Preface

All students can learn mathematics with understanding! It is through the teacher's actions that every student can have this experience. We believe that teachers must create a classroom environment in which students are given opportunities to solve problems and work together, using their ideas and strategies, to solve them. Effective mathematics instruction involves posing tasks that engage students in the mathematics they are expected to learn. Then, by allowing students to interact with and productively struggle with *their own mathematical ideas* and *their own strategies*, they will learn to see the connections among mathematical topics and the real world. Students value mathematics and feel empowered to use it.

Creating a classroom in which students design solution pathways, engage in productive struggle, and connect one mathematical idea to another is complex. Questions arise, such as, "How do I get students to wrestle with problems if they just want me to show them how to do it? What kinds of tasks lend themselves to this type of engagement? Where can I learn the mathematics content I need in order to be able to teach in this way?" With these and other questions firmly in mind, we have several objectives in the ninth edition of this textbook:

1. Illustrate what it means to teach mathematics using a problem-based approach.
2. Serve as a go-to reference for all of the mathematics content suggested for grades pre-K–8 as recommended in the Common Core State Standards (CCSSO, 2010) and in standards used in other states, and for the research-based strategies that illustrate how students best learn this content.
3. Present a practical resource of robust, problem-based activities and tasks that can engage students in the use of significant mathematical concepts and skills.
4. Report on technology that makes teaching mathematics in a problem-based approach more visible, including links to classroom videos and ready-to-use activity pages, and references to quality websites.

We hope you will find that this is a valuable resource for teaching and learning mathematics!

NEW to this Edition

Excitedly, we are able to offer this edition in an **electronic format**. Our Pearson eText*:

- allows us to hyperlink users to teaching and classroom videos that support concept development,
- includes downloadable Activity Pages for many activities in the text,
- offers Blackline Masters as upsized pages and accessible at point of use,
- includes Expanded Lessons with corresponding Blackline Masters, as appropriate, to support the lesson, and
- provides Self-Checks and feedback at the ends of major chapter sections to these and Writing to Learn assessment questions.

We briefly describe new features below, along with the substantive changes that we have made since the eighth edition to reflect the changing landscape of mathematics education. The following are highlights of the most significant changes in the ninth edition.

*Please note that eText enhancements such as video clips and Self-Check quizzes are only available in the Pearson eText, and not other third-party eTexts such as CourseSmart or Kindle.

Blackline Masters, Activity Pages and Teacher Resource Pages

More than 130 ready-to-use pages have been created to support the problems and Activities throughout the book. With one click on the link, you can download these to practice teaching an activity or to use with K–8 students in classroom settings. Activities that previously required cards or pages now include these resources as links. Some popular charts in the text have also been made into printable resources and handouts such as reflection questions to guide culturally relevant instruction.

Activities at a Glance

By popular demand, we have prepared a matrix (Appendix D) that lists all Section II activities, the mathematics they develop, which CCSS standards they address, and the page where they can be found. We believe you will find this an invaluable resource for planning instruction.

Self-Assessment Opportunities for the Reader

As we know, learners benefit from assessing their understanding along the way especially when there is a large amount of content to comprehend. To support teacher learning, each chapter begins with a set of learning outcomes that identify the goals of the chapter and link to Self-Check quizzes. Pearson eText users can click to open a Self-Check, input their answers, and receive immediate feedback. Self-Checks fall at the end of every major text section. Also, at the end of each chapter the popular Writing to Learn section now has end-of-chapter questions that can be answered online. Pearson eText users will get feedback for answers.

Expanded Lessons

Every chapter in Section II has at least one Expanded Lesson linked to an Activity. You may recognize some of these from the Field Experience Guide; the eText has allowed us to integrate these into the text! These lessons focus on concepts central to elementary and middle school mathematics and include (1) NCTM and CCSSO grade-level recommendations, (2) adaptation suggestions for English language learners (ELLs) and students with special needs, and (3) formative assessment suggestions.

Increased Focus on Common Core State Standards for Mathematics and Mathematical Practices

What began in the eighth edition is even stronger in the ninth edition. The CCSS are described in Chapter 1 along with other standards documents, and the Standards for Mathematical Practices are integrated into Chapter 2. In Section II, CCSS references are embedded in the text and every Activity lists the CCSS content that can be developed in that Activity. Standards for Mathematical Practice margin notes identify text content that shows what these practices look like in classroom teaching.

Reorganization and Enhancement to Section I

If you are a seasoned user of this book, you will immediately note that Chapters 2 through 4 are dramatically different. Chapter 2 has Activity Pages for each of the tasks presented and the chapter has been reorganized to move theory to the end. Chapter 3 now focuses exclusively on worthwhile tasks and classroom discourse, with merged and enhanced discussion of problems and worthwhile tasks; the three-phase lesson plan format (*before*, *during*, and *after*) has been moved to the beginning of Chapter 4. Chapter 4, the planning chapter, also underwent additional, major revisions that include (1) adding in the lesson plan format, (2) offering a refined process for planning a lesson (now eight steps, not ten), and (3) stronger sections on differentiating instruction and involving families. Chapter 4 discussions about ELLs and students with special needs have been moved and integrated into Chapter 6. Chapter 7, on technology, no longer has content-specific topics but rather a stronger focus on emerging technologies. Content chapters now house technology sections as appropriate.

Major Changes to Specific Chapters

Basic Facts (Chapter 10)

There are three major changes to this chapter. First, there is a much stronger focus on assessing basic facts. This section presents the risks of using timed tests and presents a strong collection of alternative assessment ideas. Second, chapter discussions pose a stronger developmental focus. For example, the need to focus first on foundational facts before moving to derived facts is shared. Third, there is a shift from a focus on mastery to a focus on fluency (as described in CCSS and in the research).

Developing Strategies for Addition and Subtraction (Chapters 11 and 12)

In previous editions there was a blurry line between Chapter 11 on place value and Chapter 12, which explored how to teach students to add and subtract. Although these topics overlap in many ways, we wanted to make it easier to find the appropriate content and corresponding activities. So, many components formerly in Chapter 11 (those that were explicitly about strategies for computing) have been shifted to Chapter 12 on addition and subtraction. This resulted in 15 more activities in Chapter 12, seven of which are new.

Fraction Operations (Chapter 16)

Using learning trajectories and a developmental approach, the discussion of how to develop meaning for each operation has been expanded. For example, the operation situations presented in Chapter 9 are now connected in Chapter 16 to rational numbers. In particular, multiplication and division have received much more attention, including more examples and activities. These changes are in response to the many requests for more support in this area!

Developing Concepts of Data Analysis (Chapter 21)

Look for several important changes in Chapter 21. There are 12 new activities that emphasize topics in CCSS. There also is more discussion on the shape of data, variability, and distribution. And, there is a notable increase in middle grades content including attention to dot plots, sampling, bivariate graphs, and, at the suggestion of reviewers, mean absolute deviation (MAD).

Additional Important Chapter-Specific Changes

The following substantive changes (not mentioned above) include

Chapter 1: Information about the new NCTM *Principles to Actions* publication with a focus on the eight guiding principles

Chapter 2: A revised and enhanced Doing Mathematics section and Knowing Mathematics section

Chapter 3: A new section on Adapting Tasks (to create worthwhile tasks) and new tasks and new authentic student work

Chapter 4: Open and parallel tasks added as ways to differentiate

Chapter 5: A more explicit development of how to use translation tasks to assess students' conceptual understanding

Chapter 6: Additional emphasis on multi-tiered systems of support including a variety of interventions

Chapter 7: Revisions reflect current software, tools, and digital apps as well as resources to support teacher reflection and collaboration

Chapter 8: Addition of Wright's progression of children's understanding of the number 10 and content from the findings from the new Background Research for the National Governor's Association Center Project on Early Mathematics

Chapter 9: An expanded alignment with the problem types discussed in the CCSS document

Chapter 13: Expanded discussion of the written records of computing multiplication and division problems including lattice multiplication, open arrays, and partial quotients

Chapter 14: A reorganization to align with the three strands of algebraic thinking; a revamped section on Structure of the Number System with more examples of the connection between arithmetic and algebra; an increased focus on covariation and inequalities and a decreased emphasis on graphs and repeating patterns, consistent with the emphasis in CCSS

Chapter 15: Many fun activities added (with manipulatives such as Play-Doh, Legos, and elastic); expanded to increase emphasis on CCSS content, including emphasis on number lines and iteration

Chapter 17: Chart on common misconceptions including descriptions and examples

Chapter 18: Major changes to the Strategies section, adding tape diagrams and expanding the section on double number lines; increased attention to graphing ratios and proportions

Chapter 19: An increased focus on converting units in the same measurement system, perimeter, and misconceptions common to learning about area; added activities that explore volume and capacity

Chapter 20: The shift in organizational focus to the four major geometry topics from the precise van Hiele level (grouping by all level 1 components), now centered on moving students from level to level using a variety of experiences within a given geometry topic

Chapter 22: Major changes to activities and figures, an expanded focus on common misconceptions, and increased attention to the models emphasized in CCSS-M (dot plots, area representations, tree diagrams)

Chapter 23: A new section on developing symbol sense, expanded section on order of operations, and many new activities

What You Will Find in This Book

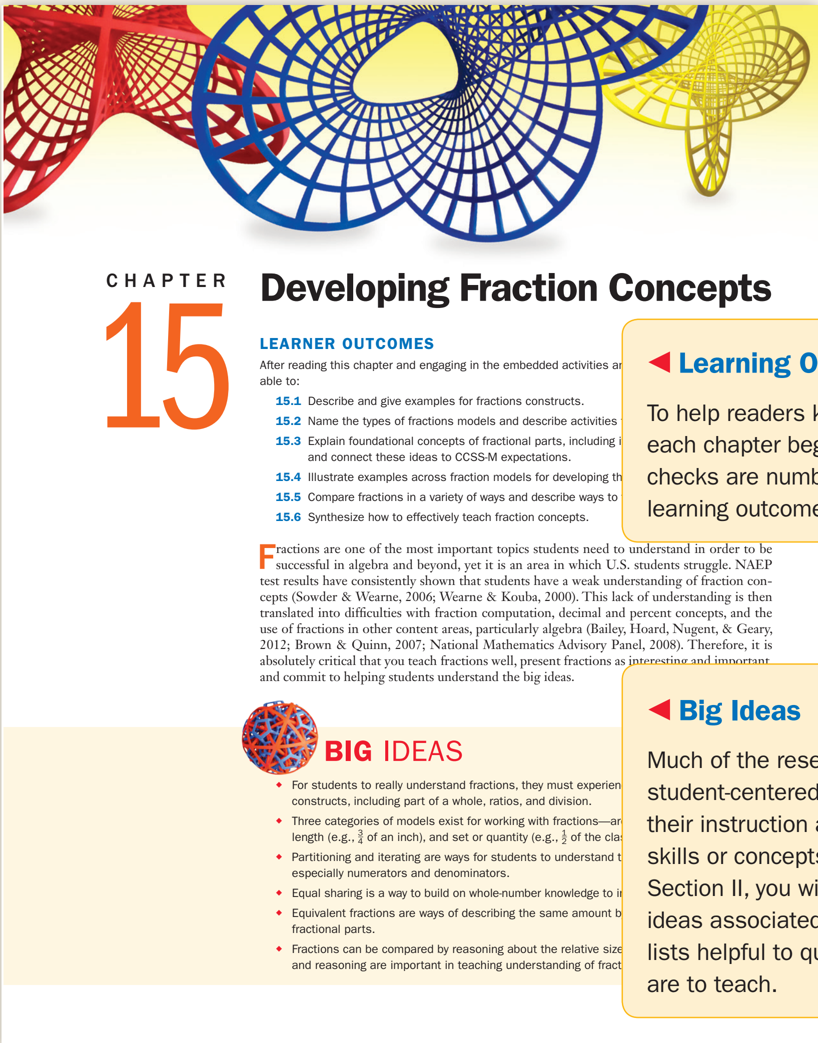
If you look at the table of contents, you will see that the chapters are separated into two distinct sections. The first section consists of seven chapters and covers important ideas that cross the boundaries of specific areas of content. The second section, consisting of 16 chapters, offers teaching suggestions and activities for every major mathematics topic in the pre-K–8 curriculum. Chapters in Section I offer perspectives on the challenging task of helping students learn mathematics. Having a feel for the discipline of mathematics—that is, to know what it means to "do mathematics"—is critical to learning how to teach mathematics well. In addition, understanding constructivist and sociocultural perspectives on learning mathematics and how they are applied to teaching through problem solving provides a foundation and rationale for how to teach and assess pre-K–8 students.

You will be teaching diverse students including students who are English language learners, are gifted, or have disabilities. In this text, you will learn how to apply instructional strategies in ways that support and challenge *all* learners. Formative assessment strategies, strategies for diverse learners, and effective use of technological tools are addressed in specific chapters in Section I (Chapters 5, 6, and 7, respectively), and throughout Section II chapters.

Each chapter of Section II focuses on one of the major content areas in pre-K–8 mathematics curriculum. It begins with identifying the big ideas for that content, and also provides guidance on how students best learn that content through many problem-based activities to engage them in understanding mathematics. Reflecting on the activities as you read can help you think about the mathematics from the perspective of the student. As often as possible, take out pencil and paper and try the problems so that you actively engage in *your learning* about *students learning* mathematics. In so doing, we are hopeful that this book will increase your own understanding of mathematics, the students you teach, and how to teach them well.

Some Special Features of This Text

By flipping through the book, you will notice many section headings, a large number of figures, and various special features. All are designed to make the book more useful as a long-term resource. Here are a few things to look for.


CHAPTER 15

Developing Fraction Concepts

LEARNER OUTCOMES

After reading this chapter and engaging in the embedded activities ar
able to:

15.1 Describe and give examples for fractions constructs.
15.2 Name the types of fractions models and describe activities
15.3 Explain foundational concepts of fractional parts, including i
and connect these ideas to CCSS-M expectations.
15.4 Illustrate examples across fraction models for developing th
15.5 Compare fractions in a variety of ways and describe ways to
15.6 Synthesize how to effectively teach fraction concepts.

Fractions are one of the most important topics students need to understand in order to be successful in algebra and beyond, yet it is an area in which U.S. students struggle. NAEP test results have consistently shown that students have a weak understanding of fraction concepts (Sowder & Wearne, 2006; Wearne & Kouba, 2000). This lack of understanding is then translated into difficulties with fraction computation, decimal and percent concepts, and the use of fractions in other content areas, particularly algebra (Bailey, Hoard, Nugent, & Geary, 2012; Brown & Quinn, 2007; National Mathematics Advisory Panel, 2008). Therefore, it is absolutely critical that you teach fractions well, present fractions as interesting and important and commit to helping students understand the big ideas.

BIG IDEAS

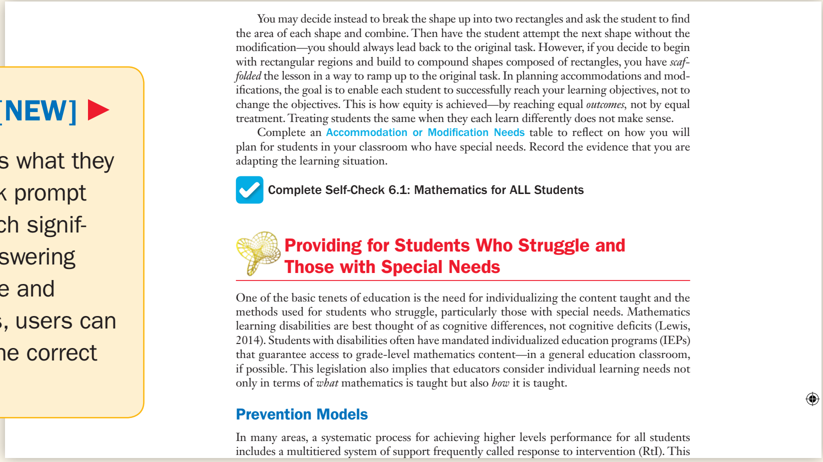
- For students to really understand fractions, they must experien
constructs, including part of a whole, ratios, and division.
- Three categories of models exist for working with fractions—ar
length (e.g., $\frac{3}{4}$ of an inch), and set or quantity (e.g., $\frac{1}{2}$ of the clas
- Partitioning and iterating are ways for students to understand t
especially numerators and denominators.
- Equal sharing is a way to build on whole-number knowledge to i
- Equivalent fractions are ways of describing the same amount b
fractional parts.
- Fractions can be compared by reasoning about the relative size
and reasoning are important in teaching understanding of fract

◀ Learning Outcomes [NEW]

To help readers know what they should expect to learn, each chapter begins with learning outcomes. Self-checks are numbered to cover and thus align with each learning outcome.

