Chapter 1

The Dissimilar Learner and Mathematics Instruction

A Problem and a Solution

A fourth-grade student approached his teacher one day to tell her that he finally “got math.” When the teacher asked him what that meant, he said, “I know that whatever I think will be wrong. So now I don’t think or say anything. I just wait for the teacher to tell me what to do and if I do it, then I know I’ll be right.” Of course, teachers plan for effective instruction and assessment to prevent students from losing confidence in their mathematics reasoning. In fact, mathematics teachers often ask two important questions when developing lessons: “How do I successfully teach mathematics concepts and skills so they are understood and remembered?” and “How can I effectively teach this math content?” Rather than focusing solely on content, however, instructional questions must also relate to the child’s needs. The child must be the focus of any pedagogical decision being made because a learner’s cognitive, behavioral, and physical needs vary widely and have great impact on achievement. For learners to succeed, teachers must assess their individual abilities and characteristics and then choose appropriate and effective instructional strategies.

The purpose of this book is to address the cognitive, behavioral, and physical needs of first- through sixth-grade students who underachieve when learning mathematics. Certainly, not all students find success when taught with uniform instructional approaches or within the same behavioral model. The traditional educational system may “give up” on these students. However, we propose that a Mathematics Improvement Plan (MIP), based on a targeted, functional assessment of each learner’s strengths and areas of concern, can be a proactive instructional tool. The MIP serves to guide a wide range of intervention strategies that connect to the Response to Intervention (RTI) model. The National Association of State Directors of Special Education defines RTI as the practice of providing high quality instruction and intervention based on a student’s needs, changing instruction and/or goals through frequent monitoring of progress, and applying the student response data to important educational decisions (South Dakota Department of Education Learning Leadership Services, 2010). The MIP supports the RTI model with specific intervention techniques that, when applied to Tier II students, fill in their gaps of understanding and mathematics procedural skills. Teaching techniques and activities included in the MIP pertain to both informal and formal factors that influence learning. The next section describes those factors and their implications for targeted instruction.
The Learner-Centered Approach

The principle of “learner at the center” implies that more than one teaching approach is necessary to achieve the desired results. The term dissimilar learner was developed for students who have not succeeded when engaged in “one size fits all” instruction (Cooper, Lingg, Puricelli, & Yard, 1995). Dissimilar learners do not fit the traditional instructional mold and are not served well at the Tier I level. They are often rebellious and dysfunctional in a learning environment that does not adequately address various teaching options and learning styles. They lack resilience (Bernard, 1995), which is defined as the “capacity to successfully overcome personal vulnerabilities and environmental stressors, to be able to ‘bounce back’ in the face of potential risks, and to maintain well-being” (Wang, 1998, p. 12). Dissimilar learners can also be thought of as struggling learners who have traditionally needed support but whose needs were not addressed until they could “demonstrate significant gaps in their learning” (Cooper, et al., 1995, p. 15). This text will provide strategies and approaches targeted to assess needs in a variety of behavioral and academic settings, so that students’ issues are dealt with in a timely and appropriate manner, forestalling greater learning problems. Research in the nature of resilience has identified certain factors, including school-related factors—termed protective factors—that can counteract these risks (Bernard, 1995, p. 3). Bernard found that schools fostering resilience also provide opportunities for children to develop the internal assets necessary for resilience, such as problem-solving skills; autonomy; a purposeful, constructive, and optimistic outlook on the future; and effective communication and relationship skills. Bernard further summarized the contributions to these external protective factors made by schools and teachers under three main categories: caring and supportive relationships, positive and high expectations, and opportunities for meaningful participation.

Werner and Smith (1998) also address protective factors in the school setting. They believe that the absence of the characteristics just described can result in a lack of academic success for students and can set in motion a cycle of inaccurate diagnosis and remediation, and, sometimes, the child’s withdrawal from the educational system. The primary characteristics of dissimilar learners are represented in several areas and listed in Figure 1.1. The types of support characteristics are listed in Figure 1.2.

Systematic Instruction

The most effective pedagogical approach that benefits the dissimilar learner is both multidimensional and systematic, as is suggested in the RTI model. It is

| • Concrete in thought processes | • Prefers group performance rather than individual performance |
| • Physical or direct confrontation yields negative results | • Low sense of self-security, especially when environment is radically different from his or her norm |
| • The visual modality is the primary intake style for learning | • Views educational system as a threat to self-preservation |
| • Tactile involvement with the environment | • Reacts negatively to rigid order |
| • Communication style is high in word usage, low in word meaning | • Emotionally fragile and volatile |
| • Adverse to written language | • Loyalties are strong but bonding is slow |

FIGURE 1.1
Primary Characteristics of Dissimilar Learners
Home Problems
- Lack of learning structure in the home
- Lack of respect for parents on the part of student
- Parents lack respect for educational system

Visual Learner
- Does not like to read
- Prefers hands-on activities
- Thought processes are more concrete than abstract

Low Skill Development
- Social skills are low
- Dissimilar learners are verbally impaired—have difficulty effectively communicating feelings

Aggressive—Verbally and Physically
- Aggressive toward adults and peers
- Verbal explosions
- Abusive language
- Physically aggressive toward peers and teachers
  - Fights
  - Throws objects
  - Makes inappropriate gestures
  - Reacts violently to being touched

Low Skill Development
- Social skills are low
- Dissimilar learners are verbally impaired—have difficulty effectively communicating feelings

Low Skill Development
- Social skills are low
- Dissimilar learners are verbally impaired—have difficulty effectively communicating feelings

Low Skill Development
- Social skills are low
- Dissimilar learners are verbally impaired—have difficulty effectively communicating feelings

Poor Personal Habits
- Poor hygiene
- Poor nutrition
- Misses classes, tardy
- Disorganized
- Complains of physical discomfort

Tactile
- Seeks excessive physical attention
- Can’t keep hands or feet to self

Poor Peer Relationships
- Has difficulty making friends
- Teases other students
- Prefers to interact with older or younger age groups
  - Often prefers to be alone
  - Isolated by peers
  - Physically threatens peers

Poor Work Habits
- Has trouble working independently
  - Needs frequent attention from the teacher
  - Needs directions to be repeated
  - Wants demands to be met immediately
  - Needs constant supervision and reminders

FIGURE 1.2
Support Characteristics of Dissimilar Learners
a “system of instructional tiers” (Riccomini & Witzel, 2009, p. 5) structured so that students receive instructional supports related to their depth of knowledge. Students classified as Tier III need the most instructional support, Tier II students need more intensive and direct instruction, and Tier I students are taught in the core instructional program, where less intervention is necessary than in the other two tiers. The Data Analysis System (DAS) provides the necessary data to determine the mathematics content and teaching approaches for Tier II students. The DAS examines all the conditions within and surrounding the child, such as the curriculum content, context of the classroom, academic and social behavior, and ways in which students process information and respond to feedback. A comprehensive approach to teaching and remediating mathematics is then identified in the MIP, which specifies and systematizes a number of factors that are applied in an orderly and logical manner. If any one of these components is ignored or overlooked, academic success is jeopardized. The problems that teachers and students experience when handling any of these factors are interrelated and may negatively affect the teacher’s attempts to bring about positive change. Therefore, a system that considers all factors when designing and targeting teaching approaches is most advantageous for students.

Mathematics strategies for an effective RTI framework include fundamental principles and standards for quality mathematics instruction. For example, small-group work, use of concrete and diagrammatic models, focus on a balance of procedures, and frequent targeted assessments are keys to success. Teachers should increase instructional time whenever possible to include steps for Tier II students who are struggling with specific topics.

Teachers must recognize that no one tool will be effective in every circumstance or environment. Planning for maximum flexibility, setting manageable classroom time periods, and incorporating a broad spectrum of teaching and behavioral approaches must be the rule rather than the exception. Effective teachers will assess the students’ success and make necessary, differentiated changes for all learners, especially dissimilar learners. Those changes will direct the teacher from a Tier I universal or total group instructional level approach to the Tier II small group instructional level. Teachers will work with those students we call dissimilar and those we believe need more specific learning analysis, as described later in the discussions of the DAS and MIP process (National Research Center on Learning Disabilities, 2007). Our approaches are most applicable at the Tier II level. Learners who will benefit most by the MIP model are those who have not experienced success in whole group or universal group instruction. These students benefit from greater analysis of their specific learning needs. These analyses will aid the teacher in considering multiple factors impacting students’ learning—context, content, process, behavior, and reinforcements.

As previously noted, two data gathering tools are useful for analyzing students’ progress and related instructional decisions. The DAS is used to document strengths and concerns for Tier II students. The MIP is used to identify and categorize specific learning strategies to differentiate instruction, based upon data recorded in the DAS. As instructors implement the DAS/MIP model and related RTI strategies, mathematics teaching will focus on students with specific learning needs, students who we have previously referred to as dissimilar learners.

**Why Do Students Struggle with Mathematics?**

The core of the child-centered, systematic teaching approach is content. The discipline of mathematics presents many challenges to dissimilar learners, yet it is a vital discipline for gauging a student’s abilities. Mathematics has often been termed the gatekeeper of success or failure for high school graduation
and career success (National Research Council [NRC], 2001). It is essential that "mathematics . . . become a pump rather than filter in the pipeline of American education" (NRC, 2001, p. 7). A lack of sufficient mathematical skill and understanding affects one’s ability to make critically important educational, life, and career decisions.

Students fall below their expected level of mathematics achievement for a variety of reasons. When asked why they were not successful in learning mathematics, many people reply that they “never understood math,” or “never liked it because it was too abstract and did not relate to them.” These reasons and others are very child-specific and can be categorized, in general, as environmental or personal, individualized factors.

**Environmental Factors That Affect Student Success**

**Instruction:** Mathematics instruction must provide students with many opportunities for concept building, relevant challenging questions, problem solving, reasoning, and connections within the curriculum and real-world situations. If taught in a way that relies too heavily on rote memorization isolated from meaning, students have difficulty recognizing and retaining math concepts and generalizations.

