category, we filter and interpret our subsequent experiences with that person in terms of that category, glossing over (or even entirely failing to recognize) important information that doesn’t fit with our expectations. After identifying Linda as learning disabled, Lorena seemed to feel that she’d adequately accounted for Linda’s work even though she hadn’t actually tried to understand Linda’s mathematical thinking about the problem at all. Could it be that, instead of being organizationally challenged, Linda had a hunch
that the solution to the problem depended on separating out trips taken by adults and children and that the color coding was an insightful and sophisticated way of playing out her intuition? Categories can obscure as well as reveal.

In this chapter, we practice suspending the tendency to categorize by focusing on first describing students’ mathematical work and then interpreting the mathematical thinking that might be behind it.

**Why Focus on Evidence?**

We’re wired as humans to interpret what we perceive. Like Lorena, we tend to fold descriptions of what we observe into our interpretations of the observations, often skipping over the evidence on which we based our assessments. When we do this, we often treat interpretations as if they were fact without necessarily even noticing that we’ve made the leap from describing to interpreting. Despite the tendency to merge these two processes, there are several advantages to being deliberate about distinguishing—and separating—description from interpretation when we examine classroom artifacts. For one, when we identify the evidence on which we base our interpretations, we can critically examine the bases for our claims—and make it possible for others to understand how we arrived at our interpretations.

Another reason to focus on grounding interpretations of classroom artifacts in evidence is that sticking close to the evidence helps to sharpen your focus on the specifics of students’ mathematical thinking. The more practice you get at describing the particulars of students’ work and interpreting the thinking that was likely to produce it, the better you’ll get at seeing subtleties in students’ thinking and at identifying the strengths as well as the weaknesses in their ideas. Since a big part of our jobs as teachers is to figure out what our students do and don’t understand, being able to hone in on important aspects of their thinking is central to promoting further learning. We don’t want to spend a lot of time working on mathematics that students already grasp, nor do we want lessons to whiz over their heads because we think they are grasping ideas when they’re actually barely hanging on.

Of course, the idea of attending to what your students do and don’t know isn’t likely to be new to you—you look for information about whether your students are “getting it” on a regular basis. What might be new is going beyond checking whether a student’s work is correct or not, taking a deeper look at how students seem to be thinking and why they might be doing so. Or maybe even this perspective isn’t new, but the chance to get more practice using the perspective is what will motivate your work in this chapter.

**Creating a Description of Student Work**

There are several reasons to practice describing students’ work. One is that it helps to sharpen your focus on the important mathematical ideas of a lesson. In some cases,
sharpening this focus might involve actually analyzing mathematics that seem simple to us as adults but that are actually rather complex and sophisticated for those students who are just learning it. (For example, think about how it is entirely second nature for adults to accurately count a group of objects. Then remember that Piaget demonstrated over 75 years ago that a major cognitive accomplishment of the young child is the discovery of the one-to-one correspondence necessary for counting [Piaget & Szeminska, 1952].)

A second reason to practice describing students’ work is that it hones your ability to recognize mathematically important elements in the work itself. We often assume that a correct answer means that students understood the relevant mathematical ideas and/or procedures, and we often further assume that correct answers mean that our students think about the math the same way that we do. Both of these assumptions can contribute to a mind-set that leads to our overlooking (or failing to gather) information relevant to students’ mathematical understanding.

In fact, students often think about mathematical situations differently than we do. For example, while adults typically use subtraction to solve “unknown addend” problems like the one below, primary school children frequently use addition. Ask a colleague to tell you how he or she figured out the answer for the following problem:

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**Apriel wants to read 15 books over the summer. So far, she’s read seven books. How many does she still have to read to reach her goal?**

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Chances are, your colleague subtracted 7 from 15 to get 8. If you ask second graders to solve the same problem, many of them will either count up from 7 to get 15 or ask themselves “what plus 7 makes 15?”

It’s therefore useful for us, as teachers, to practice looking for aspects of students’ thinking that are different from those of adults. Sometimes students simply have different ways of solving problems; sometimes their approaches may not seem elegant or even all that sensible from an adult perspective but nonetheless have a certain mathematical logic and coherence for the student, whose ideas are still under development. The importance of understanding the developmental trajectories of students’ mathematical thinking and the value of learning to recognize and characterize evidence of developing thinking in students’ work underlies the work of many mathematics educators and researchers (Carpenter, Franke, & Levi, 2003; Fosnot & Dolk, 2001; Mason, 2002; Schifter, 1996; Schifter, Bastable, & Russell, 2010a, 2010b [and other titles in the Developing Mathematical Ideas series]; Sherin, Jacobs, & Philipp, 2011; Stein, Smith, Henningsen, & Silver, 2009).

A third reason for describing students’ work is that it provides evidence you can use in interpreting student understanding—and in considering whether you agree with
others’ interpretations. A description should be objective; different people should be able to agree on a description. Consider, for example, third-grader Nate’s solution to the problem “20 paws. How many cats?” (see Figure 2.1).

Everyone should be able to agree that Nate drew a picture of four cats, that each cat has four legs, and that his answer to the question “How many cats” was 5. We might wonder about why he drew four and answered “five,” and we can even offer some conjectures. (Maybe he drew four cats, counted paws, and knew that he needed only one cat’s worth of paws more, so he didn’t bother drawing the last cat. Or perhaps he made tally marks on scratch paper to find his answer but made a mistake when representing his solution on the worksheet. Or maybe Nate felt unsure of his own answer, which was actually four cats,

Figure 2.1 Nate’s solution

Source: Miller, E.D., (2000). Read it! Draw it! Solve it! Grade 3. Student Workbook. p. 156. Copyright ETA/Cuisenaire. The names and logos of ETA/Cuisenaire, its divisions, or its subsidiaries may not be used in advertising or publicity pertaining to distribution of this information without specific, written prior permission.
and wrote “5” because that was his table partner Ellen’s answer, and Ellen usually gets answers right in math class.) At the point of making such conjectures, though, we are moving away from a simple description to possible interpretations of Nate’s thinking. Describing his work should remain in the realm of the observable, not in the realm of speculation.

Creating a description of a student’s work is also different than creating a blow-by-blow narrative of every feature of it. For our purposes here, we don’t want to know everything that a student did; we want a description of those aspects of the solution that are relevant to students’ mathematical thinking. For example, we don’t need to note that Nate drew smiling cats with curly tails and spiky whiskers (although these features do make his work particularly endearing).

Since we’re looking to tie the elements of the description to the key mathematical ideas and skills that underlie the problem, it’s of central importance to understand what the key mathematical ideas and skills are. Therefore, the first step in describing students’ work is to be clear about the mathematics at play in the problem. The more developed your understanding of the mathematics is, the more you will be able to capture the highlights of the mathematical ideas and skills you see the student calling on. You can then use your descriptions of the work to provide evidence for your interpretation of students’ understanding of the relevant mathematics.

We’ve included a variety of exercises in this chapter so you can practice describing and interpreting students’ thinking using different kinds of artifacts. We’ll start with a relatively simple one—a written vignette that describes a student working on a problem. Exercise 2 is also relatively simple—a short video clip of a student solving a single problem. Exercise 3 uses students’ written solutions to a word problem. We’ve also included two additional video-based exercises at the end of the chapter (Exercises 4 and 5). These exercises provide you with more experience analyzing videos of students at work.

**Exercise 1: Danny**

Imagine that you are watching a 5-year-old kindergartner named Danny work on the problem his teacher has just posed for the class:

If there are seven children on the playground and eight more come over to play, how many children are there on the playground?

Here’s what you see:

Danny and his table-mates James, Ben, and Ellen work individually. Each table has containers of snap cubes, base-10 blocks, popsicle sticks, and counting bears for children to use. Danny hums to himself as he grabs several handfuls of snap cubes. He comments that he’s got a lot of red cubes and not so many yellows. He silently
moves cubes from the big pile to a smaller grouping; when he’s got seven, he stops and pushes them together into a bunch to his right. He looks back at the problem and then pulls out eight more from the larger pile. He pushes these eight together and puts the remaining snap cubes back in the container.

