Preface

Learning Mathematics in Elementary and Middle Schools is organized around one guiding principle: understanding how children process math concepts. This sixth edition extends this principle a step further by also providing clear guidance on recognizing how this affects children’s ability to learn those concepts and what you can do to support the development of their understanding. This edition focuses on classroom practice. In addition to numerous videos of children and teachers and an innovative preassessment feature, it includes more than 60 artifacts of real children’s work to provide you a uniquely valuable opportunity to develop your teaching skills by connecting with the work of real children.

The Pearson eText for this title is an affordable, interactive version of the print text that includes videos, additional student work samples, and links to related content on the web. The play button appears where video is available, and hyperlinked words provide access to additional student work samples and other related websites.

To learn more about the enhanced Pearson eText, go to www.pearsonhighered.com/etextbooks.

New to This Edition

Users of earlier editions of Learning Mathematics in Elementary and Middle Schools will notice important revisions that reflect constant changes in mathematics education. Changes include:

- Integration of the Common Core State Standards (CCSS) for Mathematics, including using these standards in instructional planning
- Learning outcomes at the beginning of each chapter, which help focus readers’ attention to the key points of the chapter
- An updated learning theory chapter, which addresses the sociocultural approach and how to connect learning principles with the CCSS mathematical practices
- More than 60 integrated samples1 of real students’ work, illustrating concepts with real-world examples and providing an opportunity to examine students’ reasoning
- Videos, a Pearson eText feature, provide footage of students, teachers, and administrators discussing and applying chapter concepts, which offer the opportunity to watch students solve problems, listen to teacher interviews, and observe the real world of teaching

Major Themes

Teaching mathematics today is an exciting prospect, but it can be an overwhelming idea, as well. As a teacher, you need to be comfortable enough with your own mathematical understanding to teach these concepts to elementary and middle school students. You need to recognize what students learn about each math concept, how they think about solving problems, and how you can modify your teaching to adapt to your students’

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120 student work samples appear in the print version, and an additional 42 are available in the eText.
cognitive needs. You also need to be familiar with effective teaching methods, with how and when to use manipulatives, and with the best uses of technology in the classroom. Finally, in addition to all this, you must know how to align your teaching with the Common Core State Standards for Mathematics or the standards used in your state.

Daunting? That is why we have written *Learning Mathematics in Elementary and Middle Schools: A Learner-Centered Approach*—to ease your mind and give you a sound background in basic math concepts and the tools you need to help all your students grasp these concepts. By focusing on how students think and work through problem solving, this text prepares you to teach mathematics effectively so that all of your students are successful. This learner-centered approach places children squarely at the center of their own learning. To help your growth as an effective teacher of mathematics, this edition emphasizes four important themes throughout:

1. A focus on children’s thinking, including artifacts of student work and activities built around videos that explore student problem solving
2. Elements of assessment that are interwoven throughout each chapter, including conducting preassessments, analyzing assessments, and using the results of assessments to plan instruction, in order to help you further understand children’s thinking as they work through mathematical processes
3. A focus on classroom practice, including examining classroom practice through videos and vignettes and strategies to reach all learners, and developing effective lesson plans
4. Alignment with the CCSS for Mathematics

## Organization and Features

Chapters 1–4 provide base knowledge in understanding the theory behind math instruction and the CCSS that guide that instruction. Chapters 1–4 provide you with the understanding you need to create a learner-centered environment that draws children and adolescents into mathematics problems and guides them as they construct mathematical principles. Chapters 5–17 break down the key instructional concepts in mathematics, following a common structure and feature set:

- **A strong CCSS focus** runs through all chapters. In addition to the opening *Connecting with the Common Core State Standards* feature that discusses CCSS related to the chapter, a standards matrix near the beginning of each chapter clearly outlines the developmentally appropriate content standards covered. In addition, CCSS boxes throughout identify specific CCSS standards and align them with mathematical concepts addressed in the text.

### CCSS Box: 2–1

<table>
<thead>
<tr>
<th>CCSS Code</th>
<th>CCSS Description</th>
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<tbody>
<tr>
<td>2.NBt.1</td>
<td>Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases: a. 100 can be thought of as a bundle of ten tens—called a ‘hundred.’ b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).</td>
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<tr>
<td>2.NBt.2</td>
<td>Count within 1,000; skip-count by 5s, 10s, and 100s.</td>
</tr>
<tr>
<td>2.NBt.3</td>
<td>Read and write numbers to 1,000 using base-ten numerals, number names, and expanded form.</td>
</tr>
<tr>
<td>2.NBT.4</td>
<td>Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using &gt;, =, and &lt; symbols to record the results of comparisons. Represent addition and subtraction with objects, fingers, mental images, drawings, use place value understanding and properties of operations to add and subtract.</td>
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<tr>
<td>2.NBT.5</td>
<td>Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.</td>
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<tr>
<td>2.NBT.6</td>
<td>Add up to four two-digit numbers using strategies based on place value and properties of operations.</td>
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<tr>
<td>2.NBT.7</td>
<td>Add and subtract within 1,000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. Relate strategies to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.</td>
</tr>
<tr>
<td>2.NBT.8</td>
<td>Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.</td>
</tr>
<tr>
<td>2.NBT.9</td>
<td>Explain why addition and subtraction strategies work, using place value and the properties of operations. (CCSS, 2013, p. 18)</td>
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In order to become familiar with the mathematics concepts and procedures discussed in this chapter, take a few minutes and complete the following Chapter Assessment. Answer each question to the best of your ability and think about how you got the answer. Then think about how elements of the Phases of Children's Thinking about each problem and any possible misunderstandings some children may have had about the concepts. Then review your responses, identify areas where you need to improve or strengthen your understanding, and refine your approach to future assessments.

**Part 1: Addition Facts**

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**Part 2: Multiplication Facts**

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### Examining Classroom Practice

Sections focus on videos of real teachers, real children, and real classrooms learning mathematics concepts; these sections give you opportunities to examine children's thinking and the decisions teachers make in practice.

### Analyzing Children's Understanding

Sections examine real children's work and explore what their work says about their reasoning skills and math development.

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### Building on Children's Understanding

Sections identify steps teachers might take to extend children's thinking, based on the analysis of their understanding.

### A Wealth of Problem-Based Activities

Activities throughout each chapter help you understand and practice chapter content and implement effective classroom applications into your own instruction.

### Literature Links

Sections throughout the book serve as extension activities that connect mathematics and literacy and illustrate how to use trade books to develop mathematical ideas and concepts.

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**Activity 3-2: Double Sidedness**

**Materials:**

A set of 12 ten-frame cards to be used by each child, one for each set.

**Procedure:**

1. Ask children to find the ball that shares its largest neighboring number.
2. Place the ball with the largest neighboring number on the table and ask children to describe its size.

**Variation:**

Ask children to find the ball that is shaded on its side.

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**Literature Link 3-2: Anno’s Magic Seeds**


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A good story may place a mathematical problem in the context of familiar situations. The problem-solving context is much more convincing when the development of mathematical topics is considered as part of the story. The story of mathematics is overlaid on a story where it does not normally arise (or at least not as prominently). *Anno’s Magic Seeds* demonstrates how an interesting story and an engaging mathematics problem can create a wonderful mathematical situation.

- **Provide a variety of experiences in which children apply several different problem-solving strategies to make mathematical discoveries.** Have children recognize and select different strategies and reasons which strategies might work better for certain kinds of problems and why.
- **Develop a chart to record the way the number of golden beans in the story changes.** Help children to identify key elements of the mathematical situations, such as what year it is (year 1, 2, 3, or 4) and the number of seeds either come in or planted.

Note: this problem is set in a real-world context. The number of seeds follows a mathematical pattern until some seismic change (for example, Jack gets married and has a child, changing the number of seeds eaten each year, and there is a hurricane that wipes out most of the golden beans). These real-world occurrences make the mathematics in the story real, showing that real world problems are sometimes “magic.”

For another example of an interesting problem situation, read *Shoebox* from the Second Grade Grit: 60 Wks. to Frost the Name of Eldery by Norton Gasbrown.
• A Sample Lesson Plan at the end of each chapter models how to use preassessment results to plan CCSS-based instruction to meet children’s needs.

Sample Lesson

A Lesson on What Is More?

Grade level: Kindergarten
Materials: At least one manipulative, such as Unifix cubes, or assorted objects to count
Lesson objective: Students will use objects to correctly represent and compare whole numbers.
Standards link: K.CC.6: Identify whether the number of objects in one group is greater than, less than, or equal to the number as “7 is more than 5” (next), make a drawing of the two sets, write the numeral representing the number in each group, and either write a sentence stating which number is more than the other or circle the larger set.

Next ask two students to come to the front of the room. Each student should pick up a few cubes and hold them up. Then ask the class “Who has more, and how do you know?” Discuss ways to determine which is more. Then sketch the two sets, write the numeral representing the number in each group, as “7 is more than 5,” and circle the larger set.

Display two groups of objects, such as one group of 5 cubes and another group of 7 cubes. Ask students to make a one-to-one correspondence between objects in both sets; the larger set will have objects that don’t have a partner with objects in the other set. Then state what they’ve observed regarding the size of the groups, such as “7 is more than 5.” Have them repeat this process twice: (1) Each student selects a set of cubes, (2) both students circle the larger set and explain how they know, and (3) using the same sheet of paper, each student sketches their set, and together they determine which set is larger, and (3) using the same sheet of paper, each student sketches their set, and together they determine which set is larger.

After bringing together the whole class, select one pair of students to come to the front of their room and have them display their sets in front of the class, and ask the class “Who has more, and how do you know?” Discuss ways to determine which is more. Then sketch the two sets, write the numeral representing the number in each group, as “7 is more than 5,” and circle the larger set.

What might the next lesson focus on, and why? Consider any cautions regarding selecting the manipulatives to use in this lesson. Would other manipulatives work well? Might you want to avoid using certain manipulatives or objects? Why? What would be the advantages and/or disadvantages of each manipulative?

In Practice activities at the end of each chapter provide activities or assignments that apply the chapter’s key concepts to classroom practice. Often, these activities include opportunities to analyze additional student work samples (an eText feature).

Blackline Masters, included in the appendix, are book specific and provide templates for creating cardboard manipulatives and more.

Instructor Resources

The following supplements to the textbook are available for download under the “Educator” tab at www.pearsonhighered.com. Simply enter the author, title, or ISBN and then select this textbook. Click on the “Resources” tab to view and download the supplements detailed below.

Instructor’s Resource Manual and Test Bank

The Instructor’s Resource Manual and Test Bank (0-13-356259-X) provides concrete suggestions to promote interactive teaching and actively involve students in learning. Each chapter contains chapter learning outcomes, a discussion of key concepts, helpful instructional tips, and activities.

TestGen

The computerized test bank software, Test Gen (0133828816), allows instructors to create and customize exams for classroom testing and for other specialized delivery options, such as over a local area network or on the web. A test bank typically contains a large set of test items, organized by chapter and ready for your use in creating a test, based on the associated textbook material. The tests can be downloaded in the following formats:

• TestGen Testbank file—PC
• TestGen Testbank file—MAC
• TestGen Testbank—Blackboard 9
• TestGen Testbank—Blackboard CE/Vista (WebCT)
• Angel Test Bank
• D2L Test Bank
• Moodle Test Bank
• Sakai Test Bank
PowerPoint™ Presentations

Ideal for lecture presentations or student handouts, PowerPoint™ Presentations (ISBN 0-13-382882-4) for each chapter include key concept summaries.

Acknowledgments

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N. Bezuk
C. Cathcart
Y. Pothier
J. Vance
Learning Outcomes
When you have finished studying this chapter, you should be able to do the following:

- Describe some of the factors that influence the teaching of mathematics.
- Discuss some of the current directions in teaching mathematics.
- Explain how national and state mathematics standards influence classroom teaching.

How should mathematics be taught? What should children know about mathematics, and what should they be able to do? As a teacher, you are expected to know what mathematics should be taught to children and at what grade level. Even as a mathematics curriculum will mostly be provided to you through a school’s math textbooks or online programs, you still need to know what math concepts are typically taught at each grade level and what each of your students personally understands about mathematics.

Think about these expectations as you listen to Gretchen, a second-grader who has been asked to solve the problem $70 - 23$. In this clip, Gretchen solves the problem in three different ways and discusses her answers and solution strategies.

Gretchen begins by using the standard subtraction algorithm, and she gets 53 as the answer. When first presented with the problem, she comments, “That’s easy.” Then the interviewer asks her to use base-ten blocks, representing ones and tens, to solve the same problem. This time Gretchen gets 47 as the answer. She seems to notice that this is a different answer than the one she got with the algorithm, so she retraces her steps, both with the blocks and with the algorithm. When rechecking the algorithm, she says, “Zero take-away three, we know that’s three.” When asked by the interviewer which answer she thinks is right, Gretchen points to her first method, using the algorithm.

The interviewer then asks Gretchen if there is another way to solve the problem, asking her if she has ever used a hundreds chart. Gretchen uses a hundreds chart to solve the problem a third time, and once again she gets an answer of 47. As soon as she realizes that she got 47 again, she says, “But I don’t get it.” The interviewer then asks her what she got this time (47), what she got with the base-ten blocks (47), and what she thinks the answer should be, to which Gretchen responds, “Fifty-three.”

Gretchen justifies her answer of 53 by going back and repeating the steps she used originally. Then she says, “Forty-seven, umm, like couldn’t be right. Because . . . like it has to be fifty-three.”

What is the “bug” in Gretchen’s subtraction algorithm? (In other words, what does she do wrong when using the subtraction algorithm?) Why might she make this mistake?
She consistently subtracts 3 from 0 and gets the result of 3 rather than recognize that one way to solve this would be to regroup so that she has 10 ones. Then she can subtract 3, resulting in 7. This is a very common error children make.

Gretchen seems to put more faith in the procedures she tries to use than in her own understanding of the concept. Where or how might she have gotten the message that learning the procedure was more important than understanding the underlying concept? Why do you think Gretchen trusts the symbolic algorithm more than the other two methods she used? If you were Gretchen’s teacher, what might you do next to help her?

Let’s revisit our first questions: How should mathematics be taught? What should children know about mathematics, and what should they be able to do? Keep these questions in mind as you read this chapter. We will come back to these issues later in the chapter.

According to the Mathematical Sciences Education Board and National Research Council (1990), “More than ever before, Americans need to think for a living; more than ever before, they need to think mathematically” (p. 3). Mathematics permeates all facets of our lives. Jennifer organizes her collection of baseball cards into a 5-by-8 array and wonders how many cards she has. Marco counts his change to be sure he has received the correct amount after buying his brother a birthday present. Mom asks daughter Jasmine to mentally calculate 15% of the family’s restaurant bill for a tip and figure out how much money they should leave to pay the bill including the tip. Elementary school children decide how much money they need to earn at a fund-raiser in order to purchase new playground equipment for their school.

Often the mathematics in real-life situations is not recognized until after one stops and reflects. Children need help in recognizing that mathematics is all around them. They need the right kinds of experiences to appreciate the fact that mathematics is a common human activity and that it is important to their present and future well-being.

Teaching mathematics is both challenging and stimulating because significant changes are taking place in mathematics education. New insights, new materials, and, of course, children who are growing up in an ever-changing society dictate a different approach to the teaching of mathematics. This chapter is about the various factors that influence the principles, practices, and future direction of mathematics and mathematics instruction.

Influences on Mathematics Education

Many factors influence what and how mathematics is taught. Such factors range in their spheres of influence from as localized as the quality of a classroom teacher to as global as social and economic issues. It is helpful to be aware of these influences to better understand the current state of the art and to put future directions in mathematics instruction into perspective. The following sections address some of the major influences, including 21st-century skills, learner influences, and national and international studies.

21st-Century Skills and Science, Technology, Engineering, and Mathematics (STEM) Learning

According to the National Research Council, “Americans have long recognized that investments in public education contribute to the common good, enhancing national prosperity and supporting stable families, neighborhoods, and communities. Education
is even more critical today, in the face of economic, environmental, and social challenges” (NRC, 2012, p. 1). To be successful in meeting said challenges in our rapidly evolving technological, global society, average citizens must be able to process and make decisions based on the vast quantities of information available today. They must be able to think mathematically and to access information provided in a variety of new and visual ways, including graphs, tables, grids, and charts.

In addition, to be successful, average citizens must possess what has commonly become known as “21st-century skills.” While technically not new skills for some students, the current 21st-century skills movement emphasizes the deliberate teaching of these 21st-century skills as a necessary means toward educational and occupational success for all. Charged with clarifying the term 21st-century skills, the NRC (2012) identified the following three domains of competence:

• The cognitive domain includes three clusters of competencies: cognitive processes and strategies, knowledge, and creativity. These clusters include competencies such as critical thinking, information literacy, reasoning and argumentation, and innovation.
• The intrapersonal domain includes three clusters of competencies: intellectual openness, work ethic and conscientiousness, and positive core self-evaluation. These clusters include competencies such as flexibility, initiative, appreciation for diversity, and metacognition.
• The interpersonal domain includes two clusters of competencies: teamwork and collaboration and leadership. These clusters include competencies such as communication, collaboration, responsibility, and conflict resolution.

In short, the concern that motivates the current 21st-century skills movement is that those who do not have the ability to think mathematically, encompassing 21st-century skills such as problem-solving, reasoning, communication, connection, and representation skills, have limited career possibilities.

Another concern is that the United States is not producing enough new scientists, mathematicians, and engineers for our nation’s needs. Mathematics ability is vital for persistence and success in STEM careers. According to the American Association for the Advancement of Science (AAAS), “Because mathematics plays such a central role in modern culture, some basic understanding of the nature of mathematics is requisite for scientific literacy. To achieve this, students need to perceive mathematics as part of the scientific endeavor, comprehend the nature of mathematical thinking, and become familiar with key mathematical ideas and skills” (1989, p. 33). Mathematics is the chief language of science and comprises the rules for analyzing scientific ideas and data.

Mathematics has two diverse personalities: It is at the same time logical and practical, creative and productive. Scientists and engineers use mathematics to solve real-world problems, while mathematicians and others appreciate and enjoy mathematics for its beauty and intellectual challenge. A challenge for teachers is to help children understand and appreciate the dual uses of mathematics—to go beyond seeing mathematics as a tool and to find a way to appreciate its variety of uses. Such an approach will encourage children to aspire to, persist, and be successful in STEM careers.

The Diversity of Our World: Learner Influences

Children, regardless of their gender, race, ethnicity, language ability, or other factors, are capable of understanding mathematics and deserve full access to high-quality mathematics teaching. The Principles and Standards for School Mathematics recommends that “all students should have access to an excellent and equitable mathematics program that provides solid support for their learning and is responsive to their prior knowledge, intellectual strengths, and personal interests” (NCTM, 2000, p. 13).