**Curricular Materials:** Spiraling the curriculum provides opportunities for learners to deal with content developmentally over time. Concepts can be built upon and related to previous learning throughout the curriculum as students become more proficient and experienced in mathematics. However, the same content must not be taught in the same manner of delivery when children experience initial failure. Students who do not “get it” the first time are not likely to “get it” the next several times either if the lessons are taught in the same manner. Moreover, underachieving students are frequently assigned repetitious and uninteresting **skill-and drill worksheets** to “learn the basics.” This type of work, which often represents a narrow view of mathematical foundations and a low level of expectation of students’ abilities, limits students’ opportunities to reason and problem solve.

**Gap Between Learner and Subject Matter:** When the mathematics content being taught is unconnected to students’ ability level and/or experiences, serious achievement gaps result. This situation may occur if students are frequently absent or transfer to another school during the academic year. A transferring student may find the mathematics curriculum at the new school to be more advanced or paced differently than what was being taught in the previous school. Without intervention strategies, these students could remain “lost” for the duration of their education.

When students have too few life experiences, such as trips to neighborhood stores or opportunities to communicate with others about numbers through practical life examples, math can become irrelevant. Gaps exist, therefore, not only in the curriculum but between the learner and the perceived usefulness of the subject matter.

**Personal or Individualized Factors That Affect Student Success**

**Locus of Control in Learning Mathematics:** Some students believe that their mathematical achievement is mainly attributable to factors beyond their control, such as luck. These students think that if they score well on a mathematics assignment, it is only because the content happens to be easy. These students do not attribute their success to understanding or hard work. Their locus is external because they believe achievement is due to factors beyond their control and do
not acknowledge that diligence and a positive attitude play a significant role in accomplishment. These students might also believe that failure is related to either the lack of innate mathematical ability or a low level of intelligence. They view their achievement as accidental and poor progress as inevitable. In doing so, they limit their capacity to study and move ahead (Beck, 2000; Phillips & Gully, 1997).

**Memory Ability:** Some students lack well-developed mental strategies for remembering how to complete algorithmic procedures and combinations of basic facts. However, they can be taught strategies to improve their capacities for remembering facts, formulas, or procedures. Repetition games such as calling out fact combinations, having each student solve them, and then repeat those that were called before their turn can help. For example, the teacher might call out “3 × 5” and a student would respond with “15.” That student would then ask a number question such as “7 – 5” of the group. The responder would reply, “3 × 5 = 15 and 7 – 5 = 2.” The game continues as each player calls out a new fact and each responder answers with all the previous combinations as well as the new answer. Students’ abilities to organize their thinking and use it to recall data will affect their success throughout the curriculum.

**Attention Span:** Students may be mentally distracted and have difficulty focusing on multistep problems and procedures. Dealing with long-term projects or a number of variables or pieces of information at one time can interfere with students’ achievement. Therefore, effective teachers should use attention getters such as drawings and learning aids. Tier II students who work in pairs can help each other stay on task. Task length should also be a serious consideration because attention span is an issue. The longer the task, the greater the probability of off-task behavior. Student support is very effective for those who may lack confidence to solve their own problems or are still unsure of fundamental concepts and/or procedures, such as renaming or finding common denominators (Riccomini & Witzel, 2009).

**Understanding the Language of Mathematics:** Students are often confused by words that also have special mathematical meaning, such as *volume*, *yard*, *power*, and *area*. Lack of understanding of mathematical terms such as *divisor*, *factor*, *multiple*, and *denominator* seriously hampers students’ abilities to focus on and understand terms and operations for algorithms and problem solving. However, simply memorizing these terms without meaning and contextual association is not productive.

### How Is Mathematics Taught Effectively for All Students?

#### Developing Mathematics Proficiency

Students who complete algorithms with little understanding quickly forget or confuse the procedures. Algorithms are “step by step procedures” (Cathcart, et al., 2011, p. 173) such as the rules needed to solve a long-division computation problem. Suppose, for example, students cannot recall if they are supposed to divide the denominator into the numerator, or the reverse, to find a decimal equivalent. They can carry out the procedure and long divide either way, but they do not understand why the process yields a decimal number and cannot explain their reasoning. Similarly, students may understand the concept of decimal numbers but may have trouble remembering the steps to complete the division algorithm. Long-term understanding and skill achievement are established
together when students successively build upon concepts in a guided discovery process (Bruner, 1977).

Understanding fundamental concepts and accurately completing algorithms contribute to becoming numerate, or mathematically proficient. These terms describe “what it means for anyone to learn mathematics successfully” (NRC, 2001, p. 4). More broadly, according to a national review of all relevant research on mathematics learning (NRC, 2001), mathematics learning is composed of five interrelated strands of thought: “the comprehension of ideas (conceptual understanding), flexible and accurate skills and procedures (procedural fluency), ability to formulate and solve problems (strategic competence), capacity to reflect and evaluate one’s knowledge and ability to reason (adaptive reasoning), [and] a habitual inclination to make sense of and value what is being learned (productive disposition)” (NRC, 2001, p. 5).

The dissimilar learner is one who has experienced little success in all five strands or may, in fact, lack development of an entire strand. However, mathematical proficiency can be expected and achieved as adaptations are made to the curriculum in light of the learners’ characteristics, such as relating problems to daily life interests or providing more time for cooperative strategic thinking. Students reach higher levels of proficiency when they engage in lessons that are developmentally structured. The time taken to move from one step to another varies according to students’ progress, and two steps may be combined in one lesson. The general framework is as follows.

**Structuring Lessons for Success**

**Step 1** Learners connect new concepts to those with which they are familiar and are actively engaged at a concrete level of understanding. Objects such as counters and base-10 blocks are manipulated to solve questions that represent authentic and interesting problems. For example, students are asked to demonstrate how many more cookies they need to bake for a class party if eight are already baked for the class of 15 students (each student is to get one cookie). When studying multiplication, learners gain a conceptual understanding of the operation by forming three groups of four counters, each representing Halloween candy collected by children. Connections are also made to previous lessons, such as relating long division to the mathematical idea of repeated subtraction. Ask students to discuss their understanding of these mathematical concepts.

**Step 2** Students represent their understanding with pictures or diagrams. For example, the sets of cookies appear as follows:

![Illustration of cookies](image)

How many more do I need to bake to get 15 cookies?

Put that many in this box.

Three sets of four may be drawn like this:

![Illustration of three sets of four cookies](image)
Chapter 1

Twelve divided by four is related to repeated subtraction in a diagram such as:

![Diagram of twelve divided by four]

**Step 3** Students attach numerals and number sentences to the drawings in this way:

A) \[ 8 + \square = 15 \]

because I need 7 more than 8 to get 15

B) \[ 3 \times 4 = 12 \]

\( \square \text{ groups} \) \( \square \text{ in each group} \)

C) \[ \frac{3}{4} \text{ groups of 4} \]

\( \square \text{ in each group} \)

**Step 4** Students practice skills and algorithmic procedures through a variety of activities and reinforcement lessons. Teachers should provide continuous and targeted feedback at each learning step, so that conceptual or procedural errors can be corrected quickly and effectively.

Finding Use and Daily Value in Learning Mathematics

Students should also value learning and the use of mathematics in their daily lives. Too often, people find it socially acceptable to say, “I am not a math person.” Although they would find it embarrassing to claim that they are not good readers, innumeracy is readily admitted. Problems and examples that relate to students’ interests and daily experiences establish relevance and value, particularly for students disengaged from studying mathematics. Presenting concepts in situations to which students connect, such as sports, currency, places of interest, and their own classrooms and schools, establishes the importance of mathematics on a personal level. Students will more readily participate and inquire when they care about content.

Meeting Standards for Dissimilar Learners

The National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (NCTM, 2000) is a guide for teaching mathematics in a way that addresses all learners’ needs to understand, use, connect, and make mathematics important in their daily lives (NCTM, 2000). The document presents a comprehensive set of standards for teaching mathematics from kindergarten through 12th grade. Dealing with the importance of meeting the needs of dissimilar learners, the Council states, “students exhibit different talents, abilities, achievements, needs, and interests in mathematics... [Nevertheless], all students must have access to the highest-quality mathematics instructional programs” (NCTM, 2000, p. 4).
Further, the *Principles and Standards* identify six principles, which are statements “reflecting basic precepts that are fundamental to a high-quality mathematics education” and are “perspectives on which educators can base decisions that affect school mathematics” (NCTM, 2000, p. 16). These principles are Equity, Curriculum, Teaching, Learning, Assessment, and Technology. Three principles (Equity, Teaching, and Learning) have particular relevance to dissimilar learners in a systems model for assessment and instructional purposes.

**The Equity Principle:** The Equity Principle states “excellence in mathematics education requires equity—high expectations and strong support for all students” (NCTM, 2000, p. 11). A major theme of this book is that all children can learn mathematics and deserve the opportunity to do so. Equity “demands that reasonable and appropriate accommodations be made as needed to promote access and attainment” (NCTM, 2000, p. 12).

Accommodations to make learning equitable should include:

- Alternative approaches with manipulatives
- A wide variety of divergent questions
- Games
- Authentic materials such as calculators, menus, maps, spinners, and measuring tools
- More time on task
- Peer tutoring

A more comprehensive list of suggestions to increase achievement is found in Appendix A, including ideas for affecting the content, behavior, and emotional environment for learners. Teachers can adapt the environment and deal with the wide range of abilities and experiences in such a way that one or a combination of methods will stimulate achievement.

**The Teaching Principle:** “Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well” (NCTM, 2000, p. 17). The Teaching Principle highlights the practice of designing instruction from students’ points of view (a child-focused model). Teachers connect new concepts and procedures to students, who associate them with what is already known. Instruction begins with prior knowledge, in terms of learners’ ability to understand language and the foundational concepts and experiences offered. Conceptual development is guided and established by the instructional sequence. It assists all students with well-designed and connected lessons, step by step, toward deeper understanding and greater skill.