Danny looks over at James, who has made two piles of counting bears and is counting the second pile, softly saying “11, 12 . . .” Danny tells James, “I’m gonna make a snake out of mine!” as he begins lining up the snap cubes from the pile of eight. “Now I’m making it longer!” he says as he adds cubes from the pile of seven. He doesn’t snap the cubes together as he makes his “snake,” but he does make sure that each cube he adds touches its neighbor. He occasionally straightens the line as he continues to add the cubes from the second pile. When he is done, he counts the line of snap cubes, beginning with the leftmost one. He touches each as he counts aloud. Danny skips over the 11th cube, counts to 14, and writes \(11 + 4\) on his worksheet.

Now follow the steps below to first build a mathematically focused description of his work and then use the description to draw some interpretations of what Danny does—and perhaps doesn’t—understand.

**Process for Analyzing Danny’s Work.** The process outlined below is one we will use in analyzing all the classroom artifacts in this book. In general, throughout the book we first ask you to consider the math that is captured in the artifact and then to work with the artifact itself. In the case of this practice exercise, the artifact is the vignette depicting Danny’s work. Follow the steps below to complete the exercise. Read through all the steps before you begin in order to get an overview of the process.

**Step 1:** Identify key math ideas and skills. Ask yourself about what a kindergartner has to know to solve the playground problem. Try to be more specific than “he has to know addition” or “he has to know his number facts.”

**Step 2:** Describe Danny’s work. Reread the vignette, looking for the parts that seem relevant to how Danny solved the problem (you may want to underline them). Use them to build a description of Danny’s solution strategy.

**Step 3:** Use evidence to build an interpretation. How would you interpret Danny’s thinking? Use the portions of the description you highlighted as evidence for your ideas about how Danny’s mathematical understanding connects to the key ideas you identified earlier.

**Step 4:** Reflect on your work. If you are working with others, take a few minutes to share and discuss your descriptions of Danny’s work. Be sure you can support your descriptions with evidence from the vignette. Use the study questions on page 21 to help you reflect on your work (and, if you are working with others, to start your discussion). You can also refer to the commentary that follows to stimulate your reflections and discussion.
Commentary on Danny’s Work. Below are thoughts about some of the key mathematical ideas that the “playground” problem taps and comments on how descriptions of Danny’s work can support interpretations of his mathematical thinking with respect to these key ideas.

Key mathematical ideas. For young children like Danny, an important idea that’s not directly related to counting or addition is the more general notion that you can represent one situation with another. In this case, the idea is that objects like snap cubes or counting bears (or even marks on a paper) can stand in for the children on the playground. In addition, a number of ideas about counting and adding are likely to be relevant to his work on the problem:

- When you’re counting the number of objects in a collection, the last number in your count tells you the number of objects (this is the idea of cardinality).
- You can think about addition as the joining of collections (subsets) of objects; the sum is the total number of objects in the new, joined set.
- The order in which you add sets doesn’t make any difference (this is the commutative property of addition).
- You can add by starting with the size of one of the subsets and then continue objects. This is often called “counting on” (see Figure 2.2).

Figure 2.2 Counting on

This group has 6 7 8 9 10

Study Questions

- What aspects of the vignette did you include in your description?
- How did the aspects you included connect to the key mathematical ideas you identified?
- How did you interpret Danny’s thinking?
  - What do you think he understood and was able to do?
  - What do you think is still under development for Danny?
- What aspects of the vignette did you exclude? What about them made you exclude them from your description?

CCSS—Kindergarten
Represent addition and subtraction with objects, fingers, mental images, drawings, [and other media]. p. 11.

CCSS—Grade 1
Use addition and subtraction within 20 to solve word problems involving situations of adding to taking from, putting together, taking apart, and comparing, with unknowns in all positions. p. 15.
Section 1  
Attention to Thinking

- It’s more efficient to “add on” to the larger subset because you have fewer objects to count.
- Because addition involves joining sets in any order, you can also decompose and recompose sets (and numbers) into configurations that make for easy addition (this idea calls on both the associative and the commutative properties of addition). For example,

\[
8 + 7 = 8 + (5 + 2) \\
= 8 + (2 + 5) \\
= (8 + 2) + 5 \\
= 10 + 5 \\
= 15
\]

*Description of Danny’s work.* Given that our goal in looking at Danny’s work is to consider his understanding of the mathematics involved in the “playground” problem, we see the following aspects of the description as particularly relevant:

- He chooses a manipulative (snap cubes) to help model the problem.
- He accurately counts out the two groups representing the addends (a group of seven and a group of eight).
- He puts these two groups together, starting with the pile of eight blocks.
- He counts out the total, starting from “1.”
- He miscounts the total.

We would *not* include the following in our description:

- He uses snap cubes instead of other available manipulatives. In some cases, choice of manipulatives might be important. If Danny were solving a problem using larger quantities, such as \(47 + 78\), then his using snap cubes instead of base-10 blocks, for example, might suggest that he’s not using knowledge of the structure of the number system to decompose the numbers into groupings that are easier to deal with \((40 + 7\) and \(70 + 8\)). Since Danny is working on a problem with small numbers here, his specific choice of manipulative is probably not all that informative.
- He begins his work by taking out a bunch of cubes, from which he counts out the set of seven and the set of eight. If we were examining Danny’s estimation skills, this observation might be of central importance, but it seems less mathematically important for this problem.
- He comments on the color of the cubes.
- He comments to James about making a snake.
- He makes sure that, as he lines up the cubes, they touch each other.
- He writes the “4” backward. While Danny’s teacher may want to help him write numbers correctly, his error is pretty common and isn’t an issue about his understanding of mathematical ideas.

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**CCSS—Kindergarten**

Understand addition as putting together and adding to . . . p. 10.

Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted. p. 10.

Understand the relationship between numbers and quantities; connect counting to cardinality. p. 11.

**CCSS—Grade 1**

Interpretation of Danny’s work. While the purpose of describing students’ work is to provide an evidentiary base for interpreting their thinking, you have probably noticed that there’s some element of interpretation happening even during the step of creating a description. This is inevitable (as we’ve been emphasizing) since what you notice is influenced by what you know. Our goal in describing his work is to have evidence that will support our interpretations of the strengths and possible limitations of his understanding about number and operations.

His accurate counting out of the group of seven and the group of eight, for example, suggests that he has come some way toward mastery of key ideas about counting (one-to-one correspondence, ordinality, and cardinality). We don’t know how accurately he can count larger numbers, but he seems to get the idea that each object gets counted once and that each count has to be associated with a number. Given that Danny seems comfortable and competent with counting, we would focus our attention on how he’s building an understanding of addition.

We think that his process of reorganizing the separate sets of blocks into a single group suggests that he understands that addition can be thought of as joining sets. However, the fact that he finds the total by recounting all of the blocks suggests that Danny doesn’t yet understand that he can use the cardinality of one set as the starting place for “counting on” the second set. Given his accurate counting of the groups of seven and eight, his miscount of the total may be a “careless” error, or it may be an indication that keeping track of the correspondence between number and object gets strained when the set he’s counting is somewhat larger—say, greater than 10 or so. We’d be curious to see whether he could accurately recount the blocks in his snake or perhaps another, somewhat smaller group of maybe 12 or 13 objects.

Danny’s work doesn’t offer evidence of his having addition strategies based on number facts, so we might also want to explore this. We could, for example, ask him whether he could think of any other ways to solve the problem and keep an eye out for whether he offered strategies that relied on decomposing and recomposing numbers—or whether he understood such strategies when suggested by someone else (e.g., finding combinations of 10 or using doubles).

Exercise 2: Melissa

Exercise 2 is based on a video clip of a portion of an interview with Melissa, a fifth grader. In this short clip, Melissa compares an improper fraction \( \frac{2}{3} \) and a mixed number \( 1 \frac{2}{3} \).