◀ Big Ideas

Much of the research and literature espousing a student-centered approach suggests that teachers plan their instruction around big ideas rather than isolated skills or concepts. At the beginning of each chapter in Section II, you will find a list of the key mathematical ideas associated with the chapter. Teachers find these lists helpful to quickly envision the mathematics they are to teach.


You may decide instead to break the shape up into two rectangles and ask the student to find the area of each shape and combine. Then have the student attempt the next shape without the modification—you should always lead back to the original task. However, if you decide to begin with rectangular regions and build to compound shapes composed of rectangles, you have *scaffolded* the lesson in a way to ramp up to the original task. In planning accommodations and modifications, the goal is to enable each student to successfully reach your learning objectives, not to change the objectives. This is how equity is achieved—by reaching equal *outcomes*, not by equal treatment. Treating students the same when they each learn differently does not make sense.

Complete an Accommodation or Modification Needs table to reflect on how you will plan for students in your classroom who have special needs. Record the evidence that you are adapting the learning situation.

Complete Self-Check 6.1: Mathematics for ALL Students

Providing for Students Who Struggle and Those with Special Needs

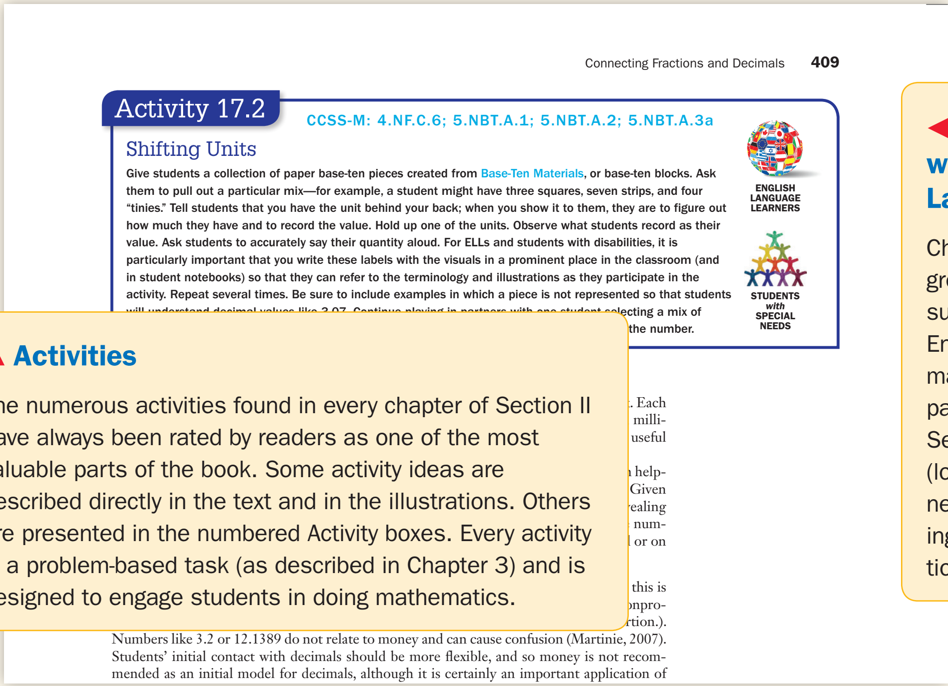
One of the basic tenets of education is the need for individualizing the content taught and the methods used for students who struggle, particularly those with special needs. Mathematics learning disabilities are best thought of as cognitive differences, not cognitive deficits (Lewis, 2014). Students with disabilities often have mandated individualized education programs (IEPs) that guarantee access to grade-level mathematics content—in a general education classroom, if possible. This legislation also implies that educators consider individual learning needs not only in terms of *what* mathematics is taught but also *how* it is taught.

Prevention Models

In many areas, a systematic process for achieving higher levels performance for all students includes a multitiered system of support frequently called response to intervention (RtI). This

Self-Check Prompts [NEW] ▶

To help readers self-assess what they have just read, a self-check prompt is offered at the end of each significant text section. After answering these quiz questions online and submitting their responses, users can review feedback on what the correct response is (and why).


Connecting Fractions and Decimals 409

Activity 17.2 CCSS-M: 4.NF.C.6; 5.NBT.A.1; 5.NBT.A.2; 5.NBT.A.3a

Shifting Units

Give students a collection of paper base-ten pieces created from Base-Ten Materials, or base-ten blocks. Ask them to pull out a particular mix—for example, a student might have three squares, seven strips, and four "tinies." Tell students that you have the unit behind your back; when you show it to them, they are to figure out how much they have and to record the value. Hold up one of the units. Observe what students record as their value. Ask students to accurately say their quantity aloud. For ELLs and students with disabilities, it is particularly important that you write these labels with the visuals in a prominent place in the classroom (and in student notebooks) so that they can refer to the terminology and illustrations as they participate in the activity. Repeat several times. Be sure to include examples in which a piece is not represented so that students will understand decimal values like 3.07. Continue playing in partners with one student selecting a mix of ... the number.

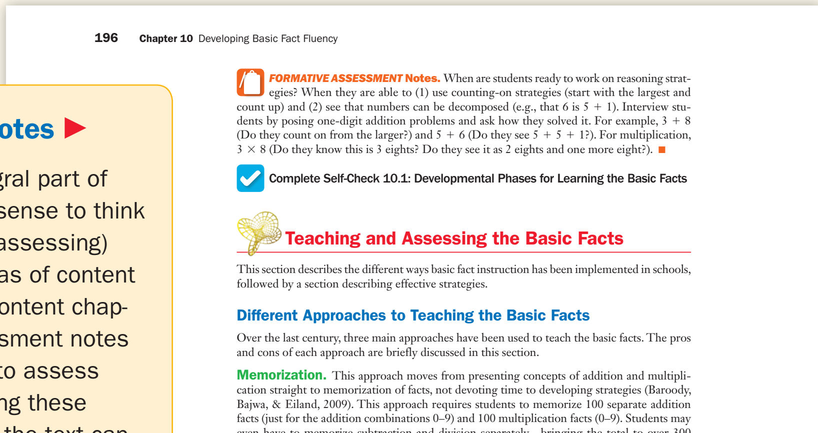
Numbers like 3.2 or 12.1389 do not relate to money and can cause confusion (Martinie, 2007). Students' initial contact with decimals should be more flexible, and so money is not recommended as an initial model for decimals, although it is certainly an important application of

◀ Adaptations for Students with Disabilities and English Language Learners

Chapter 6 provides detailed background and strategies for how to support students with disabilities and English language learners (ELLs). But, many adaptations are specific to a particular activity or task. Therefore, Section II chapters offer activities (look for the icon) that can meet the needs of exceptional students including specific instructions with adaptations directly within the Activities.

▲ Activities

The numerous activities found in every chapter of Section II have always been rated by readers as one of the most valuable parts of the book. Some activity ideas are described directly in the text and in the illustrations. Others are presented in the numbered Activity boxes. Every activity is a problem-based task (as described in Chapter 3) and is designed to engage students in doing mathematics.


196 Chapter 10 Developing Basic Fact Fluency

FORMATIVE ASSESSMENT Notes. When are students ready to work on reasoning strategies? When they are able to (1) use counting-on strategies (start with the largest and count up) and (2) see that numbers can be decomposed (e.g., that 6 is 5 + 1). Interview students by posing one-digit addition problems and ask how they solved it. For example, 3 + 8 (Do they count on from the larger?) and 5 + 6 (Do they see 5 + 5 + 1?). For multiplication, 3 × 8 (Do they know this is 3 eights? Do they see it as 2 eights and one more eight?). ■

Complete Self-Check 10.1: Developmental Phases for Learning the Basic Facts

Teaching and Assessing the Basic Facts

This section describes the different ways basic fact instruction has been implemented in schools, followed by a section describing effective strategies.

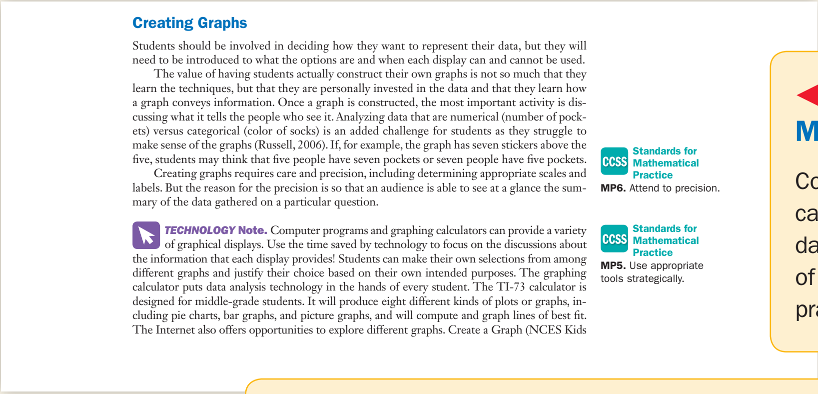
Different Approaches to Teaching the Basic Facts

Over the last century, three main approaches have been used to teach the basic facts. The pros and cons of each approach are briefly discussed in this section.

Memorization. This approach moves from presenting concepts of addition and multiplication straight to memorization of facts, not devoting time to developing strategies (Baroody, Bajwa, & Eiland, 2009). This approach requires students to memorize 100 separate addition facts (just for the addition combinations 0–9) and 100 multiplication facts (0–9). Students may even have to memorize subtraction and division separately—bringing the total to over 300

Formative Assessment Notes ▶

Assessment should be an integral part of instruction. Similarly, it makes sense to think about what to be listening for (assessing) as you read about different areas of content development. Throughout the content chapters, there are formative assessment notes with brief descriptions of ways to assess the topic in that section. Reading these assessment notes as you read the text can also help you understand how best to assist struggling students.



Creating Graphs

Students should be involved in deciding how they want to represent their data, but they will need to be introduced to what the options are and when each display can and cannot be used.

The value of having students actually construct their own graphs is not so much that they learn the techniques, but that they are personally invested in the data and that they learn how a graph conveys information. Once a graph is constructed, the most important activity is discussing what it tells the people who see it. Analyzing data that are numerical (number of pockets) versus categorical (color of socks) is an added challenge for students as they struggle to make sense of the graphs (Russell, 2006). If, for example, the graph has seven stickers above the five, students may think that five people have seven pockets or seven people have five pockets.

Creating graphs requires care and precision, including determining appropriate scales and labels. But the reason for the precision is so that an audience is able to see at a glance the summary of the data gathered on a particular question.

CCSS Standards for Mathematical Practice
MP6. Attend to precision.

***TECHNOLOGY* Note.** Computer programs and graphing calculators can provide a variety of graphical displays. Use the time saved by technology to focus on the discussions about the information that each display provides! Students can make their own selections from among different graphs and justify their choice based on their own intended purposes. The graphing calculator puts data analysis technology in the hands of every student. The TI-73 calculator is designed for middle-grade students. It will produce eight different kinds of plots or graphs, including pie charts, bar graphs, and picture graphs, and will compute and graph lines of best fit. The Internet also offers opportunities to explore different graphs. Create a Graph (NCES Kids

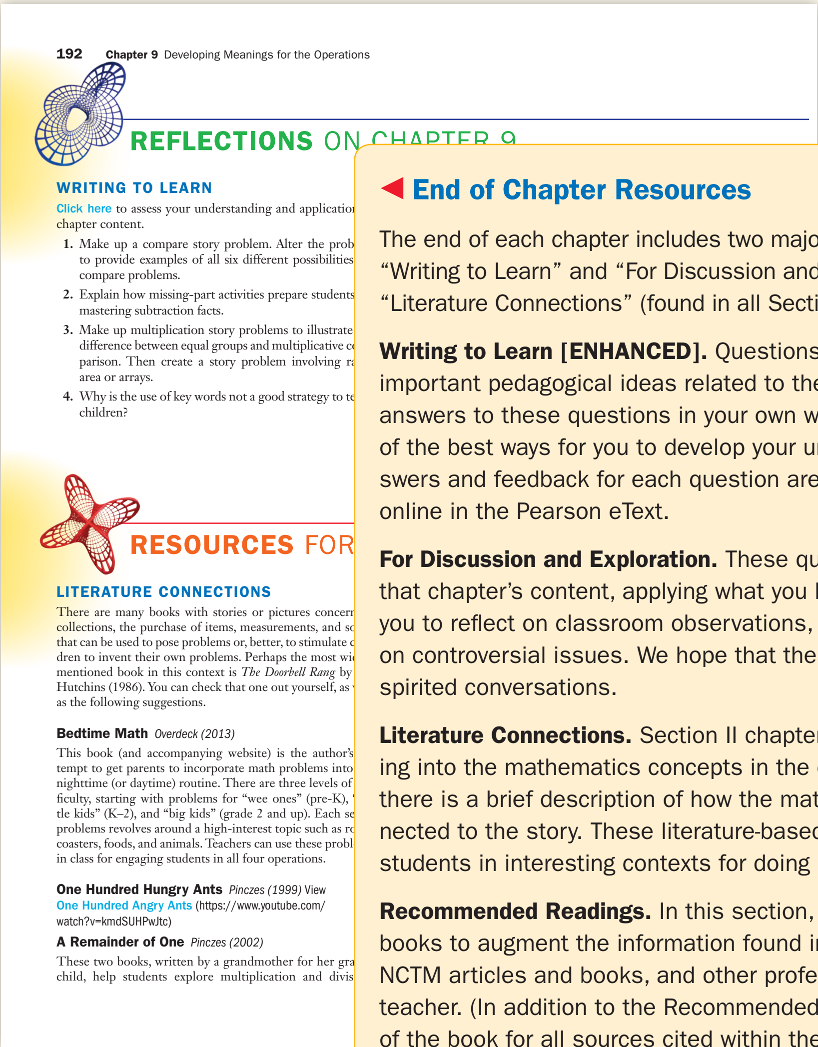
CCSS Standards for Mathematical Practice
MP5. Use appropriate tools strategically.

◀ Standards for Mathematical Practice Margin Notes [NEW]

Connections to the eight Standards of Mathematical Practice from the Common Core State Standards are highlighted in the margins. The location of the note indicates an example of the identified practice in the nearby text.

▲ Technology Notes

Infusing technological tools is important in learning mathematics, as you will learn in Chapter 7. We have infused technology notes throughout Section II. A technology icon is used to identify places within the text or activity where a technology idea or resource is discussed. Descriptions include open-source (free) software, applets, and other Web-based resources, as well as ideas for calculator use.



192 **Chapter 9** Developing Meanings for the Operations

REFLECTIONS ON CHAPTER 9

WRITING TO LEARN

Click here to assess your understanding and applicatio
chapter content.

1. Make up a compare story problem. Alter the prob to provide examples of all six different possibilities compare problems.
2. Explain how missing-part activities prepare students mastering subtraction facts.
3. Make up multiplication story problems to illustrate difference between equal groups and multiplicative c parison. Then create a story problem involving r area or arrays.
4. Why is the use of key words not a good strategy to te children?