Dissimilar learners are often presented with practice materials for the majority of their instructional time in class in an effort to make certain they know “the basics.” By doing so, these students are negatively affected. They typically lack credible foundational concepts, so they have difficulty working with reinforcement practice materials that assume understanding. Lacking those concepts, students often forget procedures and then are remediated with more of the same misunderstood work, repeating a cycle of inaccurate or soon-forgotten practice of uninteresting skill-based assignments. The learner is not the center of instruction in terms of needs; rather, the drill and practice is the focus, which does not provide much long-term success.

The Teaching Principle is implemented when students engage in authentic, interesting, and challenging lessons. They stimulate interest and relate to real academic needs. Learners build a strong foundation on their understanding of ideas and number sense.
The Learning Principle: The Learning Principle, which is closely related to the Teaching Principle, calls for students to “learn mathematics with understanding, actively building new knowledge from experience and prior knowledge” (NCTM, 2000, p. 20). Presenting students with alternative approaches to learning the content of mathematics provides valuable opportunities to build upon those that are most closely related to students’ thinking and previous experiences. For example, students are given place value blocks to bundle into groups of tens when modeling the regrouping necessary to subtract 9 from 35. When the hundreds place is introduced in the problem 235 – 79, students again work with place value blocks to actively connect previously learned concepts to regrouping hundreds into groups of tens.

Building understanding, which is often found missing in the experiences of dissimilar learners, is essential as content becomes more complex. For example, students are often able to invent and utilize their algorithms as they make sense of and understand mathematics for themselves. Discussions of ideas and thinking about patterns and strategies provide further exploration into a variety of problem-solving methods. Establishing an environment of trust in which novel ideas and alternative ways of thinking are acceptable is essential. The ultimate goal is to find strategies directly related to the students’ assessed learning characteristics and to use that knowledge to build understanding, skill proficiency, and confidence for mathematical success.

The Common Core Standards for Mathematics (Council of Chief State School Officers and National Governors Association Center for Best Practices, 2010) is a recent national effort to establish the same standards in each of the 50 states for teaching mathematics. The Common Core mathematical principles highlight, as do the NCTM Principles and Standards (NCTM, 2000), the importance of instruction reaching all students. Whereas the Common Core standards do not define specific intervention strategies, the document stresses that “all students must have the opportunity to learn and meet the same high standards if they are to access the knowledge and skills necessary in their post-school lives” (Council of Chief State School Officers and National Governors Association Center for Best Practices, 2010, p. 4). The Standards were designed to ensure that the broadest population of students be offered “appropriate accommodations to ensure maximum participation” (Council of Chief State School Officers and National Governors Association Center for Best Practices, 2010, p. 4). Instruction, as described in the Common Core State Standards for Mathematics Principles, should support students’ mathematics sense making, reasoning, modeling and using appropriate tools, structure, and patterning to solve problems.

Curriculum Focal Points for Prekindergarten Through Grade 8 Mathematics: A Quest for Coherence (NCTM, 2006) provides guidelines for specifying the most significant mathematical concepts and skills at each grade level. The document also deals with issues related to students who struggle with learning mathematics by emphasizing that “instruction focused on a small number of key areas of emphasis provides extended experience with core concepts and skills” (NCTM, 2006, p. 5). Teachers and learners benefit from organized instruction, as identified by the focal points, that “assumes that the learning of mathematics is cumulative, with work in the later grades building on and deepening what students have learned in the earlier grades, without repetitious and inefficient re-teaching. A curriculum built on focal points has the potential to offer opportunities for the diagnosis of difficulties and immediate intervention” (NCTM, 2006, p. 13). The NCTM Principles and Standards and Curriculum Focal Points stress the importance of establishing relevance, need, and number sense and connecting mathematics teaching and learning to related content and everyday experiences. The DAS and MIP included in this text are designed to guide teachers in their work to focus on students’ assessed needs and provide a coherent, integrated mathematics curriculum for learners.
Identifying and Meeting the Needs of Dissimilar Learners

The authors of this text have created a three-step systematic approach to improving dissimilar learners’ mathematics achievement.

**Step 1 Assessing strengths and areas of concern.** The first step in planning for instructional needs is conducting an assessment. Mathematics content knowledge, environmental factors that may affect learning, learning style, behavior, and the manner in which reinforcement typically occurs are considered. Assessment data is gathered from anecdotal records of daily observations; performance on classroom assignments; informal classroom mathematics tests, quizzes, and homework; and in-class work and/or formal standardized test results. This information is recorded on a DAS. The value of assessment, in general, is that it leads to an overall perception of the functional abilities of a learner’s strengths and areas of concern. Data collected for a DAS informs instruction and prescribes a more accessible environment to influence future learning. A template for the DAS is found in Table 1.1. A brief description of the areas in the table follows:

- **Context:** This refers to the physical environment in which students exist. The setting includes the classroom, hallway, lunchroom, art room, gym, and bus. The contextual environment includes all environments in which the school holds administrative authority over the child.
- **Content:** This includes the curriculum and the current course of study in which the child is engaged.
- **Process:** This refers to methods, strategies, and tools that students prefer for accepting and expressing information, such as listening, speaking, writing, or drawing.
- **Behavior:** This includes academic and social behaviors such as whether the student enjoys learning through print material and/or in group settings; does or does not like to correct and complete assignments; is willing to ask questions of other students in a group; and is willing to socialize, communicate, and work well with teachers and classmates.
- **Reinforcement:** This refers to responses from the environment that cause inappropriate or appropriate behavior to reoccur (Sperbar, Premack, & Premack, 1996).

**Recording Behavior Patterns:** High-probable behavior is described as a behavior that is likely to occur and will occur on a consistent basis. It might include the desire to play mathematics games or use the computer. Low-probable behavior

<table>
<thead>
<tr>
<th>TABLE 1.1 A Sample Data Analysis Sheet</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data Analysis Sheet</td>
</tr>
<tr>
<td>Student Name:</td>
</tr>
<tr>
<td>Team Members:</td>
</tr>
<tr>
<td>Data Analysis Record</td>
</tr>
<tr>
<td>Context</td>
</tr>
<tr>
<td>Input</td>
</tr>
<tr>
<td>+</td>
</tr>
<tr>
<td>+</td>
</tr>
<tr>
<td>+</td>
</tr>
<tr>
<td>Note: The + symbols indicate strengths and the – symbols indicate areas of concern.</td>
</tr>
</tbody>
</table>
describes behavior that is very likely to occur below an average rate or at a very minimal level (Sperber et al., 1996). Low-probable behavior could be the students’ rate of studying alone or studying with others; completing drill pages or solving problems are other behaviors that occur less often in dissimilar learners.

For example, a classroom climate (context) that is conducive to student achievement would be classified as a “+” symbol. However, if a student is having difficulty in the physical environment of the classroom, the teacher would mark this category with a “−” symbol. Likewise, students’ unacceptable (low occurring) classroom behavior during the mathematics lesson is coded as a “−.” By collecting and reviewing this information, the teacher recognizes which classroom activities foster high-probable behavior and which do not. Also, teachers can focus on a low-occurring positive behavior in content, for example, as a starting point for lesson planning. If bundling tens during place value lessons is unfamiliar to students and is coded as a “−,” teachers should begin with reviewing a skill that could be considered a “+,” such as counting.

An additional dimension of this assessment approach is that context refers to all environments that students use in a typical school day. Often, the teacher records behavior only in the classroom, yet it is widely known that low-occurring behaviors most often occur outside of the classroom (Sperber et al., 1996). Student actions in all environments are reflective of learning achievement. Social and emotional behaviors, language usage, spatial awareness, and other academic functions recorded in any part of a school environment provide valuable assessment data. The DAS generates a large bank of strengths (plusses), which become available to teachers when looking for reinforcers in academic endeavors. Anything that the student likes to do and does well in the learning environment is a reinforcer; conversely, anything the student does at a low rate of occurrence is something the student will avoid and is not reinforcing. This data collection provides the information for the actual DAS.

Step 2 Completing the DAS. DAS information provides teachers with current behavioral data, collected from a real environment, and an informed foundation upon which to diagnose difficulties. The DAS process evaluates the child against herself and generates diagnostic and prescriptive information that is ready to use. The result frames the remediation plan for the MIP.

Utilizing the data listed for each of the areas—context, content, process, behavior, and reinforcement—the DAS is prepared for each student several times during the school year as learning conditions and/or a child’s characteristics change. High- and low-probable-occurring behaviors (HPBs, LPBs) are reviewed and recorded for each category.

Completing the DAS: Instructions for completing the DAS are as follows:

1. Identify the data as collected.
2. Record the data for each category as a strength (HPB) or a concern (LPB), including information from all environments. Define what a student does rather than what is not done. For example, record that a student “can count to 10” rather than “child cannot count to 10 or higher quantities.”
3. Record behavioral information, including more than one data sample; for example, record that a student raised his hand, worked well in a group, or received feedback favorably as a “+,” and record examples of low-occurring behaviors as a “−.”
4. Record reinforcement activities such as giving verbal praise or stickers for feedback.