Typically, when we talk about working with video artifacts we mean footage of classroom work—whole-class discussions, for example, or excerpts of small-group work. Because we want you to have the chance to zero in on students’ thinking with a minimum of distraction for these first video exercises, we’ve chosen video of individual students.
The video, as well as the videos for Exercises 4 and 5 at the end of the chapter, capture a portion of a one-on-one “math interviews” with elementary students. These interviews were specifically designed to elicit students’ thinking as they solved problems, not to help students get correct answers or fix errors they might have made during the course of their problem solving. Because the interviews were explicitly about not affecting the students’ thinking, the video captures a kind of interaction that you may find foreign. In fact, the goal of “just” listening for students’ ideas rather than of intervening to promote learning may strike you as decidedly unteacherly. But if you think about it, it’s hard to be a responsive teacher without taking the time to understand how your students are actually thinking. In your own classroom, you probably check in informally with your students on a fairly regular basis as they work, waiting to make comments or suggestions or to pose challenges until you get a sense of how they’re approaching a task. Similarly, when students come to you for help, you probably take a minute to gather some information about how they’ve already approached the problem or what they find difficult or confusing before you weigh in. It’s that “data gathering” mind-set rather than an “intervening” one that we’re tapping into in the video exercises.

Process for Viewing and Analyzing the Melissa Video. Follow the steps below as you complete the exercise. Read through all the steps before you begin in order to get an overview of the process.

**Step 1:** Identify key math ideas. Take a few minutes to think about Melissa’s task (compare $\frac{5}{3}$ and $\frac{12}{3}$). What are important mathematical ideas involved in representing numbers as improper and mixed fractions and in comparing them? What kinds of difficulties might you expect students to encounter when making such comparisons?

**Step 2:** Prepare to watch the video. Make a copy of the blank Chapter 2 worksheet (see Figure 2.3; a blank worksheet is located in the Appendix and as a writeable PDF on PDToolKit). Think about the kind of information you’ll need to collect to fill out the worksheet; have some paper on hand for making notes about the details of Melissa’s work as you watch the video.

**Step 3:** View the Melissa clip. Find the Melissa clip on PDToolKit and watch it at least once. Remember that your purpose in viewing the video is to practice describing and interpreting Melissa’s work; you may want to make notes while you’re viewing to remember things that seem particularly interesting or important. If you do, be sure to note the associated time code on the video clip so that you can easily refer back to your evidence.
Chapter 2 Worksheet: Describing and Interpreting Artifacts

Key Mathematical Ideas:

<table>
<thead>
<tr>
<th>My Descriptions</th>
<th>My interpretations and supporting evidence</th>
<th>Alternative interpretations?</th>
<th>Questions I’d like to ask</th>
</tr>
</thead>
<tbody>
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</tbody>
</table>

Step 4: **Complete the “My Descriptions” column of the worksheet.** Remember that your goal here is to describe, objectively, what Melissa does (and says) that you think is relevant to her understanding of the math. You may want to replay segments of the video or even watch it through a second time from beginning to end—Melissa’s explanation goes by pretty quickly, and there might be parts of it that you’d like to revisit as you complete the worksheet. You can also refer to the video transcript (Figure 2.4) as you complete the worksheet.

Step 5: **Complete the “My Interpretations” column of the worksheet.** Use the “My Descriptions” column to provide evidence for your interpretations.

Step 6: **Complete the rest of the worksheet.** See if you can come up with alternative interpretations for Melissa’s work. Often, generating alternatives helps us think more broadly about a student’s knowledge and skill. Think, too, about questions that Melissa’s work raises for you: what do you wonder about her math understanding? What questions would you like to ask her?
Interviewer: Can you say those two for me?

Melissa: Five thirds and one and two thirds.

Interviewer: Okay.

Melissa: And I think they’re equal. Because that’s five thirds, and I was thinking maybe times those together, or divide them. That’s two. I said, “Five times 3 is 15.” So I thought maybe that’s 5, and that’s 1. One, one, five here. Fifteen. And then that’s—times those together, and that’s 5. I mean, 15.

Interviewer: Oh, okay. So you multiplied 3 times—

Melissa: Five.

Interviewer: Five.

Melissa: And that’s fifteen.

Interviewer: Fifteen.

Melissa: Is equal to. And then 1, and plus these—add these together.

Interviewer: You added those two.

Melissa: And that’s 15.

Interviewer: Okay, very good.

Source: Philipp, R., Cabral, C., & Schappelle, B. (2011). Searchable IMAP Video Collection: Children’s Mathematical Thinking Clips Copyright © 2011 by San Diego State University Research Foundation. All rights reserved except those explicitly licensed to Pearson Education, Inc. Video and transcript were created during IMAP project by Randolph Philipp, Bonnie Schappelle and Candace Cabral.

Step 7: Reflect on your work. If you are working with others, take a few minutes to share and discuss your descriptions and interpretations. Be sure to support your interpretations with evidence from the video—your notes and worksheet comments will help you do this. Use the study questions on page 27 to help you reflect on your observations (and, if you are working with others, to start your discussion). You can also use the completed sample worksheet (Table 2.1 on page 28) to help you think about your own responses and to stimulate your reflections and discussion.
Study Questions

- What aspects of Melissa’s work did you include in your description? What details did you intentionally leave out?
- What evidence did you use in formulating your interpretations of her math understanding?
- What alternative interpretations did you make?
- What were some strengths and weaknesses you found in Melissa’s work?
- What do you think she does and doesn’t understand?
- If you had been the interviewer, what questions would you have liked to ask Melissa to find out more about her thinking?

Commentary on Melissa’s Interview. This video clip may serve as a reminder that students may answer problems correctly even though their reasoning is faulty (and vice versa—they may also get problems wrong even if their basic conceptual reasoning is sound). It can also serve as a reminder that many of us have a tendency to “fill in the blanks” of another person’s thinking, making the assumption that they think the same way we do and that correct answers imply correct thinking. If the interviewer had simply accepted Melissa’s solution without probing her thinking, it would have been tempting to conclude that she had a clear and mature understanding of how to move between different representations of fractions larger than one—she did not hesitate as she named both numbers and offered her observation that they were equal to the interviewer without prompting. This clip reminds us of the importance of holding the “correct answers = correct thinking” assumption at bay and checking in with students to make sure that correct answers do indeed reflect solid mathematical thinking.

Key mathematical ideas. We see the following mathematical ideas as relevant for Melissa’s task (see also Table 2.1).

- There are many different ways to represent a number.
- In writing a fraction, the denominator represents the number of parts into which a whole has been equally divided; the numerator represents how many parts of the whole are under consideration.
- Whole numbers can be expressed as a fraction. For example, \( \frac{3}{3} = \frac{4}{4} = \frac{240}{240} = 1 \).
- When we write an improper fraction like \( \frac{5}{3} \), we mean “five one-thirds,” or \( \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \).
### Key Mathematical Ideas:
- There are many different ways to represent a number.
- In writing a fraction, the denominator represents the number of parts into which a whole has been equally divided; the numerator represents how many parts of the whole are under consideration.
- Whole numbers can be expressed as a fraction.
- When we write an improper fraction like $\frac{5}{3}$ we mean “five one-thirds.”