RESOURCES FOR

LITERATURE CONNECTIONS

There are many books with stories or pictures concern collections, the purchase of items, measurements, and s that can be used to pose problems or, better, to stimulate dren to invent their own problems. Perhaps the most wi mentioned book in this context is *The Doorbell Rang* by Hutchins (1986). You can check that one out yourself, as as the following suggestions.

Bedtime Math *Overdeck (2013)*

This book (and accompanying website) is the author's tempt to get parents to incorporate math problems into nighttime (or daytime) routine. There are three levels of ficulty, starting with problems for "wee ones" (pre-K), tle kids" (K–2), and "big kids" (grade 2 and up). Each se problems revolves around a high-interest topic such as r coasters, foods, and animals. Teachers can use these probl in class for engaging students in all four operations.

One Hundred Hungry Ants *Pinczes (1999)* View One Hundred Angry Ants (https://www.youtube.com/watch?v=kmdSUHPwJtc)

A Remainder of One *Pinczes (2002)*

These two books, written by a grandmother for her gra child, help students explore multiplication and divis

◀ End of Chapter Resources

The end of each chapter includes two major subsections: *Reflections*, which includes "Writing to Learn" and "For Discussion and Exploration," and *Resources*, which includes "Literature Connections" (found in all Section II chapters) and "Recommended Readings."

Writing to Learn [ENHANCED]. Questions are provided that help you reflect on the important pedagogical ideas related to the content in the chapter. Actually writing out the answers to these questions in your own words, or talking about them with peers, is one of the best ways for you to develop your understanding of each chapter's main ideas. Answers and feedback for each question are now provided when answering these questions online in the Pearson eText.

For Discussion and Exploration. These questions ask you to explore an issue related to that chapter's content, applying what you have learned. For example, questions may ask you to reflect on classroom observations, analyze curriculum materials, or take a position on controversial issues. We hope that these questions will stimulate thought and cause spirited conversations.

Literature Connections. Section II chapters contain great children's literature for launching into the mathematics concepts in the chapter just read. For each title suggested, there is a brief description of how the mathematics concepts in the chapter can be connected to the story. These literature-based mathematics activities will help you engage students in interesting contexts for doing mathematics.

Recommended Readings. In this section, you will find an annotated list of articles and books to augment the information found in the chapter. These recommendations include NCTM articles and books, and other professional resources designed for the classroom teacher. (In addition to the Recommended Readings, there is a References list at the end of the book for all sources cited within the chapters.)

Supplements for Instructors

Qualified college adopters can contact their Pearson sales representatives for information on ordering any of the supplements described below. These instructor supplements are all posted and available for download (click on Educators) from the Pearson Instructor Resource Center at www.pearsonhighered.com/irc. The IRC houses the following:

- **Instructor's Resource Manual** The Instructor's Resource Manual for the ninth edition includes a wealth of resources designed to help instructors teach the course, including chapter notes, activity suggestions, and suggested assessment and test questions.
- **Electronic Test Bank** An electronic test bank (TB) contains hundreds of challenging questions as multiple-choice or short-answer questions. Instructors can choose from these questions and create their own customized exams.
- **PowerPoint™ Presentation** Ideal for instructors to use for lecture presentations or student handouts, the PowerPoint presentation provides ready-to-use graphics and text images tied to the individual chapters and content development of the text.

Acknowledgments

Many talented people have contributed to the success of this book, and we are deeply grateful to all those who have assisted over the years. Without the success of the first edition, there would certainly not have been a second, much less nine editions. The following people worked closely with John on the first edition, and he was sincerely indebted to Warren Crown, John Dossey, Bob Gilbert, and Steven Willoughby, who gave time and great care in offering detailed comments on the original manuscript.

In preparing this ninth edition, we have received thoughtful input from the following mathematics teacher educators who offered comments on the eighth edition or on the manuscript for the ninth. Each reviewer challenged us to think through important issues. Many specific suggestions have found their way into this book, and their feedback helped us focus on important ideas. Thank you to Jessica Cohen, Western Washington University; Shea Mosely Culpepper, University of Houston; Shirley Dissler, High Point University; Cynthia Gautreau, California State University in Fullerton; Kevin LoPresto, Radford University; Ryan Nivens, East Tennessee State University; Adrienne Redmond-Sanogo, Oklahoma State University; and Douglas Roebuck, Ball State University. We are indebted to you for your dedicated and professional insight.

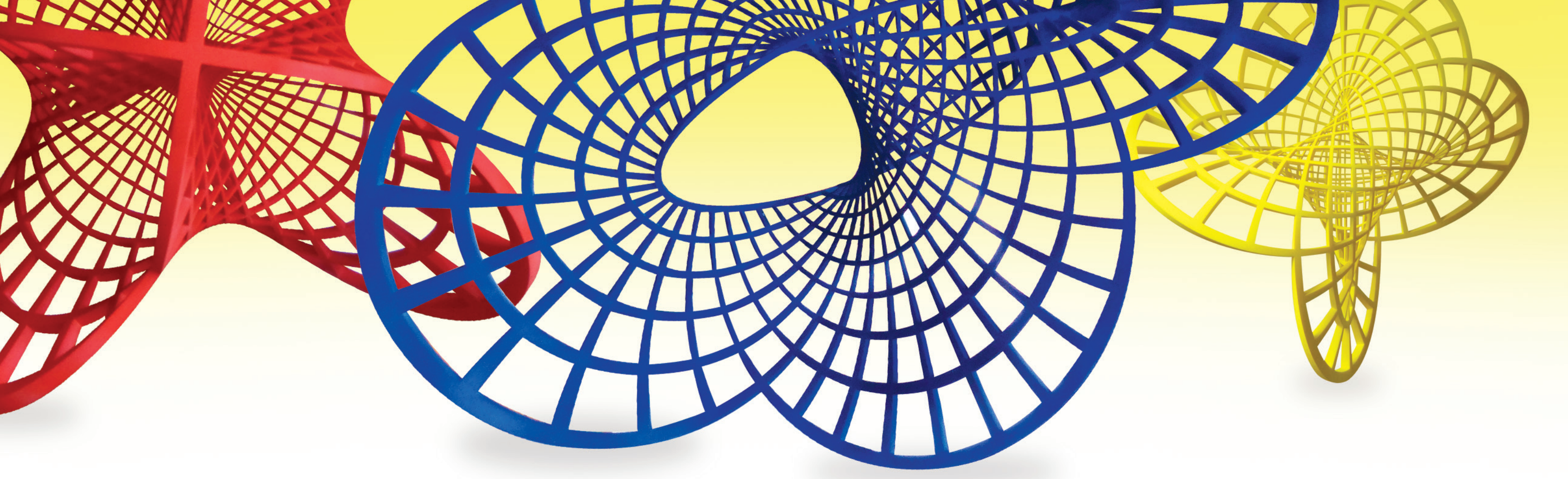
We received constant and valuable support and advice from colleagues at Pearson. We are privileged to work with our development editor, Linda Bishop, whose positive demeanor and upbeat responses on even the tightest of deadlines was most appreciated. Linda consistently offered us sound advice and much encouragement. We are also fortunate to work with Meredith Fossel, who has helped us define the direction of this edition, and helped us with the important decisions that would make the book a better product for pre-service and in-service teachers. We also wish to thank the production and editing team at MPS North America LLC, in particular Katie Watterson, who carefully and conscientiously assisted in preparing this edition for publication. Finally, our sincere thanks goes to Elizabeth Todd Brown, who helped write some of the ancillary materials.

We would each like to thank our families for their many contributions and support. On behalf of John, we thank his wife, Sharon, who was John's biggest supporter and a sounding board as he wrote the first six editions of this book. We also recognize his daughters, Bridget (a fifth-grade teacher in Chesterfield County, Virginia) and Gretchen (an associate professor of psychology and associate dean for undergraduate education at Rutgers University–Newark). They were John's first students, and he tested many ideas that are in this book by their sides. We can't forget those who called John "Math Grandpa": his granddaughters, Maggie, Aidan, and Gracie.

From Karen Karp: I would like to express thanks to my husband, Bob Ronau, who as a mathematics educator graciously helped me think about decisions while offering insights and encouragement. In addition, I thank my children, Matthew, Tammy, Joshua, Misty, Matt, Christine, Jeffrey, and Pamela for their kind support and inspiration. I also am grateful for my wonderful grandchildren, Jessica, Zane, Madeline, Jack and Emma, who have helped deepen my understanding about how children think.

From Jennifer Bay-Williams: I am forever grateful to my supportive and patient husband, Mitch Williams. My children, MacKenna (12 years) and Nicolas (9 years), along with their peers and teachers, continue to help me think more deeply about mathematics teaching and learning. My parents, siblings, nieces, and nephews have all provided support to the writing of this edition.

Most importantly, we thank all the teachers and students who gave of themselves by assessing what worked and what didn't work in the many iterations of this book. In particular for the ninth edition, we thank teachers who generously tested activities and provided student work for us: Kimberly Clore, Kim George, and Kelly Eaton. We continue to seek suggestions from teachers who use this book so please email us at teachingdevelopmentally@gmail.com with any advice, ideas, or insights you would like to share.



CHAPTER 1

Teaching Mathematics in the 21st Century

LEARNER OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

1.1 Summarize the factors that influence the teaching of mathematics.

1.2 Describe the important documents that are a part of the movement toward a set of shared expectations for students.

1.3 Explore the qualities needed to learn and grow as a professional teacher of mathematics.

Someday soon you will find yourself in front of a class of students, or perhaps you are already teaching. What general ideas will guide the way you will teach mathematics? This book will help you become comfortable with the mathematics content of the pre-K–8 curriculum. You will also learn about research-based strategies for helping students come to know mathematics and be confident in their ability to do mathematics. These two things—your knowledge of mathematics and how students learn mathematics—are the most important tools you can acquire to be successful.

Becoming an Effective Teacher of Mathematics

Before we get started, think back to when you were in pre-K–8 classrooms as a student. What are your remembrances of learning mathematics? Here are some thoughts from in-service and pre-service teachers of whom we asked the same question. Which description do you resonate with?

> I was really good at math in lower elementary grades, but because I never understood why math works, it made it very difficult to embrace the concepts as I moved into higher grades. I started believing I wasn't good at math so I didn't get too upset when my grades reflected that. *Kathryn*
>
> As a student I always felt lost during mathematics instruction. It was as if everyone around me had a magic key or code that I missed out on getting. *Tracy*
>
> I remember math being very challenging, intimidating, and capable of making me literally sick to my stomach. Math was a bunch of rules and formulas I was expected to memorize, but not to understand. *Mary Rebekah*

> I consider myself to be really good at math and I enjoy mathematics-related activities, but I often wonder if I would have been GREAT at math and had a completely different career if I cared about math as much as I do now. Sometimes I feel robbed. *April*
>
> Math went from engaging, interactive instruction that I excelled at and loved, to lecture-style instruction that I struggled with. I could not seek outside help, even though I tried, because the teacher's way was so different from the way of the people trying to help me. I went from getting all As to getting low Bs and Cs without knowing how the change happened. *Janelle*
>
> Math class was full of elimination games where students were pitted against each other to see who could answer a math fact the fastest. Because I have a good memory I did well, but I hated every moment. It was such a nerve-wracking experience and for the longest time that is what I thought math was. *Lawrence*
>
> Math was never a problem because it was logical, everything made sense. *Tova*

As you can see these memories run the gamut with an array of emotions and experiences. The question now becomes, what do you hope your students will say as they think back to your mathematics instruction? The challenge is to get all of your students to learn mathematics with understanding and enthusiasm. Would you relish hearing your students, fifteen years after leaving your classroom, state that you encouraged them to be mathematically minded, curious about solving new problems, self-motivated, able to critically think about both correct and incorrect strategies, and that you nurtured them to be a risk takers willing to try and persevere on challenging tasks? What will your legacy be?

As part of your personal desire to build successful learners of mathematics, you might recognize the challenge that mathematics is sometimes seen as the subject that people love to hate. At social events of all kinds—even at parent–teacher conferences—other adults will respond to the fact that you are a teacher of mathematics with comments such as "I could never do math," or "I can't even balance my checking account." Instead of dismissing these disclosures, consider what positive action you can take. Would people confide that they don't read and hadn't read a book in years? That is not likely. Families' and teachers' attitudes toward mathematics may enhance or detract from students' ability to do math. It is important for you and for students' families to know that mathematics ability is not inherited—anyone can learn mathematics. Moreover, learning mathematics is an essential life skill. You need to find ways of countering these statements, especially if they are stated in the presence of students, pointing out that it is a myth that only some people can be successful in learning mathematics. Only in that way can the chain of passing apprehension from family member to child, or in rare cases teacher to student, be broken. There is much joy to be had in solving mathematical problems, and you need to model this excitement and nurture that passion in your students.

Your students need to ultimately think of themselves as mathematicians in the same way as many of them think of themselves as readers. As students interact with our increasingly mathematical and technological world, they need to construct, modify, communicate or integrate new information in many forms. Solving novel problems and approaching circumstances with a mathematical perspective should come as naturally as reading new materials to comprehend facts, insights, or news. Consider how important this is to interpreting and successfully surviving in our economy and in our environment.

The goal of this book is to help you understand the mathematics methods that will make you an effective teacher. As you dig into the information your vision and confidence will grow.

A Changing World

In his book *The World Is Flat* (2007), Thomas Friedman discusses the need for people to have skills that are lasting and will survive the ever-changing landscape of available jobs. These are specific categories within a larger group that are called "untouchables" as regardless of the shifting landscape of job options—they will be successful in finding jobs. He is the one who defined these broad categories—such as math lover. Friedman points out that in a world that is digitized and surrounded by algorithms, math lovers will always have career opportunities and options. This is important as science, technology, engineering, and math (STEM) jobs, because of a skills gap, take more than twice as long to fill as other jobs in the marketplace (Rothwell, 2014). This is also aligned with the thinkers who believe students need to not just be college ready but innovation ready (Wagner, 2012).

Now it becomes the job of every teacher of mathematics to prepare students with skills for potential careers and develop a "love of math" in students. Lynn Arthur Steen, a well-known mathematician and educator, stated, "As information becomes ever more quantitative and as society relies increasingly on computers and the data they produce, an innumerate citizen today is as vulnerable as the illiterate peasant of Gutenberg's time" (1997, p. xv).

The changing world influences what should be taught in pre-K–8 mathematics classrooms. As we prepare pre-K–8 students for jobs that possibly do not currently exist, we can predict that there are few jobs for people where they just do simple computation. We can also predict that there will be work that requires interpreting complex data, designing algorithms to make predictions, and using the ability to approach new problems in a variety of ways.

As you prepare to help students learn mathematics for the future, it is important to have some perspective on the forces that effect change in the mathematics classroom. This chapter addresses the leadership that you, the teacher, will develop as you shape the mathematics experience for your students. Your beliefs about what it means to know and do mathematics and about how students make sense of mathematics will affect how you approach instruction and the understandings and skills your students take from the classroom.

Factors to Consider

For more than two decades, mathematics education has constantly undergone change. There have been significant reforms that reflect the technological and informational needs of our society, research on how students learn mathematics, the importance of providing opportunities to learn for all students, and ideas on how and what to teach from an international perspective. Just as we would not expect doctors to be using the exact same techniques and medicines that were prevalent when you were a child, teachers' methods are evolving and transforming via a powerful collection of expert knowledge about how the mind functions and how to design effective instruction (Wiggins, 2013).

