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Teaching Mathematics for Understanding

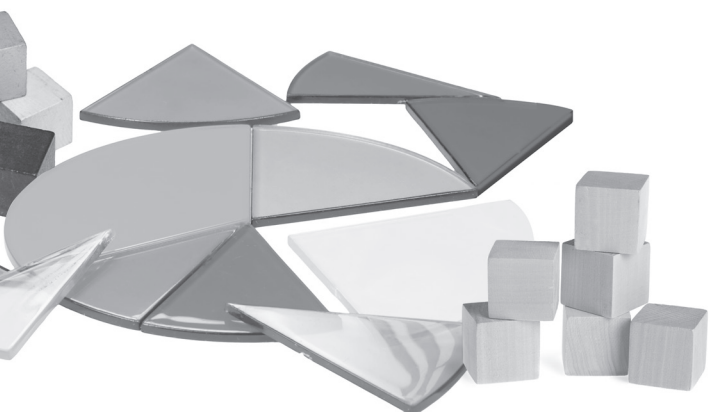
An understanding can never be "covered" if it is to be understood.

Wiggins and McTighe (2005, p. 229)

Teachers generally agree that teaching for understanding is a good thing. But this statement begs the question: What is *understanding?* Understanding is being able to think and act flexibly with a topic or concept. It goes beyond knowing; it is more than a collection of information, facts, or data. It is more than being able to follow steps in a procedure. One hallmark of mathematical understanding is a student's ability to justify why a given mathematical claim or answer is true or why a mathematical rule makes sense (CCSSO, 2010). Although students might *know* their multiplication basic facts and be able to give you quick answers to questions about these basic facts, they might not *understand* multiplication. They might not be able to justify the correctness of their answer or provide an example of when it would make sense to use this basic fact. These tasks go beyond simply knowing mathematical facts and procedures. Understanding must be a primary goal for all of the mathematics you teach.

Understanding and Doing Mathematics

Procedural proficiency, a main focus of mathematics instruction in the past, remains important today, but conceptual understanding is an equally important goal (CCSSO, 2010; National Research Council, 2001; NCTM, 2000). Numerous reports and standards emphasize the need to address skills and understanding in an integrated manner; among these are the *Common Core State Standards* (CCSSO, 2010), a state-led effort coordinated by the National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO) that has been adopted by nearly every state



and the District of Columbia. This effort has resulted in attention to *how* mathematics is taught, not just *what* is taught.

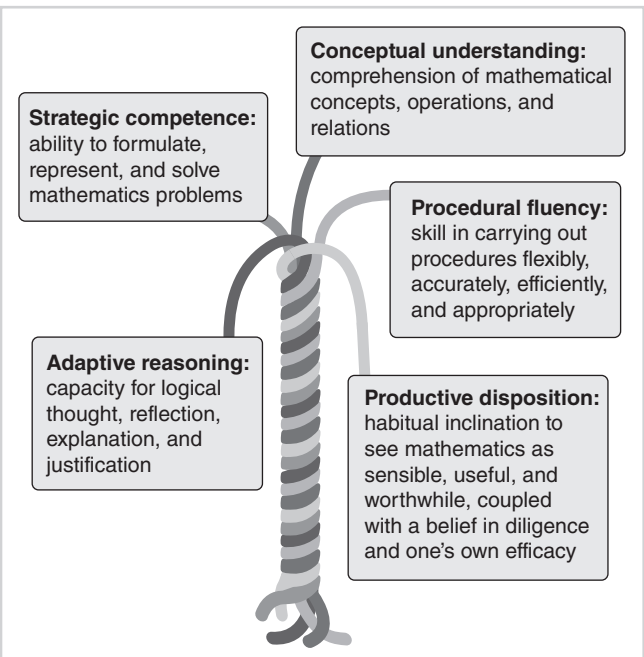
The National Council of Teachers of Mathematics (NCTM, 2000) identifies the process standards of problem solving, reasoning and proof, representation, communication, and connections as ways to think about how students should engage in learning mathematics content as they develop both procedural fluency and conceptual understanding. Students engaged in the process of *problem solving* build mathematical knowledge and understanding by grappling with and solving genuine problems as opposed to completing routine exercises. They use *reasoning and proof* to make sense of mathematical tasks and concepts and to develop, justify, and evaluate mathematical arguments and solutions. Students create and use *representations* (e.g., diagrams, graphs, symbols, and manipulatives) to reason through problems. They also engage in *communication* as they explain their ideas and reasoning verbally, in writing, and through representations. Students develop and use *connections* between mathematical ideas as they learn new mathematical concepts and procedures. They also build *connections* between mathematics and other disciplines by applying mathematics to real-world situations. By engaging in these processes, students learn mathematics by *doing* mathematics. Consequently, the process standards should not be taught separately from but in conjunction with mathematics as ways of learning mathematics.