<table>
<thead>
<tr>
<th>My Descriptions</th>
<th>My Interpretations and Supporting Evidence</th>
<th>Alternative Interpretations?</th>
<th>My Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Melissa reads both numbers without hesitation.</td>
<td>She’s familiar with fractions.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>She says that she thinks the numbers are equal and then goes on to explain her thinking without prompting.</td>
<td>Until Melissa offered her explanation, it seemed that she knew that the two numbers were equal. Generally, lack of hesitation suggests that someone is confident of their ideas; is this a good interpretation in this case—does that tiny laugh suggest lack of certainty in her work/answer?</td>
<td>She anticipated that the interviewer would want her to explain her answer. She might be a quick thinker—she comes up with an explanation on the spot that manages to “find” a 15 on both sides of the equals sign even though the explanation does not make sense mathematically.</td>
<td>How confident is Melissa of her answer? Is she used to thinking through (and explaining) her answers in school? Does she know that she’s provided an explanation that isn’t mathematically valid? Is she troubled by it? (What does her lack of hesitation mean in terms of her own attention to mathematical ideas and concepts?)</td>
</tr>
<tr>
<td>She says that you might multiply or divide the $\frac{5}{3}$.</td>
<td>She’s trying to remember what her teacher told her about how to compare fractions.</td>
<td>She’s thinking about how to put each number into a form that will allow comparison.</td>
<td>What does she understand about the meaning of the value in the numerator and in the denominator? Does she understand that $1 = \frac{1}{1}$? Could she represent these values with drawings or manipulatives?</td>
</tr>
<tr>
<td>After she talks about multiplying/dividing, she says “that’s two.”</td>
<td></td>
<td></td>
<td>What did she mean by “that’s two”? (Or was it maybe “that’s too,” even though it didn’t seem like she was making a comparison or starting a sentence that she didn’t finish?) Is this statement mathematically important?</td>
</tr>
<tr>
<td>She gets 15 with the mixed number by adding the 2 and 3 from the $\frac{5}{3}$ and writing the 5 to the right of the 1 from the mixed number.</td>
<td>She doesn’t seem to be engaging her sense making here and is either ignoring what she knows about place value or is confused about the value of the digits—she treats the “1” in $1\frac{5}{3}$ as both a unit and a 10 (when she renames the number 15). She doesn’t seem to see either fraction ($\frac{5}{3}$ or $\frac{1}{2}$) as representing a value itself but as made up of two individual parts (numerator and denominator) that can be operated on (multiplying in one case and adding in the other).</td>
<td>She doesn’t remember the algorithm and is trying to recall the operations she’s been told to use—she knows that when you rename fractions, you do some combination of multiplying and adding, so that’s what she’s doing.</td>
<td>Does Melissa think that math can make sense? Is her difficulty deeply conceptual, or is she “just” having trouble with the notation and “naked” computation? Would she be able to represent and compare $\frac{5}{3}$ and $1\frac{5}{3}$ if she’d been asked to draw them with diagrams or build them with objects?</td>
</tr>
</tbody>
</table>
**Description of Melissa’s work.** We see the following aspects of Melissa’s work as salient in terms of her mathematical understanding and skill (see also the sample worksheet).

- Melissa doesn’t hesitate to engage in the problem.
- On her own initiative, Melissa states that the two quantities are equal and offers a reason.
- She multiplies numerator and denominator of the improper fraction (to get 15) but adds the numerator and denominator portion of the mixed number (to get 5).
- She rewrites the mixed number, writing the 1 and then the 5 (which she got from adding numerator and denominator) and says “15.”

**Interpretations (with evidence) and questions about Melissa’s work.** We were left with a lot more questions about Melissa’s understanding than we were with interpretations of it. That said, we do have two interpretations to offer—perhaps you had others of your own.

- The ease with which she reads the two numbers and volunteers the fact that they are equal suggests that Melissa has some familiarity with fractions and that she understands that they can be named in different ways. She may also understand that it’s possible to describe a quantity in terms of a fraction that’s larger than 1.
- Melissa’s explanation suggests that she’s encountered but not mastered the algorithms for converting mixed numbers to improper fractions and vice versa: she starts her work by saying she needs to multiply (or maybe divide). She then renames 5 by multiplying numerator and denominator (5 × 3); later on, she adds numerator and denominator (2 + 3) from the mixed number. These are operations that are part of the “conversion” algorithms for mixed and improper fractions, but the way she uses them lacks a mathematical logic. She operates on both numbers (5 and 13) to demonstrate equivalence rather than leaving one as is and working on the other in order to express both numbers in the same form—either two mixed numbers or two improper fractions. She also operates on numerator and denominator differently in the two cases and seems to be performing computations on any of the numbers that will make the equivalence “come out.” She doesn’t hesitate as she offers her explanation, but as she proceeds, she looks at the camera in a way that could be interpreted as self-conscious.

After observing Melissa’s video, we’re left with the following questions about what she might and might not understand:

- Is Melissa aware of the lack of logic to her answer?
- Does she have the expectation that her math work should make sense?
- This short clip of Melissa’s work doesn’t allow us much insight into what she understands about fractions more generally. Some possible follow-up questions about her understanding could include the following:
• Can she identify equivalent proper fractions when they are presented numerically?
• Does she understand that $\frac{1}{2} = \frac{2}{4}$?
• Can she accurately draw (or build with manipulatives) representations of $\frac{5}{3}$ and $1\frac{2}{3}$? If so, can she accurately compare these?

**Reflecting on Exercise 2.** Take a minute to think about your work on this exercise. Were you able to separate out descriptions of Melissa’s mathematical work from interpretations of her thinking? Were you able to support your interpretations with evidence?

We want you to focus on the distinction between description and interpretation because we want you to get used to recognizing where, in your own work with students, you’re making decisions based on evidence about their thinking and where you’re filling in those blanks and making decisions based on your assumptions about their understanding. Teachers can’t do their work without interpreting their students’ work, but it’s important to check these interpretations. We want to neither overestimate our students’ understanding—as someone might have done with Melissa—nor underestimate it.

**Using Artifacts in Your Own Classroom**

We’ll return to collecting video-based artifacts at the end of this section, but first want to highlight some kinds of data that you can consider collecting as part of activities that are already part of your regular instruction and require little or no extra effort to gather. Below are two examples of classroom activities that can yield artifacts to mine for data about your students’ thinking; we’re sure that you will think of other kinds of artifacts that can be informative as well.

**Warm-Ups**

Does your textbook suggest a warm-up activity at the beginning of a lesson? Warm-ups are a nice time for practice, review, or as a lead-in to the main part of your lesson. They can also offer you an opportunity to do some quick checks on the robustness of students’ skill and knowledge. If you use warm-ups that involve written work, you can collect students’ papers and review them more carefully after class (see the next section below). If warm-ups are oral, you can still jot down notes about interesting or surprising responses—answers that seem way off, for example, or common errors that you find students making. You can use these notes to reflect on the strengths and weaknesses you see in students’ responses and to use these observations to support future planning.
It may be tempting to try to fix issues that emerge during a warm-up right then and there, but this will take time away from your main lesson by turning the warm-up into a lesson of its own. Instead, think about using the warm-up to keep track of concepts or skills that you feel you need to revisit in future classes or on a more individual basis with particular students.

Homework, Quizzes, and Tests

The triumvirate of homework, quizzes, and tests offers you a constant stream of data about your students. In addition to providing information about whether students are doing their assignments, whether they seem to be studying for exams, and how well they are performing, with a little extra work on your part you can use homework, quizzes, and tests to get a sense of the kind of thinking students are applying to their work.

For example, try taking a homework assignment (or a quiz or test) and sorting the papers into three piles: a “gets the idea” pile, a “somewhat gets it” pile, and a pile for students who seem to be really struggling (or whose solutions raise lots of questions for you). Which pile is the largest? Is it the pile that you expected? If so, you probably have a pretty solid grasp of students’ understanding. It’s not uncommon, though, for teachers to think that students have a better handle on the math they’re been studying than they really do. It can be somewhat surprising to find that it’s the second or third piles that are bigger than the first or second. You can give this sorting a try in Exercise 3, where you’ll practice with the homework samples from a fourth-grade class.
Exercise 3: Sorting Homework

The work you’ll use in this exercise was collected from fourth-grade students in February, about halfway through the school year. They were asked to solve the following word problem:

Carolyn’s mom brought eight brownies to Carolyn’s dance group. That day six girls were there. If they share the brownies equally, how much does each girl get?

(The structure of the problem is identical to the one Myrna works on in Exercise 5.)

**Process for Sorting the Homework Samples.** Follow the steps below as you complete the exercise. Read through all the steps before you begin in order to get an overview of the process.

**Step 1:** Identify key math ideas. Take a minute to think about the problem. What are the key mathematical ideas for fourth-grade students? How would you solve the problem? What might you expect the range of work to look like for a fourth-grade class?

**Step 2:** Study the student work samples. These are found in the Appendix and as PDFs on PDToolKit. You may want to make copies of the work samples or perhaps just take notes on the ones in the book. If you do the latter, you might find sticky notes useful. Be sure to make brief notes about your interpretation of each student’s work and supporting descriptive evidence. With 14 work samples to consider, you don’t need extensive notes, but you do want to remember what thoughts went into your sorting decisions.