A column becomes a “+” when it contains more cells with a “+” for high-probable behavior than a “−” for low-probable behavior. A sample DAS, with collected data, is shown in Table 1.2.
### TABLE 1.2 How to Fill in the Data Analysis Sheet

<table>
<thead>
<tr>
<th>Content Assessment</th>
<th>Process</th>
<th>Behavior</th>
<th>Reinforcement</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input</strong></td>
<td><strong>Output</strong></td>
<td><strong>Academic</strong></td>
<td><strong>Social</strong></td>
</tr>
<tr>
<td><strong>Context</strong></td>
<td><strong>Context</strong></td>
<td><strong>Behavior</strong></td>
<td><strong>Reinforcement</strong></td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Information should cover the strengths (+) and areas of concern (–) of a student’s environment and his or her reactions to it, such as:</td>
<td>Information should cover the strengths (+) and areas of concern (–) representative of specific skills associated with learning mathematics, such as:</td>
<td>Information should cover the strengths (+) and areas of concern (–) in which the student takes in information when learning and puts out the information gained from his or her learning, such as:</td>
<td>Information should cover the strengths (+) and areas of concern (–) in which the student reacts to reinforcement methods used in learning, such as:</td>
</tr>
<tr>
<td><strong>Learned Concepts I</strong></td>
<td><strong>Learned Concepts II</strong></td>
<td><strong>Learned Concepts III</strong></td>
<td><strong>Learned Concepts IV</strong></td>
</tr>
<tr>
<td>Reads and recognizes numerals</td>
<td>Knows subtraction involving zero</td>
<td>Knows how to borrow across a zero</td>
<td>Can't sit for a long period of time; doesn't communicate well in groups</td>
</tr>
<tr>
<td>Knows proper alignment of digits</td>
<td>Knows that place value rules exist</td>
<td>Sits by self or in group, or sits near teacher</td>
<td>Doesn't like being in front of class or getting to talk with teacher one-on-one</td>
</tr>
<tr>
<td>Error Pattern I</td>
<td>Error Pattern II</td>
<td>Error Pattern III</td>
<td>Error Pattern IV</td>
</tr>
<tr>
<td>Records more than a single digit for place value</td>
<td>Borrows from wrong place</td>
<td>Doesn't listen well in class, to directions, or to other peers; does not like to ask for help; does not take time to read directions, or reads directions in a hurry</td>
<td>Doesn't like made-up work or extra credit work</td>
</tr>
<tr>
<td>Unable to express multidigit numerals using place value system</td>
<td>Does not regroup</td>
<td>Doesn't listen well in class, to directions, or to other peers; does not like to ask for help; does not take time to read directions, or reads directions in a hurry</td>
<td>Doesn't like tangible reinforcements, such as stickers, candy, or free time</td>
</tr>
<tr>
<td>Error Pattern III</td>
<td>Error Pattern IV</td>
<td>Error Pattern V</td>
<td>Error Pattern VI</td>
</tr>
<tr>
<td>Borrows from wrong place</td>
<td>Does not connect numerals to concept underlying the collecting and trading principles</td>
<td>Slow to complete assignments; doesn't complete assignments at all, or completes assignments late</td>
<td>Argues with peers; defiant with teacher; behaves shyly with peers or teacher, or only does well in one-on-one interactions</td>
</tr>
<tr>
<td>Does not connect numerals to concept underlying the collecting and trading principles</td>
<td>–</td>
<td>Can't stay focused, can't complete assignments or other duties, or has difficulty with directions</td>
<td>Becomes frustrated and angry when she makes errors, or upset when others don't do their share of group work</td>
</tr>
<tr>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Note: The + symbols indicate strengths and the – symbols indicate areas of concern.
Case studies that contain additional examples of completed DAS forms are included in each chapter.

**Step 3 Designing the MIP.** Mathematics instruction should focus on all factors that affect learning while simultaneously building on students’ mathematical strengths and recognizing students’ error patterns. As discussed, this information is gathered systematically on a DAS as an organizing tool for individualizing instructional needs. The teacher considers the DAS information, noting the particular type of classroom or learning environment (i.e., works best alone or with groups), reinforcement, and other factors that will affect the mathematics lesson to be designed. Based on recognizing how students learn in the classroom, the MIP is developed to recognize and generate more HPBs than LPBs in remediation activities that address error patterns. The totality of the mathematics instruction must be considered rather than a repetition of lessons not learned well without a change in approach. A template for the MIP is given in Table 1.3.*

**Completing the MIP Form:** To complete an MIP, teachers must:

1. Review DAS information related to the learner’s environment.
2. Diagnose a mathematics error pattern for a concept or skill within a particular topic, such as place value, whole-number computation, rational numbers, and problem solving.
3. Prescribe mathematics remediation strategies that encourage high-occurring behaviors and build upon low-occurring behaviors, including any content error patterns. If a student is regrouping whole numbers incorrectly when subtracting, the teacher may identify the specific misunderstandings as conceptual errors. Strategies involving manipulatives and drawings, as well as methods to encourage HPBs, such as working in small groups and receiving frequent written reinforcement, are then selected and listed in the MIP.

The following is an example of typical information teachers might record under each column.

4. Complete the MIP with plans for each cell that will elicit HPBs. Build each cell’s activities on “+” behaviors, which are those behaviors that students can do or will like in terms of how they learn best. Spend no

---

**TABLE 1.3 A Sample Mathematics Improvement Plan**

<table>
<thead>
<tr>
<th>Time</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Context</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Content</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Process</td>
<td>Input</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Output</td>
<td></td>
</tr>
<tr>
<td>Behavior</td>
<td>Academic</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Social</td>
<td></td>
</tr>
<tr>
<td>Reinforcement</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*For your use, a second MIP template is provided in Appendix B.

---

Note: The + symbols indicate strengths and the – symbols indicate areas of concern.
more than 30 minutes on lessons or activities that will pose a great cognitive mathematics challenge to students.

5. Implement the prescribed remediation strategies in the context of the learning climate and student behavior.

In this way, the MIP is based on the DAS information and serves as a guide for a well-planned, child-centered, focused learning approach. As new topics are introduced, mathematics achievement is again assessed and a new MIP for the content component is written.

See Table 1.4 for a sample of a completed MIP with collected data. A completed MIP form is included in each chapter.
Conclusion

Mathematics fluency has been defined in terms of understanding and skill levels, the ability to reason and solve problems, and a positive disposition toward learning. Many pupils who have not succeeded as well as they, or their parents or teachers, would expect are considered dissimilar learners. They do not progress satisfactorily when taught with traditional, one-dimensional instructional approaches. Environmental and personal reasons for underachievement also affect learning performance. A targeted responsive intervention approach that assesses students’ needs and abilities, the environment, classroom, behavior, learning style, and means of reinforcement is essential for designing a successful strategy. Strategies that consider all aspects of the learner’s instructional needs can more accurately address the needs of dissimilar learners and can lead to a global approach to success.

This book is based on the premise that teachers should systematically diagnose students’ specific mathematics error patterns, organize them in two to three levels, prescribe plans of action, and implement those techniques in the context of students’ learning environments. Mathematics teaching and learning that stresses the relevance of the discipline to students’ lives; connects rules to conceptual understanding; offers challenging, authentic, and interesting problems to all students; and encourages students to express their reasoning in a positive classroom climate lead to achievement.
Discussion Questions

1. What are the advantages of addressing the learning needs of dissimilar learners with a systematic instructional approach?

2. What characteristics are often found in students referred to as dissimilar?

3. Why is assessment so important in developing a useful DAS or MIP, as in developing an individualized education program, for students?

4. Describe an activity that relates primary (grades K–3) and also upper elementary (grades 4–6) students’ everyday experiences to a lesson on the value of learning mathematics.

5. How does focusing mathematics teaching on specific HPBs encourage students to achieve?

6. What mathematical thinking processes do the NCTM publications Principles and Standards and Curriculum Focal Points recommend be taught through mathematics instruction to all students, including those who are struggling with mathematics? How do thinking processes enhance and make mathematics learning more effective?

7. Which teaching principles have the most impact, in your experience? Explain your reasoning.

8. Interview three classmates or acquaintances. Ask them if they had difficulty learning mathematics. If so, what reasons could they provide for those experiences? Identify their reasons as environmental or personal. Explain your choices.

9. How does learning mathematics in the context of understanding and then practicing the concepts differ from how you were taught mathematics? Provide specific examples.

References


Chapter 1


Chapter 2

Place Value

The Importance of Place Value

Place value is often identified as the most fundamental concept imbedded in the elementary and middle school mathematics curriculum. Place value concepts and procedures underpin and make possible efficient counting of quantities in groups, singles, and bases and computation of multi-digit whole and rational numbers. "It is absolutely essential that students develop a solid understanding of the base ten numeration system and place-value concepts by the end of grade 2. Students need many instructional experiences to develop their understanding of the systems including how numbers are written" (NCTM, 2000, p. 81). The Common Core State Standards Initiative for Mathematics (CCS) identifies place value concept development as early as kindergarten, when students "work with numbers 11–19 to gain foundations for place value in the strand of ‘Number and Operations in Base Ten’" (Common Core Standards, 2010). Throughout the elementary curriculum, students work with place value systems to read, write, search for patterns, and ultimately "perform operations with multi-digit whole numbers and with decimals to hundredths" (CCS, 2010).

Effective place value instruction makes it possible for students to progress from counting by ones to larger groups, bases; trading those base groups, such as 10 or 60; and correctly solving problems with computation involving renaming.

Instruction that builds a strong foundation upon which rules and procedures are applied should begin early in the K-6 mathematics curriculum. Tier I students and those considered Tier II benefit from targeted lessons in which students make sense of quantity by using materials and drawings to connect with recording digits to represent the amount of objects seen and handled. Students begin with groups of objects and are asked how to determine the total amount. Ask students how they grouped objects and why using 5 or 10 as a benchmark is helpful. As students realize the efficiency of identifying totals by grouping materials in familiar subsets such as 10, the ideas of place value or grouping become established. This work then leads to trading groups of the same value and learning to record large quantities accurately and efficiently. Students, who may be grouped as Tier II students, should be encouraged to learn to count using number names and subsets handling ungrouped objects one at a time. Conceptual development is then enhanced with increased use of modeling, emphasis on vocabulary development, and audio and video supports. Connecting materials to symbols and words with additional time and the use of technology, where available, is very helpful and should be used for both introductory conceptual and reinforcement (procedural) Tier I and II lessons.

This chapter will deal with typical multi-digit place value errors of both conceptual understanding and procedure. Appropriate remediation activities will be described for each place value error pattern.
What Is Place Value?

Place value systems are also termed *positional systems* because the value of a number is determined in part by the position or place it holds. In a decimal place value system, for example, each digit represents a group or base of 10. Place value “pertains to an understanding that the same numeral represents different amounts depending on which position it is in” (Charlesworth & Lind, 2003, pp. 308–309). The place value concept enables us to represent any value using 10 symbols (0–9) and compute using whole numbers. Other positional or place value systems include those based on groups of 12, as seen in clock time for counting hours, or groups of 60, for minutes in the hour.