Achieving excellence and equity in our classrooms and schools is not only recommended but mandated by federal and state requirements (No Child Left Behind (NCLB), 2002). The Elementary and Secondary Education Act (ESEA) reauthorization proposal “Blueprint for reform” explains that the federal commitment to meeting the needs of all students will “continue and strengthen” (U.S. Department of Education, 2010): “Our proposal will help ensure that teachers and leaders are better prepared to
meet the needs of diverse learners, that assessments more accurately and appropriately measure the performance of students with disabilities, and that more districts and schools implement high-quality, state- and locally-determined curricula and instructional supports that incorporate the principles of universal design for learning to meet all students’ needs” (p. 6).

Mathematics instruction is the great equalizer. Children who are successful in mathematics have access to a wide range of careers, while those who are not successful have only limited opportunities. The key factor in the achievement of all children is the quality of the teaching. All children can succeed in mathematics if their teachers are aware of how to modify instructional experiences to best meet their needs.

**Gender considerations.** A major stereotype about mathematics is that it is a male domain (Lindberg, Hyde, Petersen, & Linn, 2010; Fennema & Sherman, 1977; Hyde, Fennema, Ryan, Frost, & Hopp, 1990; Nosek, et al., 2009). The stereotype is widespread. When it comes to the subject of mathematics, children perceive men to be better than women (Steele, 2003), parents perceive their sons to be better than their daughters (Furnham, Reeves, & Budhani, 2002), and teachers overestimate the math ability of boys (Li, 1999; Lindberg, Hyde, & Hirsch, 2008).

The impacts of the stereotype are equally widespread. Teachers tend to interact more with boys than girls, calling on them more often and giving them more criticism and praise, and have higher achievement expectations of boys. The stereotype impacts students’ beliefs about their own abilities (Boucher & Harter, 2005; Frome & Eccles, 1998; Keller, 2001; Tiedemann, 2000), which subsequently impacts students’ (1) interests in and decisions to select and/or persist in mathematical activities, course work, and/or STEM careers (Bandura, 1997; Bussey & Bandura, 1999; Eccles, 1994; Jacobs, Davis-Kean, Bleeker, Eccles, & Malanchuk, 2005) and (2) performance (Steele, 1997; Steele & Aronson, 1995). This stereotype also can impact policy decisions (Arms, 2007). Teachers should carefully consider the following facts related to the impact of the stereotype:

- Girls tend to have less confidence than boys in their mathematical competence, even when they have equal ability.
- Girls (and their parents) are more likely to attribute their struggles to lack of ability, whereas boys attribute their struggles to a lack of work.
- Girls have lower enrollment rates in physics courses, narrowing their career choices.

Despite this stereotype, a recent analysis of 242 studies published between 1990 and 2007 representing more than 1.2 million people revealed no significant differences in mathematical performance between girls and boys (Lindberg, Hyde, & Petersen, 2010). When variations across U.S. ethnic groups were examined, only minor gender differences were reported in favor of Caucasian males, with negligible gender differences for ethnic minority groups combined (Lindberg, Hyde, & Petersen, 2010). Cross-national data show significant correlation between gender gaps related to mathematical performance and gender inequality (Else-Quest, Hyde, & Linn, 2010; Guiso, Monte, Sapienza, & Zingales, 2008; Penner, 2008). A major implication of these findings is that we must dispel the stereotype that mathematics is a male domain.

Overall, it is clear that, in the U.S. and some other nations, girls have reached equality with boys in mathematics performance. It is crucial that teachers know this and share it with parents to counteract stereotypes about female math inferiority held by gatekeepers such as parents and teachers, and by students themselves. (Lindberg, Hyde, & Petersen, 2010, p. 15)

**What can teachers do?** Perhaps more than anyone else, teachers can counteract the negative consequences of the “mathematics is a male domain” stereotype to better ensure equity within our schools. They can help girls’ confidence in their mathematics abilities and interests in future mathematical activities, courses, and STEM careers by taking these steps:

- Interact more with girls on high-cognitive-level mathematics activities, encourage them to engage in independent learning, ensure that they attend to their tasks, and expect them to be successful.
- Make relevant connections between math and girls’ lives.
• Place more emphasis on cooperative mathematics activities (which increase girls’ achievement) and less emphasis on competitive activities. Include team-building activities.
• Provide girls with opportunities to learn about and interact with female role models in math-based careers.

**Students with special needs.** All students need to understand mathematics, including students with special needs. Students with special needs include students with learning difficulties as well as gifted students. In the past, some educators believed that students with learning difficulties could be successful only in rote memorization of mathematics facts and procedures and that, with lots of practice, students would eventually become competent in performing low-level tasks. However, today’s federal requirements (No Child Left Behind [2001]; ESEA Reauthorization [US Department of Education, 2010]; and the Individuals with Disabilities Education Act [U.S. Department of Education, 2004]), NCTM recommended principles and standards for school mathematics (NCTM, 2000), and widely adopted Common Core State Standards for Mathematics (CCSS, 2010) call for more rigorous mathematics curricula and instruction for all students that emphasize mathematical practices/processes for developing conceptual understanding.

Similarly, the expectation that all students be exposed to curricula and instruction that emphasize conceptual understanding is supported within the literature on gifted education (Malloy, 2004; NCTM, 1989, 2000; National Research Council, 2001). Gifted students need to be challenged; they need to be supported in developing higher-order thinking skills and understanding the meaning and applications of mathematics.

**What can teachers do?** To support the learning of children with special needs, current thinking centers on providing high-quality mathematics instruction that is differentiated to meet individual needs while building upon unique strengths.

The following are suggestions for meeting the needs of students with disabilities:

• For students with cognitive challenges, consider doing the following:
  ° “Model the mathematics processes being taught,
  ° Provide concrete representations of the concept before moving on to pictorial or abstract representations,
  ° Make use of available technology, such as computers, software, calculators, and overhead projectors,
  ° Provide a variety of “authentic” experiences for the construction and reinforcement of concepts,
  ° Encourage students to provide answers in several formats—written, verbal, and pictorial,
  ° Use cooperative learning groups, and
  ° Develop daily routines” (Truelove, Holaway-Johnson, Leslie, & Smith, 2007, p. 337–338)

• For students with emotional and behavioral difficulties, consider doing the following:
  ° “Encourage students to stay in class and in school,
  ° Establish clear classroom rules and procedures and post them in your classroom,
  ° Establish a reinforcement system to recognize appropriate behavior,
  ° Create a positive classroom environment and positive teacher-student interactions,
  ° Use appropriate corrective strategies, preferably strategies that are nonaggressive, and avoid removing students from the classroom,
  ° Give students ample opportunity to practice newly presented material, and
  ° Present students with one item to complete at a time to avoid overwhelming them” (Truelove, Holaway-Johnson, Leslie, & Smith, 2007, p. 338–339)

• For students with physical challenges, consider doing the following:
  ° “Label all diagrams with clear, bold letters,
• Prepare study guides and vocabulary guides,
• Include [videos] with closed captioning,
• Make sure that rulers and other measuring devices have large, readable scales or raised numbers and marks,
• Use a variety of teaching methods, and
• Train aides and service providers in the unique qualities of mathematics vocabulary and processes” (Truelove, Holaway-Johnson, Leslie, & Smith, 2007, p. 339–340)

• Utilize co-teaching strategies, including
  • Team teaching in which “teachers take turns instructing the whole class,”
  • Alternative teaching in which “one teacher work with small group for pre-teaching, re-teaching, supplementary work, or enrichment while other teacher instructs the larger group,”
  • One-teach/one-assist in which “one takes the lead, and the other circulates, observes, and assists students,” and
  • Parallel teaching in which “teachers plan jointly, but each teacher instructs half the class” (Treahy & Gurganus, 2010, p. 486)

Often, gifted students are asked to help others, are given extra computational work, or, sadly, asked to patiently wait for extended periods during designated mathematics instructional time while their peer complete their work (Galbraith, 1998; Winebrenner, 2001). While helping others and extra practice are not always problematic, these tasks do not adequately challenge gifted students. The following are suggestions for meeting the needs of gifted students:

• Utilize enrichment activities that correspond to the whole-class unit. The Mathematics Investigation Center (MIC) offer nine such activities which can be assigned to, but are not limited to, gifted students (Wilkins, Wilkins, & Oliver, 2006):
  • Integrating mathematics and science with a short science experiment or research project,
  • Writing about mathematics,
  • Integrating mathematics and social studies with a problem “based in everyday family, cultural, or citizenship activities” (p. 9):
  • Literature and mathematics connection,
  • Mathematics game,
  • Logic problem,
  • Building project, including paper folding, block building, and collage making,
  • Problem solving, and
  • Data project

• Explain new material—gifted students need explanations of new topics as well
• Praise privately
• Involve students, avoiding statements such as “I know you know the answer; I would like to hear from others in the class” (Barger, 2009)
• Promote multiple solutions
• Allow exploration
• Encourage questions: know that it is fine if you do not know the answer immediately, encourage students to try to figure out the answers, and consider the best times to answer the questions related to the engagement of other students (Barger, 2009)

**Students who are culturally, ethnically, and/or linguistically diverse.** Student populations are growing increasing culturally, ethnically, and linguistically diverse. Unfortunately, much of the current research links traditional mathematics education with perpetuating culturally based inequities (Averill, Anderson, Easton, Maro, Smith, & Hynds, 2009). To ensure high-quality mathematics instruction for all students, teachers are encouraged to ‘draw learning contexts from students’ culturally located expertise,
knowledge, interests, and experiences” (p. 159). By drawing on students’ cultural and ethnic experiences, students will more likely engage in activities developed to attain mathematical understanding. Integrating literature is one way teachers can connect mathematics instruction to different cultures (Nelson, 2012).

Teachers must remember that just because students have difficulty understanding the language, they may have strong mathematics abilities. So while students who are learning English while they are learning mathematics have special challenges, they can be successful if involved in high-quality instruction. English language learners (ELLs) need support and patience because learning mathematics while also learning English takes time, and communicating in a second language can seem risky (Meyer, 2000) and create high levels of anxiety (Cady, Hodges, & Brown, 2010).

ELLs often make good progress in computational skills but have difficulty with word problems. Even though ELLs may have developed conversational fluency in English, they may still need additional support in developing proficiency in the more specialized language of mathematics. Teachers should carefully vary sentence complexity, problem context, and response complexity in word problems to help ELLs be successful (Garrison, Ponce, & Amaral, 2007). In general, teachers should repeat directions, speak slowly, simplify sentence structures, and give directions using language consistently (Cady, Hodges, & Brown, 2010).

Other strategies that are effective in supporting ELLs include visual scaffolding, real-world objects, manipulative materials, cooperative learning, advance organizers, preview/review, modeled talk, attribute charting, and word walls (Herrell & Jordan, 2008; Cady, Hodges, & Brown, 2010). Pairing an ELL with a English-speaking peer can be helpful, but when doing so, make sure to give volunteers guidelines and to support and rotate peer volunteers so they do not become overburdened (Cady et al., 2010). We will revisit these strategies throughout this book as we focus on helping children learn specific mathematics topics.

**What can teachers do?** Teachers can modify instruction to make learning mathematics more accessible to English language learners by:

- Using demonstrations and modeling
- Using manipulative materials
- Using technology
- Using graphic organizers and pictures
- Connecting symbols with words
- Using cooperative group work and peer tutoring
- Simplifying, clarifying, and paraphrasing instructional language
- Directly teaching instructional vocabulary and using vocabulary in meaningful contexts
- Encouraging students to “retell”
- Building on children’s prior experiences and knowledge and posting problems in familiar situations
- Using dramatization or acting out problem situations
- Creating mathematics language banks, such as a “Math Word Wall,” where mathematics terms are posted on a bulletin board or wall
- Focusing on meaning and developing understanding rather than memorization

**National and International Studies**

Near the end of the 20th century, much criticism was leveled at school mathematics. Large-scale national and international studies showed that children in the United States did not fare very well on tests of mathematics proficiency compared with children in some other countries (Lapointe, Mead, & Phillips, 1989; Schmidt, 2004; Travers & McKnight, 1984). While there have been improvements since that time in some areas, results of these large-scale assessments still evoke concern related to our students’ mathematical competencies.
TIMSS. In 2011, over 50 countries participated in the fifth administration of the Trends in International Mathematics and Science Study (TIMSS), which examines the mathematics and science achievement of children in fourth and eighth grades. The 2011 TIMSS results showed that U.S. fourth-graders scored above the international average but were outperformed by students in 8 of the participating countries. Similarly, U.S. eighth-graders scored above the international average but below students in 11 of the participating countries. From the 1995 to the 2011 administration, U.S. students have made gains (fourth-graders gained 23 and eighth-graders gained 17 points), but there still is room for improvement (Plisko, 2004). Additional 2011 TIMSS results can be accessed at http://nces.ed.gov/timss/results11.asp.

PISA. The Programme for International Student Assessment (PISA) is an international study, coordinated by the Organisation for Economic Co-operation and Development, which examines reading, mathematics, and science skills and knowledge of 15-year-old students in more than 70 countries. Every 3 years, randomly selected students participate. In 2000, 2003, and 2006, students were assessed on reading, mathematics, and science, respectively. In 2009, this same cycle of testing resumed with reading. Results of the 2003 assessment revealed that U.S. students performed lower than the average for most OECD countries for each subscale of the mathematics assessment; 2012 results will be available in December of 2013 (http://nces.ed.gov/surveys/pisa).

NAEP. The National Assessment of Educational Progress (NAEP), often referred to as “The Nation’s Report Card,” periodically examines the achievement of U.S. children in nine content areas—including mathematics—at grades 4, 8, and 12. The mathematics assessment includes five mathematics content areas: (1) number properties and operations, (2) measurement, (3) geometry, (4) data analysis and probability, and (5) algebra. NAEP results are useful in identifying mathematics topics in which students are doing well and those “in which students could do better” (Kloosterman & Lester, 2007). Data from 2009 revealed that only 39% of fourth-graders and 34% of eighth-graders scored at the proficient or advanced level. 2011 data revealed that a higher percentage of fourth- and eighth-graders performed at or above the proficient level than in any other assessment year—specifically, 40% of fourth-graders and 35% of eighth-graders (http://nationsreportcard.gov/math_2011/summary.aspx).

Importance of Teacher Quality

How can we help all children succeed in mathematics? Perhaps the most important influence on what mathematics children learn and on how that knowledge is constructed is an enthusiastic, understanding, and knowledgeable teacher. What does a teacher need to know and be able to do in order for all children to succeed?

Disposition. Particularly for female students, teachers with high mathematics anxiety transmit this anxiety to their students. Such anxiety can begin very early, persist throughout life, and impede problem solving and critical thinking, both necessary for success in mathematics (Sparks, 2011). Therefore, it is necessary that you become comfortable and excited about mathematics, so to transmit a positive, low-stress attitude about mathematics.

Content and Pedagogical Knowledge. The federal No Child Left Behind Act of 2001 mandates that every teacher must be “highly qualified” in each subject he or she teaches. This legislation recognizes the importance of teachers’ knowledge. To help children, teachers need to have a deep understanding of the mathematics they are teaching, as well as an understanding of how to help children construct mathematical understandings.

The view that children should construct their own mathematical knowledge does not imply that you should sit back and wait for it to happen. Rather, you must actively observe and listen to children as they engage in and talk about their mathematical explorations. You must be skilled in detecting seeds of mathematical concepts and in providing experiences that will enable those seeds to grow into mature understandings. You must develop lessons that align with standards but also build on children’s thinking.

A key feature of this text book is its link to children’s thinking, as you saw at the beginning of this chapter, when you viewed the video clip of Gretchen solving a subtraction
Instructional programs from prekindergarten through grade 12 should enable all students to—

• Build new mathematical knowledge through problem solving.
• Solve problems that arise in mathematics and in other contexts.
• Apply and adapt a variety of appropriate strategies to solve problems.
• Monitor and reflect on the process of mathematical problem solving.

• Recognize reasoning and proof as fundamental aspects of mathematics.
• Make and investigate mathematical conjectures.
• Develop and evaluate mathematical arguments and proofs.
• Select and use various types of reasoning and methods of proof.

• Organize and consolidate mathematical thinking through communication.
• Communicate mathematical thinking coherently and clearly to peers, teachers, and others.
• Analyze and evaluate the mathematical thinking and strategies of others.
• Use the language of mathematics to express mathematical ideas precisely.

• Recognize and use connections among mathematical ideas.
• Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
• Recognize and apply mathematics in contexts outside of mathematics.
• Create and use representations to model and interpret physical, social, and mathematical phenomena.

There is much to learn about how children think about and come to understand different mathematics concepts. As you read through this book, pay particular attention to how children are thinking about mathematics and how teachers can help children extend their understandings.

Curriculum Guided by Standards

NCTM Principles and Standards. In an effort to meet the needs of all students for success in the rapidly evolving global, technological world, NCTM developed a set of guiding principles and standards for prekindergarten–grade 12 mathematics education described in Principles and Standards for School Mathematics (NCTM, 2000).

According to the NCTM (2000):

The Principles describe particular features of high-quality mathematics education. The Standards describe the mathematical content and processes that children should learn. Together, the Principles and Standards constitute a vision to guide educators as they strive for the continual improvement of mathematics education in classrooms, schools, and educational systems. (p. 11)

NCTM’s six principles—equity, curriculum, teaching, learning, assessment, and technology—highlight important issues that are related to all aspects of school mathematics programs and are fundamental in high-quality math education. These principles guide educators in making decisions about teaching and learning and in creating a classroom environment conducive to learning. It is important that you consider these principles when planning mathematics instruction and designing mathematics learning environments.

In addition to these principles, the NCTM Principles and Standards document identifies five content standards, which describe the mathematics content children should know, and five process standards, which describe the mathematical processes children should be able to use in prekindergarten through grade 12. The process standards are listed and described in Table 1-1.

<table>
<thead>
<tr>
<th>Problem Solving</th>
<th>Reasoning and Proof</th>
<th>Communication</th>
<th>Connections</th>
<th>Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Build new mathe-</td>
<td>Recognize reasoning</td>
<td>Organize and</td>
<td>Recognize</td>
<td>Create and</td>
</tr>
<tr>
<td>matical knowledge</td>
<td>and proof as funda-</td>
<td>consolidate</td>
<td>and use con-</td>
<td>use represen-</td>
</tr>
</tbody>
</table>
| through problem | mental aspects of  | mathematical  | nections amon-
| solving.        | mathematics.        | thinking      | g content   |
| Solve problems  | Make and investigate| coherently    | g and com-   |
| that arise in    | mathematical        | clearly to    | municate   |
| mathematics and  | conjectures.        | peers, teach-  | mathe-  |
| in other        |                     | ers, and oth-  | matical    |
| contexts.       |                     | ers.          | ideas.     |
| Apply and adapt | Develop and evalu-   | Analyze and   | Understand  |
| a variety of     | ate mathematical    | evaluate the  | how mathe-  |
| appropriate      | arguments and       | mathematical  | matical    |
| strategies to    | proofs.             | thinking and  | ideas inter-|
| solve problems.  |                     | strategies     | connect   |
| Monitor and re- | Select and use      | of others.    | and build   |
| flect on the    | various types of    | Use the lan-   | on one     |
| process of       | reasoning and       | guage of    | another to  |
| mathematical     | methods of proof.   | mathematics   | produce a   |
| problem solving. |                     | to express     | coherent    |
|                  |                     | mathematical   | whole.     |

Common Core State Standards

In 2010, states across the United States began adopting the Common Core State Standards (CCSS). The National Governors Association (NGA) and the Council of Chief State School Officers (CCSSO), in collaboration with teachers, researchers, and experts, developed the standards “to be robust and relevant to the real world, reflecting the knowledge and skills that our young people need for success in college and careers” in order that “our communities be best positioned to compete successfully in the global economy” (www.corestandards.org).