**Step 3:** Sort the work samples. We suggest you try three piles: (1) basically gets it, (2) somewhat gets it, and (3) needs more work. However, if you have categories that you prefer, try those too. Make sure that you can justify your sorting decisions with evidence for your interpretations of students’ work.

**Step 4:** Refine your sorts into subpiles. Now that you have done a rough sort, try sorting even further within each pile. A second sort will challenge you to be more explicit about the aspects of students’ work that led you to your categorization. Look more carefully at the first pile. What kinds of differences do you see among this group of students, all of whom are basically able to successfully do the mathematics? For example, do you see differences in the representations that different students use, in the specifics of the approaches they take to their solutions, in the sophistication of their thinking, or in the
care they take to check (and possibly correct) their work? Do the same for the other piles.

**Step 5: Reflect on your work.** If you’re working with others, take a few minutes to share and discuss how you sorted the samples and why you did so. Be sure to support your interpretations with evidence from the work samples. Use the study questions below to help you reflect on your observations (and, if you are working with others, to start your discussion). We’ve also provided a short commentary below.

<table>
<thead>
<tr>
<th>Study Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>▶ What was your evidence for sorting work into particular piles?</td>
</tr>
<tr>
<td>▶ What’s common to the work samples within a pile? What kinds of differences did you find?</td>
</tr>
<tr>
<td>▶ Do you feel that the second sort helped you make more fine-grained distinctions between students’ likely thinking? For example, how did you sort Aiden, Andrew, Annie, Lexi, and Patrick? Were they all in the same pile on the first sort? Were they in the same subpiles on the second sort? Why or why not?</td>
</tr>
<tr>
<td>▶ How might you use this approach to quickly get a sense of your students’ understanding?</td>
</tr>
</tbody>
</table>

**Commentary on Exercise 3.** The way someone sorts the work depends, of course, on his or her mathematical goals. One reasonable goal would be for students to find a solution where all the brownies are divided among the six girls—this goal requires expressing the solution as a fraction (or decimal). If you were to sort on the basis of this criterion, then your “gets it” pile might have included all solutions that were expressed as one brownie and a fraction of another (whether or not the answer was actually right or wrong), and your “needs more work” pile might include responses expressed in terms only of one brownie and a whole-number remainder. In this case, you might not have a “kind of gets it” pile with this group of students at all.

Your goal might have been more exacting. For example, you might have been looking for correct solutions expressed as mixed numbers or even solutions where the fraction was expressed in lowest terms. If so, then your “gets it” pile would be substantially smaller. But for any of these goals (and others, too), chances are that you were able to find enough commonality among students’ work that you could sort them into a relatively small number of piles according to criteria based on levels of understanding division and/or fractions.
When you go one step further and refine your sorts into subcategories, you can begin to identify groups of students with common strengths and weaknesses in their understanding. This lets you think about how to provide other learning experiences for these groups that can be more specifically targeted to develop the weaknesses and build on their strengths. In this way, you can efficiently differentiate instruction. For example, Aiden’s, Andrew’s, and Annie’s sketches of the problem solution all seem to demonstrate a conceptual understanding of the task, but their numerical answers suggest that they may have experienced some difficulty connecting different solution representations. If you were their teacher, you might think about a lesson or two that provide opportunities for these students to make these connections explicit.

Conducting a more careful analysis of students’ work can potentially help you with another teaching challenge—deciding when it is okay to move on to another idea or topic. You can use the same analyses you’ve done to better understand the needs of individual students to help you think about the class as a whole. For example, you can ask yourself whether most of the students in the “needs more work” pile are the ones who are always pretty much playing catch-up or whether there are more students in this pile than is typical. If the latter is the case, you might want to spend more time consolidating students’ understanding before moving on. Then again, you might feel that even the students who need more experiences have some basic understanding on which you could continue to build as you proceed with the curriculum, and you would decide to move on.

Regardless of your decision, as long as there are students in your class who are still struggling with the concept, you face the considerable challenge of figuring out what kinds of experiences you need to provide them so that they can continue learning the math. Your close observation and analysis of their work can help you plan additional learning opportunities that will go beyond “reteaching” to target specific requisite ideas and skills that need scaffolding or developing. (In many cases, reteaching is about as effective as trying to communicate with a non–English speaker by simply repeating what you’ve said, only slower and louder. Articulating more clearly and at a higher volume still isn’t going to make things any more sensible to someone totally lacking knowledge of English. Similarly, retracing the same mathematical ground isn’t likely to help students who are standing in an entirely different part of the mathematical landscape.)

When you take the extra time to closely examine an assignment, you can get a clearer sense of your students’ progress and also those aspects of understanding and skill that still need developing. While a deeper analysis of students’ work is a worthwhile use of time, you don’t have to do it every day. This kind of analysis does take extra time, and we all know that teachers’ time is at a premium. Instead, reserve your closer look at students’ work for
an assignment that focuses on an idea you see as central to the current unit or make
the time to reflect more deeply on a lesson that left you with questions about what your
students were thinking. Instead of doing an exercise like this with work from the whole
class, you might decide to examine the work of one or two students you’re wondering
about. (If you’re wondering how students approached a particular task, you might make
such a “depth sounding” a one-time event. If, on the other hand, you’re not sure about a
child’s mathematical thinking across the board, you may want to collect that student’s
work on a more regular basis and analyze it over time to build a better understanding of
his or her overall grasp of mathematical ideas and skills.) Remember, too, that the more
practice you get in attending to students’ ideas, the better you’re likely to be in zeroing
in on important thinking as it’s happening in the bustle of classroom lessons.

Making Videos of Your Own

Because video can capture the math classroom in all of its dynamism and complexity,
it offers an unparalleled opportunity to practice looking for students’ mathematical
thinking in a variety of classroom contexts—students working individually, solving
problems in small groups, and sharing and discussing their work as a whole class. With
video, you can capture students’ solutions as they unfold over time and can inquire into
both verbal and nonverbal elements of their problem solving. You can practice analyzing
students’ work in real time and can also slow the action down, stopping and revisiting
aspects of the class as you see fit. The more video you study, the better you become at
noticing the mathematically important details of students’ work and at interpreting how
these details reflect students’ mathematical thinking. Working with video outside the
classroom is great for developing an eye (and ear) for the mathematical thinking that
goes on during class activities and discussions.

Now that video cameras are relatively inexpensive, it’s pretty easy to make videos
of your own lessons for later review and reflection. (In fact, if you have a cell phone that
records video, you could even think about making short recordings with it.) This new
video accessibility offers a wonderful opportunity to take a more careful look at the
mathematical work that goes on in your class and to take more time to think about how
to build toward greater student understanding in subsequent lessons. But working with
video of your own classroom can also be distracting, so be prepared. You might find,
for example, that it’s hard to look past elements of your own teaching (or your haircut or
your posture) to focus in on how students are working with the mathematical ideas of
the lesson. Try to take the same generous and inquiring approach to your own class that
we ask you to take throughout this book when analyzing the work of others. Remember,
you’re working with artifacts in order to learn to explore students’ mathematical
thinking more deeply and, by extension, to take advantage of this learning in your own
teaching. This work is not about labeling or passing judgment on others or on ourselves.
We have a few technical suggestions about filming. You're likely to get more out of the video if you can focus the camera on different speakers during discussion and move about the room during individual and/or small-group work to capture some of the conversation and work in progress. If you can swing it, you might even try to round up several cameras so that you can set up a fixed camera to capture whole-group discussions and also have a roaming camera to capture individual or small-group work.

We also advise trying to get someone else to do the video recording if at all possible. With someone else as cameraperson, the video is likely to capture more of the work going on in the lesson because you don’t have to add “filmmaker” to your other responsibilities. You'll be free to continue to function in your normal role as teacher while someone else documents the work. You might arrange a quid pro quo with another teacher in your building, videotaping his or her class in exchange for video of yours. Having a colleague do the videography (another teacher in the building or perhaps a math coach or specialist) has the added advantage of offering a new pair of eyes in the class and most likely a fresh look at your own students. You also gain the advantage of being able to discuss the lesson with another colleague who was there to experience it firsthand. If a colleague isn’t available, you might try getting a video production student from a local high school.