The following are examples of regrouping in base-10 and base-12 systems in whole-number algorithms. Recent literature (Ma, 1999) uses the term *regrouping*. It applies to the exchanges of base groups in the four operations of addition, subtraction, multiplication, and division. For example, in

\[
\begin{array}{c}
\hline
& 1 \quad 745 \\
+ & 389 \\
\hline
& 1134 \\
\end{array}
\]

10 ones in the sum of the ones column, 14, are regrouped to the tens column as “1 ten.”

Likewise, for the following example:

\[
\begin{array}{c}
6 & 14 \\
7 \text{ ft} & 2 \text{ in.} \\
- & 4 \text{ ft} & 8 \text{ in.} \\
\hline
& 2 \text{ ft} & 6 \text{ in.} \\
\end{array}
\]

“7 feet” is renamed to “6 feet 12 inches.” The quantity of “12 inches” is combined with 2 inches to be named as “14 inches” when computing in a place value system based on groups of 12.

The *Hindu-Arabic Numeration System*: The Hindu-Arabic place value numeration system is based on the principle of collection and exchange of groups of 10. Remember that counting is the concept upon which place value is built. In this system, 10 ones can be traded and represented by one group of 10, 10 groups of 10 each can be exchanged and represented as 100, 10 groups of 100 each can be regrouped and represented as 1,000, and so on. This mechanism of collection and exchange makes possible a system in which only 10 unique symbols are necessary to express any quantity.

Multiplying each quantity by the value of its position or place and then adding all those values together determine the total value of a number. The following example indicates how the total value is found for 47 and for 385.

\[
\begin{align*}
(4 \times 10) + (7 \times 1) &= 47 \\
(3 \times 100) + (8 \times 10) + (5 \times 1) &= 385
\end{align*}
\]

Several important properties of the base-10 place value system include:

1. Ten unique symbols (0–9) express any numerical quantity.
2. The value of each base-10 place is multiplied by 10 as the digits move to the left from the ones place.
3. The decimal point is a symbol that enables the system to express parts of numbers. Each place is divided by 10 (tenths, hundredths, thousandths, and so on) as one moves to the right of the decimal point in a number. For example, the value of each place is as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>.01</th>
<th>.001</th>
<th>.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. The zero symbol (0) is a placeholder that represents a set that has no members or elements and is integral to expressing and computing quantity.

### Why Do Students Struggle with Place Value?

#### Common Errors

Misunderstanding and errors are evident in student work when place value concepts and procedures are learned in isolation from previous knowledge and with little meaning (Baroody, 1990, 2001). For example, not remembering from which direction to count over the number of places when multiplying decimals can be a result of not understanding why the decimal point is placed in a specific spot in the product.

#### Conceptual Errors

Conceptual misunderstandings often occur when students lack fundamental understanding and experience with positional systems (Kamii, 1985). Learners struggle with trading groups for collections of groups, such as regrouping 10 tens for 1 hundred. There is a lack of understanding of the place value structure, that is, multiplying each place value position to the left of a number by the base (such as 10) and dividing each place to the right of the decimal point by the base.

If students’ errors are diagnosed as conceptual in nature, remediation begins with using manipulative materials. These might include place value blocks, counters of any type, and place value charts. Use of more than print-based activity is critical for students needing less-traditional approaches (Clausen-May, 2005). Research indicates that students' experience using physical models can be effective early in the curriculum. The materials should "help them [students] think about how to combine quantities and eventually how this process connects with written procedure" (Kilpatrick, Swafford, & Findell, 2001, p. 198). However, "merely having manipulatives available does not ensure that students will think about how to group the quantities and express them symbolically" (NCTM, 2000, p. 80). Rather, students must use manipulatives to represent groups of tens in classroom discussions and in authentic, cooperative activities.

#### Procedural Errors

Some students become confused, however, when using place value rules to group objects, trade, and compute, as in renaming values to subtract from zero, for example. Content gaps and language barriers are common reasons that students make a variety of procedural errors. These are mistakes in reading and writing small and large quantities as well as in recording whole number and decimal computation. “Understanding and fluency are related . . . and there is some evidence that understanding is the basis for developing procedural fluency” (Kilpatrick et al., 2001, p. 197). You can promote understanding by providing students with examples that connect to their daily lives and have
solutions that can be determined as sensible. For example, students could determine if an answer is reasonable when comparing multi-digit enrollments in various classrooms because the numbers used are related to students’ experiences and, therefore, students can discuss, explain, and justify their answers.

Remediation activities for rule-based or procedural errors do not necessarily have to involve manipulative materials in all cases. Focus lessons on drawing and/or representing objects with tallies. Use color coding to make students aware of the place value positions. Make notations as reminders. For example, when subtracting, students can draw an arrow over the “2” if that helps them remember where to start. Or, pupils could circle the ones column in each example, prior to computing, in order to remember to regroup that place and not the tens place.

\[
\begin{array}{c}
\downarrow \\
432 \\
-29 \\
\end{array}
\begin{array}{c}
402 \\
-20 \\
\end{array}
\]

The most common errors specifically related to place value include those in Figure 2.1.

**About the Student: Colin**

Colin is a composite picture of a fourth grader with specific learning needs and abilities who demonstrates difficulty with fundamental place value concepts and the procedural skill of expressing quantity.

Colin is a bright fourth grader who likes school, especially the learning activities and the social aspect. He is open and friendly with peers but is more reticent around adults. He has a very strong sense of fairness and is well liked by both boys and girls. He has excellent coordination and prefers an active life, which makes it difficult for him to sit still for long periods and stay focused on routine tasks. He generally follows school and classroom rules and rarely receives a reprimand. He does not like to make mistakes and is slow to complete assignments because he tries to get everything done perfectly. Rather than turn in incomplete work, he will put it in his desk. Colin does not like to ask for help because he does not want to look “stupid.”

His reading skills are appropriate for fourth grade. However, Colin dislikes reading and delays tasks involving reading. When avoidance tactics are unsuccessful, he reads quickly and with little attention to nuances and detail. Consequently, he misses important information. Colin’s reading style causes him to misinterpret directions. His reading style also causes him to think he has read one thing when that is not what was written, which results in his answers being
incorrect. He then becomes frustrated and angry with himself but does not acknowledge the connection between his approach to reading and the resulting errors in his work.

Colin quickly memorizes math facts. He knows addition, subtraction, and multiplication facts and is learning division facts. Colin’s rapid reading and skimming of details causes problems for him in math. He may not attend to all the steps in multistep directions, which results in errors. Colin’s poor reading habits make solving word problems difficult. When confronted with his errors he becomes frustrated and makes self-deprecating comments.

Colin enjoys helping his peers, but especially enjoys helping adults. He likes to feel useful, but quickly sees through “made up” jobs. He likes to earn free time, which he usually spends drawing or playing games with friends. Colin does not respond to tangible reinforcers, such as stickers or points, unless the points lead to free time or helping time. Colin prefers learning tasks that require active participation or a hands-on approach. He enjoys class discussion and is an active participant in group work.

### Error Patterns: Diagnosis, Prescription, Remediation

The following error patterns are those made by Colin on different days, when he worked on computations that required place value understanding and skills. These mistakes are typical of many children’s thought patterns. The next sections include an analysis of each type of error and a diagnosis of its origin as either conceptual or procedural in nature. Based on that information, as well as the factors that affect Colin’s learning in general, remediation activities for each error pattern are described.

**Place Value Error Pattern I for Colin: Notating Multi-Digit Numerals**

The first type of place value error is found in Student Work Sample I for Colin. Colin responded to six questions asking him to express, with numerals, various quantities that are written with number words.

**Diagnosing the Error:** The teacher examines Colin’s work and identifies the type of mistake he is making as conceptual or procedural. Mathematical strengths are also noted. These might include Colin’s ability to write numerals, order numerals, identify places correctly, and more. Record your own analysis of the error pattern you detect in the top half of the box and the strengths you see in the bottom half of the box:

<table>
<thead>
<tr>
<th>Colin’s Error Patterns(s):</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Colin’s Strengths:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

A diagnosis of Colin’s work reveals that Colin can read and recognize single-digit numerals. However, he does not understand the rules for writing multiplace numerals using place value notation.
He expresses each digit as a separate cardinal value rather than multiplying each numeral by its place value (4 × 1000). For example, he writes “50” instead of “5” to indicate the number of tens in “57.” Colin lacks a conceptual understanding of place value in that his notation is not connected to any sense of a positional system.

An example of a completed DAS for Colin’s error is shown in Table 2.1. Characteristics, both positive and negative, are listed for the context in which Colin learns. The table also contains information about Colin’s input and output processes, academic and social behaviors, and the reinforcers that are most effective to promote overall achievement.

**Prescription:** Colin should work with both nonproportional and proportional materials to build collections of tens to trade and represent with numerals. Non-proportional manipulatives could include pennies, dimes, Popsicle sticks, coffee stirrers, and buttons. These items can be grouped in bundles of 10 to be traded. However, none of the objects is proportionally 10 times the size of the others. Each stick or coin is about the same size. Each single item represents one unit.