The CCSS mathematics standards describe both the mathematical content and mathematical practices that should be developed in all students. The mathematical content standards are divided into different domains by grade level (see Table 1-2). The mathematical practices (see Table 1-3) were developed based on NCTM’s five process standards. Please note that while most states have adopted the CCSS, some states still rely on their own standards either exclusively or in tandem with the CCSS. You can find your state’s standards by accessing your state’s Department of Education: www2.ed.gov/about/contacts/state/index.html. While both sets of standards are important, this book focuses on the CCSS.

Directions in Mathematics Education

The influences described above steer the once teacher-centered, answer-oriented, traditional mathematics classroom toward a more innovative classroom articulated in the expectations of the CCSS. Although many changes have occurred as a result of the preceding influences, looking into the future is difficult. It seems reasonable, however, to predict that mathematics instruction in which children make sense of, reason about, discuss, and solve mathematical problems will continue to be emphasized. Further, equity with regard to achievement, the use of technology, an appropriate role for computation and estimation, and authentic assessment will be implemented. In addition, the issue of parent involvement to enhance children’s learning must be considered. The following sections discuss several important directions in mathematics education.

Problem Solving

Problem solving has always been an important part of a mathematics program. In recent years, however, the importance of problem solving has been reemphasized, with its inclusion as one of the NCTM process standards and as the focus of CCSS Mathematical Practice 1. Recent recommendations offered by the National Research Council (2012) encouraged curriculum developers to provide “sustained instruction to develop expertise in problem solving” (p. 10), deliberately teaching and assessing problem solving, and integrating it within mathematics courses. These recommendations are being implemented, evident by the increased emphasis on problem solving in textbook series and government curriculum guides, which now include many good problem-solving activities. The NCTM’s Principles and Standards describes the problem-solving process standard (see Table 1-1), as does CCSS Mathematical Practice 1 (see Table 1-3).

Researchers have been actively trying to document the characteristics of good problem solvers. Likewise, teachers have been experimenting with strategies (often called heuristics) that develop problem-solving skills in children. Go to the www.teachingchannel.org website. Search the site for, and watch, the “Persistence in Problem Solving” video clip. What strategies does this teacher use to develop problem-solving skills in her students? Describe this problem-solving classroom environment. Identify what the students and teachers are doing and saying that make this a problem-solving classroom. Because problem solving is at the heart of school mathematics, it is discussed in great detail in Chapter 3.

Reasoning

Reasoning is very important in understanding and doing mathematics. Rather than merely emphasize memorization of basic facts, rules, and principles, mathematics programs today place more emphasis on mathematical reasoning and other higher-order thinking skills such as application, analysis, synthesis, and evaluation. You may recall that the NRC (2012)
<table>
<thead>
<tr>
<th>Domain</th>
<th>Grade</th>
<th>Mathematically Proficient Students Will:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counting and Cardinality</td>
<td>Kindergarten</td>
<td>Know number names and the count sequence.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Count to tell the number of objects.</td>
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<tr>
<td></td>
<td></td>
<td>Compare numbers.</td>
</tr>
<tr>
<td>Operations and Algebraic</td>
<td>Kindergarten</td>
<td>Understand addition and understand subtraction.</td>
</tr>
<tr>
<td>Thinking</td>
<td>Grade 1</td>
<td>Represent and solve problems involving addition and subtraction.</td>
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<tr>
<td></td>
<td></td>
<td>Understand and apply properties of operations and the relationship between addition and subtraction.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Add and subtract within 20.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Work with addition and subtraction equations.</td>
</tr>
<tr>
<td></td>
<td>Grade 2</td>
<td>Represent and solve problems involving addition and subtraction.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Add and subtract within 20.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Work with equal groups of objects to gain foundations for multiplication.</td>
</tr>
<tr>
<td></td>
<td>Grade 3</td>
<td>Represent and solve problems involving multiplication and division.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Understand properties of multiplication and the relationship between multiplication and division.</td>
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<tr>
<td></td>
<td></td>
<td>Multiply and divide within 100.</td>
</tr>
<tr>
<td></td>
<td>Grade 4</td>
<td>Use the four operations with whole numbers to solve problems.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Gain familiarity with factors and multiples.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Generate and analyze patterns.</td>
</tr>
<tr>
<td></td>
<td>Grade 5</td>
<td>Write and interpret numerical expressions.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Analyze patterns and relationships.</td>
</tr>
<tr>
<td>Number and Operations in Base</td>
<td>Kindergarten</td>
<td>Work with numbers 11–19 to gain foundations for place value.</td>
</tr>
<tr>
<td>Ten</td>
<td>Grade 1</td>
<td>Extend the counting sequence.</td>
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<tr>
<td></td>
<td></td>
<td>Use place value understanding and properties of operations to add and subtract.</td>
</tr>
<tr>
<td></td>
<td>Grade 2</td>
<td>Understand place value.</td>
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<tr>
<td></td>
<td></td>
<td>Use place value understanding and properties of operations to add and subtract.</td>
</tr>
<tr>
<td></td>
<td>Grade 3</td>
<td>Use place value understanding and properties of operations to perform multidigit arithmetic</td>
</tr>
<tr>
<td></td>
<td>Grade 4</td>
<td>Generalize place value understanding for multidigit whole numbers.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Use place value understanding and properties of operations to perform multidigit arithmetic.</td>
</tr>
<tr>
<td></td>
<td>Grade 5</td>
<td>Understand the place value system.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Perform operations with multidigit whole numbers and with decimals to hundredths.</td>
</tr>
<tr>
<td>Measurement and Data</td>
<td>Kindergarten</td>
<td>Describe and compare measurable attributes.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Classify objects and count the number of objects in each category.</td>
</tr>
</tbody>
</table>

(Continued)
<table>
<thead>
<tr>
<th>Domain</th>
<th>Grade</th>
<th>Mathematically Proficient Students Will:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Grade 1</td>
<td>Measure lengths indirectly and by iterating length units.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Tell and write time.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Represent and interpret data.</td>
</tr>
<tr>
<td>Grade 1</td>
<td></td>
<td>Measure and estimate lengths in standard units.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Relate addition and subtraction to length.</td>
</tr>
<tr>
<td>Grade 2</td>
<td></td>
<td>Work with time and money.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Represent and interpret data.</td>
</tr>
<tr>
<td>Grade 3</td>
<td></td>
<td>Solve problems involving measurement and estimation.</td>
</tr>
<tr>
<td>Grade 4</td>
<td></td>
<td>Represent and interpret data.</td>
</tr>
<tr>
<td>Grade 5</td>
<td></td>
<td>Convert like measurement units within a given measurement system.</td>
</tr>
<tr>
<td>Geometry</td>
<td>Kindergarten</td>
<td>Identify and describe shapes.</td>
</tr>
<tr>
<td></td>
<td>Grade 1</td>
<td>Reason with shapes and their attributes.</td>
</tr>
<tr>
<td>Grade 2</td>
<td></td>
<td>Reason with shapes and their attributes.</td>
</tr>
<tr>
<td>Grade 3</td>
<td></td>
<td>Reason with shapes and their attributes.</td>
</tr>
<tr>
<td>Grade 4</td>
<td></td>
<td>Draw and identify lines and angles, and classify shapes by properties of their lines and angles.</td>
</tr>
<tr>
<td>Grade 5</td>
<td></td>
<td>Graph points on the coordinate plane to solve real-world and mathematical problems.</td>
</tr>
<tr>
<td>Grade 6</td>
<td></td>
<td>Classify two-dimensional figures into categories based on their properties.</td>
</tr>
<tr>
<td>Grade 7</td>
<td></td>
<td>Solve real-world and mathematical problems involving area, surface area, and volume.</td>
</tr>
<tr>
<td>Grade 8</td>
<td></td>
<td>Draw construct, and describe geometrical figures and describe the relationships between them.</td>
</tr>
<tr>
<td>Grade 1</td>
<td></td>
<td>Understand congruence and similarity using physical models, transparencies, or geometry software.</td>
</tr>
<tr>
<td>Grade 2</td>
<td></td>
<td>Understand and apply the Pythagorean Theorem.</td>
</tr>
<tr>
<td>Grade 3</td>
<td></td>
<td>Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.</td>
</tr>
<tr>
<td>Domain</td>
<td>Grade</td>
<td>Mathematically Proficient Students Will:</td>
</tr>
<tr>
<td>-------------------------------</td>
<td>-------</td>
<td>-------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Number and Operations—Fractions</td>
<td>Grade 3</td>
<td>Develop understanding of fractions as numbers.</td>
</tr>
<tr>
<td></td>
<td>Grade 4</td>
<td>Extend understanding of fraction equivalence and ordering. Build fractions from unit fractions.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Understand decimal notation for fractions, and compare decimal fractions.</td>
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<tr>
<td></td>
<td>Grade 5</td>
<td>Use equivalent fractions as a strategy to add and subtract fractions.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Apply and extend previous understandings of multiplication and division.</td>
</tr>
<tr>
<td>Ratios and Proportional Relationships</td>
<td>Grade 6</td>
<td>Understand ratio concepts and use ratio reasoning to solve problems.</td>
</tr>
<tr>
<td></td>
<td>Grade 7</td>
<td>Analyze proportional relationships and use them to solve real-world and mathematical problems.</td>
</tr>
<tr>
<td>Number System</td>
<td>Grade 6</td>
<td>Apply and extend previous understandings of multiplication and division to divide fractions by fractions.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Compute fluently with multidigit numbers and find common factors and multiples.</td>
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<tr>
<td></td>
<td></td>
<td>Apply and extend previous understandings of numbers to the system of rational numbers.</td>
</tr>
<tr>
<td></td>
<td>Grade 7</td>
<td>Apply and extend previous understandings of operations with fractions.</td>
</tr>
<tr>
<td></td>
<td>Grade 8</td>
<td>Know that there are numbers that are not rational, and approximate them by rational numbers.</td>
</tr>
<tr>
<td>Expressions and Equations</td>
<td>Grade 6</td>
<td>Apply and extend previous understandings of arithmetic to algebraic expressions.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Reason about and solve one-variable equations and inequalities.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Represent and analyze quantitative relationships between dependent and independent variables.</td>
</tr>
<tr>
<td></td>
<td>Grade 7</td>
<td>Use properties of operations to generate equivalent expressions.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Solve real-life and mathematical problems using numerical and algebraic expressions and equations.</td>
</tr>
<tr>
<td></td>
<td>Grade 8</td>
<td>Work with radicals and integer exponents.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Understand the connections between proportional relationships, lines, and linear equations.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Analyze and solve linear equations and pairs of simultaneous linear equations.</td>
</tr>
<tr>
<td>Functions</td>
<td>Grade 8</td>
<td>Define, evaluate, and compare functions.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Use functions to model relationships between quantities.</td>
</tr>
<tr>
<td>Statistics and Probability</td>
<td>Grade 6</td>
<td>Develop understanding of statistical variability.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Summarize and describe distributions.</td>
</tr>
<tr>
<td></td>
<td>Grade 7</td>
<td>Use random sampling to draw inferences about a population.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Draw informal comparative inferences about two populations.</td>
</tr>
<tr>
<td></td>
<td>Grade 8</td>
<td>Investigate chance processes and develop, use, and evaluate probability models.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Investigate patterns of association in bivariate data.</td>
</tr>
</tbody>
</table>

includes reasoning among the identified 21st-century competencies. In addition, reasoning is the focus of three of the CCSS Mathematical Practices: 2 (“Reason abstractly and quantitatively”), 3 (“Construct viable arguments and critique the reasoning of others”), and 8 (“Look for and express regularity in repeated reasoning”). The NCTM’s Principles and Standards describes the reasoning and proof process standard (see Table 1-1), as do CCSS Mathematical Practices 2, 3, 7, and 8 related to reasoning (see Table 1-3).

Reasoning skills are often included in problem-solving activities. Problems such as the following help children develop reasoning skills:

Sarah is younger than Alyssa. She is also older and shorter than Patrick. Alyssa is taller and younger than Juan. Juan is taller than Patrick.

1. Arrange the four people by age.
2. Arrange the four people by height.

Encourage children to solve this problem using whatever strategy makes sense to them. After they have solved the problem, ask children to share their solution processes. Follow this with a discussion about the problem and the solution strategies children used. Having children share their solution strategies helps them learn from each other and models different types of reasoning.

Communication

Communication is another important direction in mathematics education. A major factor in shaping mathematics programs and teaching in the next decade concerns mathematics as communication. Children need an opportunity to reflect on and explain or justify their ideas and solutions both orally and in writing. There are two aspects to mathematical communication. First, mathematics is a language. Like English, Spanish, or any other language, mathematics has words (symbols) and semantic and syntactical rules; meaning is conveyed through mathematical symbols and their associated rules. A second aspect of mathematical communication involves the use of language within mathematics. This can be a powerful determinant of what is learned and how it is learned.

McKenzie (1990) draws a parallel (and highlights some differences) between reading for meaning and solving a mathematics problem with meaning. Both processes require the use of prior knowledge. Indeed, in both processes, children are continually predicting, sampling, confirming, self-correcting, and reprocessing—further evidence that reading is not an isolated subject to be taught at a particular time of day. Rather, reading for meaning is a process that must permeate all subject areas. Literature Link 1-1 is an example of how children’s literature may be used in the mathematics classroom.

Talking, reading, writing, listening, and representing are important components of communication in mathematics. Children need to engage in all of them. Figure 1-1 suggests a variety of activities for each component that serves to reinforce each component’s role in mathematical communication. In addition, asking thought-provoking questions will encourage quality communication.

NRC (2012) also suggests that curriculum developers stress “elaboration, questions, and explanation” (p. 9). As with problem solving, curriculum designers are considering such recommendations. For example, an early childhood problem-based curriculum called Mentoring Young Mathematicians incorporates an adaptation of Chapin, O’Connor, & Anderson’s (2009) “talk moves,” simple tools a teacher can utilize to ensure greater communication. The adapted talk moves include repeat and check, agree/disagree and why, partner talk, add on, and think time (Gavin, Casa, Chapin, & Sheffield, 2010).

Both NCTM and the CCSS stress communication. CCSS Mathematical Practice 6 (“Attend to precision”) requires students to learn the language of mathematics and to be able to communicate mathematically, accurately, and precisely. In addition, only if students are able to communicate mathematically will they be able to master CCSS Mathematical Practice 3, related to constructing arguments and critiquing their peers’ reasoning. The NCTM’s Principles and Standards describes the communication process standard (see Table 1-1), as do CCSS Mathematical Practices 3 and 6, which are related to reasoning (see Table 1-3).
## TABLE 1-3 Common Core Standards for Mathematical Practice

<table>
<thead>
<tr>
<th>Standard</th>
<th>Mathematically Proficient Students Can:</th>
</tr>
</thead>
<tbody>
<tr>
<td>CCSS.Math.Practice.MP1: Make</td>
<td>• Explain the meaning of a problem</td>
</tr>
<tr>
<td>sense of problems and persevere</td>
<td>• Contemplate and/or determine multiple entry points to finding solutions</td>
</tr>
<tr>
<td>in solving them.</td>
<td>• Derive a plan/strategy for solving the problem, such as considering a similar, simpler problem to gain insight into finding the solution, looking for trends/patterns</td>
</tr>
<tr>
<td></td>
<td>• Monitor and evaluate progress and adapt the original plan if necessary</td>
</tr>
<tr>
<td></td>
<td>• Explain relationships between multiple representations: equations, verbal descriptions, tables, and graphs</td>
</tr>
<tr>
<td></td>
<td>• Check solutions using alternative methods</td>
</tr>
<tr>
<td></td>
<td>• Can assess the validity of a solution by questioning, “Does this make sense?”</td>
</tr>
<tr>
<td>CCSS.Math.Practice.MP2: Reason</td>
<td>• Make sense of quantities and their relationships</td>
</tr>
<tr>
<td>abstractly and quantitatively.</td>
<td>• Can create mathematical representations of problems</td>
</tr>
<tr>
<td></td>
<td>• Can use properties and operations to solve problems</td>
</tr>
<tr>
<td>CCSS.Math.Practice.MP3: Construct</td>
<td>• Understand and use stated assumptions, definitions, and previously established results in justifications and explanations</td>
</tr>
<tr>
<td>viable arguments and critique</td>
<td>• Make conjectures and support conjectures with a logical progression of statements</td>
</tr>
<tr>
<td>the reasoning of others.</td>
<td>• When logic fails, recognize that conjecture was false and explain why false</td>
</tr>
<tr>
<td></td>
<td>• Use cases to analyze problems and counterexamples to test and/or defend solutions/conjectures</td>
</tr>
<tr>
<td></td>
<td>• Justify and/or explain solutions to others; critique/respond to others’ arguments</td>
</tr>
<tr>
<td>CCSS.Math.Practice.MP4: Model</td>
<td>• Apply mathematics to solve problems arising in everyday life, society, and the workplace</td>
</tr>
<tr>
<td>with mathematics.</td>
<td>• Make assumptions and approximations to simplify a complicated situation</td>
</tr>
<tr>
<td></td>
<td>• Identify important quantities in a practical situation and use tools such as diagrams, two-way tables, graphs, flowcharts and formulas to assist in analyzing and drawing conclusions</td>
</tr>
<tr>
<td></td>
<td>• Reflect on reasonableness of a result/conclusion; improve upon model if result not reasonable</td>
</tr>
<tr>
<td>CCSS.Math.Practice.MP5: Use</td>
<td>• Consider the available tools when solving a mathematical problem</td>
</tr>
<tr>
<td>appropriate tools strategically.</td>
<td>• Determine which tool will be most useful in solving a problem</td>
</tr>
<tr>
<td></td>
<td>• Use estimation to detect errors</td>
</tr>
<tr>
<td></td>
<td>• Use technology and other mathematical resources to visualize, explore, and compare information; to deepen understanding of concepts; and pose and solve problems</td>
</tr>
<tr>
<td>CCSS.Math.Practice.MP6: Attend</td>
<td>• Communicate precisely to others</td>
</tr>
<tr>
<td>to precision.</td>
<td>• Use clear definitions in discussions and reasoning</td>
</tr>
<tr>
<td></td>
<td>• State the meaning of symbols and use symbols appropriately</td>
</tr>
<tr>
<td></td>
<td>• Specify units of measure and label axes</td>
</tr>
<tr>
<td></td>
<td>• Calculate accurately and efficiently and express numerical answers with a degree of precision appropriate for the problem context</td>
</tr>
<tr>
<td>CCSS.Math.Practice.MP7: Look</td>
<td>• Identify patterns or structures</td>
</tr>
<tr>
<td>for and make use of structure.</td>
<td>• Can shift perspective to see complicated things or as being composed of several objects</td>
</tr>
<tr>
<td></td>
<td>• Notice if calculations are repeated and look both for general methods and for shortcuts</td>
</tr>
<tr>
<td>CCSS.Math.Practice.MP8: Look</td>
<td>• When solving problems, can maintain oversight of the process, while attending to details</td>
</tr>
<tr>
<td>for and express regularity in</td>
<td>• Evaluate the reasonableness of results/solutions</td>
</tr>
<tr>
<td>repeated reasoning.</td>
<td></td>
</tr>
</tbody>
</table>

Imagine a World Full of “Math Curses”!