There are several significant factors in this transformation. One factor is the public or political pressure for change in mathematics education due largely to information about student performance in national and international studies. These large scale comparisons of student performance continue to make headlines, provoke public opinion, and pressure legislatures to call for tougher standards backed by testing. The pressures of testing policies exerted on schools and ultimately on teachers may have an impact on instruction. These studies are important because international and national assessments provide strong evidence that mathematics teaching *must* change if our students are to be competitive in the global market and able to understand the complex issues they must confront as responsible citizens.

National Assessment of Education Progress (NAEP). Since the 1960s, at regular intervals, the United States gathers data on how fourth-, eighth-, and twelfth-grade students are doing in mathematics on the NAEP. These data provide an important tool for policy makers and educators to measure the overall improvement of U.S. students over time in what is called the "Nation's Report Card." NAEP uses four achievement levels: below basic, basic, proficient, and advanced, with proficient and advanced representing substantial grade-level achievement. The criterion-referenced test is designed to reflect the current curriculum but keeps a few stable items from 1982 for purposes of comparison (Kloosterman, Rutledge, & Kenney, 2009b). In the most recent assessment in 2013, less than half of all U.S. students in grades 4 and 8 performed at the desirable levels of proficient and advanced (42 percent in fourth grade and 35 percent in eighth grade) (National Center for Education Statistics, 2013). Despite encouraging gains in the NAEP scores over the last 30 years due to important shifts in instructional practices (particularly at the elementary level) (Kloosterman, Rutledge, & Kenney, 2009b), some U.S. students' performance still reveals disappointing levels of competency.

Trends in International Mathematics and Science Study (TIMSS). In the mid-1990s, 41 nations participated in the Third International Mathematics and Science Study, the largest study of mathematics and science education ever conducted. Data were gathered in grades 4, 8, and 12 from 500,000 students as well as from teachers. The most widely reported results revealed that U.S. students performed above the international average of the TIMSS countries at the fourth grade, below the average at the eighth grade, and significantly below average at the twelfth grade (National Academy Press, 1999; U.S. Department of Education, 1997).

TIMSS studies were repeated often with the most recent in 2011 in which 57 countries participated. For details, please visit the TIMSS website. The 2011 TIMSS found that U.S. fourth and eighth graders were above the international average but were significantly outperformed at fourth-grade level mathematics by education systems in Singapore, Republic of Korea, Hong Kong, Chinese Taipei, Japan, Northern Ireland, Belgium, Finland, England, and the Russian Federation and outperformed at the eighth-grade level by education systems in Republic of Korea, Singapore, Chinese Taipei, Hong Kong, Japan, Russian Federation, Israel, and Finland.

One of the most interesting components of the study was the videotaping of eighth-grade classrooms in the United States, Australia, and five of the highest-achieving countries. The results indicate that teaching is a cultural activity and, despite similarities, the differences in the ways countries taught mathematics were often striking. In all countries, problems or tasks were frequently used to begin the lesson. However, as a lesson progressed, the way these problems were handled in the United States was in stark contrast to high-achieving countries. Analysis revealed that, although the world is for all purposes unrecognizable from what it was 100 years ago, the U.S. approach to teaching mathematics during the same time frame was essentially unchanged (Stigler & Hiebert, 2009). Other countries incorporated a variety of methods, but they frequently used a problem-solving approach with an emphasis on conceptual understanding and students engaged in problem solving (Hiebert et al., 2003). Teaching in the high-achieving countries more closely resembles the recommendations of the National Council of Teachers of Mathematics, the major professional organization for mathematics teachers, discussed next.

National Council of Teachers of Mathematics (NCTM). One transformative factor is the professional leadership of the National Council of Teachers of Mathematics (NCTM). The NCTM, with more than 80,000 members, is the world's largest mathematics education organization. This group holds an influential role in the support of teachers and an emphasis on what is best for learners. Their guidance in the creation and dissemination of standards for curriculum, assessment, and teaching led the way for other disciplines. For an array of resources, including the Illuminations component which consists of a set of exciting instructional experiences for your students, visit the NCTM website.

Complete Self-Check 1.1: A Changing World

The Movement toward Shared Standards

The momentum for reform in mathematics education began in earnest in the early 1980s. The main impetus was a response to a need for more problem solving as well as the research of developmental psychologists who identified how students can best learn mathematics. Then in 1989, NCTM published the first set of standards for a subject area in the *Curriculum and Evaluation Standards for School Mathematics*. Many believe that no other document has ever had such an enormous effect on school mathematics or on any other area of the curriculum.

NCTM followed in 1991 with a set of standards for teaching that articulated a vision of teaching mathematics for all students, not just a few. In 1995, NCTM added to the collection the *Assessment Standards for School Mathematics*, which focused on the importance of integrating assessment with instruction and indicated the key role that assessment plays in implementing change (see Chapter 5). In 2000, however, NCTM released *Principles and Standards for School Mathematics* as an update of its original standards document. Combined, these documents prompted a revolutionary reform movement in mathematics education throughout the world.

As these documents influenced teacher practice, ongoing debate continued about the U.S. curriculum. In particular, many argued that instead of hurrying through several topics every year, the curriculum needed to address content more deeply. Guidance was needed in deciding what mathematics content should be taught at each grade level and, in 2006, NCTM released *Curriculum Focal Points*, a little publication with a big message—the mathematics taught at each grade level needs to be focused, provide depth, and explicitly show connections.

In 2010, the Council of Chief State School Officers (CCSSO) presented the Common Core State Standards, which are grade-level specific standards which incorporate ideas from *Curriculum Focal Points* as well as international curriculum documents. A large majority of U.S. states adopted these as their standards. In less than 25 years, the standards movement transformed the country from having little to no coherent vision on what mathematics should be taught and when, to a more widely shared idea of what students should know and be able to do at each grade level.

In the following sections, we discuss three significant documents critical to your work as a teacher of mathematics.

Principles and Standards for School Mathematics

The *Principles and Standards for School Mathematics* (NCTM, 2000) provides guidance and direction for teachers and other leaders in pre-K–12 mathematics education. This is particularly true in states and regions where they have developed their own standards.

The Six Principles.

One of the most important features of *Principles and Standards for School Mathematics* is the articulation of six principles fundamental to high-quality mathematics education. These principles must be blended into all programs as building excellence in mathematics education involves much more than simply listing content objectives.

The Equity Principle. The strong message of this principle is there should be high expectations and intentional ways to support all students. All students must have the opportunity and adequate support to learn mathematics regardless of their race, socioeconomic status, gender, culture, language, or disability. This principle is interwoven into all other principles.

The Curriculum Principle. The curriculum should be coherent and built around "big ideas" in the curriculum and in daily classroom instruction. We think of these big ideas as "important" if they help develop other ideas, link one idea to another, or serve to illustrate the discipline of mathematics as a human endeavor. Students must be helped to see that mathematics is an integrated whole that grows and connects across the grades rather than a collection of isolated bits and pieces.

The Teaching Principle. What students learn about mathematics depends almost entirely on the experiences that teachers provide every day in the classroom. To provide high-quality mathematics education, teachers must (1) understand deeply the mathematics content they are teaching; (2) understand how students learn mathematics, including common misconceptions; and (3) select meaningful instructional tasks and generalizable strategies that will enhance learning.

The Learning Principle. This principle is based on two fundamental ideas. First, learning mathematics with understanding is essential. Mathematics today requires not only computational skills but also the ability to think and reason mathematically to solve new problems and learn to respond to novel situations that students will face in the future. Second, students *can* learn mathematics with understanding. Learning is enhanced in classrooms where students are required to evaluate their own ideas and those of others, make mathematical conjectures and test them, and develop their reasoning and sense-making skills.

The Assessment Principle. Ongoing assessment highlights the most important mathematics concepts for students. Assessment that includes ongoing observation and student interaction encourages students to articulate and, thus, clarify their ideas. Feedback from daily assessment helps students establish goals and become more independent learners. By continuously gathering data about students' understanding of concepts and growth in reasoning, teachers can better make the daily decisions that support student learning. For assessment to be effective, teachers must use a variety of assessment techniques, understand their mathematical goals deeply, and have a research-supported notion of students' thinking or common misunderstandings.

The Technology Principle. Calculators, computers, and other emerging technologies are essential tools for learning and doing mathematics. Technology permits students to focus on mathematical ideas, to reason, and to solve problems in ways that are often impossible without these tools. Technology enhances the learning of mathematics by allowing for increased exploration, enhanced representation, and communication of ideas.

The Five Content Standards. *Principles and Standards* includes four grade bands: pre-K–2, 3–5, 6–8, and 9–12. The emphasis on preschool recognizes the need to highlight the critical years before students enter kindergarten. There is a common set of five content standards throughout the grades:

- Number and Operations
- Algebra
- Geometry
- Measurement
- Data Analysis and Probability

Each content standard includes a set of goals applicable to all grade bands followed by specific expectations for what students should know at each grade band. Although the same five content standards apply across all grades, you should not infer that each strand has equal weight or emphasis in every grade band. Number and Operations is the most heavily emphasized strand from pre-K through grade 5 and continues to be important in the middle grades, with a lesser emphasis in grades 9–12. This is in contrast to Algebra, which moves from an emphasis related to number and operations in the early grades and builds to a strong focus in the middle and high school grade bands.

The Five Process Standards. The process standards refer to the mathematical processes through which pre-K–12 students acquire and use mathematical knowledge. The process standards should not be regarded as separate content or strands in the mathematics curriculum, rather, they are integral components of all mathematics learning and teaching. The five process standards and ways you can develop these elements in your students can be found in Table 1.1.

Members of NCTM have free online access to the *Principles and Standards* and nonmembers can sign up for 120 days of free access to the full document on the NCTM website under the tab Standards and Focal Points.

Common Core State Standards

As noted earlier, the dialogue on improving mathematics teaching and learning extends beyond mathematics educators. Policymakers and elected officials considered previous NCTM standards documents, international assessments, and research on the best way to prepare students to be "college and career ready." The National Governors Association Center for Best Practices and the Council of Chief State School Officers (CCSSO) collaborated with other professional groups and entities to develop shared expectations for K–12 students across states, a focused set of mathematics content standards and practices, and efficiency of material and assessment development (Porter, McMaken, Hwang, & Yang, 2011). As a result, they created the Common Core State Standards for Mathematics (CCSS-M) which can be downloaded for free at http://www.corestandards.org/math. At this time more than 40 states, Washington, D.C., four territories, and Department of Defense Schools have adopted the Common Core State Standards. This represents the largest shift of mathematics content in the United States in more than 100 years. A few states did not opt to participate in the adoption of the standards from the start of their development and at this time others are still deciding their level of participation or reevaluating their own standards against CCSS-M.

TABLE 1.1 THE FIVE PROCESS STANDARDS FROM *PRINCIPLES AND STANDARDS FOR SCHOOL MATHEMATICS*

Process Standard	How Can You Develop These Processes in Your Students?
Problem Solving	• Start instruction with a problem to solve—as problem solving is the vehicle for developing mathematical ideas. • Select meaningful mathematical tasks. • Set problems in a situation to which students can relate. • Use a variety of strategies to solve problems. • Have students self-assess their understanding of the problem and their strategy use.
Reasoning and Proof	• Have students consider evidence of why something is true or not. • Create opportunities for students to evaluate conjectures—do they hold true? • Encourage students to use logical reasoning to see if something always works or their answers make sense. • Demonstrate a variety of ways for students to justify their thinking through finding examples and counterexamples to use in a logical argument.
Communication	• Invite students to talk about, write about, describe, and explain their mathematical ideas as a way to examine their thinking. • Give students opportunities to share ideas so that others can understand and actively discuss their reasoning. • Share examples of student work, so students can compare and assess others' thinking. • Present precise mathematical language and notation so that the word usage and definitions can act as a foundation for students' future learning.
Connections	• Emphasize how mathematical ideas explicitly connect to students' prior mathematical knowledge and future learning. • Assist students in developing the relationships between the mathematics being learned and real world contexts and in other subject areas.
Representation	• Encourage students to use multiple representations to explore relationships and communicate their thinking. • Create opportunities for students to move from one representation of an mathematical concept or idea to another to add depth of understanding. • Provide problems where students can use mathematical models to clarify or represent a situation.

Source: Adapted with permission from NCTM (National Council of Teachers of Mathematics). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

The document articulates an overview of *critical areas* for each grade from K–8 to provide a coherent curriculum built around big ideas. These larger groups of related standards are called *domains*, and there are eleven that relate to grades K–8 (see Figure 1.1).

The Common Core State Standards go beyond specifying mathematics content expectations to also include Standards for Mathematical Practice. These are "'processes and proficiencies' with longstanding importance in mathematics education" (CCSSO, 2010, p. 6) that are based on the underlying frameworks of the NCTM process standards and the components of mathematical proficiency identified by NRC in their important document *Adding It Up* (National Research Council, 2001). Teachers must develop these mathematical practices in all students (CCSSO, 2010, pp. 7–8) as described briefly in Table 1.2. A more detailed description of the Standards for Mathematical Practice can be found in Appendix B.

Kindergarten	Grade 1	Grade 2	Grade 3	Grade 4	Grade 5	Grade 6	Grade 7	Grade 8
Counting and Cardinality								
Operations and Algebraic Thinking						Expressions and Equations		
Number and Operations in Base Ten						The Number System		
Measurement and Data						Statistics and Probability		
Geometry								
			Number and Operations—Fractions			Rations and Proportional Relationships		Functions

FIGURE 1.1 Common Core State Standards domains by grade level.

TABLE 1.2 THE STANDARDS FOR MATHEMATICAL PRACTICE FROM THE COMMON CORE STATE STANDARDS

Mathematical Practice	K–8 Students Should Be Able to:
Make sense of problems and persevere in solving them.	• Explain what the problem is asking. • Describe possible approaches to a solution. • Consider similar problems to gain insights. • Use concrete objects or drawings to think about and solve problems. • Monitor and evaluate their progress and change strategies if needed. • Check their answers using a different method.
Reason abstractly and quantitatively.	• Explain the relationship between quantities in problem situations. • Represent situations using symbols (e.g., writing expressions or equations). • Create representations that fit the problem. • Use flexibly the different properties of operations and objects.
Construct viable arguments and critique the reasoning of others.	• Understand and use assumptions, definitions, and previous results to explain or justify solutions. • Make conjectures by building a logical set of statements. • Analyze situations and use examples and counterexamples. • Justify conclusions in a way that is understandable to teachers and peers. • Compare two possible arguments for strengths and weaknesses to enhance the final argument.
Model with mathematics.	• Apply mathematics to solve problems in everyday life. • Make assumptions and approximations to simplify a problem. • Identify important quantities and use tools to connect their relationships. • Reflect on the reasonableness of their answer based on the context of the problem.
Use appropriate tools strategically.	• Consider a variety of tools and choose the appropriate tool (e.g., manipulative, ruler, technology) to support their problem solving. • Use estimation to detect possible errors and establish a reasonable range of answers. • Use technology to help visualize, explore, and compare information.
Attend to precision.	• Communicate precisely using clear definitions and appropriate mathematical language. • State accurately the meanings of symbols. • Specify appropriate units of measure and labels of axes. • Use a level of precision suitable for the problem context.
Look for and make use of structure.	• Identify and explain mathematical patterns or structures. • Shift viewpoints and see things as single objects or as comprised of multiple objects. • Explain why and when properties of operations are true in a context.
Look for and express regularity in repeated reasoning.	• Notice if patterns in calculations are repeated and use that information to solve problems. • Use and justify the use of general methods or shortcuts by identifying generalizations. • Self-assess as they work to see whether a strategy makes sense, checking for reasonableness prior to finalizing their answer.

Source: Based on Council of Chief State School Officers. (2010). *Common Core State Standards*.