The craft sticks shown are bundled in groups of 10. Each stick is identical in size. Groups of 10 can only be represented with these materials when 10 of the sticks, or 10 of the tens groups, are bundled together.
### TABLE 2.1 Data Analysis Sheet

**Student: Colin**

**Team Members: Sherman, Richardson, and Yard**

<table>
<thead>
<tr>
<th>Context</th>
<th>Content Assessment</th>
<th>Process</th>
<th>Behavior</th>
<th>Reinforcement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Input</td>
<td>Output</td>
<td>Academic</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>• Likes school</td>
<td>Learned Concepts I</td>
<td>• Reading skills are grade appropriate</td>
<td>• Enjoying learning activities</td>
<td>• Likes being with peers</td>
</tr>
<tr>
<td>• Open and friendly with peers</td>
<td>• Reads and recognizes numerals</td>
<td>• Memorizes quickly</td>
<td>• Likes concrete activities</td>
<td>• Enjoying helping adults</td>
</tr>
<tr>
<td>• Likes to be clustered with his peers</td>
<td>• Reads numerals from left to right</td>
<td>• Concrete in his thinking</td>
<td>• Wants to complete his activities</td>
<td>• Likes to feel useful</td>
</tr>
<tr>
<td>• Enjoys all cooperative learning activities</td>
<td>• Correctly identifies each place value in numerals</td>
<td>• Excellent coordination; prefers an active lifestyle</td>
<td>• Strong sense of fairness</td>
<td>• Free time is important to him</td>
</tr>
<tr>
<td>• Enjoys any and all group participatory activities</td>
<td>Learned Concepts II</td>
<td>• Retains information he has learned</td>
<td>Well liked by boys and girls</td>
<td>• Likes drawing and playing games with his friends</td>
</tr>
<tr>
<td>• Likes being in front of class</td>
<td>• Reads and recognizes numerals</td>
<td>• Enjoys group discussions</td>
<td>Follows school rules</td>
<td>• Points help if they lead him to free time or helping time</td>
</tr>
<tr>
<td></td>
<td>• Knows that place value rules exist</td>
<td>Is good at putting ideas into written form</td>
<td>• Likes being in front of class</td>
<td>• Likes being in front of class</td>
</tr>
<tr>
<td></td>
<td>Learned Concepts III</td>
<td>• Once he gets started he seldom fails to complete his work</td>
<td>• Likes giving oral reports</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Reads and recognizes numerals</td>
<td>• Likes being with peers</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Reads numerals from left to right</td>
<td>• Likes helping adults</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Correctly identifies each place value in numerals</td>
<td>• Likes to be isolated in the classroom</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>• Reticent around adults</td>
<td>Error Pattern I</td>
<td>• Does not like to ask for help</td>
<td>• Becomes frustrated and angry when he makes errors</td>
<td>• Doesn't like made-up work or tangible reinforcers such as stickers and candy</td>
</tr>
<tr>
<td>• Can’t sit for a long period of time</td>
<td>• Unable to ex- press multi-digit numerals using place value system</td>
<td>• Does not like reading</td>
<td>• Becomes upset when he believes a group member is not doing his share</td>
<td></td>
</tr>
<tr>
<td>• Doesn’t like to work with his teacher on a one-on-one basis</td>
<td>Error Pattern II</td>
<td>• Misinterprets directions</td>
<td>• Avoids independent reading</td>
<td>• Doesn’t like being with teacher one-on-one</td>
</tr>
<tr>
<td>• Doesn’t like to work alone</td>
<td>• Cannot determine order, inequalities, equalities among multi-digit numbers</td>
<td>• Thinks he has read something when he hasn’t</td>
<td>• Doesn’t like being with teacher one-on-one</td>
<td></td>
</tr>
<tr>
<td>• Doesn’t like to be isolated in the classroom</td>
<td>Error Pattern III</td>
<td>• Doesn’t do well with multitask assignments</td>
<td>• Can’t stay focused for a long period of time</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Does not regroup in bases of 12 and 60</td>
<td>• Slow to complete assignments</td>
<td>• Doesn’t like to make mistakes</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Uses groups of 10 incorrectly</td>
<td>• Refuses to submit incomplete assignments</td>
<td>• Unable to connect his reading problems to his errors</td>
<td></td>
</tr>
</tbody>
</table>

*Note: The + symbols indicate strengths and the – symbols indicate areas of concern.*
**Proportional Materials:** In contrast, proportional materials are specifically designed so that one object in the collection is proportionally 10 times the size of another. Place value blocks are proportional materials. A set of these includes unit cubes that represent the ones place, longs that are actually 10 times the length of one unit cube, flats that represent the size of 10 longs, and a block that is the size of 10 hundreds flats stacked in a column. The place value materials are shown below.

![Diagram of place value blocks](image)

Having Colin manipulate materials and cooperate in a group setting will build on his strengths, according to his DAS. Activities should incorporate those characteristics and should progress from using hands-on manipulatives to representing the objects with drawings and numerals. Colin first collects groups of 10 ones, then trades them for longs (groups of 10), and continues in this manner with 10 longs traded for a hundreds block, and so on. The overall plan for Colin will be to provide opportunities to develop a strong conceptual foundation of place value and number sense by using materials to express amounts in real-world problems posed to him.

**Remediation:** The following remediation activities, beginning with the use of hands-on materials, are described in this section and listed in the MIP in Table 2.2.
### Table 2.2: Mathematics Improvement Plan I for Colin: Writing Multiplace Numerals Using Place Value Notation

<table>
<thead>
<tr>
<th>Time</th>
<th>30 Minutes</th>
<th>20–30 Minutes</th>
<th>15–20 Minutes</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Context</strong></td>
<td>Classroom activities of a cooperative learning style, Colin will work with four of his peers on a group activity (+)</td>
<td>Independent seatwork (+)</td>
<td>Colin will work in the classroom, with a classmate whom he likes, on this activity (+)</td>
</tr>
<tr>
<td><strong>Content</strong></td>
<td>Trades nonproportional and proportional blocks of groups of 10 to answer authentic, real-world questions (+)</td>
<td>Independently rolls cubes to fill place value chart reflecting solutions to real-world problems (+)</td>
<td>Works with a student on “Reach the Target” and placing index cards on multi-digit numerals to express them correctly (+)</td>
</tr>
<tr>
<td><strong>Process</strong></td>
<td><strong>Input</strong></td>
<td>Teacher gives multiple instructions for the task at hand (−)</td>
<td>The work has visual and written examples of the task at hand (+)</td>
</tr>
<tr>
<td></td>
<td><strong>Output</strong></td>
<td>Group engages in manipulative activities that require a concrete outcome (+)</td>
<td>Is expected to write his results (+)</td>
</tr>
<tr>
<td><strong>Behavior</strong></td>
<td><strong>Academic</strong></td>
<td>Group produces a single written project, and all students have input (+)</td>
<td>Completes work for teacher feedback (+)</td>
</tr>
<tr>
<td></td>
<td><strong>Social</strong></td>
<td>Direct group to work well together and have a sense of responsibility (+)</td>
<td>Could be easily frustrated (−)</td>
</tr>
<tr>
<td><strong>Reinforcement</strong></td>
<td>If the group does well, members will get free time to play games with a partner (+)</td>
<td>For every problem he gets right he will get to help the teacher with a needed classroom task (+)</td>
<td>Teacher gives oral and written comments (+)</td>
</tr>
</tbody>
</table>

Note: The + symbols indicate strengths and the – symbols indicate areas of concern.

**Introductory Activity**: The teacher should begin with two-digit numbers in real-world problems to solve, such as:

Our school is holding a physical fitness day for all students. Our class is supposed to tell the principal how many of us can run one mile in less than a half hour. How can I use the place value chart to help show the number of students who do that, and also write the correct number for the school office and the awards ceremony?

Colin is given 27 single Popsicle sticks and the place value chart. He places all 27 in the ones column of the chart, because each represents the quantity of “one,” a single unit. Colin is encouraged to then bundle the sticks in groups of 10 to help him visualize what groups of 10 look like.
He should also report situations in which he notices groups of 10 in various authentic, real-world settings; including the fact that he has 10 fingers and that the U.S. currency system uses base-10 groups of coins and paper money.

Colin counts out 10 sticks, puts a rubber band around that group, and places it in the tens section of the place value chart. He sees he made one group of 10 from the 27 with which he began. Colin continues counting by repeating to count and group by 10s and placing the bundle in the 10s column. Leftover single sticks (7) are placed in the ones column.

To record quantity, Colin writes the number of bundles of 10 he sees in the tens column slot and the number of single sticks he has in the ones column area of the place value chart. It would then look like this:

<table>
<thead>
<tr>
<th>Tens</th>
<th>Ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>⬃ ⬃ ⬃ ⬃ ⬃ ⬃ ⬃</td>
<td>⬃ ⬃ ⬃</td>
</tr>
</tbody>
</table>

Record: 2    Record: 7

Colin reads the number or is helped by the teacher to say “27.” The teacher asks Colin why 20 is not written as a “2” and a “0” in the tens column and whether or not the “2” in the tens place represents both digits in the numeral “20.”

Using Place Value Blocks: Working with proportional materials, Colin lines up 10 units next to a base-10 long. He gently glides 10 units aside, using the long, so that he has the 1 long in front of him. He writes, in his math journal, “1 ten = 10 ones.” He can illustrate the base-10 materials as well to help him remember what that equation means. Colin then trades 10 of the longs for a hundred flat by measuring 10 longs next to the flat. Because the flat and 10 longs take up the same amount of space, he records that “1 hundred = 10 tens,” and draws a diagram by tracing them in his journal to show the equivalence:

Long

Concept/Skill-Building Games: Colin plays the following “Reach the Target” game with the class and teacher to continue with the trading principle using proportional materials. A target number of 27, for example, is announced by the leader and a wooden number cube marked with digits 0 to 5 is rolled. Working with another student to build confidence, Colin and his partner place the number of ones units in the ones column according to the number called. If “3” were called, they would place three single units in the ones column. When 10 units are eventually placed in the ones column by any of the players, they trade them for one rod to be placed in the tens column. The first group that has gathered two longs in the tens column and seven units in the ones column, for example,
by virtue of filling the ones column and trading a long for 10 ones, wins. The chart would look as follows:

<table>
<thead>
<tr>
<th>Tens</th>
<th>Ones</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The winning team must record the value of “27” on the board, explaining to the class why “2” in the tens column represents two groups of 10 and the “7” represents 7 ones. The game continues for players to reach additional targets such as 37, 46, and so on.