Children's literature provides a context through which mathematical concepts, patterns, problem solving, and real-world contexts may be explored. Many of us take the mathematics in the world around us for granted. In *Math Curse*, the main character thinks of everything in life as a math problem. How can you engage your students in a similar manner?

- Keep a math journal for one day, recording all of the mathematical problems you encounter. Be as creative as the narrator in the book and think about ways mathematics may be hidden in typical activities. Create a class book of children's “math curse” experiences.
- Record or bring cut-out examples from magazines, the newspaper, or the Internet of numbers and symbols used in everyday life. Examples might include (1) graphs or other statistics presented in a newspaper, (2) the dollar sign and decimal used in our monetary system, or (3) pictures of repeating or tessellating patterns in various designs. Design a class bulletin board called “Mathematics in the World Around Us.”
- Communicate using the vocabulary of mathematical terms and symbols in the book. For example, investigate the Mayan numeral system of counting presented in the story. Discuss why the mathematics teacher in the book is named “Mrs. Fibonacci.”
- Model and solve some of the mathematical puzzles in the book and determine which ones are simply nonsense.
- Investigate the mathematical conversions, tables, measures, and terms illustrated on the endpapers of the book.
- Books such as *Math in the Bath (and Other Fun Places, Too!)* (Atherlay, 1995) for younger children and *Counting on Frank* (Clement, 1991), whose witty narrator will amuse older children, show children how mathematics is a part of their everyday experiences.

Source: Dr. Patricia Moyer-Packenham, Utah State University.

Connections

In the past, mathematics was often considered a subject unto itself. Frequently, it was broken down internally into many unrelated parts. In the future, however, you can integrate mathematics throughout the curriculum and punctuate it with real-world applications. The NCTM's *Principles and Standards* describes the connections process standard (see Table 1-1).

Figure 1-1

**Components of Mathematics as Communication**

- **MATH as Communication**
- **WRITING**
  - logs, letters, journals, reports
- **REPRESENTING**
  - graphs, tables, charts, words, symbols, manipulatives
- **LISTENING**
  - others' solutions, class reports, directions, others' strategies
- **READING**
  - tables, charts, statistics on sports page, menu, children’s literature where math is involved
- **TALKING**
  - class presentations, discussion of strategies, results of surveys, cooperative group work

Integration with other school subjects. When children recognize that mathematics can be used in other subject areas, it becomes more relevant to them. For example, graphing is a skill that children can apply to problems in social studies and science. In art class, geometric concepts such as slides, flips, and turns can be applied to create a variety of interesting designs. And finally, as a language-learning assignment, children can write about the way they solved a problem, how they feel about the mathematics they are doing, or what successes or difficulties they experience in understanding mathematics.

Consider the following example: Ms. Jacobs’s kindergarten students discussed how they got to school each day. As part of their discussion, they made a graph showing how they arrived that morning. Some children walked, some rode the bus, some rode their bikes, and others came in a car, as illustrated here:

![Graph showing modes of transportation](image)

The class used this graph to notice that most of the children arrived by car, and fewer walked or rode their bikes.

Integration with real-world settings. In the real world, people solve mathematics problems that arise from a particular setting. Pilots use mathematics for navigational problem solving, firefighters apply measurement concepts and processes when they fight fires, interior designers employ mathematics when they order carpeting and wallpaper, and so on. The 1995 NCTM yearbook, *Connecting Mathematics across the Curriculum*, focuses on mathematics in the real world. Mathematics is holistic in the sense that integrative threads that connect other content areas in the curriculum will be explicitly identified so that children can “see” the connections. Some connections are mentioned in subsequent chapters of this book. One example, a connection between elementary and secondary levels, is illustrated here. A simple number, 7,425, familiar to elementary school-aged children, is written in expanded form and, through a series of generalizations, transformed into a polynomial, familiar to secondary school students.

\[
7,425 = 7 \times 1,000 + 4 \times 100 + 2 \times 10 + 5
\]

\[
7 \times x^3 + 4 \times x^2 + 2 \times x + 5
\]

\[
7x^3 + 4x^2 + 2x + 5
\]

\[
ax^3 + bx^2 + cx + d
\]

Representation of Mathematical Ideas

Much in mathematics is abstract, and making it meaningful to children has been a continuing challenge for teachers. In the past, mathematics was taught at an abstract level, even in elementary school, where children are not yet fully able to make the kinds of abstractions expected for understanding. Because we know now that children learn in different ways, it makes sense for teachers and children to represent mathematical concepts in different ways as well.

Considerable emphasis is placed on representing mathematical ideas with concrete materials: Blocks, counters, and many other physical apparatuses that children can manipulate have been used to embody mathematical ideas. This emphasis will continue, but, currently, there is a shift to a more multirepresentational approach that includes...
spoken language, concrete objects, pictures, real-life situations, and written symbols. Observing and making relationships within and among these representations helps children develop understanding (Behr, Lesh, Post, & Silver, 1983; Cuoco, 2001; Hiebert, 1990). The NRC (2012) supports and encourages the use of “multiple and varied representations” as a necessary component to helping students develop 21st-century skills (p. 9).

The NCTM’s *Principles and Standards* discusses how children should be able to use representations (see Table 1-1), as do CCSS Mathematical Practices 4 and 5, which are related to representations (see Table 1-3). This topic is discussed in more detail in Chapter 2.

**Technology**

We believe that the use of calculators and other technologies will continue to increase, particularly due to the prevalence of mobile devices, such as smart phones and tablet devices. This prediction stems from the following reasons:

- Calculators and other forms of technology continue to be used extensively in the home and office.
- The cost of calculators and other forms of technology continues to decrease, while their power and functions continue to increase.
- Curriculum documents increasingly encourage the use of calculators and other forms of technology.
- Some tests allow and even encourage calculator use.

**Computation and Estimation**

In the past, a heavy emphasis was placed on computation and computational procedures in elementary schools. But according to the Mathematical Sciences Education Board and NRC (1989), “Mathematics today involves far more than calculation; clarification of the problem, deduction of consequences, formulation of alternatives, and development of appropriate tools are as much a part of the modern mathematician’s craft as are solving equations or providing answers” (p. 5).

Currently, although the need for children to learn how to perform paper-and-pencil computations is recognized, the focus is on less complex calculations. More complex computations (for example, multiplying two three-digit numbers) are more realistically done on a calculator than with paper and pencil. Also, the current emphasis is on solving real problems that may require the use of a calculator, rather than computation for computation’s sake. The focus will continue to be on supporting children in choosing the appropriate computational tool in solving a problem. That is, is an estimate sufficient? If not, is mental computation feasible? Can this be done easily with pencil and paper, or should a calculator be used? Figure 1–2, adapted from the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), reflects this philosophy.

In addition, emphasis is placed on estimation (approximating the answer) and mental computation (performing an exact computation mentally). Estimation and mental computation are useful in helping children develop number and operation sense. The renewed emphasis on estimation and mental computation can be traced, at least in part, to the advent of the calculator. The calculator will display a result when keys are pressed, but were the correct keys pressed? Were they pressed in the right sequence? An estimate will tell you whether your answer is reasonable. (Estimation and mental computation are discussed more fully in Chapter 9.)

**Assessment**

According to the *Principles and Standards*, “Assessment should support the learning of important mathematics and furnish useful information to both teachers and students” (NCTM, 2000, p. 22). The nature of assessment and strategies for assessing student learning are changing markedly. This topic is discussed in depth in Chapter 4. We mention it here, however, because it is another important area in which significant change
is occurring. Assessment must be more than just a score on a test, rather it should be an ongoing part of instruction that guides you in making instructional decisions.

**Parent Involvement**

Collaboration between teachers and parents is an effective strategy for increasing children's success in mathematics. According to Bezuk, Whitehurst-Payne, and Aydelotte (2000), “collaboration between teachers and parents is critically important to increase student achievement in order to achieve the goal of all students succeeding in mathematics” (p. 148).

There are many ways you can involve parents to enhance a child's learning. Some of these strategies include the following:

- Help parents learn more about what their child is learning about mathematics and how their child is learning mathematics.
- Provide activities parents can do with their children at home to reinforce and extend their children’s learning.
- Discuss why mathematics is important for future success, including noting careers that involve using mathematics.

Suggestions and ideas for activities that can be sent home with children to do with their parents are included throughout this textbook.

**Conclusion**

This chapter has described a number of factors that have influenced the course of mathematics education in schools. These factors have changed and will continue to change both the mathematics curriculum and how mathematical ideas are taught.

Think back to Gretchen, the second-grader in the opening video, and the questions posed there:

- How should mathematics be taught?
- What should children know about mathematics, and what should they be able to do?
Now that you’ve read this chapter, has your thinking about these questions changed? What do you want to know more about related to the teaching and learning of mathematics? As a teacher of mathematics, how will these influences affect your teaching?

Sometimes influences pull in opposite directions, making it difficult to maintain a balance. Educators often describe a need for a balance in school mathematics among three needs: the needs of the child, the needs of society, and the needs of the subject. If we overemphasize any one area, such as the computation component of mathematics, we may tend to neglect its application in society and also the needs of the child, resulting in an imbalance.

The suggestions and activities in subsequent chapters of this book will enable you to devise a mathematics program for the children in your classroom that develops mathematical ideas in a nontrivial way, makes applications to everyday situations apparent, and carefully considers the needs of the child, making allowances for differences in background, learning style, and motivation to learn mathematics. Throughout the instructional process, the teacher is the most important factor in determining the strength of the mathematics program. Your challenge as a future teacher is to learn as much as you can about how to help children learn mathematics.

IN PRACTICE

Complete the following activities to include in your professional portfolio.

1. Browse through issues of *Teaching Children Mathematics* and *Mathematics Teaching in the Middle School*. Begin collecting articles you find interesting and useful.

2. Interview elementary or middle school children. Ask them what they like and don’t like about mathematics class. Describe what you as a teacher might do to change any negative attitudes they have.

3. Consult a copy of your state mathematics standards. Compare them with the NCTM Principles and Standards.

4. Think about the mathematics of daily life. List all the ways mathematics is used in one day of life.

LINKS TO THE INTERNET

**National Council of Teachers of Mathematics**

*www.nctm.org*

- Contains information about the NCTM standards and other publications as well as news releases related to mathematics teaching and learning.

**Math Forum**

*http://mathforum.org*

- Contains Student Center, Teachers’ Place, Research Division, and a section for parents and other citizens. Also includes Ask Dr. Math, where you can ask questions about K–12 mathematics.

**Teachers’ Net**

*http://teachers.net*

- Contains many different types of resources for teachers, including curriculum resources, lesson plans, chat boards, and mail rings.

**Math.com**

*www.math.com*

- Contains many different types of resources for teachers (in the Teacher tab), including lesson plans, classroom resources, standards, and free stuff.

**Teaching Channel**

*www.teachingchannel.org*

Contains many videos of effective and inspiring teaching.

RESOURCES FOR TEACHERS

**Practitioner Journal Articles to Assist in Differentiated Instruction**


**Reference Books: Increasing Equity in Learning Mathematics**


**Children’s Literature**


Learning Outcomes

When you have finished studying this chapter, you should be able to do the following:

• Describe how the constructivist and sociocultural approaches inform the teaching of mathematics.

• List and give examples of the five different representations of mathematics concepts and/or processes.

• Discuss the importance of discourse and communication in the mathematics classroom.

• Explain how teachers might scaffold experiences designed to engage students in the Common Core State Standards (CCSS) Mathematical Practices and give an example for each of the eight practices.

Connecting with the Common Core State Standards

Children’s learning in mathematics has received significant attention over several decades—locally, nationally, and even internationally. A broader view of learning envisions children developing “mathematical power” (National Research Council [NRC], 2001, p. 34). Mathematical power involves reasoning logically, solving nonroutine problems, connecting mathematical ideas, and communicating about math. Mathematical power is connected with the vision of mathematical proficiency that the NRC promotes and in the mathematical practices proposed by the CCSS Initiative for Mathematics (CCSSI-M) (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010).

How do students achieve this mathematical power by engaging in the mathematical practices? The National Council of Teachers of Mathematics (NCTM) Principles and Standards for School Mathematics (2000) promotes this learning principle: “Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge” (p. 20). One of the most important tasks elementary teachers can accomplish is to help children develop a deep understanding of mathematics. This chapter focuses on providing teachers with strategies and suggestions for helping children learn mathematics with understanding, thus developing mathematical power.

As children know about mathematics, and what should they be able to do? How do children best learn mathematics? Think about these questions as you view a clip of June, an exemplary teacher who engages her students in reasoning mathematically. For this video, June was asked to teach her students a lesson from a state-adopted textbook in which the focus is on learning a procedure for converting mixed numbers to improper fractions. Her students were then assessed, and several of them were videotaped solving problems. Five weeks later, June taught another lesson focusing on the same topic, only this time she focused on developing understanding of the concept. Again, her students, including Rachel, were assessed and videotaped.
What do you think Rachel learned from the first lesson focusing on learning the rule? After this lesson, Rachel wasn’t able to convert $\frac{37}{8}$ to an improper fraction. She commented, “We did this before. But I don’t really remember it as well, because I didn’t figure it out for myself.” When asked by the interviewer what she meant by that, Rachel said, “Well, when she (the teacher) tells us the answer to something, then I try and find out how she got it. And so when I figure that out, it’s easier. And once I figure it out, it stays there, because I was the one who brought it there.”

When asked if that’s how the teacher usually does it, Rachel responded affirmatively, saying, “And then this little time, it was different. And it was harder.”

Then June taught a lesson on the same topic conceptually, focusing on developing children’s understanding. She commented that “when students create their own algorithms, it is something that they keep forever.”

In the fourth video segment, Rachel was asked to solve the same problem, converting $\frac{37}{8}$ to an improper fraction. She began by applying a rule incorrectly, multiplying the whole number 3 by the numerator 3 and adding 8, then putting that over 8, for a result of $\frac{17}{8}$, which is incorrect. Rachel’s work is shown here:

$$3 \times \frac{3}{8} + \frac{9}{8} = \frac{17}{8}$$

Then Rachel offered another way to solve this problem, drawing one circle cut into eight parts, then drawing two more circles and then three-eighths, as shown below, but she had difficulty finishing the problem.

![Circle diagram](image)

The interviewer asked her how many parts she had in each circle (8), and Rachel was able to correctly determine that there are 27 pieces, which is $\frac{27}{8}$. The interviewer noted that Rachel got two different answers and asked her which one she thought was correct. Rachel thought her second answer, $\frac{27}{8}$, was correct. With some questioning from the interviewer, Rachel developed a correct rule for converting $\frac{37}{8}$ to an improper fraction.

After solving this problem, the interviewer asked Rachel why she began solving this problem with a correct procedure even though she had recently been taught how to solve such problems conceptually. Rachel attributed that to the fact that she was taught the rule first. She said that she’d prefer learning the second lesson, the conceptual lesson, first, because she felt that she “would have remembered how to do it the right way, and the correct way.”

What do you think children need to know about the concept of the lesson: converting between mixed numbers and improper fractions? It seemed that Rachel learned best when she was expected to know more than the rule and when she was expected to understand the concepts behind the process.

Based on this video clip, it seems that Rachel knows that she learns best when she is able to make sense of what she’s doing. What does June think is important in helping children learn mathematics? Did this video clip confirm or challenge any of your beliefs about what’s important in mathematics teaching or learning?
Learning Theories

Theories about how children learn have changed over the past. These theories have a significant bearing on what mathematics is taught and how it is taught. For example, a predominant theory in the late 19th century, mental discipline, viewed the mind as a kind of muscle that required a reasonable amount of exercise to keep it properly tuned. In mathematics, lengthy or complex computations like dividing 5,234 by 378 were carried to multiple decimal places as a major form of exercise. Instruction stressed ways to perform these computations accurately. During much of the 20th century, the behaviorist approach spotlighted children’s external and observable behaviors. It emphasized drill and practice as a means for reinforcing the mathematical ideas much like the mental discipline theory. However, it viewed learning mathematics as occurring through a hierarchy. Children needed to learn and be proficient in addition with the single-digit numbers (1 through 9) before moving to two-digit addition. For children to learn the addition facts to 18, they needed to understand the concepts of addition and place value, the commutative property, and be able to decompose numbers. For example, to solve $13 + 12$, one could decompose 13 into $10 + 3$ and 12 into $10 + 2$ and then find $10 + 10$ first, find $2 + 3$, and then combine $20 + 5$ to find that the sum is 25. Considering what children need to understand and be able to do in order to be successful in solving a problem is helpful when planning instruction. More recent theories fall into two general camps: the cognitive/constructivist approach and sociocultural approach. Each approach provides complementary perspectives on teaching mathematics. The cognitive/constructivist approach emphasizes children’s thinking as they internalize the mathematical ideas; the sociocultural approach focuses on the influence of the classroom environment and learning experiences in shaping children’s understandings.

The Cognitive/Constructivist Approach

Mathematics teaching and learning has been influenced by cognitive or constructivist approaches to education. Because we have opted for very broad categories of learning theories in this section, cognitive and constructive theories are discussed together. For the purposes of this text, it is adequate to consider constructivism as an extension of the cognitive approach.

Constructivism is based on the premise that knowledge is actively constructed by the individual, not passively received from an outside source (Goldin, 1998, 2003; Fosnot, 1996; von Glasersfeld, 1990). Constructivists believe that children must be actively involved in constructing their own understandings of a concept. In other words, according to constructivists, children must play an active role in developing their own understandings rather than passively receiving knowledge. This activity helps them connect new knowledge to their existing knowledge as they internalize their understanding (Kamii, 2004).