If you can’t arrange to get help with the videotaping, you can still set up a stationary camera someplace in the room where you can capture most of the action for whole-group work (include the board in the shot). If your lesson includes individual or small-group work, you may find a way to take the camera with you as you circulate. You might also think about videotaping one-on-one sessions with individual students, particularly if you’re puzzled about how they’re thinking or approaching their work.

Make sure that you also capture at least some of your students’ written work as well, either recording work samples on camera or by taking notes. If you regularly use chart paper to record solutions, you’re all set. Having the written work is often helpful for following students’ comments—you might be surprised by how often students’ verbal descriptions of their solutions use ambiguous referents (what is the “it” in “then I timesed it by . . .”) and how much the corresponding written work can help to clarify some of the ambiguity.

Finally, you can use copies of the blank Chapter 2 worksheet to help you describe and interpret your students’ mathematical thinking. Feel free to use it as is or to modify it to make it work better for you when describing and interpreting artifacts of your own. While you obviously have much more background knowledge about your own students and the history of their mathematical experiences in your class to inform your analysis, try to focus primarily on describing the mathematical work you see and on pointing to evidence in the artifact to support your interpretations of their thinking. What mathematical strengths do you see in your students? Weaknesses? What questions about their understanding would you like to pursue?
Wrapping Up

The goal of this chapter is to help you to attend more closely to your students’ mathematical thinking, distinguishing between making descriptions of their work and drawing interpretations of their understanding on the basis of evidence from those descriptions. Our research and that of others suggests that with practice, teachers learn to notice important and subtle aspects of their students’ work and to become more adept at interpreting the thinking behind the work (Goldsmith & Seago, 2011; Santagata, Zannoni, & Sigler, 2007; Star & Strickland, 2008; van Es, 2011; van Es & Sherin, 2008). We hope that you will take the opportunities we present in this chapter, including the exercises following this section, to practice honing your skills.

Additional Exercises: Video Clips of Two Math Interviews

In these exercises, you’ll work with video clips of two other math interviews to practice describing and interpreting student work. In Exercise 4, third-grader Kasage solves a multidigit addition problem; Myrna, a second grader, solves a division problem in Exercise 5. Both exercises include completed sample worksheets to support your work.

Exercise 4: Kasage

In this clip, Kasage, a third grader, works on the “naked” addition problem, \(638 + 476\). The interviewer asks her to solve it using another student’s strategy. The strategy involves working left to right and, presumably, is different from the one that Kasage uses spontaneously.

**Process for Viewing and Analyzing the Kasage Video.** Follow the steps below as you complete the exercise. Read through all the steps to get an overview of the process.

**Step 1:** Identify key math ideas. Take a few minutes to think about key mathematical ideas involved in adding multidigit numbers. What kind of understanding and skill are involved in doing the computation from left to right? What kinds of difficulties might you expect Kasage to encounter when using this strategy (which is likely not her own)?

**Step 2:** Prepare to watch the video. Make a copy of the blank Chapter 2 worksheet (located in the Appendix and as a writeable PDF on PDToolkit). Think about the kind of information you’ll need to collect to fill out the worksheet; have some paper on hand for making notes about the details of Kasage’s work as you watch the video.
Step 3: **View the Kasage clip.** Find the *Kasage* clip on PDToolKit and watch it at least once. Recall that your purpose in viewing the video is to practice describing and interpreting Kasage’s work. You may want to view the clip once to get an overall sense of her work and a second time to gather more detail as you complete your worksheet. Keep scratch paper handy to jot down notes and time codes so that you can easily refer back to the evidence that you used to build your interpretations. We’ve included a transcript of the interview for you to supplement your video viewing (Figure 2.5); you may want to consult it while completing the worksheet.

**Figure 2.5**

*Transcript of Kasage Video*

**Interviewer:** Now let’s give you a little more difficult problem that I want you to use the same approach that Julio used, OK? Can you write this problem down for me?

**Kasage:** Um hum

**Interviewer:** It’s six hundred and thirty-eight plus four hundred and seventy-six.

**Kasage:** I didn’t really get it.

**Interviewer:** You didn’t get it?

**Kasage:** I think it was 34 or something. No, I think it was a thousand something.

**Interviewer:** OK, why do you think it was a thousand something?

**Kasage:** Because I, those are bigger numbers. They can’t add up to 34.

**Interviewer:** OK, so can you explain to me what you did here.

**Kasage:** Six plus four equals 10, so I put the ten there. And seven plus, seven—seven plus three equals 10, so I put the ten there and I—eight plus 6 equals 14, so I put the 14 there.

**Interviewer:** OK. Can you tell me what this 6 represents here?

**Kasage:** The hundreds.

**Interviewer:** The hundreds? OK, so is that really a six, or is it maybe a bigger number?

**Kasage:** It’s a bigger number?

**Interviewer:** Can you tell me what number it is?

*Continued*
| **Kasage:** | 600? |
| **Interviewer:** | 600? OK, and what does this four represent here? |
| **Kasage:** | 400? |
| **Interviewer:** | OK, so what would be the answer to this plus this? If this equals 600 and this equals 400. . . |
| **Kasage:** | One thousand? |
| **Interviewer:** | OK, so would it be ten or would it be the thousand? |
| **Kasage:** | Thousand. |
| **Interviewer:** | Thousand? OK. Do you maybe want to try working it out that way? |
| **Kasage:** | Yeah. |
| **Interviewer:** | OK. |
| **Kasage:** | Do I still leave the ten and the fourteen there? |
| **Interviewer:** | Well, I don’t know. Would this be—what does this three represent here? |
| **Kasage:** | Tens. |
| **Interviewer:** | But what—how many tens? |
| **Kasage:** | Thirty. |
| **Interviewer:** | Thirty? OK. What does this seven represent here? |
| **Kasage:** | Seventy. |
| **Interviewer:** | OK, so what would those two numbers add up to? |
| **Kasage:** | Ten hundred? |
| **Interviewer:** | Ten hundred? Well, we said this was what? What did you say the seven was? |
| **Kasage:** | Seventy. |
| **Interviewer:** | OK. And what’s this one? |
| **Kasage:** | Thirty. |
| **Interviewer:** | OK, so what are those two numbers added together? |
| **Kasage:** | Another one thousand? |
| **Interviewer:** | Another one thousand? Really? OK. And what’s—this plus that is 14? So do you want to refigure your answer, or do you want to leave it at 34? |
| **Kasage:** | I want to redo it. |
| **Interviewer:** | Redo it. OK. Did my explaining help you a little bit, or no? |
| **Kasage:** | A little bit. |
| **Interviewer:** | OK. . . K, so it’s 3400. OK. |

*Source: Philipp, R., Cabral, C., & Schappelle, B. (2011). Searchable IMAP Video Collection: Children’s Mathematical Thinking Clips. Copyright © 2011 by San Diego State University Research Foundation. All rights reserved except those explicitly licensed to Pearson Education, Inc. Video and transcript were created during IMAP project by Randolph Philipp, Bonnie Schappelle and Candace Cabral.*
Step 4: Complete the “My Descriptions” column of the worksheet. Remember that your goal here is to describe, objectively, things that Kasage does and says that you think are noteworthy and relevant to her understanding of the math.

Step 5: Fill in the “My Interpretations” column of the worksheet. Use the “My Descriptions” column to provide evidence for your interpretations.

Step 6: Complete the rest of the worksheet. See if you can come up with alternative interpretations for Kasage’s work. Also note any kinds of questions you’d like to ask Kasage if you had the chance to do so.

Step 7: Reflect on your work. If you are working with others, take a few minutes to share and discuss your descriptions and interpretations. Be sure to support your interpretations with evidence from the video—your notes and worksheet comments will help you do this. Use the study questions to help you reflect on your observations (and, if you are working with others, to start your discussion). You can also use the completed sample worksheet (Table 2.2) to help you think about your own responses and to stimulate your reflections and discussion.