Watch this video (https://www.youtube.com/watch?v=GlJ44te7jrw) to get a good overview of the CCSS-M from teachers and authors. Additionally, the Illustrative Mathematics Project website provides tools and support for the Common Core State Standards. It includes multiple ways to look at the standards across grades and domains as well as provides task and problems that will illustrate individual standards.

Learning Progressions. The Common Core State Standards were developed with strong consideration given to building coherence through the research on what is known about the development of students' understanding of mathematics over time (Cobb & Jackson, 2011). The selection of topics at particular grades reflects not only rigorous mathematics but also what is known from current research and practice about learning progressions which are sometimes referred to as *learning trajectories* (Clements & Sarama, 2014; Confrey, Maloney, & Corley, 2014; Daro, Mosher, & Corcoran, 2011; Maloney, Confrey, Ng & Nickell, 2014). It is these learning progressions that can help teachers know what came before as well as what to expect next as students reach key points along the (Corcoran, Mosher, & Rogat, 2009) road to learning mathematics concepts. These progressions identify the interim goals students should reach on the pathway to desired learning targets (Daro, Mosher, & Corcoran, 2011). Although these paths are not identical for all students, they can inform the order of instructional experiences which will support movement toward understanding and application of mathematics concepts. There is a website for the "Progressions Documents for the

Common Core Math Standards" where progressions for the domains in the Common Core State Standards can be found.

Assessments. The initial idea was to have new summative assessments developed through two major consortia, Partnership for Assessment of Readiness for College and Careers (PARCC) and Smarter Balanced Assessment Consortium, are developing assessments which will align to the *Common Core State Standards*. These assessments will focus on both the grade-level content standards and the standards for mathematical practice. This process is being put into place to eliminate the need for each state to develop unique assessments for the standards, a problem that has existed since the beginning of the standards era. Yet, there are states developing their own approaches to end-of-year assessment as well.

Principles to Actions

NCTM has developed a publication that capitalizes on the timing of the adoption of the Common Core State Standards to explore the specific learning conditions, school structures, and teaching practices which will be important for a high quality education for all students. The book uses detailed classroom stories and student work samples to illustrate the careful, reflective work required of effective teachers of mathematics through 6 guiding principles (see Table 1.3). A series of presentations (webcasts), led by the authors of the publication, explore several of the guiding principles and are available on the *Principles to Actions* portion of the NCTM's website.

TABLE 1.3 THE SIX GUIDING PRINCIPLES FROM THE *PRINCIPLES TO ACTIONS*

Guiding Principle	Suggestions for Classroom Actions That Align with the Principles
Teaching and Learning	• Select focused mathematics goals. • Use meaningful instructional tasks that develop reasoning, sense making, and problem solving strategies. • Present and encourage a variety of mathematical representations that connect the same ideas or concepts. • Facilitate student discussions and conversations about important mathematical ideas. • Ask essential questions that are planned to be a catalyst for deeper levels of thinking. • Use a strong foundation of conceptual understanding as a foundation for procedural fluency. • Encourage productive struggle—as it is a way to deepen understanding and move toward student application of their learning. • Generate ways for students to provide evidence of their thinking through discussions, illustrations, and written responses.
Access and Equity	• Establish high expectations for all students. • Provide supports targeted to student needs (equity not equality). • Provide instructional opportunities for students to demonstrate their competence in different ways—creating tasks with easy entry points for students who struggle and extension options for those who finish quickly. • Identify obstacles to students' success and find ways to bridge or eliminate those barriers. • Develop all students' confidence that they can do mathematics. • Enhance the learning of all by celebrating students' diversity.
Curriculum	• Build connections across mathematics topics to capitalize on broad themes and big ideas. • Look for both horizontal and vertical alignment to build coherence. • Avoid thinking of a curriculum as a checklist or disconnected set of daily lessons.
Tools and Technology	• Include an array of technological tools and manipulatives to support the exploration of mathematical concepts, structures, and relationships. • Think beyond computation when considering the integration of technology. • Explore connections to how technology use for problem solving links to career readiness.
Assessment	• Incorporate a continuous assessment plan to follow how students are performing and how instruction can be modified and thereby improved. • Move beyond test results that just look at overall increases and decreases to pinpoint specific student needs. • Consider the use of multiple assessments to capture a variety of student performance. • Encourage students to self-assess sometimes by evaluating the work of others to enhance their own performance. • Teach students how to check their work.
Professionalism	• Develop a long-term plan for building your expertise. • Build collaborations that will enhance the work of the group of collaborators as you enhance the performance of the students in the school. • Take advantage of all coaching, mentoring and professional development opportunities and be a life-long learner. • Structure in time to reflect on and analyze your instructional practices.

Pause & Reflect

Take a moment now to select one or two of the six guiding principles that seem especially significant to you and are areas in which you wish to develop more expertise. Why do you think these are the most important to your teaching?

Complete Self-Check 1.2: The Movement toward Shared Standards

An Invitation to Learn and Grow

The mathematics education described in this book may not be the same as the mathematics content and the mathematics teaching you experienced in grades K–8. As a practicing or prospective teacher facing the challenge of teaching mathematics from a problem solving approach, this book may require you to confront some of your personal beliefs—beliefs about what it means to *do mathematics*, how one goes about *learning mathematics*, how to *teach mathematics*, and what it means to *assess mathematics*. Success in mathematics isn't merely about speed or the notion that there is "one right answer." Thinking and talking about mathematics as a means to sense making is a strategy that will serve us well in becoming a society where all citizens are confident in **their ability to do math** (https://www.youtube.com/watch?v=0gW9g8Ofi8A).

Becoming a Teacher of Mathematics

This book and this course of study are critical to your professional teaching career. The mathematics education course you are taking now as a pre-service teacher or the professional development you are experiencing as an in-service teacher will be the foundation for much of the mathematics instruction you do in your classroom for the next decade. The authors of this book take that seriously, as we know you do. Therefore, this section lists and describes the characteristics, habits of thought, skills, and dispositions you will need to succeed as a teacher of mathematics.

Knowledge of Mathematics. You will need to have a profound, flexible, and adaptive knowledge of mathematics content (Ma, 1999). This statement is not intended to scare you if you feel that mathematics is not your strong suit, but it is meant to help you prepare for a serious semester of learning about mathematics and how to teach it. The "school effects" for mathematics are great, meaning that unlike other subject areas, where students have frequent interactions with their family or others outside of school on topics such as reading books, exploring nature, or discussing current events, in the area of mathematics what we do in school is often "it" for many students. This adds to the earnestness of your responsibility, because a student's learning for the year in mathematics will likely come only from you. If you are not sure of a fractional concept or other aspect of mathematics content knowledge, now is the time to make changes in your depth of understanding and flexibility with mathematical ideas to best prepare for your role as an instructional leader. This book and your professor or instructor will help you in that process.

Persistence. You need the ability to stave off frustration and demonstrate persistence. Dweck (2007) has described the brain as similar to a muscle—one that can be strengthened with a good workout! As you move through this book and work the problems yourself, you will learn methods and strategies that will help you anticipate the barriers to students' learning and identify strategies to get them past these stumbling blocks. It is likely that what works for you as a learner will work for your students. As you conduct this mental "workout," if you ponder, struggle, talk about your thinking, and reflect on how these new ideas fit or don't fit with your prior knowledge, then you will enhance your repertoire as a teacher. Remember as you model these characteristics for your students, they too will begin to value perseverance more than speed. In fact, Einstein did not describe himself as intelligent—instead he suggested he was just someone who continued to work on problems longer than others. Creating opportunities for your students to productively struggle is part of the learning process (Stigler & Hiebert, 2009).

Positive Disposition. Prepare yourself by developing a positive attitude toward the subject of mathematics. Research shows that teachers with positive attitudes teach math in more successful ways that result in their students liking math more (Karp, 1991). If in your heart you say, "I never liked math," that mindset will be evident in your instruction (Beilock, Gunderson & Levine, 2010; Maloney, Gunderson, Ramirez, Levin & Beilock, 2014). The good news is that research shows that changing attitudes toward mathematics is relatively easy (Tobias, 1995) and that attitude changes are long-lasting (Dweck, 2006). Additionally math methods courses have been found to be effective in reducing mathematics anxiety (Tooke & Lindstrom, 1998). Expanding your knowledge of the subject and trying new ways to approach problems, you can learn to enjoy doing and presenting mathematical activities. Not only can you acquire a positive attitude toward mathematics, as a professional it is essential that you do.

To explore your students' attitudes toward mathematics consider using this interview protocol. Here you can explore how the classroom environment may affect their attitudes.

Readiness for Change. Demonstrate a readiness for change, even for change so radical that it may cause disequilibrium. You may find that what is familiar will become unfamiliar and, conversely, what is unfamiliar will become familiar. For example, you may have always referred to "reducing fractions" as the process of changing $\frac{2}{4}$ to $\frac{1}{2}$, but this is misleading as the fractions are not getting smaller. Such terminology can lead to mistaken connections. Did the reduced fraction go on a diet? A careful look will point out that *reducing* is not the term to use; rather, you are writing an equivalent fraction that is simplified or in lowest terms. Even though you have used the language *reducing* for years, you need to become familiar with more precise language such as "simplifying fractions."

On the other hand, what is unfamiliar will become more comfortable. It may feel uncomfortable for you to be asking students, "Did anyone solve it differently?" especially if you are worried that you might not understand their approach. Yet this question is essential to effective teaching. As you bravely use this strategy, it will become comfortable (and you will learn new strategies!).

Another potentially difficult shift in practice is toward an emphasis on concepts as well as procedures. What happens in a procedure-focused classroom when a student doesn't understand division of fractions? A teacher with only procedural knowledge is often left to repeat the procedure louder and slower, "Just change the division sign to multiplication, flip over the second fraction, and multiply." We know this approach doesn't work well if we want students to fully understand the process of dividing fractions, so let's consider an example using $3\frac{1}{2} \div \frac{1}{2} = \_\_\_\_$. You might start by relating this division problem to prior knowledge of a whole number division problem such as $25 \div 5 = \_\_\_\_$. A corresponding story problem might be, "How many orders of 5 pizzas are there in a group of 25 pizzas?" Then ask students to put words around the fraction division problem, such as "You plan to serve each guest $\frac{1}{2}$ a pizza. If you have $3\frac{1}{2}$ pizzas, how many guests can you serve?" Yes, there are seven halves in $3\frac{1}{2}$ and therefore 7 guests can be served. Are you surprised that you can do this division of fractions problem in your head?

To respond to students' challenges, uncertainties, and frustrations you may need to unlearn and relearn mathematical concepts, developing comprehensive conceptual understanding and a variety of representations along the way. Supporting your mathematics content knowledge on solid, well-supported terrain is your best hope of making a lasting difference in your students' learning of mathematics—so be ready for change. What you already understand will provide you with many "Aha" moments as you read this book and connect new information to your current mathematics knowledge.

Life-Long Learning, Make Time to Be Self-Aware and Reflective. As Steve Leinwand wrote, "If you don't feel inadequate, you're probably not doing the job" (2007, p. 583). No matter whether you are a pre-service teacher or an experienced teacher, there is always more to learn about the content and methodology of teaching mathematics. The ability to examine oneself for areas that need improvement or to reflect on successes and challenges is critical for growth and development. The best teachers are always trying to improve their practice through the reading latest article, reading the newest book, attending the most recent conference, or signing up for the next series of professional development opportunities. These teachers don't say, "Oh, that's what I am already doing"; instead, they identify and celebrate each new insight. The highly effective teachers never "finish" learning nor exhaust the number of new mental connections that they make and, as a result, they never see teaching as stale or stagnant. An ancient Chinese proverb states, "The best time to plant a tree is twenty years ago; the second best time is today." Explore this self-reflection chart on professional growth to list your strengths and indicate areas for continued growth.

Think back to the quotations from teachers at the beginning of this chapter. Again, what memories will you create for your students? As you begin this adventure let's be reminded of what John Van de Walle said with every new edition, "Enjoy the journey!"

Complete Self-Check 1.3: An Invitation to Learn and Grow

REFLECTIONS ON CHAPTER 1

WRITING TO LEARN

Click here to assess your understanding and application of chapter content.

1. What is meant by a *process* standard as referred to in the *Principles and Standards*? Give a brief description of each of the five process standards.
2. What are the six Standards for Mathematical Practice? How do they relate to the Common Core State Standards content expectations?

FOR DISCUSSION AND EXPLORATION

- Examine a textbook or school mathematics curriculum at any grade level of your choice. If possible, use a teacher's edition. Page through any chapter and look for signs of the five NCTM process standards or the six CCSSO Standards for Mathematical Practice. To what extent are students who are being taught from that textbook likely to be doing and learning mathematics in ways described by those processes or practices?

RESOURCES FOR CHAPTER 1

RECOMMENDED READINGS

Articles

Hoffman, L., & Brahier, D. (2008). Improving the planning and teaching of mathematics by reflecting on research. *Mathematics Teaching in the Middle School, 13* (7), 412–417.

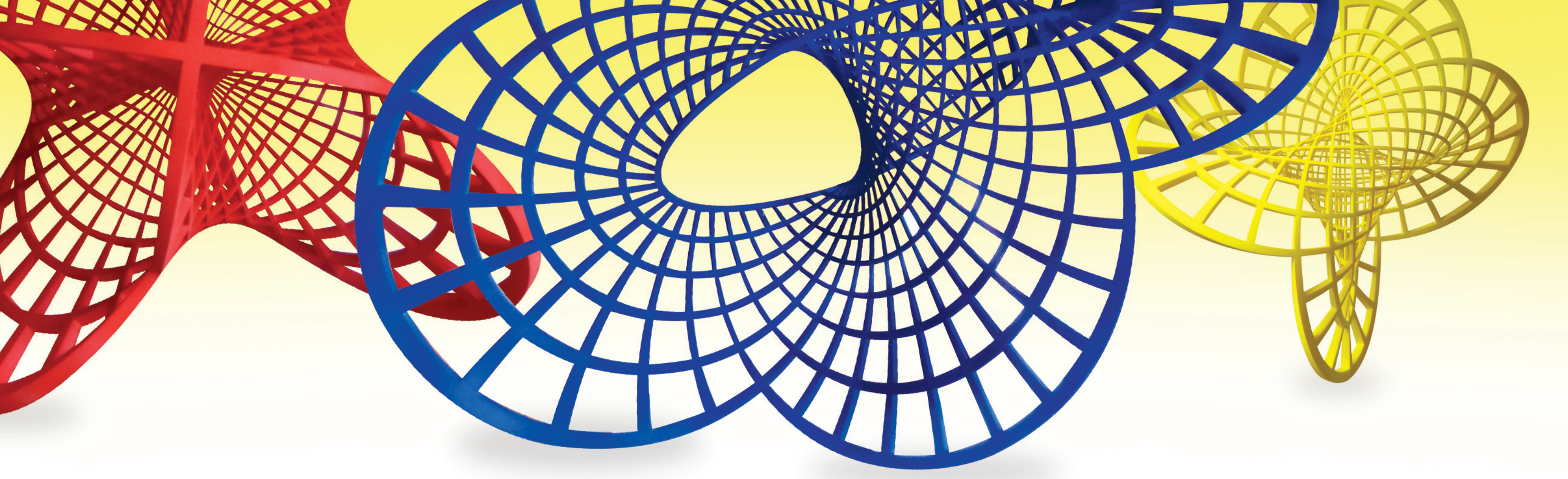
This article addresses how teachers' philosophies and beliefs influence their mathematics instruction. Using TIMSS and NAEP studies as a foundation, the authors discuss posing higher-level problems, asking thought-provoking questions, facing students' frustration, and using mistakes to enhance understanding of concepts. They suggest reflective questions that are useful for self-assessment or discussions with peers.

Books

Bush, S. & Karp, K. (2015) *Discovering lessons for the* Common Core State Standards *in grades K–5*. Reston, VA: NCTM.