The role of the teacher. One way to think of the difference between the behaviorist and constructivist views of learning is with respect to where knowledge comes from and how knowledge is acquired. The behaviorist view is that the teacher or curriculum designer is the source of knowledge and that the main task is to transmit this knowledge to the child, who is a passive recipient. A constructivist, on the other hand, believes that children construct their own knowledge. From the view of constructivists, instruction is learner centered rather than teacher centered.

The constructivist point of view suggests at least two significant implications for teaching mathematics (Kamii, 2004). To support children’s learning, you can:

- Focus on children's thinking rather than on their writing correct answers
- Encourage children to discuss, even disagree, among themselves rather than concentrate on getting right answers and correcting wrong ones

The teacher's role is to structure appropriate experiences so that the child can actively construct meaning. This structure is designed to support children in making connections among the mathematical ideas.
It is difficult to describe a classroom or lesson that would reflect a constructivist approach because there would be considerable variation, based on children's needs. The following elements are often included in constructivist lessons:

- A high level of interaction
- An emphasis on student autonomy or responsibility
- Interaction of children with materials
- Frequent group work

In constructivist classrooms, children frequently interact with the teacher, who encourages, nurtures, and provides help—but often in the form of higher-level questions to encourage children to reflect on what they have done in order to construct meaning. Questions would require children to explain and justify: “Why?” “What does that tell you?” “What can you tell me?” “Why not?” “What do you mean ‘it doesn’t work’?” (Confrey, 1990). Questions involving classifying, giving examples, generalizing, applying, and other higher-level questions would also be asked frequently. Table 2-1 provides examples of different types of questions.

**TABLE 2-1 Questions to Stimulate Communication**

<table>
<thead>
<tr>
<th>Category</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classifying</td>
<td>How are these shapes alike?</td>
</tr>
<tr>
<td></td>
<td>How are they different?</td>
</tr>
<tr>
<td>Hypothesizing</td>
<td>What if …? What could be true here?</td>
</tr>
<tr>
<td>Specializing</td>
<td>Can you give a specific example of how this works?</td>
</tr>
<tr>
<td>Generalizing</td>
<td>Can you see a pattern? Describe it.</td>
</tr>
<tr>
<td>Convincing</td>
<td>How do you know you are right?</td>
</tr>
<tr>
<td>Analyzing</td>
<td>Is this diagram correct?</td>
</tr>
<tr>
<td></td>
<td>What is this all about?</td>
</tr>
</tbody>
</table>

In a constructivist setting, the teacher is responsible for establishing a learning environment that will spark children’s interest and open up areas (topics) of study. You can do this by providing appropriate materials, activities, and reinforcement. Some children may need supplemental or different materials and activities from other children in order to help them move forward in understanding the concept.

**Learner-centered instruction.** Good instruction meets the needs of all children in the classroom. It’s a teacher’s responsibility to determine the learning needs of his or her children and design instruction that helps each child progress. When designing learner-centered instruction, consider the ability, achievement, and special needs of all the children. Throughout this textbook, you will learn more about how to assess the abilities and needs of children and how to use those abilities and needs to make instructional decisions.

**Cooperative learning.** Group work, although not essential, is very likely a feature of a constructivist learning environment. Small-group cooperative learning has been much discussed in current professional literature (Artzt & Newman, 1990a, 1990b; Davidson, 1990a, 1990b; Slavin & Lake, 2008). What is cooperative learning? (Some prefer to use the term collaborative rather than cooperative.) It is both an organization and a process in which a small group of children (usually three to five in number, heterogeneous in both ability and some personal characteristics) work together to complete a task or project or solve a problem. There are many ways to incorporate cooperative learning into the classroom, but they seem to have four elements in common (Artzt & Newman, 1990b):

First, the members of a group must perceive that they are part of a team and that they all have a common goal. Second, group members must realize that the problem they are to solve is a
group problem and that the success or failure of the group will be shared by all of the members of the group. Third, to accomplish the group's goal, all students must talk with one another—to engage in discussion of all problems. Finally, it must be clear that each member's individual work has a direct effect on the group's success. Teamwork is of utmost importance. (pp. 2–3)

NCTM's *Principles and Standards* supports the use of small-group learning in mathematics. "Working in pairs or small groups enables students to hear different ways of thinking and refine the ways in which they explain their own ideas. Having students share the results of their small-group findings gives teachers opportunities to ask questions for clarification and to model mathematical language" (NCTM, 2000, p. 128). Representing, talking, listening, writing, and reading can all be addressed in a cooperative learning setting. It is also an excellent forum for cooperatively and actively exploring a concept with concrete materials.

Davidson (1990a) concludes from research that small-group cooperative learning has a positive effect on “academic achievement, self-esteem or self-confidence as a learner, intergroup relations including cross-race friendships, social acceptance of mainstreamed children, and ability to use social skills (if these are taught)” (p. 54).

Many educators and psychologists have influenced the development of constructivism. We will emphasize the work of one: Jean Piaget. 

**Jean Piaget.** The work of Jean Piaget, a Swiss philosopher-epistemologist, has influenced our thinking about how mathematics is learned. Key concepts in Piaget's theory of learning include schema, adaptation, and operations. A schema is a cognitive structure that one constructs by putting together pieces of knowledge. For example, children might develop a “matching” schema and later an “intuitive qualitative correspondence” schema in order to determine whether two sets are numerically equivalent. For example, a “matching” schema refers to children's being able to line up two sets of objects to show the one-to-one correspondence between the elements in each set. If the elements match exactly, the sets are numerically equivalent. In Figure 2-1, each item in set A is in a one-to-one correspondence with an item in set B, with no items left over, so sets A and B are the same size.

An “intuitive qualitative correspondence” schema is less procedural and more holistic than a “matching” schema. It involves children’s making an intuitive decision about the

![Figure 2-1: Schemas for Determining Whether Sets Are Numerically Equivalent](image-url)
relative size of two sets. In diagram B in Figure 2-1, a child using the “intuitive qualitative correspondence” schema would determine that set A seems to have more elements than set B, without counting or making a one-to-one correspondence.

Schemas are developed by a process of adaptation, which can take two forms: assimilation and accommodation. Assimilation of a schema occurs when one’s existing cognitive structure requires little modification to include the new idea, and the new idea can be added to the existing structure. On the other hand, if no relevant schemas exist, new behavior sequences are built up through experimentation, instruction, or both. Piaget calls this process accommodation. The mechanism by which a schema is assimilated or accommodated into one’s cognitive structure is an operation, an internalized action that can modify knowledge. Putting things into a series (e.g., arranging sticks from shortest to longest) and constructing a classification (e.g., sorting laundry by putting the whites in one pile and the colors in another pile) are examples of operations that help modify children’s mathematical understandings.

Educational implications. Most children in the elementary grades are in what Piaget called the concrete operational stage. This means that elementary school children will learn mathematical concepts by manipulating materials and observing what happens. You must provide the kind of concrete experiences that will facilitate learning. Even for the middle school level, experience and research suggest that children are not yet able to think about many concepts at an abstract level and, in fact, still need concrete representations. Visual learners may continue to find such models helpful throughout their educational careers.

Piaget’s conservation tasks provide us with methods for assessing readiness for certain concepts. Conservation involves recognizing that a change or transformation does not change the property in question. For example, a row of 10 counters still contains 10 counters when spread out or pushed together. “Spreading out” and “pushing together” are transformations that don't change the quantity of counters, but children who are non-conservers don’t realize this. A number conservation task is described in Chapter 5, and several measurement conservation tasks are described in Chapter 15.

Piaget’s work also has implications for curriculum sequencing. For example, children seem to conserve numbers early in the concrete operational stage, while mass and volume are not conserved until the end of this stage. These findings suggest that formal measurement of area and volume should be delayed until intermediate grades.

Piaget’s observations on knowledge development suggest that the ideal learning environment is one that allows the elementary and middle school child to explore ideas. In mathematics learning, this is most effectively done with the aid of concrete manipulative materials.

The Sociocultural Approach

Learning theorists have more recently emphasized the situation within which the learning occurs—the classroom experience. Lave and Wenger (1991) and Wenger (1998) characterize the importance of social participation and active involvement in a community as an essential ingredient in learning. This situated perspective claims that how and where a child learns an idea is fundamental to what the child learns (Cobb & Bowers, 1999; Greeno, Collins, & Resnick, 1996). From this view, teachers must focus on the classroom environment and experiences in which children are learning (Putnam & Borko, 2000).

The sociocultural approach draws heavily on the work of Vygotsky (1896–1934), a Russian psychologist. Vygotsky describes children’s learning as “cultural” where they interact with others, objects and events in the classroom environment. He adds to this notion that children’s ability to solve a problem is affected by a window of opportunity that he labeled as the zone of proximal development, or ZPD (Vygotsky, 1962; Wertsch, 1991). The lower limit of this zone begins with the child’s previous knowledge, concepts, and skills and the upper limit is determined by the tasks that can be successfully completed only with step-by-step instruction.

The role of the teacher. The sociocultural perspective directs teachers’ attention to designing a classroom environment where children engage in developmental processes as individual learners. Vygotsky states that children’s learning occurs because of the social interaction that happens before their development of the ideas (Slavin, 2003).
For him, children learn through play that establishes a ZPD for each child. The teacher's role is to structure the social activities in ways that engage children in play in a community culture and leads to the development of new knowledge within the ZPD for each of the learners. It is important to recognize that the mathematical community that is established does result in both a common ZPD across the learners as well as ZPDs for individual learners (Cobb, 1994; Goos, 2004).

Bransford, Brown, and Cocking (2004) extend the view, describing teachers as coaches who scaffold children's thinking and activity. They propose that “young children come to school with many ideas about mathematics” and teachers need to engage them in thinking about their everyday experiences as springboards for further learning (p. 171). They suggest these considerations when designing the instructional experiences:

- Motivate the children's interest in the task
- Simplify the task to make it more manageable and achievable
- Provide some direction to help the children focus on meeting the goal
- Reduce the children's frustration and risks in the activities
- Model and clearly define the expectations of the activities and tasks

Vygotsky believed that as children talk, they internalize the words and begin to reason with these ideas. In support of this process, teachers need to encourage the children to share their ideas in the classroom activities as they enter their ZPD with respect to the new mathematical ideas. The communication among the children and the teacher is an essential part of helping children “construct links between their informal, intuitive notions and the abstract language and symbolism of mathematics” (NCTM, 1989, p. 26). As they engage in discussion about the mathematical ideas with others, they find themselves in a zone that is somewhere between their current understandings and their potential understandings (Vygotsky, 1962). In this process, the teacher provides the children with new information that they are able to assimilate with their current knowledge as they formalize their understandings.

**Scaffolding.** Scaffolding is an instructional approach consistent with Vygotsky's ideas for guiding children's learning in mathematics, an approach where the teacher gradually releases the learning responsibility to the children (Pearson & Gallagher, 1983). NCTM's *Principles and Standards* (2000) describes effective teaching as requiring that teachers have a solid understanding of what the learners know and need to learn. To gain this understanding, teachers must listen carefully to children's ideas and explanations and use that information when making the instructional decisions in a scaffold that supports and challenges children in the learning process. As they identify, design and implement activities and tasks for guiding children's learning, they must be aware of when support is no longer needed and when children are able to continue on their own. Anghileri (2006) proposes a research-supported hierarchy of three levels for scaffolding children's learning of mathematics:

- **Level 1:** Environmental provisions support learning without direct intervention from the teacher. This level provides an environment with multiple tools, such as building blocks for free play and grouping children for peer collaboration in solving problems. At this level, math manipulatives are particularly useful. Children have difficulties in understanding and distinguishing fraction numerators and denominators, for example in determining that one-half is larger than two-fifths. At this first level, children play with fraction tiles, building structures where they have opportunities to visualize the relationships among the tiles. In small groups, they interact in the design of multiple structures with the multiple tiles and in the process become familiar with the relationships among the tiles.

- **Level 2:** Explaining, reviewing, and restructuring is the level where the teacher's goal is to refocus the children's attention and give them further opportunities to develop their own understanding rather than relying on an explanation from the teacher. At this level, small groups might have one of each of the tiles and may be challenged to construct a rectangle using all the different tile lengths. They soon realize they need a different number of tiles for each of the colored tiles, and they must request additional tiles from a “tile distributor” in order to complete their rectangle. Here teachers need to ask children to look, touch, and verbalize what they observe and
are thinking. Teachers prompt and probe with questions in ways that ask children to explain and justify their actions for the number of additional tiles.

Echevarria, Vogt, and Short (2013) describe some teacher practices (adapted from Brown, 2008) for gradually releasing the learning responsibility to the children:

- Explain the benefits of strategy use in general and the value of using specific strategies
- Mentally model (e.g., think-aloud) to make thinking transparent to students
- Provide guided and independent practice so that student learn to use strategies when cued by a diverse array of goals, needs, task demands, and texts
- Promote independent strategy use by gradually shifting responsibility for strategy application to students

- **Level 3: Developing conceptual thinking** is where the teacher engages the children in conceptual discourse, assisting them in making connections and extending their thinking. At this level, additional representational tools support children’s moves from informal to more formal mathematical terms for expressing, communicating and reflecting on the mathematical activities. For example, instead of using tiles, the teacher might add fraction circles, pattern block fractions, or virtual representations of the fraction tiles where they are able to use sliders to adjust the denominators of the fractions. At this level, teacher questioning is critical for engaging the learners in discourse as they make connections with mathematical abstractions.

**Manipulatives.** Teachers play important roles when they make decisions for scaffolding children’s learning. Hiebert et al. (1997) encourage the use of mathematical tools to build a foundation for children's understanding. In addition to listening to children talk about their ideas, teachers gain additional information about children’s understandings by observing them manipulate physical objects such as base-ten blocks, pattern blocks, fraction tiles, and so on. Math manipulatives potentially provide concrete representations of abstract concepts to support children in making connections between their informal knowledge and experiences and mathematical abstractions. Groups of children can work together, where they talk about and explain their ideas to each other. Manipulatives provide opportunities where children can be physically engaged with the ideas, using the objects as tools for communicating their thinking.

With pattern blocks, children might explore the relationship of the green triangle to the yellow hexagon, showing how many triangles will cover the hexagon. How many of the blue rhombi are needed to cover the red trapezoid? How many different ways can they cover the yellow hexagon with combinations of the blue rhombus, the red trapezoid, and the tan kite? Such activities lead to discussions about the relationships among the multiple pattern blocks, such as “The red trapezoid is one-half of the yellow hexagon because it takes two of these red trapezoids to cover the yellow hexagon.” In other words, these blocks provide children with ways to communicate their thinking with the handheld objects where they have the responsibility for explaining the mathematical patterns and ideas.

The most challenging process for incorporating manipulatives in instruction is identifying ways to facilitate the children’s ability to transfer what they do with the manipulatives to conceptual and procedural understandings. While the handheld objects provide initial experiences, virtual representations aid in the translation between the physical objects and children’s conceptual understandings (Moyer, Bolyard, & Spikell, 2002). NCTM (2000) advises that “work with virtual manipulatives … can allow young children to extend [the] physical experience and to develop an initial understanding of sophisticated ideas like the use of algorithms” (pp. 26–27). Many virtual manipulatives or applets are available on the Internet. These virtual options often present children with a visual object (representing the handheld blocks), symbolic or numeric expressions, and verbal or written explanations simultaneously on the screen as shown in Figure 2-2. Here the Math Playground website provides multiple representations that enable children to link their informal, intuitive notions with the abstract language and symbolism of fractions. With these virtual fraction manipulatives children use the sliders to change the “number of parts” in the fractions. The problem is to find a fraction between one-half and four-fifths. First, the child determines the least common denominator and then moves the slider for the divisor for one-half to 10 and changes the divisor for four-fifths to 10. The
visual refreshes to display the tiles cut into tenths, the numeric symbols for the fractions and the graphical representations of the fractions. When the child enters $7/10$ as a fraction between the two other fractions, the location is added to the graph, demonstrating that it is a fraction between those two fractions.

The Internet provides many more virtual manipulatives for scaffolding children’s learning of mathematics concepts and processes. The National Library of Virtual Manipulatives is indexed by grade level and content, such as number and operations, algebra, geometry, measurement, data analysis, and probability. NCTM provides Illuminations with multiple virtual applications and games, as well as lessons for using them. With iPads more and more frequently available in classrooms, lots of apps provide virtual manipulatives for teachers to consider when scaffolding children’s learning in mathematics.

**Discourse.** Teaching for understanding requires that teachers have a good sense of each child’s ZPD. Recognizing that individuals have different mathematical understandings of particular concepts, teachers must differentiate the understandings. Engaging children in discourse is an important strategy for gathering their differing understandings. Remember Vygotsky’s belief that as children talk they internalize the words and begin to reason with these ideas. Engaging children in classroom discourse not only communicates the differences in their understandings, but also provides opportunities for them to formalize their understandings. Their communications and explanations assist their teachers in scaffolding the instruction through the three-level hierarchy (Anghileri, 2006). NCTM (2000) warns that “it is important to avoid a premature rush to impose formal mathematical language; students need to develop an appreciation of the need for precise definitions and for the communicative power of conventional mathematical terms by first communicating in their own words” (p. 63).

The community in which the discourse occurs might be in partners, small groups, or the whole class. Lamberg (2013) describes the different types of “math talk” where each serves a “different purpose and contributes to the development of new mathematical insights” (p. 5). The whole-class discussions often proceed in three phases: launching, where the teacher introduces the children to the problem, the tools, and the expectations; exploring, where students work on the problem with partners or in small groups; and discussing and summarizing, in a whole class discussion where various student-generated approaches are examined (Lampert, 2001; Sherin, 2002). Communication and work with partners and small groups are limited by the perspectives of those in the discussion. As a result, misconceptions may not be challenged and may lead to additional misconceptions. Combining these discussions with whole-class discussions provides opportunities for challenging misconceptions as well as presenting multiple perspectives that had not been examined in the small group talk.

![Virtual Equivalent Fraction Manipulatives from Math Playground](source: www.mathplayground.com/visual_fractions.html)
The role of the teacher in the whole-class discussion is to “pose questions and scaffold the discussion by making decisions about how to sequence or facilitate the discussion so that there is a logical progression of ideas that everyone can understand” (Lamberg, 2013, p. 6). The challenge that teachers face when orchestrating the discussions is to maintain the focus on students' thinking rather than “sanctioning particular approaches as being correct” or communicating the correct response (Stein, Engel, Smith, & Hughes, 2008, p. 315). Stein et al. (2008) identify five practices that can be planned in advance rather than resorting to improvisation in the discussion:

1. Anticipating likely student responses to cognitively demanding mathematical tasks
2. Monitoring students’ responses to the tasks during the explore phase
3. Selecting particular students to present their mathematical responses during the discuss-and-summarize phase
4. Purposefully sequencing the student response that will be displayed
5. Connecting different students’ response and between students’ response and key ideas (p. 321)

They conclude that these practices provide “a reliable process that teachers can depend on to gradually improve their classroom discussions over time” (Stein et al., 2008, p. 335). Learning to ask questions during the whole class discussion is a challenging teacher task that is essential for successful discourse where children are able to make mathematical connections in ways that support their ZPD. Review the Sample Lesson in this chapter, where the children are solving problems involving multidigit addition. Note the sample questions that the teacher proposes as possible follow-up questions for the multiple processes for multidigit addition presented by the children.