Adding It Up (National Research Council, 2001), an influential research review on how students learn mathematics, identifies the following five strands of mathematical proficiency as indicators that someone understands (and can do) mathematics:

- ***Conceptual understanding:*** Comprehension of mathematical concepts, operations, and relations
- ***Procedural fluency:*** Skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- ***Strategic competence:*** Ability to formulate, represent, and solve mathematical problems
- ***Adaptive reasoning:*** Capacity for logical thought, reflection, explanation, and justification
- ***Productive disposition:*** Habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (Reprinted with permission from p. 116 of *Adding It Up: Helping Children Learn Mathematics*, 2001 by the National Academy of Sciences, Courtesy of the National Academies Press, Washington, D.C.)

Figure 1.1

Interrelated and intertwined strands of mathematical proficiency.



Source: Reprinted with permission from Kilpatrick, J., Swafford, J., & Findell, B. (Eds.), *Adding It Up: Helping Children Learn Mathematics.* Copyright 2001 by the National Academy of Sciences. Courtesy of the National Academies Press, Washington, D.C.

This report maintains that the strands of mathematical proficiency are interwoven and interdependent—that is, the development of one strand aids the development of others (Figure 1.1).

Building on the NCTM process standards and the five strands of mathematical proficiency, the *Common Core State Standards* (CCSSO, 2010) outline the following eight Standards for Mathematical Practice (see Appendix A) as ways in which students can develop and demonstrate a deep understanding of and capacity to do mathematics. Keep in mind that you, as a teacher, have a responsibility to help students develop these practices. Here we provide a brief discussion about each mathematical practice.

1. *Make sense of problems and persevere in solving them.* To make sense of problems, students need to learn how to analyze the given information, the parameters, and the relationships in a problem so that they can understand the situation and identify possible ways to solve it. One way to help students analyze problems is to have them create bar diagrams to make sense of the quantities and relationships involved. Once students learn various strategies for making sense of problems, encourage them to remain committed to solving them. As they learn to monitor and assess their progress and change course as needed, they will solve the problems they set out to solve!

2. *Reason abstractly and quantitatively.* This practice involves students reasoning with quantities and their relationships in problem situations. You can support students' development of this practice by helping them create representations that correspond to the meanings of the quantities and the units involved. When appropriate, students should also learn to represent and manipulate the situation symbolically. Encourage students to find connections between the abstract symbols and the representation that illustrates the quantities and their relationships. For example, when fourth graders draw a bar diagram showing one tree as being four times the height of another tree, encourage them to connect their representation to the expression $4 \times h$, where h is the height of the shorter tree. Ultimately, students should be able to move flexibly between symbols and other representations.

3. *Construct viable arguments and critique the reasoning of others.* This practice emphasizes the importance of students' using mathematical reasoning to justify their ideas and solutions, including being able to recognize and use counterexamples. Encourage students to examine each others' arguments to determine whether they make sense and to identify ways to clarify or improve the arguments. This practice emphasizes that mathematics is based on reasoning and should be examined in a community—not carried out in isolation. Tips for supporting students as they learn to justify their ideas can be found in Chapter 2.

4. *Model with mathematics.* This practice encourages students to use the mathematics they know to solve problems from everyday life. For third graders, this could mean writing a multiplication or division equation to represent a given situation or using their measurement sense to determine whether a rug advertised in the newspaper would fit in a designated location in their classroom. Be sure to encourage students to determine whether their mathematical results make sense in the context of the given situation.

5. *Use appropriate tools strategically.* Students should become familiar with a variety of problem-solving tools and they should learn to choose which ones are most appropriate for a given situation. For example, fifth graders should experience using the following tools for computation with decimals: base-ten manipulatives, decimal grids, pencil and paper, calculators, and number lines. Then, if these students are asked to find the sum of 3.45 and 2.9 and provide their reasoning, they could use base-ten manipulatives or decimal grids to illustrate the meaning of each decimal and how the decimals were combined.

Teaching Tip

Research suggests that students, in particular girls, may tend to continue to use the same tools because they feel comfortable with the tools and are afraid to take risks (Ambrose, 2002). Look for students who tend to use the same tool or strategy every time they work on a task. Encourage all students to take risks and to try new tools and strategies.