Bush, S. & Karp, K. (2014) *Discovering lessons for the* Common Core State Standards *in grades 6–8*. Reston, VA: NCTM.

These two books align the lessons in articles in NCTM journals for the past fifteen years with the Common Core State Standards and the Standards for Mathematical Practices. They provide a way to see how the standards play out in instructional tasks and activities for classroom use.



CHAPTER

2

Exploring What It Means to Know and Do Mathematics

LEARNER OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

2.1 Describe what it means to do mathematics.

2.2 Design and implement strategies for solving authentic mathematics tasks.

2.3 Illustrate through content examples, what a mathematically proficient student knows and is able to do.

2.4 Compare learning theories related to mathematics, and connect the theories to effective teaching practices.

2.5 Synthesize the important theoretical and content ideas related to learning mathematics.

This chapter explains how to help students learn mathematics. To get at how to help students learn, however, we must first consider what is important *to* learn. Let's look at a poorly understood topic, division of fractions, as an opening example. If a student has learned this topic well, what will they know and what should they be able to do? The answer is more than being able to successfully implement a procedure (e.g., commonly called the "invert and multiply" procedure). There is much more to know and understand about division of fractions: What does $3 \div \frac{1}{4}$ mean conceptually? What is a situation that might be solved with such an equation? Will the result be greater than or less than 3 and why? What ways can we solve equations like this? What illustration or manipulative could illustrate this equation? What is the relationship of this equation to subtraction? To multiplication? All of these questions can be answered by a student who fully understands a topic such as division of fractions. We must lead students to this conceptual understanding.

This chapter can help you. It could be called the "what" and "how" of teaching mathematics. First, *what* does doing mathematics look like (be ready to experience this yourself through four great tasks!) and what is important to know about mathematics? Second, *how* do we help students develop a strong understanding of mathematics? By the end of this chapter, you will be able to draw strong connections between the what and the how of teaching mathematics.

What Does It Mean to Do Mathematics?

Mathematics is more than completing sets of exercises or mimicking processes the teacher explains. Doing mathematics means generating strategies for solving a problem, applying that strategy, and checking to see whether your answer makes sense. Finding and exploring regularity or order, and then making sense of it, is what doing mathematics in the real world is all about.

Doing mathematics in classrooms should closely model the act of doing mathematics in the real world. Even our youngest students can notice patterns and order. For example, post a series of problems and ask first or second graders, "What patterns to you notice?"

$$6 + 7 =$$
$$5 + 8 =$$
$$4 + 9 =$$

Think about the patterns students might notice: the first addend is going down 1, the second one is going up one, and the sums are the same. How might exploring these patterns help students to learn about addition? Also consider the next situation related to multiplication that might be explored by third to fifth graders.

Even $\times$ Even $=$ Even	Always	Sometimes	Never
Odd $\times$ Odd $=$ Even	Always	Sometimes	Never
Odd $\times$ Even $=$ Even	Always	Sometimes	Never

Exploring generalizations such as these multiplication ones provides students an opportunity to learn important relationships about numbers as they deepen their understanding of the operations. With each of these problems, you have the opportunity to have students debate which answer they think is correct and to justify (i.e., prove) their response.

In middle school, students continue to explore more advanced patterns and order, extending to negative numbers and exponents, as well as using variables. You also might ask middle school students to look for patterns comparing two solutions, as in this example:

For a fundraiser, Annie and Mac decided to sell school wristbands. They cost $.75 and they are going to sell them for $2.50. They sold 35 the first day. They each calculate the Day 1 profit differently. Who is correct? Explain.

Annie: $(35 \times 2.50) - (35 \times .75) =$

Mac: $\$1.75 \times 35 =$

In comparing these two strategies for finding profit, students are seeing relationships between the equations and the situations, noticing properties of the operations "in action," and discussing equivalencies (a major idea in mathematics!).

Engaging in the science of pattern and order, as the previous two examples illustrate, is *doing* mathematics. Basic facts and basic skills such as computation of whole numbers, fractions, and decimals are important in enabling students to be able to *do* mathematics. But if skills are taught by rote memorization or isolated practice, students will not learn to *do* mathematics, and will not be prepared to do the mathematics required in the 21st century.

Verbs of Doing Mathematics

Doing mathematics begins with posing worthwhile tasks and then creating an environment where students take risks, share, and defend mathematical ideas. Students in traditional mathematics classes often describe mathematics as imitating what the teacher shows them. Instructions to students given by teachers or in textbooks ask students to listen, copy, memorize, drill, and compute. These are lower-level thinking activities and do not adequately prepare students for the real act of doing mathematics. In contrast, the following verbs engage students in doing mathematics:

collaborate	describe	justify
communicate	develop	predict
compare	explain	represent
conjecture	explore	solve
construct	formulate	use
create	invent	verify
critique	investigate	

These verbs lead to opportunities for higher-level thinking and encompass "making sense" and "figuring out." These verbs may look familiar to you, as they are on the higher level of Bloom's (revised) Taxomony (Anderson & Krathwohl, 2001) (see Figure 2.1).

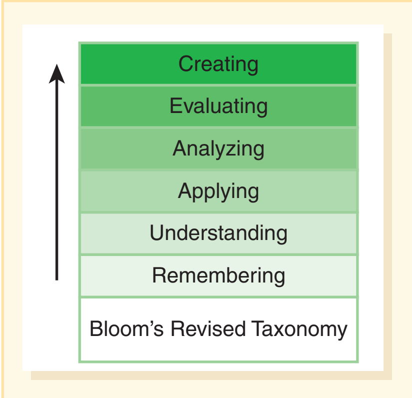


FIGURE 2.1 Bloom's (Revised) Taxonomy (Anderson & Krathwohl, 2001).

Source: Anderson, L.W., & Krathwohl, D. R. (Eds.). (2001). *A taxonomy for learning, teaching, and assessing: A revision of Bloom's Taxonomy of educational objectives: Complete Edition.* New York, NY: Addison Wesley, Longman.

In observing a third-grade classroom where the teacher used this approach to teaching mathematics, researchers found that students became "doers" of mathematics. In other words the students began to take the math ideas to the next level by (1) connecting to previous material, (2) responding with information beyond the required response, and (3) conjecturing or predicting (Fillingim & Barlow, 2010). When this happens on a daily basis, students are getting an empowering message: "You are capable of making sense of this—you are capable of doing mathematics!"

Complete Self-Check 2.1: What Does It Mean to Do Mathematics?

An Invitation to Do Mathematics

The purpose of this section is to provide you with opportunities to engage in the science of pattern and order—to *do* some mathematics. For each problem posed, allow yourself to try to (1) make connections within the mathematics (i.e., make mathematical relationships explicit) and (2) engage in productive struggle.

We will explore four different problems. None requires mathematics beyond elementary school mathematics—not even algebra. But the problems do require higher-level thinking and reasoning. As you read each task, stop and solve first. Then read the "Few Ideas" section. Then, you will be doing mathematics and seeing how others may think about the problem differently (or the same). Have fun!

Searching for Patterns

1. *Start and Jump Numbers*

Begin with a number (start) and add (jump) a fixed amount. For example, start with 3 and jump by 5s. Use the Start and Jump Numbers Activity Page or write the list on a piece of paper. Examine the list and record as many patterns as you see.

A Few Ideas. Here are some questions to guide your pattern search:

- Do you see at least one alternating pattern?
- Have you noticed an odd/even pattern? Why is this pattern true?
- What do you notice about the numbers in the tens place?
- Do the patterns change when the numbers are greater than 100?

Don't forget to think about what happens to your patterns after the numbers are more than 100. How are you thinking about 113? One way is as 1 hundred, 1 ten, and 3 ones. But, of course, it could also be "eleventy-three," where the tens digit has gone from 9 to 10 to 11. How do these different perspectives affect your patterns? What would happen after 999?

Next Steps. Sometimes when you have discovered some patterns in mathematics, it is a good idea to make some changes and see how the changes affect the patterns. What changes might you make in this problem?

Your changes may be even more interesting than the following suggestions. But here are some ideas:

- Change the start number but keep the jump number equal to 5. What is the same and what is different?
- Keep the same start number and explore with different jump numbers.
- What patterns do different jump numbers make? For example, when the jump number was 5, the ones-digit pattern repeated every two numbers—it had a "pattern length" of 2. But when the jump number is 3, the length of the ones-digit pattern is 10! Do other jump numbers create different pattern lengths?
    - For a jump number of 3, how does the ones-digit pattern relate to the circle of numbers in Figure 2.2? Are there similar circles of numbers for other jump numbers?
    - Using the circle of numbers for 3, find the pattern for jumps of multiples of 3, that is, jumps of 6, 9, or 12.

Using Technology. Calculators facilitate exploration of this problem. Using the calculator makes the list generation accessible for young children who can't skip count yet, and it opens the door for students to work with bigger jump numbers like 25 or 36. Most simple calculators have an automatic constant feature that will add the same number successively. For example, if you press 3 + 5 = and then keep pressing =, the calculator will keep counting by fives from the previous answer. This works for the other three operations. Consider demonstrating this with an online calculator or app for the white board so the class can observe and discuss the counting.

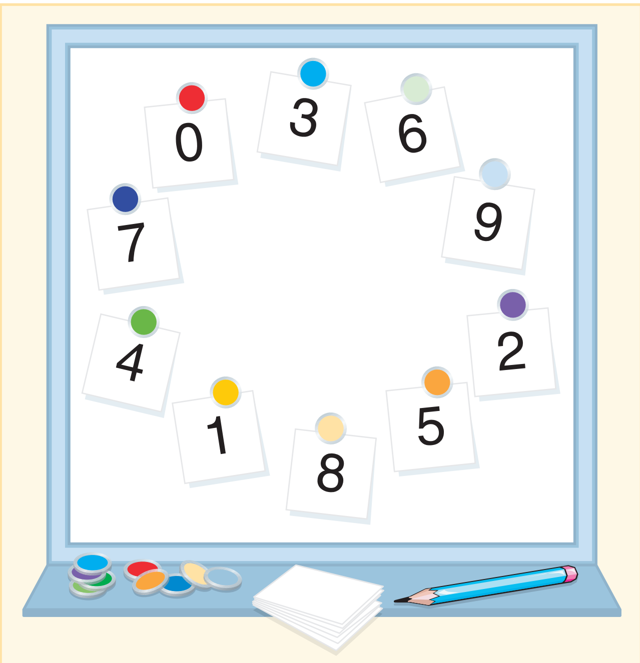


FIGURE 2.2 For jumps of 3, this cycle of digits will occur in the ones place. The start number determines where the cycle begins.

Analyzing a Situation

2. *Two Machines, One Job*

Ron's recycle shop started when Ron bought a used paper-shredding machine. Business was good, so Ron bought a new shredding machine. The old machine could shred a truckload of paper in 4 hours. The new machine could shred the same truckload in only 2 hours. How long will it take to shred a truckload of paper if Ron runs both shredders at the same time?

Use the **Two Machines, One Job** Activity Page to record your solution to this problem. Do not read on until you have an answer or are stuck. Can you check that you are correct? Can you approach the problem using a picture?

A Few Ideas. Have you tried to predict approximately how much time you think it should take the two machines? For example, will it be closer to 1 hour or closer to 4 hours? What facts about the situation led you to this estimated time? Is there a way to check your estimate? Checking a guess in this way sometimes leads to a new insight.

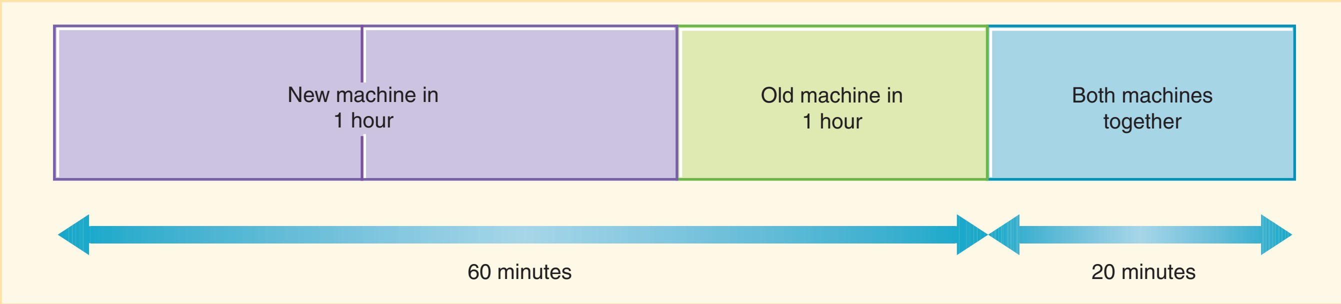


FIGURE 2.3 Cora's solution to the paper-shredding problem.

Some people draw pictures to solve problems. Others like to use something they can move or change. For example, you might draw a rectangle or a line segment to stand for the truckload of paper, or you might get some counters (chips, plastic cubes, pennies) and make a collection that stands for the truckload.

Consider Solutions of Others. There are many ways to model and solve the problem, and understanding other people's ways can develop our own understanding. The following is one explanation for solving the problem, using strips (based on Schifter & Fosnot, 1993):

> "This rectangle [see Figure 2.3] stands for the whole truckload. In 1 hour, the new machine will do half of this." The rectangle is divided in half. "In 1 hour, the old machine could shred $\frac{1}{4}$ of the paper." The rectangle is divided accordingly. "So in 1 hour, the two machines have done $\frac{3}{4}$ of the truck, and there is $\frac{1}{4}$ left. What is left is one-third as much as what they already did, so it should take the two machines one-third as long to do that part as it took to do the first part. One-third of an hour is 20 minutes. That means it takes 1 hour and 20 minutes to do it all.

As with the teachers in these examples, it is important to decide whether your solution is correct through justifying why you did what you did; this reflects real problem solving rather than checking with an answer key. After you have justified that you have solved the problem in a correct manner, try to find other ways that students might solve the problem—in considering multiple ways, you are making mathematical connections.

Generalizing Relationships

3. *One Up, One Down*

***Addition*. When you add 7 + 7, you get 14. When you make the first number 1 more and the second number 1 less, you get the same answer:**

$$\overset{\uparrow}{7} + \overset{\downarrow}{7} = 14 \quad \text{and} \quad 8 + 6 = 14$$

It works for 5 + 5 too:

$$\overset{\uparrow}{5} + \overset{\downarrow}{5} = 10 \quad \text{and} \quad 6 + 4 = 10$$

Does this work for any doubles? For what other addition problems does one up, one down work? Why does it work?

Explore this problem using the **One Up, One Down: Addition** Activity Page. Explore and develop your own conjectures.

Multiplication. Explore this problem using the **One Up, One Down: Multiplication** Activity Page, which focuses on the question, "How does one up, one down work with multiplication?"

$$\overset{\uparrow}{7} \times \overset{\downarrow}{7} = 49$$

$$8 \times 6 = 48$$

One Up, One Down results in an answer that is one less than the original problem. Does this work for any squares (e.g., 5×5)? Is it true for non-square multiplication problems? Explore and develop your own conjectures.

Explore the multiplication problem, responding to the questions posed. Notice that you are asked to develop conjectures. Developing and testing conjectures are an important aspect of mathematical reasoning (Lannin, Ellis, & Elliott, 2011).

A Few Ideas. Multiplication is more complicated. Why? Use a physical model or picture to compare the before and after products. For example, draw rectangles (or arrays) with a length and height of each of the factors (see Figure 2.4(a)), then draw the new rectangle (e.g., 8-by-6-unit rectangle). See how the rectangles compare.

You may prefer to think of multiplication as equal sets. For example, using stacks of chips, 7×7 is seven stacks with seven chips in each stack (set) (see Figure 2.4(b)). The expression 8×6 is represented by eight stacks of six (though six stacks of eight is a possible interpretation). See how the stacks for each expression compare. Consider working with one or both of these approaches to gain insights and make conjectures.?