### Connecting the Learning Principles with the CCSS Mathematical Practices

Several basic mathematical practices can be derived from the work of Piaget, Vygotsky, and many others for support when considering teaching for understanding as suggested in the CCSS:

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

The following six guidelines, based on the work of learning theorists, are essential to making mathematics instruction meaningful for all children and for supporting them in developing these mathematical practices.

1. Begin with concrete representations
2. Develop understanding
3. Encourage communication
4. Make connections
5. Motivate children
6. Provide opportunities for practice

### Begin with Concrete Representation

Making sense of problems and persevering in solving them (CCSS Mathematical Practice 1) are mathematical practices that teachers need to consider as they scaffold
children’s mathematical experiences. Children seem to learn best when learning begins with a concrete representation of a mathematical concept. In fact, it is best to provide children with multiple embodiments of mathematical concepts much like the variety of concrete representations for fractions (e.g., fraction tiles, fraction circles). Manipulatives are essential when engaging children in conceptualizing and solving problems. This does not mean that concrete manipulatives should be used exclusively. Virtual manipulatives add dynamic capabilities where children can duplicate and extend their actions with the concrete representations, thus connecting mental images with the resulting virtual images (Clements & McMillen, 1996; Moyer et al., 2002). Teachers need to scaffold the experiences with multiple representations of the concepts to support their sense making and finding of solutions to mathematical problems.

When building an understanding of the addition algorithm, have children set out bundles of popsicle sticks (or equivalent objects) and manipulate them to represent the process of addition (Figure 2-3). The use of different manipulatives at different times helps children abstract the essence of the concept and lends variety to the mathematics program. This action adds to children’s engagement in reasoning abstractly and quantitatively (Mathematical Practice 2) and in their appropriate strategic use of tools (Mathematical Practice 5).

Manipulatives do not guarantee success, but you can take steps to promote success when planning a lesson that includes manipulatives where you purposefully scaffold the instruction, as described in the three levels described by Anghileri (2006) for developing children’s understanding. Remember the importance of Level 1 of the hierarchy. Provide time for free play where children have opportunities to explore, looking for and making use of the structure and features of the manipulatives, as described in Mathematical Practice 7.

**Develop Understanding**

What is the difference between knowing how to do something and understanding it? Think about multiplication. Knowing how to multiply usually means knowing a procedure to
find the answer. But understanding multiplication means realizing what multiplication means and being able to identify situations in which multiplication could be used.

A goal of mathematics instruction is for all children to develop a deep understanding of the concepts. Scaffolding activities and experiences to guide children in a process that results in building understanding is the essence of teaching.

**Modes of representation.** One way to help children develop understanding is to carefully select the *modes of representation* they use in instruction, or the way in which mathematics concepts are represented (Behr, Lesh, Post, & Silver, 1983). Five modes of representation include real-world situations, manipulative models, pictures, oral language, and written symbols (see Figure 2-4). For example, the concept of five might be represented with five fingers (real-world situation), with five Unifix cubes (manipulative model), with a picture of five flowers (pictures), by saying the word *five* (oral language), and by writing the word *five* or the symbol 5 (written symbols).

![Figure 2-4 Modes of Representation](image)

Using the preceding example of the concept of five, you might ask a child who has displayed five fingers whether she or he could show the same number in another way, such as by drawing a picture of five objects or picking up a group of five blocks. This is called *translation*. Translation refers to asking children to represent a concept in more than one mode and is indicated by the arrows in Figure 2-4.

Why is this concept important? Research shows that instruction in which children are encouraged to translate between modes of representation enhances children's understanding. In this process, they are engaged in multiple mathematical practices—modeling what they know about the mathematics (Mathematical Practice 4), using appropriate tools (Mathematical Practice 5), and attending to precision (Mathematical Practice 6) as they communicate their understandings. Good teachers ensure that their lessons include a variety of modes of representation and opportunities for children to make translations between modes.

Hiebert (1990) states, “Meaning or understanding in mathematics comes from building or recognizing relationships either between representations or within representations” (p. 32). Here children are engaged in Mathematical Practice 7 as they look for and make use of the mathematical structure.

Building relationships between representations occurs, for example, when a child listens (spoken language) to a problem, represents and manipulates it with blocks (concrete objects), and then writes a response on paper (written symbols).

Building relationships within representations often involves recognizing patterns within the representation. Hiebert (1990) uses the base-ten blocks in a decimal context...
as an example. In this setting, children can recognize the “pattern of repeated partitioning by 10 and the corresponding decrease in the size of the blocks” (p. 33), which could go on forever if the blocks could be cut finely enough.

**Reflection** enhances children’s understanding of a concept. Reflection is often needed to observe patterns within a representational system. Teachers or children can encourage reflection by asking appropriate questions or by challenging each other’s observations. Note again the importance of communication as described in the practice of attending to precision (Mathematical Practice 6) and looking for and expressing regularity in repeated reasoning (Mathematical Practice 8). For example, you might ask a child to compare his or her solution strategy with that of another child and to describe the differences.

Understanding new concepts is more likely to occur when children understand prerequisite concepts than when they have only a superficial understanding or when they have learned previous skills and concepts by rote. Just as we would not consider building a house without proper footings, so children cannot build mathematical structures without meaningful prerequisite learning. For example, children who do not have a good understanding of the place-value concept often have considerable difficulty ordering decimal fractions. Rather, they sometimes arrange the decimals based on the number of digits (see Chapter 12).

**Making connections.** Van de Walle, Karp, and Bay-Williams (2013) point out that understanding is demonstrated when connections are formed between procedural knowledge and conceptual knowledge. **Procedural knowledge** is the knowledge of the symbolism used to represent mathematical ideas and the rules and procedures used to perform a mathematical task. **Conceptual knowledge** consists of relationships that connect a number of mathematical ideas or concepts. For example, the concepts of addition and subtraction are related; that is, knowing that $6 + 5 = 11$ helps one think about $11 - 5 = 6$. This conceptual knowledge greatly helps children acquire the procedural knowledge of addition and subtraction computation while engaging them in reasoning abstractly and quantitatively as in Mathematical Practice 2. An example that nicely illustrates the difference between knowing how and knowing why involves division of common fractions. Recalling and applying the rule “invert and multiply” is procedural knowledge; being able to explain or justify why it works is conceptual knowledge and action where they are actively involved in looking for and making use of mathematical structure (Mathematical Practice 7). Being able to see or make connections between conceptual knowledge and procedural knowledge is what Skemp (1989) calls relational understanding (Figure 2-5), an important goal to help children think mathematically.

![Figure 2-5](image)

**Encourage Communication**

Communication plays an important role in children’s mathematics learning. It “forces” children to think through a concept, often resulting in more refined understanding as they attend to precision in their explanations (Mathematical Practice 6). Highlighted in the NCTM standards, communicating in mathematics means encouraging children to engage in interactive conversations as they work through mathematical processes. In the process, they must construct viable arguments and critique the reasoning of other class members (Mathematical Practice 3). Such interactions can help them clarify what they do or do not understand about mathematical concepts or processes.
“Math talk” contributes to the development of mathematical insights and understandings (Lamberg, 2013). Communication in a mathematics classroom can take many forms. It can be oral or written. It can be from child to child or between a child and the teacher. Multiple types of these conversations serve different purposes. Partners can brainstorm ideas; small groups can share their ideas; and the whole class can talk about the differences and similarity of the ideas. The teacher’s role is to facilitate the whole-class discussion in ways that assure students learn mathematics through communicating (Lamberg, 2013). See Literature Links 2-1 and 2-2 for examples of using children's books to support communication.

Problems provide many opportunities for communication where they are able to model their ideas with mathematics (Mathematical Practice 4). In addition, talking and writing about mathematics helps children solidify their understanding of systems, such as the quipus used by people of the Inca empire in Peru. Quipus were made from colored strings made of dyed cotton or wool. Using 10 as the base, the knots in the cords appeared in the 100s, 10s, and units positions. Children can easily represent these cords with colored string or yarn while reinforcing their understanding of our own system of place value.

• Encourage children to create their own system of symbols for 1–30 and beyond. Have children describe how they will write very large numbers in their system.
• Demonstrate for children how to count using the ancient system of body counting presented in the text. How might ancient people recall which number is represented by a part of the body?
• Many objects have been used for counting throughout recorded history. Model interesting and efficient record-keeping systems, such as the quipus used by people of the Inca empire in Peru. Quipus were made from colored strings made of dyed cotton or wool. Using 10 as the base, the knots in the cords appeared in the 100s, 10s, and units positions. Children can easily represent these cords with colored string or yarn while reinforcing their understanding of our own system of place value.
• Use paint and Styrofoam cutouts of numerals in different numbers systems (such as Egyptian, Mayan, Babylonian, or Roman) to stamp numbers on cloth.
• Investigate the different ways that time has been recorded using ancient clocks (sundials, water clocks, sand clocks) and calendars (Aztec calendar). Have children collect and analyze information and draw illustrations to accompany their text information. Create a class book on the way time was recorded throughout history and call it The History of Time.

“A critical mode of representation for children is oral language. Children learn to represent and defend their mathematical ideas through the use of language. G is for Googol: A Math Alphabet Book is filled with interesting mathematics vocabulary. The definitions are child friendly while maintaining mathematical accuracy. Mathematics words are given with their meanings, and the diagrams and illustrations support the explanations presented in the text. Promoting the use of accurate mathematical language supports the CCSS mathematical practice of attending to precision in the language students use to communicate mathematical ideas.

• In addition to the mathematical terms given for each letter, other words on each page are presented for children to investigate. For example, on the “P is for probability” page, P is also for palindrome, parabola, parallel, percent, pi, point, polygon, prime number, and Pythagorean theorem. Have the children investigate and make connections to find the mathematical significance of these additional words.
• Design your own mathematics alphabet book. Select one of the additional words listed in the book and create a child-friendly mathematical definition. Along with the definitions, use drawings, diagrams, and examples that would help children to understand the meaning of the mathematical terms.
• Copy and assemble the children’s definition pages into one large alphabet book so that each child has his or her own copy for reference.

Why Do We Count?

Children use their own representations as a part of their mathematical development. Yet it is also important that they learn common representations, such as the numerals and number words used in counting and calculating in the base-ten system. In the historically accurate and beautifully illustrated The History of Counting, children can investigate the development of this precise and efficient system.

• Encourage children to create their own system of symbols for 1–30 and beyond. Have children describe how they will write very large numbers in their system.
• Demonstrate for children how to count using the ancient system of body counting presented in the text. How might ancient people recall which number is represented by a part of the body?
• Many objects have been used for counting throughout recorded history. Model interesting and efficient record-keeping systems, such as the quipus used by people of the Inca empire in Peru. Quipus were made from colored strings made of dyed cotton or wool. Using 10 as the base, the knots in the cords appeared in the 100s, 10s, and units positions. Children can easily represent these cords with colored string or yarn while reinforcing their understanding of our own system of place value.

Source: Dr. Patricia Moyer-Packenham, Utah State University.

Learning to Talk Mathematically!

A critical mode of representation for children is oral language. Children learn to represent and defend their mathematical ideas through the use of language. G is for Googol: A Math Alphabet Book is filled with interesting mathematics vocabulary. The definitions are child friendly while maintaining mathematical accuracy. Mathematics words are given with their meanings, and the diagrams and illustrations support the explanations presented in the text. Promoting the use of accurate mathematical language supports the CCSS mathematical practice of attending to precision in the language students use to communicate mathematical ideas.

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• Design your own mathematics alphabet book. Select one of the additional words listed in the book and create a child-friendly mathematical definition. Along with the definitions, use drawings, diagrams, and examples that would help children to understand the meaning of the mathematical terms.
• Copy and assemble the children’s definition pages into one large alphabet book so that each child has his or her own copy for reference.

Source: Dr. Patricia Moyer-Packenham, Utah State University.
Mathematics. They can communicate solutions via reports, stories, word problems for other children to solve, descriptions of how children solved particular problems, or entries in a math journal. Keeping a journal is generally associated with the language arts discipline, but a math journal can be used to reinforce a child’s understanding of mathematical concepts.

**Math journals.** A math journal offers an opportunity for children to think about and write about the mathematics concepts they are learning. Further, it provides teachers with an excellent assessment tool. Journals should be a regular part of mathematics class activities and can be included routinely as part of homework assignments.

Math journal prompts might include the following:

- *I think the answer is …*
- *I solved the problem by …*
- *Another way to solve the problem would be …*
- *I still have a question about …*
- *The thing I liked most was …*

Communication in a mathematics classroom provides you with valuable insights into children’s understanding and helps you plan further in scaffolding your instructions in ways that engage them in multiple mathematical practices.

**Make Connections**

When children build connections between mathematical ideas and other content areas, mathematics becomes more meaningful, and understanding is enhanced. Using a thematic approach is one way to build an integrated curriculum because it can address not only basic skills but also more open-ended and higher-level objectives. Individual interests and other individual differences may be more easily accommodated in a thematic unit. The cooperative learning approach lends itself to thematic units.

Thematic units also provide opportunities to connect mathematics to real life through field trips and related activities. For example, as part of a theme about products and consumers, children might visit a nearby shopping mall (with permission from the administration) to conduct discussions with retail shop owners to see how mathematics knowledge is useful in their jobs. They could also calculate the total cost for each member of their small group to purchase a particular snack at one of the food outlets or observe and describe geometric shapes in the mall decor; observe slides, flips, and turns in shop logos; and so on.

Even without a thematic approach, however, many opportunities to integrate mathematics with other subjects are encountered daily. For example, an art teacher might reinforce one-to-one correspondence by having a child distribute one paintbrush to every child in class. In physical education, children might measure distance and time, count when skipping, and keep track of scores and other statistics in games. Social studies and science lessons offer many opportunities for creating and interpreting graphs. Make sure to seize and discuss serendipitous opportunities so that children “see” the connection of mathematics to their in-school and out-of-school experiences. These activities engage the children in mathematical thinking as emphasized in the CCSS mathematical practices.

Connections are not automatic. You must provide experiences in which the connections are obvious or at least where they can be made explicit, as described in the previous paragraph. This will encourage children to look for other connections and eventually to recognize the pervasive nature of mathematics in the world around them.

**Motivate Children**

Motivation fuels mathematical learning. If children are motivated, they attend to instruction, strive for meaning, and persevere when difficulties arise. Competent teachers, effective instructional models, and thought-provoking activities guide the process, but children must first be motivated to learn mathematics. (Holmes, 1990, p. 101)

Motivating a child encourages the child to give attention, time, energy, and perseverance to learning. It is the willingness to accept the challenge to understand a concept or solve
a problem, as described in Mathematical Practice 1 (“Make sense of problems and persevere in solving them”). Motivation is associated with the belief that one can succeed. Almost all children begin kindergarten with this belief. As the years pass, some lose faith in their ability, especially in mathematics. Thus, the level of motivation is one of many ways in which children within the classroom differ.

Although motivation is largely internal to each child, there are strategies you can employ to increase motivation. On a general level, individuals become motivated when the concepts they are learning are meaningful and when they experience satisfaction, success, and recognition. Communication and meaningful opportunities for students to engage in mathematics conversations about real-world problems can be very motivating, in addition to enhancing children's understanding of mathematics concepts. Give children meaningful tasks and assignments at which they can be successful, and then recognize their achievements. Engage the children in discussions through carefully crafted questions to draw them into mathematical conversations where they construct viable arguments and critique the reasoning of others (Mathematical Practice 2).

More specifically, there are differences in what motivates children. So-called academically inclined children are motivated by achievement. Special challenges such as puzzles, nonroutine problems, and strategy games will capture their attention and increase motivation. Other children experience increased motivation when they can see the utilitarian value of what they are learning. Application or real-world types of activities should be designed for these children. This does not imply that certain children are given one type of experience exclusively. All children should experience different types of activities, although some may simply opt for a larger dose of one type than another. Variety in activities helps to enhance motivation.

In the literature, motivation has often been categorized as extrinsic (grades, stars, etc.) or intrinsic (internal interest and desire to learn). The goal of teachers is to engage children in motivating activities that offer extrinsic experiences and “light an intrinsic fire” related to mathematics in children.

**Attitudes.** Attitudes are an important part of motivation. Children who feel good about mathematics and their ability to do mathematics are often motivated to learn and engage in learning, as described in Piaget's assimilation and accommodation constructs. On the other hand, children who have negative attitudes about mathematics or their ability in mathematics often exhibit disinterest. Since attitude is connected with achievement in mathematics, it is important for you to provide experiences and the kind of environment that will foster positive attitudes. Minimal stress, emphasis on meaning and understanding rather than memorization, successful experiences, engagement in whole-class discussions about concepts and ideas, meaningful use of manipulatives, relating mathematics to the real world, and meaningful small group work are some generalized guidelines for fostering the development of positive attitudes while also engaging them in mathematical practices.

It is essential to add that your attitude toward mathematics also is influential in forming children's attitudes. Being positive and enthusiastic while engaging in mathematics can make a difference in children's success. Your interest in mathematics and its connections to everyday life can affect your children's motivation to learn mathematics.

**Provide Opportunities for Practice**

The belief that mathematics needs to be meaningful and the idea that children construct their own mathematical knowledge do not rule out the need for practice. Practice contributes significantly to making routine procedures automatic. Practice results in more efficient execution of a procedure and, thus, to the expenditure of less mental effort (Hiebert, 1999). Expending as little mental effort as possible on a routine procedure is important because it allows one to give more effort to a more complex task of which the routine is only a part. If a person has to devote too much effort to the routine task, attention to the major task may be lost.

Practice does not have to be dull and boring, though. Games, puzzles, riddles, little surprises, and calculators all are useful ways of providing practice where they persevere...
in solving the problems (Mathematical Practice 1). Of course, this is not to say that work-
sheets, flash cards, and other traditional means of providing practice should not be used.
The difference is in the purpose: thinking versus rote memorization. Classroom discus-
sions might be used to engage the children in a community for sharing their various
approaches to routine procedures in more complex tasks where they are also engaged in
critiquing the reasoning of others (Mathematical Practice 3). Check the follow-up ques-
tions the teacher identified for extending the children's ideas about multidigit addition in
the Sample Lesson for this chapter. The following guidelines may be helpful in selecting
activities for practice purposes. As a teacher, though, you will want to recognize that any
one activity may not meet all of the guidelines.