6. *Attend to precision.* In communicating ideas to others, it is imperative that students learn to be explicit about their reasoning. For example, they need to be clear about the meanings of operations and symbols they use, to indicate units involved in a problem, and to clearly label diagrams that accompany their explanations. As students share their ideas, emphasize this expectation and ask clarifying questions that help make the details of their reasoning more apparent. Teachers can further encourage students' attention to precision by introducing,

highlighting, and encouraging the use of accurate mathematical terminology in explanations and diagrams.

7. *Look for and make use of structure.* Students who look for and recognize a pattern or structure can experience a shift in their perspective or understanding. Therefore, set the expectation that students will look for patterns and structure and help them reflect on their significance. For example, help students notice that the order in which they multiply two numbers does not change the product—both 4×7 and 7×4 equal 28. Once students recognize this pattern in other examples, they will have a new understanding and use of a powerful property of our number system: the commutative property of multiplication.

8. *Look for and express regularity in repeated reasoning.* Encourage students to step back and reflect on any regularity that occurs in an effort to help them develop a general idea or method to identify shortcuts. For example, as students begin multiplying numbers, they will encounter situations in which a number is multiplied by 0. Over time, help them reflect on the results of multiplying any number by 0. Eventually they should be able to express that when any number is multiplied by 0, the product is always 0.

Like the process standards, the Standards for Mathematical Practice should not be taught separately from the mathematics. Instead, teachers should incorporate these practices as ways for students to learn and do mathematics. Students who learn to use these eight practices as they engage with mathematical concepts and skills have a greater chance of developing conceptual understanding. Note that learning these mathematical practices and developing understanding take time. So the common notion of simply and quickly "covering the material" is problematic. The opening quotation states it well: "An understanding can never be 'covered' if it is to be understood" (Wiggins & McTighe, 2005, p. 229). Understanding is an end goal—that is, it is developed over time by incorporating the process standards and mathematical practices and striving toward mathematical proficiency.

How Do Students Learn?

Let's look at a couple of research-based theories that can illustrate how students learn in general: constructivism and sociocultural theory. Although one theory focuses on the individual learner while the other emphasizes the social and cultural aspects of the classroom, these theories are not competing; they are actually compatible (Norton & D'Ambrosio, 2008).

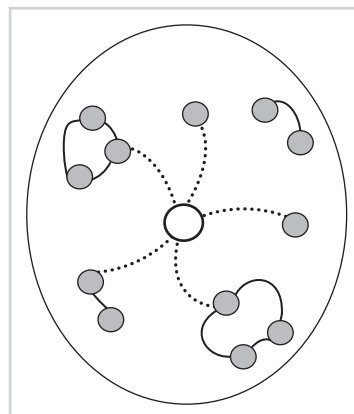
Constructivism

At the heart of constructivism is the notion that learners are not blank slates but rather creators (constructors) of their own learning. All people, all of the time, construct or give meaning to things they perceive or think about. Whether you are listening passively to a lecture or actively engaging in synthesizing findings in a project, your brain is applying prior knowledge (existing schemas) to make sense of the new information.

Constructing something in the physical world requires tools, materials, and effort. The tools you use to build understanding are your existing ideas and knowledge. Your materials might be things you see, hear, or touch, or they might be your own thoughts and ideas. The effort required to construct knowledge and understanding is reflective thought.

Through reflective thought, people connect existing idea to new information and thus modify their existing schemas or background knowledge to incorporate new ideas. Making these connections can happen in either of two ways—*assimilation* or *accommodation*. Assimilation occurs when a new concept "fits" with prior knowledge and the new information expands an existing mental network. Accommodation takes place when the new concept does not "fit" with the existing network, thus creating a cognitive conflict or state of confusion that causes what theorists call *disequilibrium*. As an example, some students assimilate fractions into their existing schemas for whole numbers. When they begin to compare fractions, they treat the numerators and denominators separately, as if they represented two whole numbers that have no relationship to each other. Such a student might mentally compare $\frac{2}{3}$ and $\frac{3}{4}$ by explaining that since $2 < 3$ (the numerators) and $3 < 4$ (the denominators), $\frac{2}{3} < \frac{3}{4}$. This student might initially confirm this erroneous thinking by using an area model that corresponds to a part-whole fraction to illustrate that $\frac{2}{3} < \frac{3}{4}$ because $\frac{2}{3}$ has less area than $\frac{3}{4}$. A teacher could then challenge this student to compare the fractions $\frac{1}{2}$ and $\frac{2}{5}$. This student's overgeneralization of whole-number ideas (e.g., $1 < 2$ and $2 < 5$, and so $\frac{1}{2} < \frac{2}{5}$) is then called into question when he or she approaches the area model illustrating that $\frac{1}{2} > \frac{2}{5}$. To settle the dissonance, the student eventually has to accommodate his or her schema for comparing fractions. It is through the struggle to resolve the disequilibrium that the brain modifies or replaces the existing schema so that the new concept fits and makes sense, resulting in a revision of thought and a deepening of the person's understanding.