Study Questions

- How did you describe Kasage’s work? What details did you intentionally leave out of your description?
- What evidence did you use in formulating your interpretations?
- What alternative interpretations did you make?
- What were some strengths and weaknesses you found in Kasage’s work?
  - What do you think she understands?
  - What do you think she still needs to work on?
  - If you had been the interviewer, what additional questions would you have liked to ask Kasage?
  - If you were Kasage’s teacher, how would you build on her strengths to address those aspects of place value/addition that you think still need work?
### Key Mathematical Ideas:

- Our base-10 number system is built on powers of ten (groupings of 1s, 10s, 100s, 1000s, and so on)
- We can decompose numbers into the values for each place—for example, $638 = 600 + 30 + 8$ and $476 = 400 + 70 + 6$
- We can find partial sums and then add these together to find the sum: $638 + 476 = (600 + 400) + (30 + 70) + (8 + 6)$

<table>
<thead>
<tr>
<th>My Descriptions</th>
<th>My Interpretations and Supporting Evidence</th>
<th>Alternative Interpretations?</th>
<th>My Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kasage represents the sums of both the “6” and the “4” (in the numbers 600 and 400) and the “7” and the “3” (in the numbers 70 and 30) as 10s, writing both partial sums directly under each other and aligned in the 100s column.</td>
<td>She can add the digits $(6 + 4 = 10$ and $7 + 3 = 10$), but she’s not clear about the value of these in terms of their “place” in the problem.</td>
<td>She doesn’t quite understand Julio’s approach, and she’s trying to solve the problem his way when it doesn’t really make sense to her. She has a somewhat shaky understanding of the place values of the different partial sums but isn’t sure about where to put the numbers to capture place value in the written format.</td>
<td>What method would Kasage use to solve the problem on her own? Could she show the value of the digits in the two numbers (638 and 476) with base-10 blocks or some other kind of representation? Is it the notation that’s challenging to her? Is the size of the problem she’s solving pushing at her level of understanding (i.e., could she solve a smaller problem successfully)?</td>
</tr>
<tr>
<td>638 + 476 10 10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>She adds 8 + 6 and aligns the answer with the other two partial sums, adding them to get 34.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>638 + 476 10 10 14</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>As Kasage finishes her answer, she says that she thinks it’s wrong because it’s too small; she estimates that the answer is in the 1000s.</td>
<td>She has—and uses—her number sense to reflect on the reasonableness of her answer.</td>
<td>She hasn’t yet fully internalized ideas about place value, and this problem is beyond her current level of comfort in dealing with place value. Since the problem involves adding two three-digit numbers, she figures that the answer will be “big,” and saying that a number in the thousands is a way of saying that the sum is likely to be a number that’s larger than the ones she typically works with.</td>
<td>Ask her why she thought the sum would be in the thousands.</td>
</tr>
</tbody>
</table>

Continued
Table 2.2  Completed Sample Worksheet: Kasage Video

<table>
<thead>
<tr>
<th>My Descriptions</th>
<th>My Interpretations and Supporting Evidence</th>
<th>Alternative Interpretations?</th>
<th>My Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>When the interviewer asks her to explain her answer, she says that 6 + 4 = 10, 7 + 3 = 10, 8 + 6 = 14.</td>
<td>She’s thinking about adding digits and isn’t taking the value of the places into account.</td>
<td>She isn’t entirely sure how to add numbers that “cross” a place value boundary.</td>
<td>Is it her understanding of addition of numbers in the 10s, 100s, or 1000s that’s a problem, or is her understanding shaky more in terms of naming or writing the sums? Would she make the same kinds of mistakes if she modeled the problem with base-10 blocks?</td>
</tr>
<tr>
<td>Her calculations with addition of single digits are correct.</td>
<td>She knows her basic number facts (sums of numbers less than 10).</td>
<td>Same questions as above.</td>
<td></td>
</tr>
<tr>
<td>When the interviewer asks her to identify the value of the digits 6 and 4, Kasage correctly identifies them and says that 600 + 400 is “one thousand.”</td>
<td>She doesn’t have a clear understanding of how to create partial sums to solve this problem.</td>
<td>Same questions as above.</td>
<td></td>
</tr>
<tr>
<td>She begins to correct her work on the problem, changing the “top” partial sum of 10 into 1000. She asks the interviewer whether she should keep the 10 and the 14 from her original solution.</td>
<td>She has some understanding of how to use partial sums: her difficulty may lie in some combination of computing the partial sums correctly and representing the sums accurately. The size of the numbers she’s dealing with seems to be adding to her difficulties.</td>
<td>Same questions as above.</td>
<td></td>
</tr>
<tr>
<td>With prompting from the interviewers, Kasage identifies the “3” and the “7” as representing 30 and 70 but isn’t sure what their sum is. She says that the sum is 1000 and, despite the interviewer’s somewhat incredulous tone, changes the second partial sum (10) to 1000.</td>
<td>While Kasage’s observation that 34 is too small an answer suggests that she can reflect on the reasonableness of an answer, she’s not applying her number sense here. Kasage is able to understand individual elements of this solution with scaffolding, but her understanding is fragile enough that she’s unable to integrate the parts into a sensible approach on her own.</td>
<td>Kasage has given up thinking for herself on this problem and is just trying to get it finished.</td>
<td>Are the interviewer’s questions helping Kasage or confusing her further? Would Kasage be able to correctly add 30 and 70 if she weren’t feeling obliged to solve this problem Julio’s way? Does she think she understands Julio’s solution process?</td>
</tr>
<tr>
<td>She leaves the 14 where it was (written directly under the original 10s) and writes 3400 as her new answer.</td>
<td>Kasage doesn’t understand how to notate partial sums in terms of “preserving” groupings in the same columns.</td>
<td>Kasage’s answer is in the ballpark of her original prediction (the sum should be “a thousand something”), so she’s okay with the solution and happy to be done with the problem.</td>
<td>Is Kasage able to make connections or find correspondences between Julio’s solution and other ways of solving the problem? Can she use Julio’s way to solve a smaller addition problem (e.g., adding two two-digit numbers)?</td>
</tr>
</tbody>
</table>
Exercise 5: Myrna

In this final exercise, you will observe Myrna, a second grader, solving a problem about sharing eight brownies among six children. Myrna is an English-language learner, and a translator assists in the interview process.

Process for Viewing and Analyzing the Myrna Video. Follow the steps below as you complete the exercise. Read through all the steps before you begin in order to get an overview of the process.

Step 1: Identify key math ideas. You will probably want to refer to your work in Exercise 3 for this step given that the problems are virtually identical. Remember, however, that Myrna is a second grader and that the written work from Exercise 3 comes from a fourth-grade class.

Step 2: Prepare to watch the video. Make a copy of the blank Chapter 2 worksheet (located in the Appendix and as a writeable PDF on PDToolKit). You may want to read over the transcript before you begin as well (see Figure 2.6); Myrna is not a native English speaker and speaks Spanish with an interpreter during much of the interview, which makes it a bit more difficult to coordinate your viewing of her actions on the video with the conversation. Think about the kind of information you’ll need to collect to fill out the worksheet; have some paper on hand for making notes about the details of Myrna’s work as you watch the video.

Step 3: View the Myrna clip. Find the Myrna clip on PDToolKit and watch it at least once. Recall that your purpose in viewing the video is to practice describing and interpreting Myrna’s work. You may want to view the clip once to get an overall sense of her work and a second (and maybe third) time to gather more detail as you complete your worksheet. Because much of the interaction is in Spanish, as you watch the video you will probably want to look closely at the transcript as well. Keep scratch paper handy to jot down notes and time codes so that you can easily refer back to the evidence you used to build your interpretations.

Step 4: Complete the “My Descriptions” column of the worksheet. Remember that your goal here is to describe, objectively, things that Myrna does and says that you think are noteworthy and relevant to her understanding of the math.

Step 5: Fill in the “My Interpretations” column of the worksheet. Use the “My Descriptions” column to provide evidence for your interpretations.