Target numbers from 11 to 19 should be avoided until several two-digit examples are used. “Fourteen” and “eleven” are unique numeral words, which do not easily lend themselves to the place value system for two-digit numbers. “Twenty-seven” is more clearly associated with the quantity it represents (2 tens and 7 ones). In fact, the teens numbers are expressed in reverse to the base-10 system. Instead of naming the tens digit first, as in “twenty-seven,” we say “nineteen,” in which case the ones digit is expressed first.

Continue this activity of trading for groups of 10 by extending the place value chart to three places. Students label the chart as before, but include an additional place for the hundreds column.

<table>
<thead>
<tr>
<th>Hundreds</th>
<th>Tens</th>
<th>Ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>Record:</td>
<td>Record:</td>
<td>Record:</td>
</tr>
</tbody>
</table>

To reach a target number of 352, for instance, number cubes are again rolled for each team. Players place the indicated number of markers in the ones column, as before. When 10 units are gathered in the ones column, they are traded for one long. When students find they have 10 longs in the center column, the group of 10 tens is traded for a flat. The teacher should pause in the activity, at several points, so that students can report the quantity they have so far grouped and record it. In this way, Colin has to write numbers such as “239” with a numeral in each place, rather than writing each place value separately, as in “200309.” Words should also be written and attached to the numerals to form the association of materials to symbols of numerals and words. It is important to record numerals with materials to establish conceptual connections.

**Writing Numbers Correctly:** Colin writes digits from 0 to 9 on a set of 10 blank index cards. An incorrect response from Student Work Sample 1, such as “507” instead of “57,” is written on the board or a paper. The number “57” is read aloud to him. Colin now knows that 50 is represented by a 5 in the tens place,
and he is asked where to put the card with the “5” on top of the “507” so the “5”
can represent a group of tens. Colin should cover the “5” and the “0” with the “5”
card, and then place the “7” card on top of the “7.” In this way, Colin uses the
numeral as it should be recorded so that each digit represents its correct place.
Another example from Colin’s Student Work Sample I is as follows:

Place Value Error Pattern II for Colin:
Order and Equivalence of Multi-Digit Numerals

The second type of place value error is found in Student Work Sample II for
Colin. Colin was asked to list quantities in order when presented with a choice
of two- or three-digit numbers.

Diagnosing the Error: Examine the mistake Colin is making but also consider his
conceptual and procedural strengths. These might include his ability to read
two-digit numbers, his understanding of the value of single-digit numbers, and
how he writes his numbers. You can list additional information in the follow-
ing box.

In terms of mathematical error patterns, Colin lacks conceptual under-
standing of the value of each digit, as determined by its place. He reads a
numeral from left to right or right to left, as if the positions of each numeral
were interchangeable if the digits in the numbers are identical. He thinks that
13 = 31. To Colin, the value of the 3 in the ones place equals the value of the
3 in the tens place. He also believes that the last digit determines the value of
a number, reading from left to right, when digits are not identical. Information
for this type of error is included in the DAS in Table 2.1.

Prescription: Colin begins with proportional base-10 blocks after complet-
ing the activities described in the previous section. If he had not done so,
he would begin the following activities with nonproportional materials. The
instructional goal is that Colin understands the meaning of each place
in a multi-digit numeral and the total value of the numeral. He needs to
STUDENT WORK SAMPLE II FOR COLIN

Write the correct answer below each question:

1. Which is larger? 13 or 31?
   \[ \text{Thay or equal} \]
   \[ 31 \]

2. Which is larger? 41 or 39
   \[ 41 \]

3. Which is larger? 543 or 215
   \[ 543 \]

4. Which is larger? 205 or 502
   \[ \text{Thay or equal} \]

5. Which is smaller? 56 or 35?
   \[ 35 \]

6. Which is smaller? 84 or 91
   \[ 84 \]

understand why one numeral represents a larger quantity than another or whether they are equal.

Remediation: The MIP in Table 2.3 briefly describes the types of remediation activities most beneficial for Colin. The teacher consults Colin’s DAS and, in light of Colin’s learning style, behavior style, and type of place value error, selects activities to address the mathematical error pattern. The teacher presents an authentic situation, such as the following.

My sister and brother are selling their comic books. Sabrina sold 25 and the other sister sold 52 books. William has been assigned to write a story about his family for the school newspaper. He wants to tell readers which sister sold the most books. Let’s help him know which way to record the numerals so that his story is correct.
Assuming Colin can use the base-10 blocks and the place value chart, he models "25" as shown:

```
Tens      Ones
|   |   |   |   |   |   |
```

[Record: 2]

Colin records a "2" beneath the base-10 rods in the tens column and a "5" in the ones column. He reads the numeral aloud.

The next step is to express "52" so that Colin can compare that number to "25." He counts 52 units and trades five groups of 10 for five longs. He places them on the place value chart. A "5" is written below the tens column on the chart. A "2" is recorded in the section below the ones column. Colin writes "52" and reads it aloud. He sees that 52 represents more tens, and therefore is larger than 25. He should continue with additional two-digit numerals.

**Playing the "Shape Game":** To extend the activity to three-digit numerals, Colin plays the "Reach the Target" game described earlier in this chapter. In a
more procedurally or practice-based activity. “The Shape Game,” Colin starts by drawing three different shapes in this way:

Each player’s goal is to record the highest or lowest digit number in the class. To begin, the class is told what numbers are written on a wooden cube. Students then decide if the winning combination should be the highest or lowest number made possible with those digits. For example, the teacher/leader indicates that the cube is marked with digits 3 to 8 and the class decides the winning number should be the highest. The cube is rolled and the number seen on the cube is called out. Students write it in one of the three shapes on their paper, trying to form a high three-digit number. The cube is rolled a second time, a digit is written in one of the other shapes, and this procedure is repeated a third time to fill the third shape. Once a number is written in a shape, it cannot be moved to another shape. If numbers 4, 7, and then 5 were rolled, the winners would have written “754” as:

The class should discuss why the winner wrote “7” in the hundreds place rather than the ones place. If the lowest possible number were to be the goal, students would have wanted to record “457.” Chance and skill are both important factors in this game.

This game helps Colin remember the value of each place because he must carefully consider the values to make his number placement decisions.

**Place Value Error Pattern III for Colin:**

**Computing in Bases 60 and 12**

The third type of place value error is found in Student Work Sample III for Colin. Colin was asked to subtract hours and minutes and feet and inches. These measurement units are built on base-60 and base-12 systems, respectively, and represent the first exercises in which Colin is working with bases other than 10.

**Diagnosing the Error:** Recognizing that Colin is not working with a base-10 system, think of strengths revealed in his work. Consider the subtraction algorithm and misconceptions that are understandable in light of previous learning.

Colin’s other mathematical strengths and error patterns can be recorded in the following chart.

<table>
<thead>
<tr>
<th>Colin’s Error Patterns(s):</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Colin’s Strengths:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>
1. Express 5 hours and 20 minutes as 4 hours and ______ minutes.

2. How much time, in hours and minutes, is between 2:30 and 4:00?

3. Express 8 hours and 10 minutes as 7 hours and ______ minutes.

4. Express 4 feet and 7 inches as 3 feet and ______ inches.

5. Express 2 feet and 6 inches as 1 foot and ______ inches.

6. Express 3 feet and 10 inches as 2 feet and ______ inches.

---

1. \[ 5 + 20 = 70 \quad 4 + 30 = 70 \]

2. \[ \begin{array}{c} 400 \text{ subtracted} \ 400 \text{ and} \ 230. \\ \hline \ 230 \end{array} \]

3. \[ 7 \text{ is one less than} \ 8 \text{ so I added} \ 10 \text{ to} \ 10 \text{ to make up.} \]

4. \[ 17 \text{ I added} \ 10. \]

5. \[ 16 \text{ I added} \ 10. \]

6. \[ 20 \text{ I added} \ 10 \text{ to make up because} \ 2 \text{ is less than} \ 3. \]
It is evident that Colin understands that renaming the numerals is required to subtract. He knows he must use some type of exchange from one place to another. That is, hours become a different number of hours, and minutes become a different number of minutes; feet become a renamed amount of feet, and inches become a renamed amount of inches. However, he is using 10 as the base number for each renaming of hours to minutes and feet to inches, rather than 60 and 12, respectively. There is no conceptual understanding that different bases, other than 10, exist. Hence, the answers are not reasonable.

An example of a completed DAS for this type of error was shown in Table 2.1.

**Prescription:** The error is both conceptual and procedural. Although Colin does understand that he should rename numbers, he does not understand how to use or record regrouping principles in non-base-10 place value systems. Working with manipulatives in the context of real-life problems associated with the computation will allow him to conceptualize reasonable answers.

**Remediation:** The MIP in Table 2.4 contains information related to this specific error. The behaviors and learning styles described for Colin should be taken into consideration when adapting the following activities. They are designed to assist Colin in working with algorithms in non-base-10 units.

**Clock Activities:** Colin begins with handmade clocks and story problems for “time” problems. The clocks are made with paper plates and handmade paper

| TABLE 2.4 Math Improvement Plan III for Colin: Understanding Bases Other than Base 10 |
|---------------------------------|----------------------------------|----------------------------------|
| **Time** | **30–35 Minutes** | **20–25 Minutes in Pairs** | **20 Minutes** |
| **Context** | Colin works with one of his peers during a group activity (+) | Works in groups of four to solve measurement problems (−) | Colin will work in the classroom, with a classmate whom he likes, on this activity (+) |
| **Content** | Uses clocks to solve real-world problems by counting minutes and hours (+) | Completes measurement problems and algorithms not based in groups of 10s (i.e., 60s, 12s, 16s) (+) | Independent work creating new problems for partner to solve involving measurement and time situations (+) |
| **Process** | Teacher provides story for students to interpret (−) | The work has visual and written examples of the task at hand (+) | The written directions are at grade level and have a singular task (+) |
| **Output** | Group is engaged in manipulative activities that require a reasonable outcome (+) | Groups are expected to write results and share them with the class (−) | Time and measurement terms are discussed and explained (−) |
| **Behavior** | Pairs solve real-world time problems (+) | Groups physically move around class while making explanations (−) | Ask questions to complete tasks (−) |
| **Social** | Pairs work well together and all students have a sense of their responsibility (+) | Colin asks question about how they got their answers (−) | His classmate is one whom he likes and trusts (+) |
| **Reinforcement** | If each pair does well, they will get free time to play games as partners (+) | For every problem Colin gets right, he will get to help the teacher with a needed task (+) | Teacher and other students give oral and written comments (−) |

**Note:** The + symbols indicate strengths and the − symbols indicate areas of concern.
arrows, attached with a paper fastener, for the hands. The clock looks like this one:

![Clock Image]

Colin is told that his friends and he are in a sports program after school. The activity begins at 3:00, and lasts for 2 hours and 15 minutes. Forty-five minutes of that time is devoted to snacks and breaks. Colin is supposed to find the amount of time that is actually spent in playing the game.