Figure 1.2
How someone constructs a new idea.



For an illustration of what it means to construct an idea, consider Figure 1.2. The gray and white dots represent ideas, and the lines joining the ideas represent the logical connections or relationships that develop between ideas. The white dot is an emerging idea—one that is being constructed. Whatever existing ideas (gray dots) are used in the construction are connected to the new idea (white dot) because those are the ideas that give meaning to the new idea. The more existing ideas that are used to give meaning to the new one, the more connections will be made.

Each student's unique collection of ideas is connected in different ways. Some ideas are well understood and well formed, while others are less so. Students' prior experiences help them develop connections and ideas about whatever they are currently learning.

Understanding exists along a continuum (Figure 1.3), from an instrumental understanding—knowing something by rote or without meaning (Skemp, 1978)—to a relational understanding—knowing what to do and why. Instrumental understanding, at the left end of the continuum, shows that ideas (e.g., concepts and procedures) are learned, but in isolation (or nearly so) to other ideas. Here you find ideas that have been memorized. Due to their isolation, these often poorly understood ideas are easily forgotten and are unlikely to be useful for constructing new ideas. At the right end of the continuum is relational understanding. Relational understanding means that each new concept or procedure (white dot)

Figure 1.3 Continuum of understanding.

is not only learned, but is also connected to many existing ideas (gray dots), so there is a rich set of connections.

A primary goal of teaching for understanding is to help students develop a relational understanding of mathematical ideas. Because relational understanding develops over time and becomes more complex as a person makes more connections between ideas, teaching for this kind of understanding takes time and must be a goal of daily instruction.

Sociocultural Theory

Like constructivism, sociocultural theory not only positions the learner as actively engaged in seeking meaning during the learning process, but it also suggests that the learner can be assisted by working with others who are "more knowledgeable." According to sociocultural theory, every learner has a unique zone of proximal development, which is a range of knowledge that may be out of reach for the individual to learn alone, but is accessible if the learner has the support of peers or more knowledgeable others (Vygotsky, 1978). For example, when students are learning about perimeter and area, they do not necessarily recognize that two rectangles can have the same perimeter but different areas. A more knowledgeable person (a peer or teacher) will know that if students explore creating different rectangles that have the same perimeter, the examples they generate will suggest this relationship between a rectangle's perimeter and area.

The most effective learning for a given student occurs when the activities of the classroom lie within his or her zone of proximal development. Targeting that zone helps teachers provide students with the right amount of challenge while avoiding boredom on the one hand and anxiety on the other when the challenge is beyond the student's current capability. Consequently, classroom discussions based on students' own ideas and solutions are "foundational to children's learning" (Wood & Turner-Vorbeck, 2001, p. 186).

Teaching for Understanding

Teaching toward Relational Understanding

To explore the notion of understanding further, let's look into a learner-centered fourth-grade classroom. In learner-centered classrooms, teachers begin *where the students are*—with *the students*' ideas. Students are allowed to solve problems or to approach tasks in ways that make sense to them. They develop their understanding of mathematics because they are at the center of explaining, providing evidence or justification, finding or creating examples, generalizing, analyzing, making predictions, applying concepts, representing ideas in different ways, and articulating connections or relationships between the given topic and other ideas.

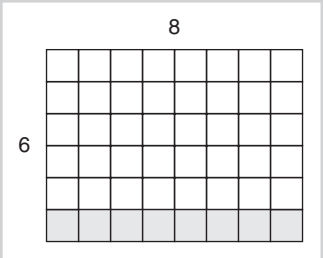
For example, in this fourth-grade classroom, students have already reviewed double-digit addition and subtraction computation and have been working on multiplication concepts and facts. They mastered most of their multiplication facts by the end of third grade but as part of an extension, the students have used contexts embedded in story problems. They are also illustrating how repeated addition can be related to the number of rows of square tiles within a rectangle. These students' combined experiences from grades 3 and 4 have resulted in a collection of ideas about tens and ones (from their work with double-digit addition and subtraction), an understanding of the meaning of multiplication as related to unitizing (i.e., a row of six as one six), a variety of number strategies for mastering multiplication facts based on the properties of the operation, and a connection between multiplication and arrays and area.

The teacher sets the following instructional objectives for her students: (1) Begin development of computational strategies for multiplication with multidigit numbers based on place value and the properties of operations. (2) Illustrate and explain multiplication calculations using equations and arrays and/or area models.