To be effective, practice activities should:
• Be based on a well-defined cognitive objective; they should not be “busy work”
• Be self-motivating and fun
• Make use of the concept or procedure being reinforced in a new and interesting form
• Be self-checking; that is, children should know when they have done it correctly
• Be adaptable for use with the whole class, a small group, or an individual child
• Provide for extension of knowledge; further exploration of an idea should be

Games. Children enjoy playing games and games provide activities where children
look for and make use of structure (Mathematical Practice 7). Games certainly are self-
motivating and fun and, in fact, meet most of the preceding guidelines for practice activ-
ities. Many games that meet specific mathematics objectives are available at educational
and general merchandise stores, and are available for computers and handheld digital
games. More importantly, you can create or adapt games that are appropriate for the
curricular objectives. For example, you might create a template (Figure 2-6) for a bingo-
like game. By changing what goes in the cells and the nature of the calling cards, you can
create bingo-type games to reinforce many different mathematical ideas and procedures.
(If templates are created with a word processor, changes can be made quickly and easily.)
Note the use of “I can” in the conventional “Free” space in Figure 2-6. Messages such as
this can help to enhance a child's self-concept with respect to mathematics.

Puzzles and riddles. Various forms of puzzles and riddles can be used to provide
interesting, often self-checking practice. Puzzles can range from simple join-the-dot
pictures to complex pattern recognition. Magic squares could also be included in this
category. The sample puzzle in Figure 2-7 involves placing numbers from 1 through 14,
3 in each box, so that the sum in each circle is 21. No number may be used more than
once. Note that 3, 6, 7, 10, and 12 have been placed as starters.

Riddles are closely related to puzzles. The Internet can help you find riddles such as
“What do you get when you cross a snowman and a wolf?” It takes only a few minutes to

Figure 2-6 Bingo Template

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</tbody>
</table>

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category. The sample puzzle in Figure 2-7 involves placing numbers from 1 through 14,
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once. Note that 3, 6, 7, 10, and 12 have been placed as starters.

Riddles are closely related to puzzles. The Internet can help you find riddles such as
“What do you get when you cross a snowman and a wolf?” It takes only a few minutes to
turn these into interesting and useful practice exercises. Children can solve these riddles virtually by completing a set of practice problems and then using a decoder you provide to convert each answer to a letter. In our riddle question, a solution, where you match the answers for multiplication problems to the letters, spells out the answer to the riddle: FROSTBITE.

**Surprises.** Many little “surprises” in mathematics can be used as warm-up activities at the beginning of a class and simultaneously involve the children in reasoning abstractly and quantitatively (Mathematical Practice 2). For example, you might give the following instructions:

1. **Write a three-digit number in which the first and third digits are different by at least 2.**
2. **Reverse the digits to create a second number. Subtract the smaller number from the larger.**
3. **Reverse the digits in the difference and add to the difference.**
4. **Did you get 1,089?**

<table>
<thead>
<tr>
<th>5 9 1</th>
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<tr>
<td>1 9 5</td>
</tr>
<tr>
<td>3 6 9</td>
</tr>
<tr>
<td>6 9 3</td>
</tr>
<tr>
<td>1,089</td>
</tr>
</tbody>
</table>

A challenge such as “Do you think this will work for all three-digit numbers?” will motivate children to do more of these problems. In the process, they will get a great deal of practice with addition and subtraction—and you don’t have to photocopy a worksheet. This challenge could also be extended to a problem-solving activity by asking children to explain why this process works where they make sense of the problem as in Mathematical Practice 1.

**The calculator as a practice tool.** The calculator can be used to provide practice with concepts as simple as counting, for estimation, and for more complex calculations. Even children in the primary grades can use the built-in constant feature (now standard on almost all inexpensive calculators) to verify a counting sequence starting at any number, to count on, to count back, and to skip count. (Chapter 5, page 114, elaborates on how this can be done.) Activity 2-1 is an example of a calculator activity providing practice with the concept of numeration.

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**Thinking about Teaching**

Teaching mathematics requires thinking about three things: how children learn, the teaching process, and what to teach. The first has already been discussed; the latter two
ACTIVITY 2-1  Place Value on a Calculator

Materials
Calculators

Procedure
1. Player A announces a number that all other participants enter into their calculators. (For example, Player A selects the number 972.)
2. Player A then announces which digit in the chosen number is to be changed to zero without changing any of the other digits. (For example, Player A wants the 7 in 972 to be changed to a zero—so the calculators would display the number 902.)
3. All remaining players attempt to change the given digit to zero by doing only one subtraction. (The correct answer in this example would be to subtract 70, because 972 - 70 = 902.)

are the focus of this section, but first we start with a discussion of understanding the needs of each child.

Understanding Individual Needs

When thinking about teaching, we must consider factors related to meeting the needs of all learners. What is the role of communication in teaching and learning mathematics? How does the language of math affect children’s understanding of math? How can we help English language learners succeed in mathematics? How can we prevent math anxiety? What are some myths about gender-related issues in learning mathematics?

Role of communication revisited. Earlier in this chapter, we stated that communication about mathematics significantly influences the mathematics curriculum and the learning of mathematics.

Communication also is an important factor in daily teaching activities and deserves additional comment here. NCTM’s Principles and Standards (2000) states:

When students are challenged to think and reason about mathematics and to communicate the results of their thinking to others orally or in writing, they learn to be clear and convincing. Listening to others’ explanations gives students opportunities to develop their own understandings. (p. 60)

Why are communication skills important in mathematics? Primarily because they help children clarify their thinking and sharpen their understanding of concepts and procedures. Representing an idea or a problem in a different form, talking about a concept or an algorithm, listening to explanations by others, writing a definition in our own words, and reading textual material all contribute to an individual’s building mathematical understandings. Allowing children to grapple with the ideas and use their own ways of communicating these ideas is important in their engagement in the ideas while simultaneously engaging them in the CCSS mathematical practices (CCSS, 2010; NCTM, 2000)

Reuille-Irons and Irons (1989) identify four sequential stages in which the development of language in mathematics can occur:

1. Child’s language. This is the natural language of the child.
2. Material language. This is language that might be associated naturally with a specific representation of a mathematical idea. “Cover up” might be an example of a material-specific expression if pictures are being used to represent a subtractive situation.
3. Mathematical language. This involves using a word or short phrase for the mathematical operation:
   - 8 apples “put with” 2 apples
   - Start with 3, “add” 5
4. Symbolic language. The words or phrases from stage 3 are now converted to symbols.
Within each stage, Reuille-Irons and Irons recommend language experiences that move from modeling aloud to creating to sharing.

The major purpose of writing in mathematics is that it “forces” one to think through a concept or process, resulting in a honing of one’s understanding.

McIntosh (1991) suggests four useful forms of writing:

1. In *learning logs*, children can reflect on what they are doing and learning. For example, children may keep track of what problems they solved and how they solved them.

2. *Journals* are similar to learning logs but often less formal and may, therefore, be more communicative than logs. They may also provide more insight into a child’s feelings about mathematics than logs. For example, a child might describe his or her feelings about learning a concept or being involved in a mathematics activity.

3. In *expository writing*, children explain an idea or process. For example, a child may explain to a new student how she or he does multiplication.

4. *Creative writing* gives children a chance to use abilities not often a part of school mathematics. Children may write poems about mathematical ideas or stories about concepts, mathematicians, etc. Here are some examples:
   - Write a story about 6.
   - Write a story about shapes.
   - Write a poem about addition.

**Language influences.** Admittedly, the language of mathematics is precise, and mathematical terms often have very specialized meanings, requiring that children attend to this precision, as in Mathematical Practice 6. For example, the expression *fairly small* may be adequate in some settings, but if you are telling mission control how much rocket fuel is stored in and present for launching the shuttle, *fairly small* is a totally inadequate expression of quantity.

However, even in mathematics, there is room for children’s own informal language. Allowing children to use their own language when first learning about concepts enables them to focus more on the concepts. Their expressions will develop into more precise language as their understanding progresses.

The language used to convey a mathematical idea has a bearing on the child’s understanding of the concept. For example, some children do not understand the term *perimeter*. If you talked about the “distance around” a shape, or if you drew a diagram and asked how much fencing would be needed to enclose the shape, many more children would understand and be able to successfully respond to the question. Likewise, the introduction of terms such as *commutative*, *associative*, and *distributive* serves no useful purpose if children have not already formed generalizations about these properties from repeated experiences examining them.

Another language-related factor of which you should be aware is children’s ability to use mathematics vocabulary without really understanding the concepts. For example, most children can talk about a triangle, but there are many who think that a figure is not a triangle unless it is equilateral, or they might say that the figure on the left (see below) is a triangle. Others might argue that the figure at the right is not a triangle because one side is not horizontal or parallel to the bottom of the page. Children’s misconceptions may be due to the visual images presented to them. Are the triangles you draw always equilateral in appearance? You must be aware of your own teaching behaviors.

**Supporting English language learners (ELLs).** Currently many children in elementary and middle schools are ELLs whose first language is a language other than English. You must be careful not to confuse limited ability to communicate in English with limited potential for learning mathematics. Many teaching strategies are available, such as sheltered instruction, ESL (English as a Second Language), and SDAIE (Specially
Designed Academic Instruction in English) to help children learn English as they learn mathematics. Strategies that many good teachers use to help all children learn mathematics, such as cooperative groups, manipulative materials, and visuals, are especially helpful for English language learners as well, as discussed in Chapter 1.

Herrell and Jordan (2011) discuss strategies for teaching English language learners. The following list contains several strategies that are particularly helpful in supporting mathematics learning:

- **Visual scaffolding**: Providing language support through visual images
- **Realia strategies**: Connecting language acquisition to the real world
- **Manipulative strategies**: Using objects to connect concepts
- **Cooperative learning**: Having groups interact to accomplish a goal
- **Advance organizers**: Getting the mind in gear for instruction
- **Preview/review**: Building vocabulary and concepts to support understanding
- **Modeled talk**: Showing while you talk
- **Attribute charting**: Organizing information to support understanding
- **Word walls**: Displaying and organizing words for easy access

These strategies are discussed in more detail, with examples, in Chapters 5 through 17.

In their research-validated SIOP (Sheltered Instruction Observation Protocol) model, Echevarria, Vogt, and Short (2013) recommend a flexible approach to providing ELL instruction. They suggest selecting from umbrella number of techniques to facilitate effective teaching of content, such as mathematics, for ELL students developing their academic English ability: cooperative learning, explicit instruction, background schema built and activated, ESL techniques, CCSS, reading and writing initiatives, technology, response to intervention, and differentiated instruction.

**Mathematics anxiety.** Mathematics anxiety, also known as math phobia, is a fear of mathematics. There is evidence that mathematics anxiety often starts in elementary school, although the symptoms often are not evident until years later. According to Burns (1998), “the way we’ve traditionally been taught mathematics has created a recurring cycle of math phobia, generation to generation, that has been difficult to break” (p. x).

Kennedy, Tipps, and Johnson (2008) list five teacher practices that contribute to mathematics anxiety: an emphasis on memorization, an emphasis on speed, an emphasis on doing one’s own work, authoritarian teaching, and lack of variety in the teaching–learning process.

So what can you do to reduce or prevent math anxiety in children? Martinez and Martinez (1996) suggest that teachers use the following instructional strategies to prevent math anxiety:

- Create an anxiety-free math class, which could include seating children in circles and small groups, with the teacher taking on the role of facilitator of learning.
- Match instruction to children’s cognitive levels.
- Plan instruction that connects mathematics to familiar situations in children’s everyday lives.
- Incorporate math games and puzzles into instruction.
- Teach math through reading and writing.
- Empower children by using technology and collaborative learning.

Social factors can contribute to an individual's attitude and motivation to learn mathematics. Placing children in nonthreatening cooperative groups can improve their self-concept, raise achievement, and increase motivation for learning. Group work is as important in mathematics as it is in social studies, science, or any other school subject.

Social considerations may also include factors such as the child’s home situation and the amount of sleep the previous night. A tense home situation can reduce a child's attentiveness and desire to learn. Furthermore, if an elementary school child was awake until midnight the night before a test, the child’s performance will likely be below expectation.
The teacher significantly influences the social situation within the classroom. However, you cannot control factors such as a child’s home situation or sleep patterns. Thus, parents, the child, and you share accountability for learning.

**Myths about learning mathematics.** According to Ginsburg and Baron (1993), there are five myths about learning mathematics. It is important that all teachers be aware of the fallacy of these myths so that they can plan appropriate instruction:

**Myth 1: Some children cannot learn math.** There is no reason all children cannot learn math, provided that they have good mathematics instruction. It is a challenge to every teacher to expect that all children will succeed in mathematics and to use a variety of instructional strategies to help them do so.

**Myth 2: Boys learn math better than girls.** There is a persistent belief in our society that boys are better at mathematics than girls are. In fact, in recent years girls have actually closed the gap, particularly at the middle school and high school levels. Differences that do occur seem to be due more to cultural influences than ability differences. It is important to provide opportunities for all children—girls as well as boys—to fully experience a variety of mathematics experiences and to have opportunities to both talk and listen in mathematics class.

**Myth 3: Poor children and children from underrepresented groups cannot learn math.** Many success stories point to the fallacy of this myth. The key components of programs that are successful for poor children and children from underrepresented groups include motivation, high expectations, role models, appropriate teaching, and real-world applications.

**Myth 4: American children have less mathematical ability than Asian children.** Differences in mathematics achievement emerge between American and Asian children after a year or two of schooling. However, these differences seem to be related not to ability but to differences in teaching and in expectations. For example, a common American notion is that “you’re either good at math or you’re not, and if you’re not good at math, there’s nothing you can do about it.” We must change our expectations to believe that all children, given good teaching, can learn mathematics.

**Myth 5: Mathematics learning disabilities are common.** There are many cases in which children do not seem to learn mathematics. However, as illustrated by the previous myths about learning mathematics, usually this lack of achievement is due not to learning disabilities but to a lack of motivation and appropriate teaching.

Clearly, good math teaching is critical. A teacher who is motivated and knowledgeable can help any child to understand and achieve success in mathematics.

**The Teaching Act**

The teaching act is a three-phase process: what the teacher does before the lesson, what the teacher does during the lesson, and what the teacher does after the lesson. This is not a totally linear process. Each phase provides input or feedback or both for the others.

**Preteaching Activities**

Before teaching a lesson, you must understand the nature of your students, diagnose what they already know, decide on an appropriate approach that will make the content meaningful, and then plan the instructional sequence and activities in more detail.

**Identifying children’s learning needs.** The first preteaching activity requires you to consider each individual child, who has unique learning needs, interests, attitudes, background, and motivation for learning. Identifying the needs of learners, including their prerequisite knowledge, should help make planning more individualized and thus more effective.

To make instructional decisions about current and future mathematical concepts children might be ready to learn, you must first determine what mathematics understandings children have already built. Only when you know what children understand and are still working on can you plan appropriate learning activities that will expand children’s knowledge.
You can diagnose children's understanding in several ways: by administering a pre-test, by conducting individual interviews of a few children in the class, or by carefully observing children involved in mathematics activities. Each of these types of assessment is described in Chapter 4. The key factor here is that rather than automatically teach the next lesson in the textbook, a good teacher decides what lesson to plan next based on what the children understand and what they need to continue working on.

Lessons should involve learners in active engagement, with experiences that meet the needs of and yet challenge some students. Planned activities might take many forms: physical as well as mental, verbal as well as written. Activity choices should foster different opportunities for involvement. Note, however, that not all mathematical ideas will be learned as a result of one lesson. Individual mathematical abilities and skills develop over time, some more slowly than others, which will require greater patience and more careful planning on your part.

Children's active involvement, like planned activities, needs to take many forms: physical as well as mental, verbal as well as written. You should plan for activities that foster, perhaps at different times, all these forms of involvement. Because success is a powerful motivator, make sure you plan learning activities in which children will be successful. This does not necessarily mean that the activity is to be easy, however. If work is too easy, children lose motivation. Plan activities to be challenging but within the range of a child's ability to complete, in other words, their ZPD.

**Using mathematics textbooks.** Consider what mathematics textbook resources are available to you. Become familiar with the wealth of resources accompanying the textbook, including suggestions for modifying lessons to enhance the concept development through the use of manipulative materials, adaptations for special needs students, and lessons for enrichment or reteaching. Keep in mind that a textbook is meant to be a resource for a thoughtful teacher, rather than a prescription that you must follow.

**Considering national, state and local standards.** Another important consideration when planning lessons is the expectations of your local school district or state board of education. Think about what children are expected to learn according to the CCSS (2010). You can keep updated on these standards and the states that are participating by visiting [www.corestandards.org](http://www.corestandards.org). How might you help children meet these standards?

**Planning.** There are two types of planning: unit planning and lesson planning. Unit planning focuses on planning a sequence of lessons on a particular concept, while lesson planning deals with daily lessons.

**Planning mathematics units.** Decisions and observations described above lead naturally into unit planning. The unit plan generally expands on one curriculum topic and includes lesson plans and teaching strategies for several days or several weeks. The unit plan might consist of a series of headings such as goals, standards, prerequisites, sequence of new skills to be introduced, developmental activities, practice activities, application of new skills, problem solving, enrichment activities, and assessing children's learning.

Another strategy for developing a unit plan is to use a graphic organizer. A graphic organizer is a pictorial organization of information about a particular topic, helping a student compress a lot of seemingly disjointed information into a structured, simple-to-read, graphic display. A graphic organizer for a unit on multiplication of a two-digit number by a one-digit number has been started in Figure 2-8. As you read more about specific instructional strategies for teaching multiplication, you could add to this graphic organizer. You must decide what type of format to use for unit planning. The process, rather than the form, is what is important.

**Lesson plan formats.** Lesson plans translate unit plans into daily activities and experiences for the children. As with unit plans, many special formats can be used. Some consist of as few as four components—objective, materials, procedures, and evaluation—some have much more complex outlines. The format will, in part, be determined by such considerations as your preferred teaching style, the nature of the activities, and the topic. Again, it is the process, rather than the format, that is important.
One format that is widely used is a three-part model. The names of the three parts vary. The following are some examples:

- Before—during—after
- Launch—explore—summarize

Whatever terminology is used, the critical elements of a three-part format are the same:

- In the first part you introduce the task, connect with prior knowledge, and motivate the children.
- The second part is the heart of the lesson, where the children are actively involved in exploring and describing mathematics concepts.
- The final part involves children’s sharing and reflecting on what they learned, while you assess their understanding.

Each of these parts is important in supporting children’s learning.