Additional Patterns to Explore. Recall that doing mathematics includes the tendency to extend beyond the problem posed. This problem lends itself to many "what if?" questions. Here are a few. If you have found other ones, great!

- Have you looked at how the first two numbers are related? For example, 7×7, 5×5, and 9×9 are all products with like factors. What if the product were two consecutive numbers (e.g., 8×7 or 13×12)? What if the factors differ by 2 or by 3?
- Think about adjusting by numbers other than one. What if you adjust two up and two down (e.g., 7×7 to 9×5)?
- What happens if you use big numbers instead of small ones (e.g., 30×30)?
- If both factors increase (i.e., one up, one up), is there a pattern?

Have you made some mathematical connections and conjectures in exploring this problem? In doing so you have hopefully felt a sense of accomplishment and excitement—one of the benefits of *doing* mathematics.

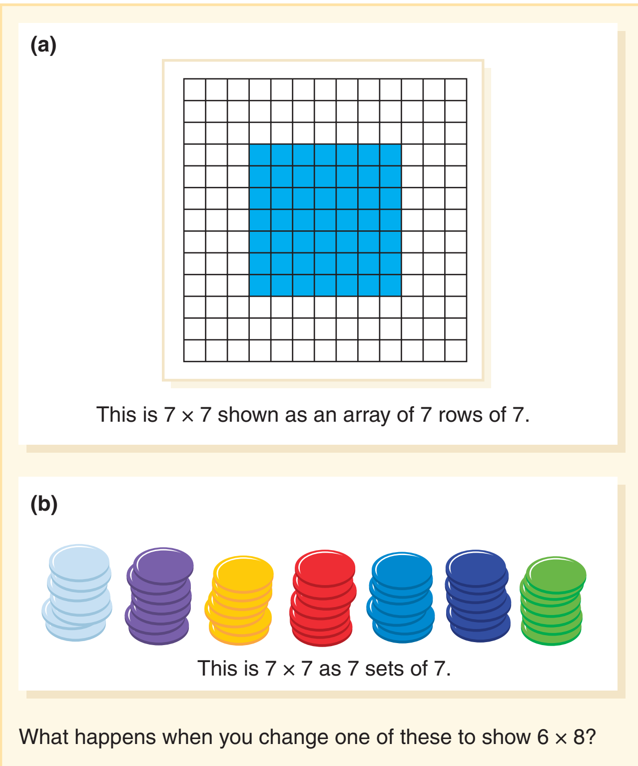


FIGURE 2.4 Two physical ways to think about multiplication that might help in the exploration.

Experimenting and Explaining

4. *The Best Chance of Purple*

Samuel, Susan, and Sandu are playing a game. The first one to spin "purple" wins! Purple means that one spin lands on red and one spins lands on blue (see Figure 2.5). Each person chooses to spin each spinner once or one of the spinners twice. Samuel chooses to spin spinner A twice; Susan chooses to spin spinner B twice;

and Sandu chooses to spin each spinner once. Who has the best chance of purple? (based on Lappan & Even, 1989)

Think about the problem and what you know. Experiment. Use the Best Chance of Purple Activity Page to explore this problem.

A Few Ideas. A good strategy for learning is to first explore a problem concretely, then analyze it abstractly. This is helpful in situations involving chance or probability. Use a paper clip with the spinners on your Activity Page, or use a virtual spinner (e.g., The NCTM Illuminations website has an Adjustable Spinner).

Consider these issues as you explore:

- Explain who you think is most likely to win and why.
- For Sandu's turn (spinner A, then spinner B), would it matter if he spun B first and then A? Why or why not?
- How might you change one spinner so that Susan has the best chance at purple?

Strategy 1: Tree Diagrams. On spinner A, the four colors each have the same chance of coming up. You could make a tree diagram for A with four branches and all the branches would have the same chance (see Figure 2.6). Spinner B has different-sized sections, leading to the following questions:

- What is the relationship between the blue region and each of the others?
- How could you make a tree diagram for B with each branch having the same chance?
- How can you add to the diagram for spinner A so that it represents spinning A twice in succession?
- Which branches on your diagram represent getting purple?
- How could you make tree diagrams for each player's choices?
- How do the tree diagrams relate to the spinners?

Tree diagrams are only one way to approach this. If the strategy makes sense to you, stop reading and solve the problem. If tree diagrams do not seem like a strategy you want to use, read on.

Strategy 2: Grids. Partition squares to represent all the possible outcomes for spinner A and spinner B. Although there are many ways to divide a square into four equal parts, if you use lines going all in the same direction, you can make comparisons of all the outcomes of one event (one whole square) with the outcomes of another event (drawn on a different square). For two independent events, you can then create lines going the other direction for the second event. Samuel's two spins are represented in Figure 2.7(a). If these two squares are overlapped, you can visually see that two parts (two-sixteenths) are "blue on red" or "red on blue." Susan's probability can be determined by layering the

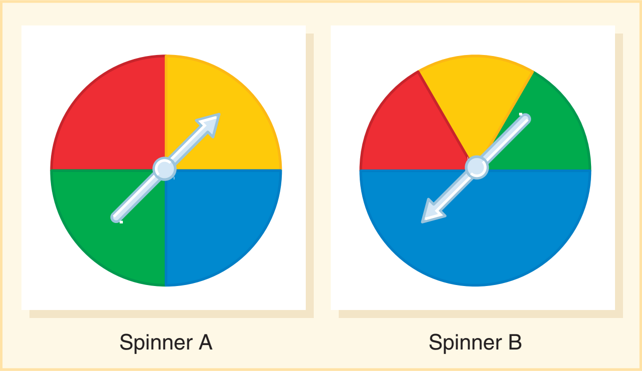


FIGURE 2.5 You may spin A twice, B twice, or A then B. Which choice gives you the best chance of spinning a red and a blue (purple)?

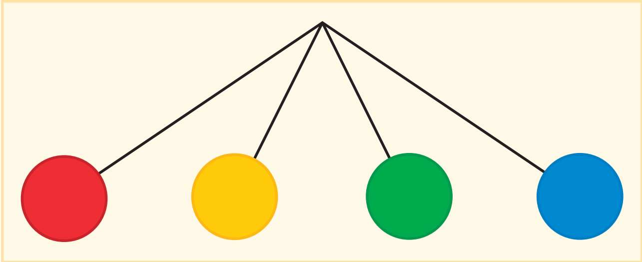


FIGURE 2.6 A tree diagram for spinner A in Figure 2.5.

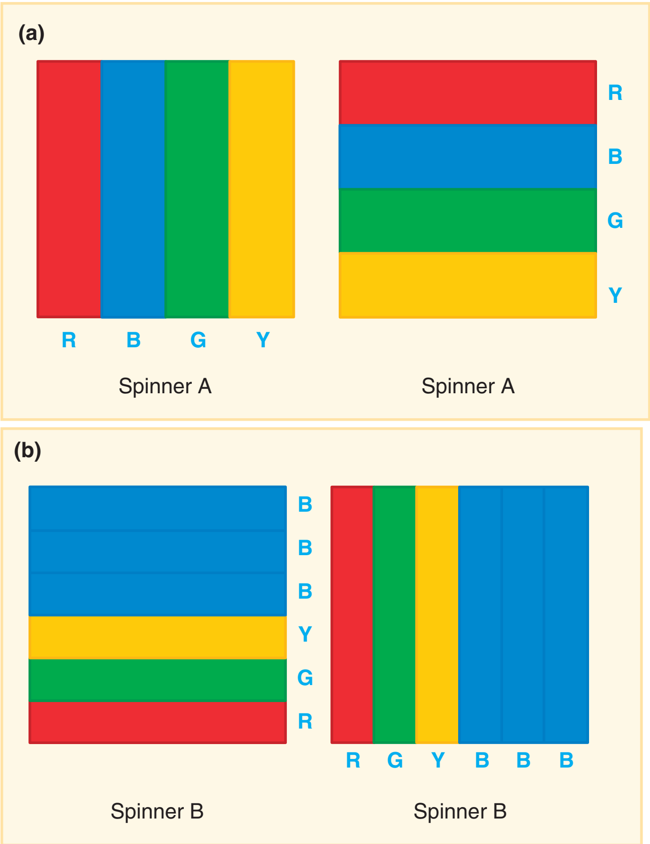


FIGURE 2.7 Grids can illustrate the chance of spinning purple with two spins.

squares in Figure 2.7(b); and Sandu's from layering one square from Figure 2.7(a) with one from Figure 2.7(b).

Why are there four parts for spinner A and 6 parts for spinner B? How is the grid strategy alike and different from the tree diagram? One strategy may make more sense to you, and one may make more sense to another. Hearing other students' explanations and reasoning for both strategies are important in building a strong understanding of mathematics.

Interesting mathematics problems such as the four presented here are plentiful. The Math Forum, for example, has a large collection of classic problems along with discussion, solutions, and extensions. NCTM teacher journals include monthly problems, and readers submit student solutions for these tasks, which appear in an issue a few months later.

Where Are the Answers?

Did you notice that no answers were shared for these four rich tasks? How do you feel about not being able to "check your answer"? You may be wondering if your answer is correct, or if there are other answers. We did this intentionally, because one aspect of becoming mathematically proficient is to be able to rely on one's own justification and reasoning to determine if an answer is correct.

Consider the message students receive when the textbook or the teacher is the source of whether an answer is correct: "Your job is to find the answers that the teacher already has." In the real world of problem solving and doing mathematics, there are no answer books. Doing mathematics includes using justification as a means of determining whether an answer is correct.

Complete Self-Check 2.2: An Invitation to Do Mathematics

What Does It Mean to Be Mathematically Proficient?

In setting learning objectives for students, we often ask, "What will students know? What will students be able to do?" The previous section addressed what they should be able to do, here we focus on several important points related to what students need to know. An important aspect of knowing is understanding.

Let's go back to fractions as an example. What is important for a student to know about fractions such as $\frac{6}{8}$? What might a fourth grader know about $\frac{6}{8}$? At what point do they know enough that they can claim they "understand" fractions? It is more complicated than it might first appear. Here is a short list of what they might know or be able to do:

- Read the fraction.
- Identify the 6 and 8 as the numerator and denominator, respectively.
- Recognize it is equivalent to $\frac{3}{4}$.
- Say that it is more than $\frac{1}{2}$ (recognize relative size).
- Draw a region that is shaded in a way to show $\frac{6}{8}$.
- Find $\frac{6}{8}$ on a number line.
- Illustrate $\frac{6}{8}$ of a set of 48 pennies or counters.
- Know that there are infinitely many equivalencies to $\frac{6}{8}$.
- Recognize $\frac{6}{8}$ as a rationale number.
- Realize $\frac{6}{8}$ might also be describing a ratio (girls to boys, for example).
- Be able to represent $\frac{6}{8}$ as a decimal fraction.

For any item on this list, how much and what a student understands will vary. For example, a student may know that $\frac{6}{8}$ can be simplified to $\frac{3}{4}$ but not understand that $\frac{3}{4}$ and $\frac{6}{8}$ represent

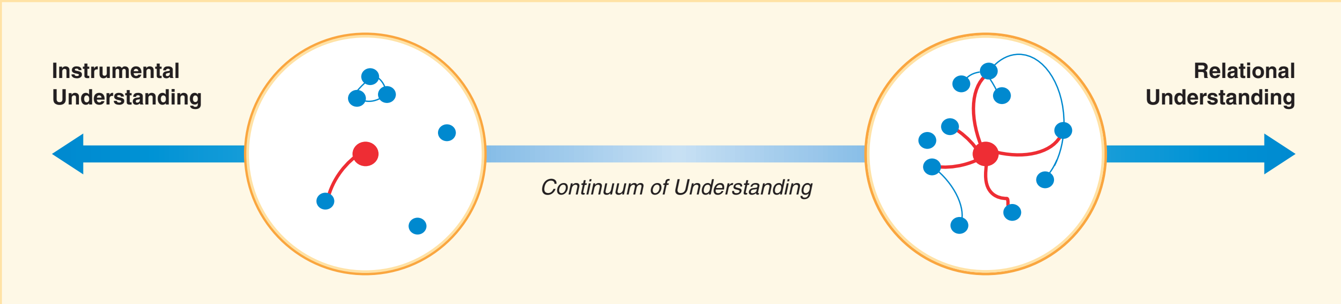


FIGURE 2.8 Understanding is a measure of the quality and quantity of connections that a new idea has with existing ideas. The greater the number of connections to a network of ideas, the better the understanding.

equal quantities, thinking that three-fourths is actually smaller. Or, they may be able to find one fraction between $\frac{1}{2}$ and $\frac{6}{8}$, but not be able to find others or think there is only one fraction between these two fractions.

Understanding can be defined as a measure of the quality and quantity of connections that an idea has with existing ideas. Understanding is not an all-or-nothing proposition. It depends on the existence of appropriate ideas and on the creation of new connections, varying with each person (Backhouse, Haggarty, Pirie, & Stratton, 1992; Davis, 1986; Hiebert & Carpenter, 1992).

Relational Understanding

One way that we can think about understanding is that it exists along a continuum from an *Instrumental understanding*—doing something without understanding (see Figure 2.8) to a *Relational understanding*—knowing what to do and why. These two terms were introduced by Richard Skemp in 1978 and continue to be an important distinction related to what is important for students to know about mathematics.

In the $\frac{6}{8}$ example, a student who only knows a procedure for simplifying $\frac{6}{8}$ to $\frac{3}{4}$ has an understanding near the instrumental end of the continuum, while a student who can draw diagrams, give examples, find equivalencies, and tell the approximate size of $\frac{6}{8}$ has an understanding toward the relational end of the continuum. Here we briefly share three important ways to nurture a relational understanding.