The lesson begins with a task that is designed to set the stage for the main part of the lesson. On a projector, the teacher shows a 6×8 rectangle made of square tiles. The bottom row of eight tiles is shaded to draw students' attention to it (see Figure 1.4). The students quickly agree that adding up 6 eights will tell how many squares are in the rectangle. The teacher asks, "But if we didn't remember that 6 rows of 8 is 48, could we slice the rectangle into two parts where we know the multiplication fact and use that to get the total mentally?" Students are given a few minutes to think of at least one way to slice the rectangle, to share the idea with a partner, and to prepare to share with the class. The students offer four ideas:

Figure 1.4

A 6 by 8 rectangle split into 5 by 8 and 1 by 8 rectangles to make it easier to find the number of square tiles.



- "We sliced one row off the bottom. The top part is 5 by 8, so 40 tiles. Forty plus the 8 tiles on the bottom row makes 48 tiles."
- "We cut the rectangle in half top to bottom. Each smaller rectangle is 6 by 4; 24 and 24 is 48." The teacher asks, "How did you add the two 24s?" One of the students from the group explained, "We used double 25 and took 2 off." A student from a different group noted, "You could also add 20 and 20 to get 40, and then add on 4 and 4 to get 48."
- "Our strategy was the same idea, but we sliced the rectangle in half the other way and got 3 times 8 or 24, and then we doubled it to get 48."
- "We used doubles. If you take 2 columns of 6, that's 12. Then double that will give you 4 columns, or 24. And then double 24 is 48."

The teacher passes out centimeter grid paper. On the board she sketches a large rectangle, labels the dimensions 8 and 24, and tells the students that she wants each of them to construct an 8- by 24-cm rectangle on the grid paper. She explains that the students' task is to figure out how many square tiles are in the rectangle without counting them. Instead, they are to slice the rectangle into two or more parts—like they did with the 6 by 8 example—and use the smaller parts to figure out how many tiles are in the entire rectangle. As is the norm in the class, the teacher expects the students to be prepared to explain their reasoning and to support it with words, numbers, and drawings.

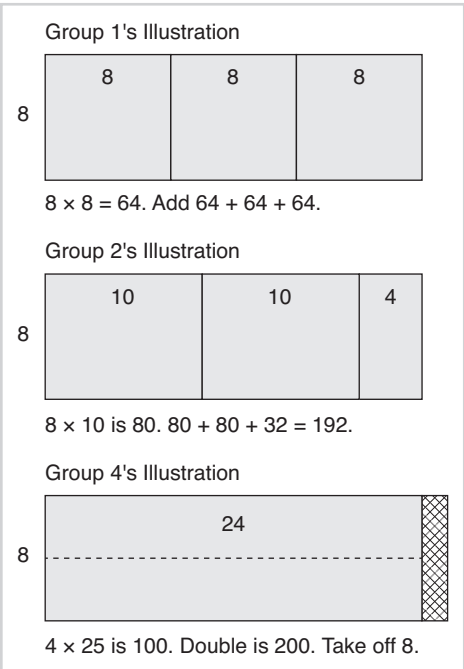
Stop and Reflect

Before reading further, solve this problem by finding two or more ways to slice the 8- by 24-cm rectangle into two or more parts. Draw a sketch for each way you can think of. Then check to see if your ways are alike or different from those that follow. ■

The students work in pairs for about 15 minutes. The teacher listens to different students talk about the task and offers a hint to a few students who are stuck: "What multiplication facts related to 8 do you know? How could you use those facts to slice the rectangle?" Soon the teacher begins a discussion by having students share their ideas and answers. As the students report, the teacher records their ideas on the board. Sometimes the teacher asks questions to help clarify ideas for others. She makes no evaluative comments, though she asks the students who are listening if they understand or have any questions to ask the presenters. The following solution strategies are common in classes where students are regularly asked to generate their own approaches. Figure 1.5 shows sketches for three of these methods.

Figure 1.5

Three different ways to slice a rectangle into smaller parts to make it easier to find its area.



- Group 1: "We know that 24, the top dimension, divided by 3 is 8, so we made three 8 by 8 squares. We know 8 times 8 is 64. We added 64 plus 64 plus 64." There is a brief discussion about how the students added the 64s mentally.
- Group 2: "We used groups of ten along the side that is 24 cm long. We sliced 2 sections of 10, and then there were 4 left at the end. Eight times 10 is 80. That makes 160 in the 2 big sections, and then the last section is 8 by 4. We added 160 and 32 in our heads."
- Group 3: "Our method was sort of like that, but we just used 8 times 20, which is 160. Then you add the 32 at the end."
- Group 4: "We didn't really slice the rectangle. Instead, we added an extra column of 8 at the end and made it 8 by 25. Then we knew that 4 rows of 25 is like a dollar, or 100. That makes 200 squares in all. But then we had to take off the 8 from the extra column that we added."