**Planning mathematics lessons.** Lesson planning involves several steps. Each step requires your careful consideration in determining how best to meet children’s needs:

1. Think about the mathematics. What should children learn about mathematics in this lesson? What is your long-term goal? What do the standards say children should understand about this topic?
2. Think about the learners. What do your children already know about this topic? What special needs should you keep in mind?
3. Choose the task. What should the children do to learn this concept?
   a. Review available resources, such as the mathematics textbook and other teacher resources (such as manipulatives and games) for tasks that will help children meet the objective.
4. Plan the three parts of the lesson and write your lesson plan. As you do, consider the following:
a. Write the lesson objectives. What will children know and be able to do as a result of this lesson?

b. What materials will you need?

c. How will you launch the lesson? On what prior knowledge will you build? How will you motivate children and get them excited about the upcoming learning activity?

d. What will children be doing in the “explore” part of the lesson? How will they construct their understanding of the topic? What do you expect that children will do? What might they have difficulty with? How will you provide guidance without “telling”? How will children communicate, reason, make connections, and solve problems? What representations will children use? In which mathematical practices and how will the children be engaged?

e. Think extra carefully about the “summarize” portion of the lesson; it must be more than just having children put away their papers. How will you help children reflect back on what they learned? How will you use this part of the lesson to assess what the children have learned and what they may need more practice with?

5. Plan how you will assess children’s learning. Lesson planning is time consuming but is crucial in providing high-quality instruction that helps all children make sense of mathematics.

A sample lesson integrating the CCSS in grade 2. The following section contains a sample lesson on numbers and operations in base-ten, appropriate for grade 2 children. Before describing the lesson, we list the goals of the lesson, using the appropriate expectations from the CCSS for grade 2. After a brief summary of the lesson, examples of the children’s work are provided. Then you will find a brief description of how each of the mathematical practices is woven into this particular lesson. Although every practice comes into play in some way during the lesson, not every aspect of each practice is addressed in this lesson.

CCSS Box 2-1 SELECTED CCSS STANDARDS FOR GRADE 2

Grade 2 Number and Operations in Base Ten

In grade 2, instructional time should focus on four critical areas: (1) extending understanding of base-ten notation; (2) building fluency with addition and subtraction; (3) using standard units of measure; and (4) describing and analyzing shapes. Students use their understanding of addition to develop fluency with addition and subtraction within 100. They solve problems within 1000 by applying their understanding of models for addition and subtraction, and they develop, discuss, and use efficient, accurate, and generalizable methods to compute sums and differences of whole numbers in base-ten notation, using their understanding of place value and the properties of operations. They select and accurately apply methods that are appropriate for the context and the numbers involved to mentally calculate sums and differences for numbers with only tens or only hundreds. (CCSS, 2010, p. 17)

Understand place value.

2.NBT.1: Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:

a. 100 can be thought of as a bundle of ten tens—called a “hundred.”

b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).

2.NBT.2: Count within 1000; skip-count by 5s, 10s, and 100s.

2.NBT.3: Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.

2.NBT.4: Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons. Represent addition and subtraction with objects, fingers, mental images, drawings

Use place value understanding and properties of operations to add and subtract.

2.NBT.5: Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

2.NBT.6: Add up to four two-digit numbers using strategies based on place value and properties of operations.

2.NBT.7: Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

2.NBT.8: Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.

2.NBT.9: Explain why addition and subtraction strategies work, using place value and the properties of operations. (CCSS, 2010, p. 19)
Sample Lesson

Solving Problems Involving Multidigit Addition

Grade level: Second

Materials: Make a variety of tools available to children, such as:
- Base-ten blocks
- Paper and pencil
- Counting frames (that have 10 metal rods with 10 beads on each rod).

Lesson objectives: The learner will:
1. Use multiple models to develop initial understandings of place value and the base-ten number system.
2. Develop a sense of whole numbers and represent and use them in flexible ways, including relating, composing, and decomposing numbers.
3. Develop and use strategies for whole-number computations, with a focus on addition and subtraction.
4. Develop fluency with multidigit addition and subtraction.
5. Use a variety of methods and tools to compute.

Standards link: 2.NBT.7: Use place value understanding and properties of operations to add and subtract. Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

Launch: Ask children if they collect anything. Have children share their responses. Pose the problem below, noting that Ian collects shells. Ask children to work together in pairs to solve the following problem in two different ways:

Ian had 186 shells in his collection. Over the summer he went to the beach and collected 149 more shells. How many shells does Ian have now?

Explore: Children may select the tool that they are most comfortable working with to solve this problem. As the children solve the problem, move around the room, observing the children as they work, listening to children's strategies, and asking questions to help children clarify their thinking.

Possible questions when observing children's work and listening to their individual strategies:
- Did you start by adding the hundreds or the ones?
- Why did you decide to add them this way?
- What if the numbers in the problem had been 200 and 350? Would your approach work?

Summarize: After children solve the problem in two ways, bring the class back together and ask children to share their solution strategies. Carefully select children to share, based on what you observed during the group work.

Potential questions when children share their ideas:
- Which approach was easiest for you to use?
- Which approach was fastest for you to use?
- What if the numbers in the problem had been 200 and 350? Which approach would you prefer to use then?

Follow-Up

Complete the following questions.
1. Did the child start by adding the hundreds or the ones?
2. Say to the child, “You have some interesting writing on your paper. Can you show that to us?”
3. As the child, “Has someone already shared an approach that was the same as yours? No? Can you explain what parts are different?”
Lesson actions. This description of the lesson shows a few of the children's solution processes and some of the teacher's planned questions.

Chris wrote:

\[
\begin{align*}
186 & \\
+149 & \\
200 & \\
120 & \\
+15 & \\
335 & \\
\end{align*}
\]

He explained:

*I added 100 and 100 to get 200; then 80 and 40 is 120, and 6 and 9 is 15. I added 200, 120, and 15 to get 335.*

The teacher asked Chris, "Why did you decide to start by adding the hundreds?"

Sarah said:

*149 is only 1 away from 150, so 150 and 100 from the 186 is 250, and 80 more is 330, and 6 more is 336. Then I have to subtract the 1, so it is 335.*

The teacher recorded on her clipboard that Sarah used estimation to solve this problem. Pat used base-ten blocks to solve the problem. Pat said:

*I took one flat for the 100 in 186 and 1 flat for the 100 in 149. I took 12 longs—8 for the 80 in 186 and 4 for the 40 in 149. I took 15 singles for the 6 in 186 and the 9 in 149. Then I counted like this, "100, 200," then the longs: "210, 220, 230, 240, 250, 260, 270, 280, 290, 300, 310, 320; then the singles: '321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335." So the answer is 335.*

The teacher recorded on her clipboard that Pat used base-ten blocks to solve this problem.

A. J. wrote:

\[
\begin{align*}
11 & \\
186 & \\
+149 & \\
335 & \\
\end{align*}
\]

She explained:

*"First I added 6 and 9 to get 15. I wrote down the 5 and carried the 1. Then I added 8 and 4 to get 12, plus 1 is 13; I wrote down the 3 and carried the 1 to get 1 and 1 and 1 is 3. So my answer is 335."*

The teacher asked A. J., "What do the ones that you wrote above the 186 mean?"

After several children shared their solution strategies, and the class asked questions to make sure they understood each other's methods; the teacher asked follow-up questions such as the following:

- Which approach was easiest for you to use?
- Which approach was fastest for you to use?
- What if the numbers in the problem had been 200 and 350? Which approach would you prefer to use then?
- Could you pick an approach that you did not use this time and use it next time to solve the next problem? How are the approaches similar? How are they different?

When the students in the class finished their task of solving the problem in two different ways, the teacher asked the students to share. One student complained that Sarah did the problem wrong, since she started with 149 and not 186. Sarah claimed that it did not matter which number she started with; she would still get the same answer. The teacher asked the students if, when they add two numbers together, they will always get the same answer, no matter which number they begin with. Some agree, and others disagree.
The teacher asked a student to restate the conjecture while the teacher wrote it on the board. The teacher decided that they will discuss and test the conjecture the following day and that this conjecture was something for the students to continue to think about.

If some students had difficulty explaining their thinking, the teacher prompted them asking follow-up questions. After several students had shared their approaches, the teacher asked about the relationships among and between the strategies:

- Who had strategies that were the same? What made them the same?
- Who had strategies that were different? How were they different?

Notice that the students in this class used a variety of representations to solve this problem. Chris and A. J. both represented the problem vertically on paper but used different approaches, and thus recorded their thinking in different ways. Pat represented the problem using base-ten blocks, whereas Sarah represented her thinking orally. The teacher encouraged the students to solve the problem in two ways to allow the children to think flexibly about the problem and also to encourage the students to use and make connections among different representations of the problem. For example, Pat’s second approach was to write down with symbols what she had done with the base-ten blocks. She wrote $100 + 100, 200 + 10, 210 + 10, 220 + 10, 230 + 10$, etc. She was thus making a connection between her work with the base-ten blocks and her work with the symbols.

**Teacher’s postlesson reflections.** After school ended that day, the teacher reflected on the lesson. She recalled that four children shared very different strategies to solve this problem, including modeling with manipulatives (Pat), estimation (Sarah), the standard algorithm (A. J.), and an informal algorithm (Chris). She recalled that all the other children in the class successfully solved the problem as well. She considered what the next lesson should focus on. Since this lesson was centered on solving a three-digit addition problem and all children were successful, the next lesson might focus on a similar subtraction problem. The teacher made a note to continue to watch a few children closely: Pat, to help him gradually move toward using symbols to solve the problem, and Chris, to keep an eye on his strategy of adding from left to right.

**Use of the Mathematical Practices in this lesson.** As the children shared their strategies, they were making sense and persevering in solving the problem (engagement in Mathematical Practice 1). Some children used the base-ten blocks, while others showed their thinking differently. The children used appropriate tools for solving the problem, as in Mathematical Practice 5. Then as the children shared their ideas, they were challenged to construct viable arguments for their approaches and critique the reasoning of others, as in Mathematical Practice 3. Can you find evidence of the children’s involvement for the other mathematical practices?

**The Process of Teaching**

There are many different models or styles of teaching. At a very simplistic level these have sometimes been described as being along a continuum from “pure telling” to “pure discovery” (Riedesel, 1990). No teacher operates at only one location along this continuum. Teachers normally have a region or range along the continuum in which they feel most comfortable. Many of the decisions made in the preplanning stage will influence the teaching style for a particular lesson.

**Models for teaching mathematics.** Several models are available for teaching mathematics, including the developmental model, the diagnostic model, the translation model, and the investigative model.

- **Developmental model.** A teacher with a constructivist theory of learning is likely to employ a developmental model of teaching in which children are actively engaged in making sense of mathematics. Even then, there are times when an explanatory approach with the whole class is appropriate.

  Riedesel (1990) identifies four aspects in which the “developmental” approach is different from the “telling” approach:

  1. The developmental approach emphasizes active learning as opposed to waiting for the teacher to explain.
2. The developmental approach builds new knowledge on experience; therefore, it is socially relevant. The explanatory approach tends to build dependence on the teacher or a textbook.

3. Developmental approaches stress children's thinking; therefore, the classroom is learner centered. In an explanatory environment, children tend to wait to see what the teacher thinks.

4. The developmental approach emphasizes a "search for relationships and patterns and leads to an understanding of mathematical structure" (p. 12).

**Diagnostic model.** A diagnostic model places assessing children's current level of mathematical understanding (such as their current ZPD) at the core of the teaching process. That knowledge is then used to structure learning activities that will help the child build onto existing mathematical knowledge. A diagnostic model developed by Ashlock, Johnson, Wilson, and Jones (1983) suggested a sequence of five types of lessons arising from a diagnostic core:

1. **Initiating:** Provides experiences with the new concept to be learned
2. **Abstracting:** Focuses on the attributes of the new concept to develop understanding
3. **Schematizing:** Focuses on interrelationships between the new concept and previously learned concepts
4. **Consolidating:** Provides practice to sharpen and clarify the new concept
5. **Transferring:** Includes problem-solving activities that show application of the new concept to new settings

**Translation model.** Earlier we referred to the claim of Behr et al. (1983) that meaning in mathematics results from building or recognizing relationships between or within representations. Building relationships between representations leads to a type of translation model for teaching. You might state a problem (spoken language) and ask the children to represent (translate) it using concrete materials. At other times pictures may be used to represent the idea. Later, translations or connections will be made between verbal, concrete, pictorial, and symbolic representations (Sawada, 1985). The translation model may be observed most often in lessons involving the operations.

**Investigative model.** Children build mathematical knowledge when they explore and experiment with ideas, processes, or data. The investigative model focuses on experimentation as well as inquiry. A possible sequence of steps in such a lesson might be as follows:

1. Structure a problem or make a statement that stimulates investigation.
2. Children do something, such as experiment or collect data.
3. Record or summarize data. Discuss and decide on an appropriate format (table, graph, chart, etc.) for the data.
4. Analyze or interpret the data. Look for patterns, relationships, etc. and describe the pattern (orally, in writing, or both).
5. Make a generalization or hypothesis and test it with other data.
6. Respond to the initial problem or statement and then write a report on the experiment/project.
7. Extend generalization to other problems, settings, or applications.

These steps are not to be interpreted rigidly. They are fluid and flexible and need to be adapted or modified for different problems and experiments. The emphasis is on exploration, experimentation, interpretation, hypothesizing, and generalizing. The investigative model fits well into a cooperative learning approach.

These models of teaching mathematics are not exclusive. Good teachers adapt a model based on their physical classroom setting, the nature of their children's needs, the children's individual differences, the mathematical topic, and their own philosophy of teaching and concepts that need to be covered.

**Postteaching Activities**

Your primary responsibilities in the postteaching phase include the ongoing activities of evaluation and reflection. Evaluation is the process of gathering information and using
it to make judgments that in turn are used to make decisions. You should evaluate the lesson and your teaching, reflect on the teaching strategies used, and assess learning. (Assessment of children’s learning is discussed in Chapter 4.)

**Evaluation of teaching.** According to NCTM (1991), the goal of evaluating mathematics teaching is to “improve teaching and enhance professional growth” (p. 72). Evaluation should be ongoing and linked to professional development. You should have opportunities to analyze your own teaching and discuss your teaching with colleagues and supervisors. Evaluation should be based on your goals and expectations for children, your plans, and evidence of children’s learning and understanding.

The NCTM (1991) teaching standards list many components of the evaluation of teaching. In general, the evaluation of teaching should focus on the teacher’s ability to (1) teach concepts, procedures, and connections; (2) promote mathematical problem solving, reasoning, and connections; (3) foster children’s mathematical dispositions; (4) assess children’s understanding of mathematics; and (5) create a learning environment that promotes the development of each child’s mathematical power.

A companion NCTM (2007) publication, *Mathematics Teaching Today*, identifies seven standards that are designed to promote your progress as a teacher:

1. Knowledge of mathematics and general pedagogy
2. Knowledge of student mathematical learning
3. Worthwhile mathematical tasks
4. Learning environment
5. Discourse
6. Reflection on student learning
7. Reflection on teaching practice

**Reflection on teaching.** Self-evaluation of teaching practices and effectiveness is strongly encouraged in the NCTM teaching standards. The last two standards in *Mathematics Teaching Today* highlight the importance of reflection—on student learning and teacher practice. *Mathematics Teaching Today* describes six major components to consider as you engage in these reflective practices as you exploring what happens in the mathematics classroom looking for opportunities for teacher improvement, particularly as you begin to implement the challenges in the CCSS:

1. Creating an environment that offers all students an equal opportunity to learn
2. Focusing on a balance of conceptual understanding and procedural fluency
3. Ensuring activity student engagement in the NCTM process standards (problem solving, reasoning, communication, connections, and representation)
4. Using technology to enhance understanding
5. Incorporating multiple assessments aligned with instructional goals and mathematical practices
6. Helping student recognize the power of sound reasoning and mathematical integrity

Reflection on your instruction is critical for your ongoing professional growth (Hart, Schultz, Najee-Ullah, & Nash, 1992), including the general approach to teaching and the type of learning activities developed for the children as well as the more overt teaching strategies. Figure 2-9 provides additional guidelines and questions for engaging in these reflective practices.

**Thinking about the Curriculum**

One can think about the mathematics curriculum at two levels: first, as the mathematics concepts, procedures, and processes to which the child is exposed; second, especially as it is experienced by the child, as all the activities and tasks the child engages in that are designed to help the child build some mathematical understanding. A beginning teacher is initially more concerned about what to teach; a more experienced teacher is probably more interested in the kinds of activities that help develop the concepts. However, every teacher needs to be concerned with both aspects.
You need to know what mathematics knowledge the children have at the start of the school year, what concepts they are expected to learn during the year, and where these concepts will lead. Consider what the CCSS expect children to understand about mathematics at this grade level. Assess children’s understanding to see what they already know about these topics. Examine the textbook series and curriculum guide used in the school district, as well as the end-of-year assessment that children are required to complete.

You must assess what mathematical understandings the children actually possess and then adapt the stated curriculum so that children understand prerequisite skills and concepts before attempting to build new ones.

The Activities

The real curriculum consists of much more than the concepts listed in a mathematics curriculum guide and/or state or district standards. It is the total of all the mathematics-related experiences a child has, both in and out of school. These experiences include playing counting games at recess, going to the store to purchase something after school, and engaging in the activities designed by the teacher. All such experiences contribute to a child’s construction of mathematics knowledge.

When selecting or developing mathematics activities for children, it is important that you keep in mind the standards (such as the CCSS and their mathematical practices) that
the children are expected to meet so that you can clearly write objectives for each lesson and goals for each unit.

Conclusion

The importance of meaning or understanding in learning mathematics cannot be overemphasized. The model shown in Figure 2-10 may help focus the picture. It shows meaning as a function of the child's exploration, often, but not exclusively, with concrete models, construction, and communication—all active, not passive, processes.

Figure 2-10 The Process of Making Meaning

Teaching mathematics requires hard work. A teacher's task is to create and help children create for themselves representations of mathematical ideas that will enable the child to build a significant mathematical knowledge structure. To do this effectively, you must:

- Be able to identify the current developmental zone for each of the children
- Understand how children learn mathematics
- Be familiar with the mathematics included in the curriculum
- Know how to design strategies and activities that will help children learn the concepts meaningfully
- Be able to assess the level of development of the concept in children

A child's attitude toward mathematics must also be assessed, and that information should be used to plan activities that will generate a positive attitude.

IN PRACTICE

Complete the following activities to include in your professional portfolio.

1. Explain the five different modes in which a mathematics concept may be represented and give an example of each for a concept of your choice.

2. Write a lesson plan for a concept of your choice. Include at least two different modes of representation in this lesson, and describe the translations that take place during the lesson.

3. Write a lesson plan that develops conceptual knowledge for a concept of your choice.

4. Write a lesson plan that includes several forms of communication to teach a mathematics concept of your choice.
LINKS TO THE INTERNET

The Educator’s Reference Desk
www.eduref.org/cgi-bin/lessons.cgi/Mathematics, click on “Free Math Lesson Plans”
Contains many resources for teachers, including more than 2,000 lesson plans.

ProTeacher
www.proteacher.com
Contains lesson plans and education news.

EducationWorld
www.educationworld.com/math/
Contains lesson plans and other resources for teachers.

RESOURCES FOR TEACHERS

Reference Books: Preventing Math Anxiety


Children’s Literature