Colin works well with others, and so taking turns in a group, he places the hands at 3:00 on his paper plate clock. He moves the hands around to show 4:00, and that 1 hour has passed, and keeps track of the minutes by counting to 60 (counting by 5s). He records in his mathematics journal that:

\[
60 \text{ minutes} = 1 \text{ hour}
\]

Colin counts another 60 minutes to find that the hands now show 5:00. Moving the hands in this way helps Colin know that 2 hours have passed.

Because Colin also knows that 45 minutes were not devoted to sports, he wants to subtract 45 from 2 hours and 15 minutes to determine the actual playing time. To move to the standard algorithm, Colin records

\[
\begin{align*}
2 \text{ hours} & \quad 15 \text{ minutes} \\
- & \quad 45 \text{ minutes}
\end{align*}
\]

and asks how to complete the algorithm. Because there are not enough minutes from which to subtract 45, Colin has to regroup minutes from the 2 hours. He changes 2 hours to 1 hour so that he can use one of the two hours for regrouping. If necessary, he is reminded that hours are counted in 60-minute groups, just as he did with his clock. The resulting computation looks like this:

\[
\begin{align*}
1 \text{ hour} & \quad 60 + 15 = 75 \text{ minutes} \\
2 \text{ hours} & \quad 45 \text{ minutes} \\
\hline
1 \text{ hour} & \quad 30 \text{ minutes}
\end{align*}
\]

This activity should be followed by a situation that does not require renaming, so that students do not assume all subtraction of time problems, or any type, require renaming the digits.

**Measurement Activities:** Renaming units in the customary system or “inch-pound” system is more complex than doing so in the metric system, which is built on the base-10 system. The former is structured so that feet are converted to inches using base 12, yards to feet in base 3, and pounds to ounces employing the base of 16. All exchanges use different bases to recall and compute. Activities that begin with direct measurement experience are essential to providing a conceptual foundation for skills and reasonable answers.

Colin works in a small group and is given a ruler or tape with which to measure items in the classroom. Students are to find and measure three objects...
in the classroom that are between 2 and 6 feet tall or wide. It is important to remind students that the end of the ruler and the end of the item must match at the “0” point to ensure accuracy. The following is an example:

![Ruler Image]

The height or width of each is recorded in the following chart:

<table>
<thead>
<tr>
<th>Item</th>
<th>Measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ________________</td>
<td>1. _____ feet _____ inches</td>
</tr>
<tr>
<td>2. ________________</td>
<td>2. _____ feet _____ inches</td>
</tr>
<tr>
<td>3. ________________</td>
<td>3. _____ feet _____ inches</td>
</tr>
</tbody>
</table>

Groups report their findings to the class so that it can be determined how much taller or wider one group’s objects are than another. For instance, one group reports that their bookcase is 5 ft, 3 in. wide. Another group reports that the case they measured is 4 ft, 9 in. wide. The class determines how to find which is wider in this way:

\[
\begin{align*}
5 & \text{ feet} & 3 & \text{ inches} \\
-4 & \text{ feet} & 9 & \text{ inches}
\end{align*}
\]

Because Colin already counted inches by 12s to measure the length of the object, he can regroup to exchange 12 in. for 1 ft. He records that trade as:

\[
\begin{align*}
4 & \text{ feet} & 12 + 3 = 15 & \text{ inches} \\
5 & \text{ feet} & 9 & \text{ inches} \\
-4 & \text{ feet} & 6 & \text{ inches}
\end{align*}
\]

Six inches was the result from counting. The computational result is confirmed from actual experience.

This type of measurement lesson/activity can be extended with situations involving the comparison of objects measured in pounds and ounces, because the base of 16 ounces is also confusing to children. Items can be weighed on classroom scales; the difference in measurement is then calculated, following the actual counting phase of the lesson. Students could bring labels to school to compare measurements and create many interesting, real-life problems for the class to solve. A student working alone can also complete these activities.
Conclusions: Understanding and Using Place Value Concepts and Skills

Conceptual understanding of place value is possible when lessons are designed in a developmental learning sequence as follows:

1. Students manipulate objects and draw related diagrams to visually express multi-digit numerals.
2. Numeric symbols are recorded to reflect quantities expressed with materials and diagrams.
3. Number names are connected to all activities in both steps “1” and “2.”

The trading aspect of learning about place value is essential to conceptual development. Students bundle objects, exchange place value blocks, and indicate the trades on place value charts. Finally, results are recorded with numerals and words. Skills can be practiced and sustained by providing students with frequent and targeted instructional feedback. The cycle is interactive in that students are able to understand and express quantities with materials, numerals, and/or words within place value systems.

<table>
<thead>
<tr>
<th>1. Materials, pictorial representations</th>
<th>2. Numerals</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. Words</td>
<td>--------------</td>
</tr>
</tbody>
</table>

Understanding place value systems, whether base 10 or non-base 10, is fundamental to computing and number sense. Remediation should be based on determining whether students’ errors are based upon conceptual or rule misunderstandings. If the former is true, bundling objects can assist students in creating groups of 10, trading them, and recording the trades in diagrams and with numerals. Questions should be framed in real-world situations and inquiry to help students personally relate to the mathematics content. If students’ errors are the results of forgetting rules, activities that focus on paper-and-pencil games can be very helpful. Examples of these useful instructional strategies are found in the following activities. Each activity is effective for work in a small group, with the whole class, or for one-on-one interaction.

### Instructional Activities

#### ACTIVITY Making 10s Treats

**Objective:** Forming groups of 10 and counting by 10s

**Materials:**
- Small paper bags, each marked “Ten”
- Pieces of cereal such as Cheerios or Fruit Loops
- Play money dimes (optional)
- Dice (regular or wooden cubes marked with numerals)
**Directions:** Tell the children that they are going to make bags of treats to sell. Taking turns, each rolls the dice and takes that number of cereal pieces. When a child has 10 pieces, they are put in a bag marked “Ten.” Each bag of 10 may be exchanged with the storekeeper for a dime. After all the children have several turns, ask each of the students to tell, and write down, how many “cereal 10s” they made.

---

**ACTIVITY** Order Game

**Objective:** Writing two- and three-digit numbers in order

**Materials:**
- Two number cubes, each marked with the digits 0 to 5
- Two number cubes, each marked with the digits 4 to 9
- Playing chart

**Directions:**
1. Students use their own charts.
2. For a two-digit game, students roll any three of the cubes. For a three-digit game, students roll all four of the cubes.
3. Player selects any two numbers rolled to record a two-digit number or combines any three cubes to record a three-digit number. In either case, players must choose the lowest two- or three-digit number they see for their turn.
4. For each turn, the cubes are rolled again and the student forms a two- or three-digit number greater than the last number he or she wrote, if possible. The new number is written in the next space.
5. Once a player cannot make a greater two- or three-digit number, he or she loses.
6. The first player to fill in all the boxes is a winner.

**Variation:** Five cubes could be rolled to play a four-digit Start game.

---

**Example**

1. Start Box
   42
2. 45
3. 57
4. 63
**ACTIVITY** Right Place

**Objective:** Identify place values for large numbers

**Materials:**
- Paper, pencil
- One set of index cards marked 1 to 9, with the word “ones” written on each one
- One set of index cards marked 1 to 9, with the word “tens” written on each one
- One set of index cards marked 1 to 9, with the word “hundreds” written on each one
- One set of index cards marked 1 to 9, with the word “thousands” written on each one

**Directions:**
1. Students record any four-digit number, or any number of digits requested. Example “3842.”
2. Playing cards are shuffled.
3. The leader calls out the numeral and the place value marked on the cards as drawn. For example, the leader says, “8 in the hundreds place.”
4. Players circle that numeral and place if recorded.
5. Numerals and places are called until a player has circled all of his/her digits. That student calls out “Right place!” and wins the game after the digits are checked against the cards drawn by the leader.
6. The game can continue until more players win, or the game can be restarted.
Discussion Questions

1. Why are only 10 symbols needed to express any value in a base-10 place value system?

2. The principle of multiplying each numeral by the base group determines the place value of a number. For example, 589 represents 5 hundreds, 8 tens, and 9 ones. Discuss and explain the next step one takes to find the total value of the numeral. What might be a real-world problem you could pose to help students understand the need to learn about place value?

3. Why are the bills and coins of U.S. currency not examples of proportional manipulatives? How might you help students understand their value using some type of hands-on teaching approach?

4. Provide two examples of proportional materials and two examples of non-proportional materials that help students model the base-10 positional system. Describe why you chose these manipulatives and how they can be effective in helping students build conceptual understanding of place value.

5. In a positional or place value system, why is it not necessary to notate 35 as 305? Explain your reasoning.

6. Why would a student make the mistake of renaming 8 feet 9 inches as 7 feet and 19 inches? Why would this error seem reasonable to a student?

7. Many errors are understandable. For example, students will count, “…twenty-nine, twenty-ten…” What techniques from your reading of the chapter might help them understand and name the "bridge" numbers correctly?

8. Why is using a positional system more efficient for expressing numerals and computing than a nonpositional system such as Roman numerals or Egyptian hieroglyphics?

References


Chapter 2


**Literature for Teaching Place Value**