Stop and Reflect

What ideas did you learn from those shared in this example? Try using some of these ideas to find the product of 6 and 38. These numbers may make you think of a method not used in the task just discussed. ■

This vignette illustrates that when students are encouraged to solve a problem in their own way (using their own particular set of gray dots, or ideas), they are able to make sense of their solution strategies and explain their reasoning. This is evidence of their development of mathematical proficiency.

During the discussion periods in classes such as this one, ideas continue to grow. The students may hear and immediately understand a clever strategy shared by a classmate that they could have used but that did not occur to them. Others may begin to create new ideas to use that build from thinking about their classmates' strategies over multiple discussions. Some students in the class may hear excellent ideas from their peers that do not make sense to them. These students are simply not ready or do not have the prerequisite concepts (gray dots) to understand these new ideas. In future class sessions there will be similar opportunities for all students to grow at their own pace based on what they already understand.

Teaching toward Instrumental Understanding

In contrast to the lesson just described, in which students are developing concepts (understanding multidigit multiplication) and procedures (the ability to multiply flexibly) and seeing the relationships between these ideas, let's consider how a lesson with the same basic objective (multidigit multiplication) might look if the focus is on instrumental understanding.

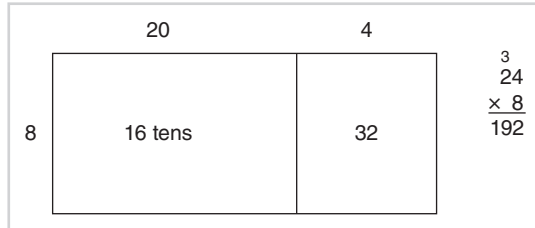
In this classroom, the teacher distributes centimeter grid paper and asks students to draw the 8- by 24-cm rectangle on their paper. On the board the teacher draws a rectangle and writes the multiplication problem 8×24 beside it. The teacher directs the students to count over to the right 20 squares and to draw a vertical line in the rectangle as she demonstrates the process on the board. Then the teacher uses a series of questions to guide

students through each step in the U.S. standard algorithm for multidigit multiplication. Students record the steps on their own paper at the same time.

- The teacher points to the small section of the rectangle and asks, "What is 8 times 4?"
- Students respond, "Thirty-two."
- The teacher notes, "We want to record the 32 in our problem." (She demonstrates how to write a 2 beneath the line in the problem and carry the 3. She also writes "32" in the small portion of the rectangle.)
- The teacher asks, "What is 8 times 2?" (Attention is directed to the 8 by 20 portion of the rectangle.)
- Students respond, "Sixteen."
- The teacher explains, "Because we are multiplying 8 by 20, we just add 0 to get 160 or 16 tens." (The teacher writes "16 tens" in the large portion of the rectangle.)
- The teacher continues the process: "We already have three tens. How much is 16 and 3?"
- Students respond, "Nineteen."
- The teacher continues the algorithm: "We record the 19 tens below the line. The final answer is 192." (See Figure 1.6.)

Figure 1.6

The U.S. standard algorithm is modeled using a rectangle partitioned into tens and ones.



Next, the students are given five similar multiplication problems. For each problem they are to sketch a small rectangle on their paper and show how it is partitioned into tens and ones. Then they record the two products in the rectangle and complete the computation on the side. The teacher circulates and helps students who are struggling by guiding them through the steps that were modeled in the first example.

In this lesson, the teacher and students use an area model on centimeter grid paper to illustrate the various partial products in the problem. After engaging in several similar lessons, most students are likely to remember, and possibly understand, how to multiply multidigit numbers by using the standard algorithm. Using an area model to illustrate the multiplication algorithm can build toward relational understanding; however, when the expectation is for all students to use one method, students do not have opportunities to apply other strategies that may help them build connections between multiplication and place value; multiplication and addition; or multiplication and estimation—connections that are fundamental characteristics of relational understanding. It is important to note that this lesson on the standard algorithm, in combination with other lessons that reinforce other approaches, *can* build a relational understanding, as it adds to students' repertoire of strategies. But if this lesson represents the sole approach to multiplying multidigit numbers, then students are more likely to develop an instrumental understanding of mathematics.

The Importance of Students' Ideas

Let's take a minute to compare these two classrooms. By examining them more closely, you can see several important differences. These differences affect what is learned and who learns. Let's consider the first difference: Who determines the procedure to use?