Step 6: Complete the rest of the worksheet. See if you can come up with alternative interpretations for Myrna’s work. Also, note any kinds of questions you’d like to ask her if you had the chance to do so.
FIGURE 2.6

Transcript of Myrna Video

**Interviewer:** If I had eight brownies . . .

**Myrna:** Brownies? What is that?

**Translator:** Brownies? They're these little cakes . . .

**Myrna:** Oh. Eight?

**Interviewer:** And I wanted to share them evenly, okay, with six people. How could I share them?

**Translator:** She wants to share them.

**Myrna:** How many are there?

**Translator:** There are eight . . .

**Myrna:** Eight. And you want to share it, how many?

**Translator:** To share them with six people. Equally. So that everyone has, the brownies, the same amount.

**Myrna:** Are there eight? [unintelligible] What was the first number? Of the little cakes?

**Translator:** Um hum. They're little chocolate cakes.

**Myrna:** With how many children?

**Translator:** With six; six people.

**Myrna:** Uh uh. (Counts under her breath.) Nhhh.

**Translator:** There are eight and they want to share them equally among six people so each person has the same amount.

**Interviewer:** So is this one brownie?

**Myrna:** Uh huh.

**Interviewer:** For one child, or one niño?

**Interviewer:** Okay. And is this 2, 3, 4, 5, 6? And how many do you have left over?

**Myrna:** (to translator) What?

**Translator:** How many are left?

*Note: Italic text indicates that the conversation was in Spanish and that the text has been translated.*
**Myrna:** Two.

**Interviewer:** Two? Okay. So how could you share those with those with these six people? How could you—now these are hard, and they’re a block, but a brownie you could cut. Could you cut it somehow?

**Translator:** If you could cut them and share them evenly among everyone, how many would you give them? To each person.

**Myrna:** How many pieces would you cut?

**Translator:** Um hum

**Myrna:** How many?

**Translator:** Um hum. Into how many would you cut? . . . six people.

**Myrna:** In three.

**Interviewer:** Three? Three.

**Interviewer:** How would you do it?

**Myrna:** (to translator): Tell her I’m going to do it.

**Interviewer:** How would you do it?

**Myrna:** How do you say “cut”?

**Interviewer:** “Cut.”

**Myrna:** Cut . . .

**Interviewer:** I’m going to pretend like this is that one brownie that you’ve got in your hand there. Okay. That’s this one. We’ll pretend that’s this extra one here. And then this is the other brownie that you have in your hand.

**Interviewer:** Okay. Now how would you cut those then so that everybody, all the other six, got an equal piece?

**Myrna:** Oops! Like this.

**Interviewer:** Good job! Very nice.

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*Source:* Philipp, R., Cabral, C., & Schappelle, B. (2011). *Searchable IMAP Video Collection: Children’s Mathematical Thinking Clips* Copyright © 2011 by San Diego State University Research Foundation. All rights reserved except those explicitly licensed to Pearson Education, Inc. Video and transcript were created during IMAP project by Randolph Philipp, Bonnie Schappelle and Candace Cabral. Transcript adapted by Lynn Goldsmith and Nahia Kassas.
**Step 7:** Reflect on your work. If you are working with others, take a few minutes to share and discuss your descriptions and interpretations. Be sure to support your interpretations with evidence from the video—your notes and worksheet comments will help you do this. Use the study questions to help you reflect on your observations (and, if you are working with others, to start your discussion). You can also use the completed sample worksheet (Table 2.3) to help you think about your own responses and to stimulate your reflections and discussion.

**Study Questions**

- How did you describe Myrna’s work? What details did you intentionally leave out of your description?
- What evidence did you use in formulating your interpretations?
- What alternative interpretations did you make?
- What were some strengths and weaknesses you found in Myrna’s work?
  - What do you think she does and doesn’t understand?
  - If you had been the interviewer, what additional questions would you have liked to ask her?
- How do you think language may have figured into Myrna’s work?
- What questions does this video raise for you about challenges involved in separating the mathematical demands of tasks from nonmathematical ones?

As a teacher, what kinds of accommodations do you make for English-language learners (or special education students who do not have specific mathematical disabilities) to help them develop mathematically?

**TABLE 2.3 Completed Sample Worksheet: Myrna Video**

**Key Mathematical Ideas:**
- Division can be thought of as sharing out equal quantities.
- If division results in a “remainder,” this remainder can be subdivided in such a way as to share it out equally as well.
  - \( \frac{1}{3} = \frac{1}{6} + \frac{1}{6} \).

<table>
<thead>
<tr>
<th>My Descriptions</th>
<th>My Interpretations and Supporting Evidence</th>
<th>Alternative Interpretations?</th>
<th>My Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Myrna asks what the first number was as she collects some snap cubes and then checks about the meaning of the word “brownie.”</td>
<td>She wants to make sure she understands the problem. This seems to have both a computational component that’s related to her collecting the right number of cubes (“what was the first number?”) and a linguistic one (“of the little cakes?”).</td>
<td>She understands that she’s going to have to share out something and, by checking about what a brownie is, may be getting information about how “shareable” it is.</td>
<td></td>
</tr>
</tbody>
</table>

*Continued*
### TABLE 2.3 Completed Sample Worksheet: Myrna Video

<table>
<thead>
<tr>
<th>My Descriptions</th>
<th>My Interpretations and Supporting Evidence</th>
<th>Alternative Interpretations?</th>
<th>My Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>She counts out some cubes off-camera and pushes some aside. She counts out six,</td>
<td>She’s using the cubes to represent brownies, and she discovers that she has one too many.</td>
<td></td>
<td>What were you planning on doing?</td>
</tr>
<tr>
<td>picks up two cubes that are snapped together, and unsnaps them. She holds one</td>
<td>She’s demonstrating “metacognitive” skills, monitoring her work to make sure that she’s got the right</td>
<td></td>
<td>What do these cubes represent?</td>
</tr>
<tr>
<td>in her left hand and puts the other down. She takes two cubes from the original</td>
<td>number of cubes to represent the problem.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>six. Myrna counts the remaining cubes on the table (she points to six, but they’re</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>not all visible in camera), looks briefly at the (three) cubes in her hand, and</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>says “uh huh.” She looks a bit puzzled. She recounts, puts one cube back, and</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>looks up at the camera.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>She pauses, saying “hmmm.”</td>
<td>She’s not sure what to do to share out the brownies.</td>
<td></td>
<td>What do the cubes on the table represent?</td>
</tr>
<tr>
<td>The interpreter repeats the problem—share the brownies out equally. The</td>
<td>Myrna agrees with the interviewer that each cube represents a child. But she counted out eight cubes, and</td>
<td></td>
<td>What do the cubes in your hand represent?</td>
</tr>
<tr>
<td>interviewer points to a cube as the interpreter begins to move away the extra</td>
<td>there are eight brownies, not eight children, in the problem.</td>
<td></td>
<td>Have you shared everything that you can share?</td>
</tr>
<tr>
<td>cubes. The interviewer helps move extra cubes, points back at one cube, and</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>asks if it’s one child. (Myrna says “yes.”)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The interviewer touches and counts each cube on the table (getting to six) and</td>
<td>Myrna agrees with the interviewer that each cube represents a child. But she counted out eight cubes, and</td>
<td></td>
<td></td>
</tr>
<tr>
<td>asks Myrna how many she has left over.</td>
<td>there are eight brownies, not eight children, in the problem.</td>
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<tr>
<td>She has a strong grasp on how to divide up the remainder and is comfortable</td>
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<tr>
<td>thinking about thirds. Instead of dividing each piece into sixths and sharing</td>
<td></td>
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<tr>
<td>out a sixth from the first brownie and a sixth from the second, she recognizes</td>
<td></td>
<td></td>
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<tr>
<td>that dividing each brownie into thirds will yield one piece for each child.</td>
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<tr>
<td>If there are six children who are sharing the brownies, why did you divide each</td>
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<td></td>
<td></td>
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<tr>
<td>piece into three?</td>
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<td></td>
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<tr>
<td>If you had divided each brownie into six pieces, would that have been fair</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sharing?</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>