In the first classroom, each student looks at the numbers in the problem, thinks about the relationships between the numbers, and then chooses a computational strategy that is based on these ideas or a preference on the facts that the student knows. They are developing several different strategies to solve multiplication problems by exploring numbers (taking numbers apart and putting them together differently); using various representations, such as arrays and area models; and thinking about connections between addition

and multiplication. The students in the first classroom are being taught mathematics for understanding—relational understanding—and are developing the kinds of mathematical proficiency described earlier.

In the second classroom, the teacher provides one strategy for how to multiply—the standard algorithm. Although the standard algorithm is a valid strategy, the entire focus of the lesson is on the steps and procedures that the teacher has outlined. The teacher solicits no ideas from individual students about how to combine the numbers. She can only find out who has or has not been able to follow her directions. And even more problematic is that the teacher shares a commonly taught "rule" that does not always work: when you multiply by 10 you just "add" a zero. (For example, consider when you multiply 14.5×10.)

When students have more choice in determining which strategies to use, as in the first classroom, they can learn more content and make more connections. In addition, if teachers do not seek out and value students' ideas, students may come to believe that mathematics is a body of rules and procedures that are learned by waiting for the teacher to tell them what to do. This view of mathematics—and what is involved in learning it—is inconsistent with mathematics as a discipline and with the learning theories described previously. Therefore, it is a worthwhile goal to transform your classroom into a mathematical community of learners who interact with each other and with the teacher as they share ideas and results, compare and evaluate strategies, challenge results, determine the validity of answers, and negotiate ideas. The rich interaction in such a classroom increases opportunities for productive engagement and reflective thinking about relevant mathematical ideas, and students develop a relational understanding of mathematics.

A second difference between the two classrooms is the learning goals. Both teachers might write "understand multidigit multiplication" as the objective for the day. What is captured in the word *understand* is very different in each setting, however. In the first classroom, the teacher's goals are for students to connect multiplication to what they already know and to see that two numbers can be multiplied in many different ways. In the second classroom, understanding is connected to being able to carry out the standard algorithm supported by a singular approach using grid paper. The learning goals and, more specifically, how the teacher interprets the meaning behind the learning goals, impact what students learn.

These lessons also differ in terms of how accessible they are—and this, in turn, affects who learns the mathematics. The first lesson is differentiated in that it meets students where they are in their current understanding. When a task is presented as "solve this in your own way," it has multiple entry points, meaning it can be approached in a variety of ways, some more sophisticated than others. Consequently, students with several different levels of prior knowledge or learning strategies can figure out a way to solve the problem. This makes the task accessible to more learners. Then, as students observe strategies that are more efficient than their own, they develop new and better ways to solve the problem. This approach also requires that the students, rather than the teacher, do the *thinking*.

In the second classroom, everyone has to do the problem in the same way. The students do not have the opportunity to apply their own ideas or to see that there are numerous ways to solve the problem. This may deprive students who need to continue working on the development of basic ideas of tens and ones, as well as students who could easily find one or more ways to do the problem if only they were asked to do so. The students in the second classroom are also likely to use the same method to multiply all numbers instead of looking for more efficient ways to multiply numbers based on the relationships between numbers. For example, they are likely to multiply 4×51 using the standard algorithm instead of thinking, "That would be 4×50 and then 4 more." Recall the importance of building on prior knowledge and learning from others. In the first classroom, student-generated strategies, multiple approaches, and discussions about the problem represent the kinds of strategies that enhance learning for a range of learners.

Students in both classrooms will eventually succeed at finding products of multidigit numbers, but what they learn about multiplication—and about doing mathematics—is quite different. Understanding and doing mathematics involves generating strategies for solving problems, applying those approaches, seeing if they lead to solutions, and checking to see if answers make sense. These activities were all present in the first classroom but not in the second. Consequently, students in the first classroom, in addition to successfully finding products of multidigit numbers, will develop richer mathematical understanding, become more flexible thinkers and better problem solvers, remain more engaged in learning, and develop more positive attitudes toward learning mathematics.

Mathematics Classrooms That Promote Understanding

Three of the most common types of teaching are direct instruction, facilitative methods (also called a *constructivist approach*), and coaching (Wiggins & McTighe, 2005). With direct instruction, the teacher usually demonstrates or models, lectures, and asks questions that are convergent or closed-ended in nature. With facilitative methods, the teacher might use investigations and inquiry, cooperative learning, discussion, and questions that are more open-ended. In coaching, the teacher provides students with guided practice and feedback that highlights ways to improve their performances.

You might be wondering which type of teaching is most appropriate if the goal is to teach mathematics for understanding. Unfortunately, there is no definitive answer because there are times when it is appropriate to engage in each of these types of teaching, depending on the instructional goals, the learners, and the situation. Some people believe that all direct instruction is ineffective because it ignores the learners' ideas and removes the productive struggle or opportunity to learn. This is not necessarily true. A teacher who is striving to teach for understanding can share information via direct instruction as long as that information does not remove the need for students to reflect on and productively struggle with the situation at hand. In other words, regardless of instructional design, the teacher should not be doing the thinking, reasoning, and connection building; it must be the students who are engaged in these activities.

Regarding facilitative or constructivist methods, remember that constructivism is a theory of learning, not a theory of teaching. Constructivism helps explain how students learn—by developing and modifying ideas (schemas) and by making connections between these ideas. Students can learn as a result of different kinds of instruction. The instructional approach chosen should depend on the ideas and relationships students have already constructed. Sometimes students readily make connections by listening to a lecture (direct instruction). Sometimes they need time to investigate a situation so they can become aware of the different ideas at play and how those ideas relate to one another (facilitative). Sometimes they need to practice a skill and receive feedback on their performance to become more accurate (coaching). No matter which type of teaching is used, constructivism and sociocultural theories remind us as teachers to continually wonder whether our students have truly developed the given concept or skill, connecting it to what they already know. By shedding light on what and how our students understand, assessment can help us determine which teaching approach may be the most appropriate at a given time.

The essence of developing relational understanding is keeping students' ideas at the forefront of classroom activities by emphasizing the process standards, mathematical proficiencies, and the Standards for Mathematical Practice. This requires that the teacher create a classroom culture in which students can learn from one another. Consider the following features of a mathematics classroom that promotes understanding (Chapin, O'Conner, &

Anderson, 2009; Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier, & Human, 1997; Hoffman, Breyfogle, & Dressler, 2009). In particular, notice who is doing the thinking, the talking, and the mathematics: students.

Teaching Tip

Listen carefully to students as they talk about what they are thinking and doing as they engage in a mathematical task. If they respond in an unexpected way, try to avoid imposing your ideas onto their ideas. Ask clarifying questions to try to make sense of the sense your students are making.

- *Students' ideas are key.* Mathematical ideas, expressed by students, are important and have the potential to contribute to everyone's learning. Learning mathematics is about coming to understand the ideas of the mathematical community.
- *Opportunities for students to talk about mathematics are common*. Learning is enhanced when students are engaged with others who are working on the same ideas. Encouraging student-to-student dialogue can help students think of themselves as capable of making sense of mathematics. Students are also more likely to question each other's ideas than the teacher's ideas.
- *Multiple approaches are encouraged.* Students must recognize that there is often a variety of methods that will lead to a solution. Respect for the ideas shared by others is critical if real discussion is to take place.
- *Mistakes are good opportunities for learning.* Students must come to realize that errors provide opportunities for growth as they are uncovered and explained. Trust must be established with an understanding that it is okay to make mistakes. Without this trust, many ideas will never be shared.
- *Math makes sense.* Students must come to understand that mathematics makes sense. Teachers should resist always evaluating students' answers. In fact, when teachers routinely respond with "Yes, that's correct," or "No, that's wrong," students will stop trying to make sense of ideas in the classroom and discussion and learning will be curtailed.

To create a climate that encourages mathematics understanding, teachers must first provide explicit instruction on the ground rules for classroom discussions. Second, teachers may need to model the type of questioning and interaction that they expect from their students. Direct instruction would be appropriate in such a situation. The crucial point in teaching for understanding is to highlight and use students' ideas to promote mathematical proficiency.

Most people go into teaching because they want to help students learn. It is hard to think of allowing—much less planning for—the students in your room to struggle. Not showing them a solution when they are experiencing difficulty seems almost counterintuitive. If our goal is relational understanding, however, the struggle is part of the learning, and teaching becomes less about the teacher and more about what the students are doing and thinking.

Keep in mind that you too are a learner. Some ideas in this book may make more sense to you than others. Others may create dissonance for you. Embrace this feeling of disequilibrium and uneasiness as an opportunity to learn—to revise your perspectives on mathematics and on the teaching and learning of mathematics as you deepen your understanding so that you can help your students deepen theirs.

Stop and Reflect

Look back at the chapter and identify any ideas that make you uncomfortable or that challenge your current thinking about mathematics or about teaching and learning mathematics. Try to determine why these ideas challenge you or make you uncomfortable. Write these ideas down and revisit them later as you read and reflect further. ■