Use and Connect Different Representations. In order for students to build connections among ideas, different representations must be included in instruction, and opportunities must be provided for students to make connections among the representations. Figure 2.9 illustrates a Web of Representations that apply to any mathematical concept. Students who have difficulty translating a concept from one representation to another also have difficulty solving problems and understanding computations (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Strengthening the ability to move between and among these representations improves student understanding and retention. For any topic you teach, you can give students the Translation Task Activity Page to complete. You can

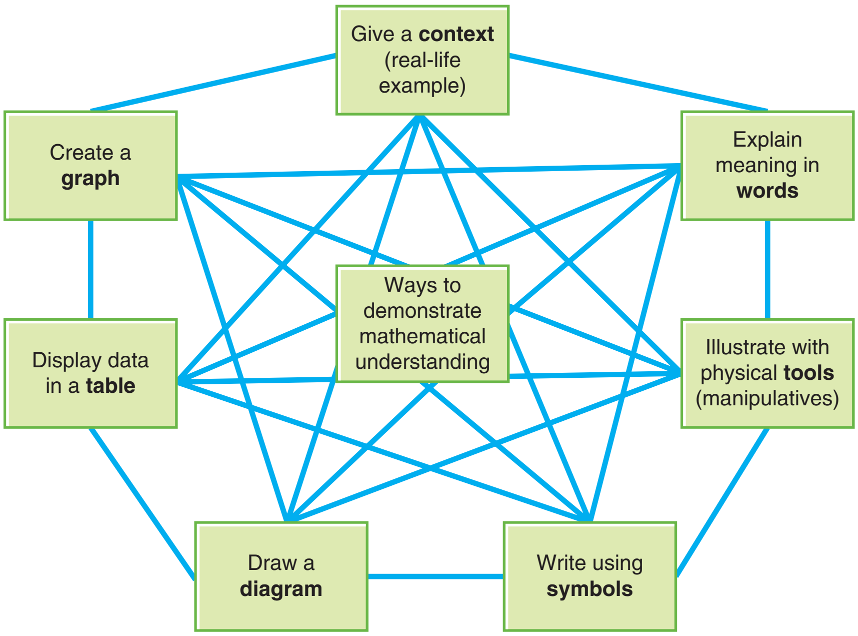
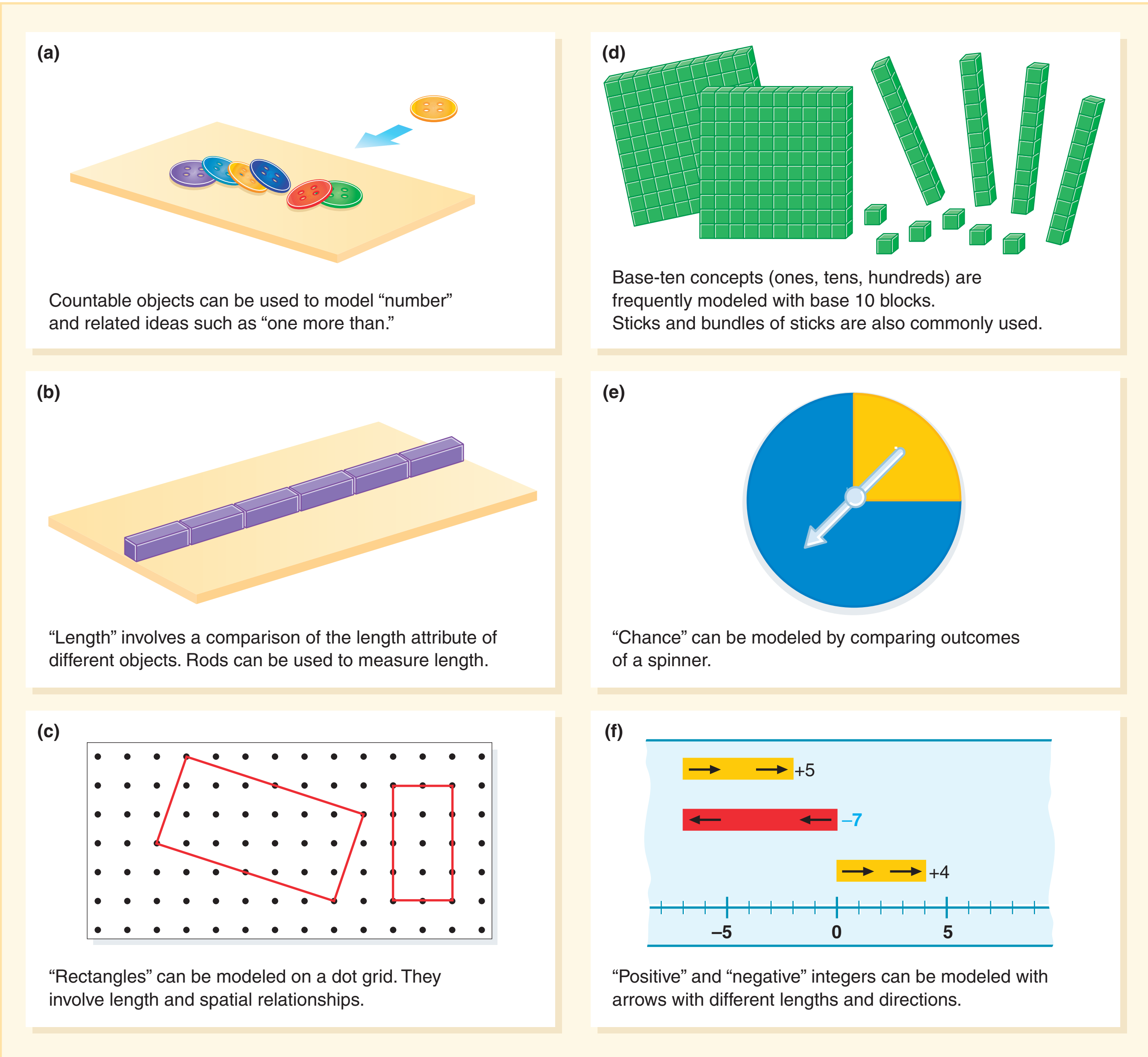


FIGURE 2.9 Web of Representations. Translations between and within each representation of a mathematical idea can help students build a relation understanding of a mathematical concept.

fill out one box and ask them to insert the other representations, or you can invite a group to work on all four representations for a given topic (e.g., multiplication of whole numbers).

Explore with Tools. A *tool* is any object, picture, or drawing that can be used to explore a concept. CCSS-M incudes calculators and manipulatives as tools for doing mathematics (CCSSO, 2010). *Manipulatives* are physical objects that students and teachers can use to illustrate and discover mathematical concepts, whether made specifically for mathematics (e.g., connecting cubes) or for other purposes (e.g., buttons). Choices for manipulatives (including virtual manipulatives) abound—from common objects such as lima beans to commercially produced materials such as Pattern Blocks. Figure 2.10 shows six examples, each representing a different concept, just to give a glimpse (Part II of this book is full of more options). More and more of these manipulatives and others (e.g., geoboards, base-ten blocks, spinners, number lines) are available in a virtual format, for example, on the National Library of Virtual Manipulatives website and the NCTM Illuminations website. Each has a range of manipulatives available.



(a) Countable objects can be used to model "number" and related ideas such as "one more than."

(d) Base-ten concepts (ones, tens, hundreds) are frequently modeled with base 10 blocks. Sticks and bundles of sticks are also commonly used.

(b) "Length" involves a comparison of the length attribute of different objects. Rods can be used to measure length.

(e) "Chance" can be modeled by comparing outcomes of a spinner.

(c) "Rectangles" can be modeled on a dot grid. They involve length and spatial relationships.

(f) "Positive" and "negative" integers can be modeled with arrows with different lengths and directions.

FIGURE 2.10 Examples of tools to illustrate mathematics concepts.

A tool does not "illustrate" a concept. The tool is used to visualize a mathematical concept and only your mind can impose the mathematical relationship on the object (Suh, 2007b; Thompson, 1994). As noted in the What Chance of Purple problem, manipulatives can be a testing ground for emerging ideas. They are more concrete and provide insights into new and abstract relationships. A variety of tools should be accessible for students to select and use appropriately as they engage in doing mathematics (CCSSO, 2010).

Before you continue, consider each of the concepts and the corresponding model in Figure 2.10. Try to separate the physical tool from the relationship that you must impose on the tool in order to "see" the concept.

The examples in Figure 2.10 are models that can show the following concepts:

a. The concept of "6" is a relationship between sets that can be matched to the words *one*, *two*, *three*, *four*, *five*, or *six*. Changing a set of counters by adding one changes the relationship. The difference between the set of 6 and the set of 7 is the relationship "one more than."

b. The concept of "measure of length" is a comparison. The length measure of an object is a comparison relationship of the length of the object to the length of the unit.

c. The concept of "rectangle" includes both spatial and length relationships. The opposite sides are of equal length and parallel and the adjacent sides meet at right angles.

d. The concept of "hundred" is not in the larger square but in the relationship of that square to the strip ("ten") and to the little square ("one").

e. "Chance" is a relationship between the frequency of an event happening compared with all possible outcomes. The spinner can be used to create relative frequencies. These can be predicted by observing relationships of sections of the spinner.

f. The concept of a "negative integer" is based on the relationships of "magnitude" and "is the opposite of." Negative quantities exist only in relation to positive quantities. Arrows on the number line model the opposite of relationship in terms of direction and size or magnitude relationship in terms of length.

While tools can be used to support relational understanding, they can be used ineffectively and not accomplish this goal. One of the most widespread misuses of tools occurs when the teacher tells students, "Do as I do." There is a natural temptation to get out the materials and show students how to use them to "show" the concept. It is just as possible to move blocks around mindlessly as it is to "invert and multiply" mindlessly. Neither promotes thinking or aids in the development of concepts (Ball, 1992; Clements & Battista, 1990; Stein & Bovalino, 2001). On the other extreme, it is ineffective to provide no focus or purpose for using the tools. This will result in nonproductive and unsystematic investigation (Stein & Bovalino, 2001).

Mathematical Proficiency

As you learned in Chapter 1, the standards used in many U.S. states are the CCSS-M (2010). Whether or not your state is one of these, the eight Standards of Mathematical Practice are worthy of attention (See Appendix B). They describe what a mathematically proficient student can do. These are not graduation expectations—these are *daily expectations* for doing mathematics, beginning in kindergarten and continuing throughout school. They are based on research on how students learn that was published by the National Research Council (NRC) in *Adding It Up* (NRC, 2001). Figure 2.11 illustrates these interrelated and interwoven strands. For more on fluency, listen to a past-president of NCTM. Go

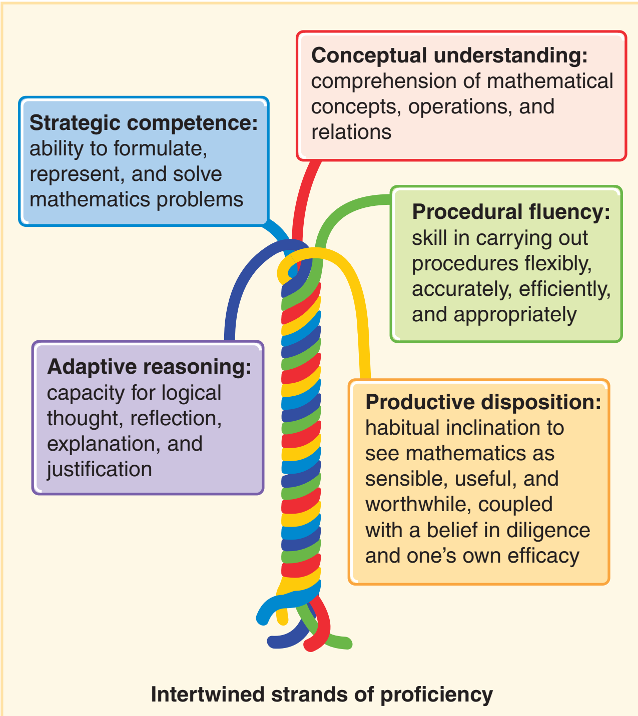


FIGURE 2.11 *Adding It Up* describes five strands of mathematical proficiency.

Source: National Research Council. (2001). *Adding It Up: Helping Children Learn Mathematics*, p. 5. Reprinted with permission from the National Academy of Sciences, courtesy of the National Academies Press, Washington, DC.

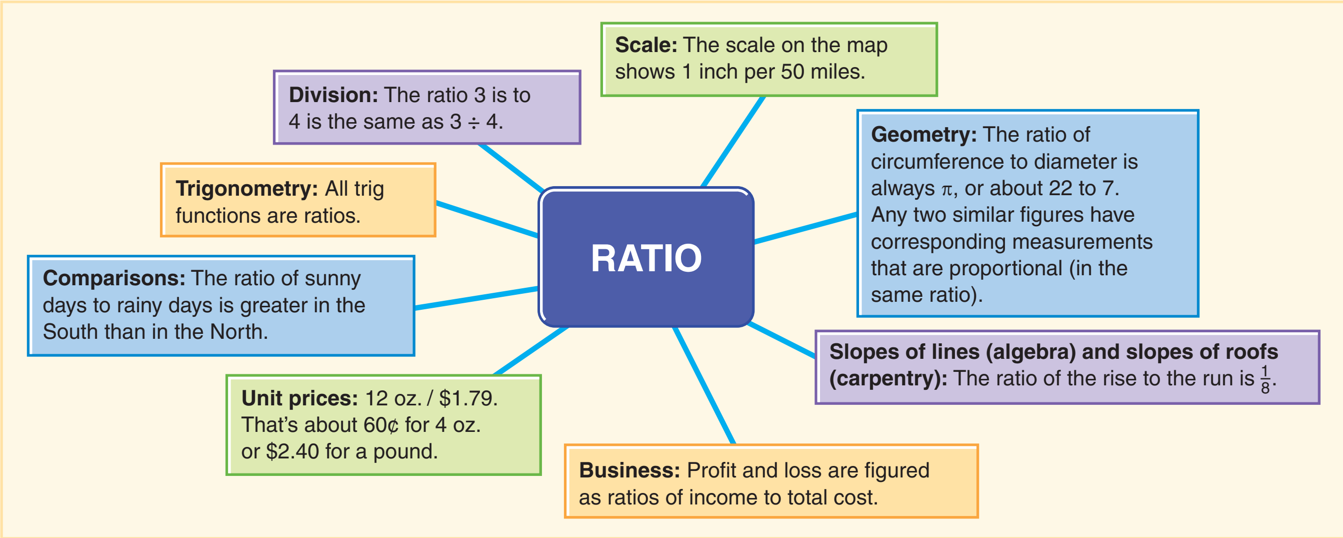


FIGURE 2.12 Potential web of ideas that could contribute to the understanding of "ratio."

to the NCTM website and search for the 2014 Annual Meeting Webcast "President's Session—Fluency . . . It's More Than Fast and Accurate."

Conceptual Understanding. Conceptual understanding is a flexible web of connections and relationships within and between ideas, interpretations and images of mathematical concepts—a relational understanding. Consider the web of associations for ratio as shown in Figure 2.12.

Students with a conceptual understanding will connect what they know about division and numbers to make sense of scaling, unit prices, and so on. Note how much is involved in having a relational understanding of ratio.

Conceptual understanding includes the network of representations and interpretations of a concepts through the use of pictures, manipulatives, tables, graphs, words, and so on (see Figure 2.9). An illustration for ratios across these representations is provided in Figure 2.13.

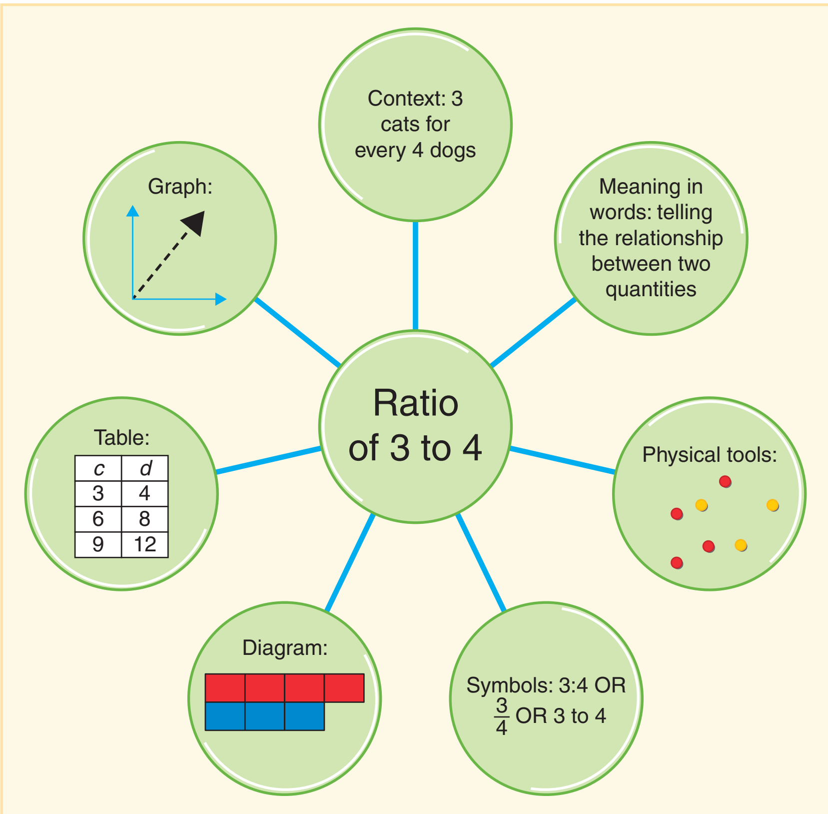


FIGURE 2.13 Multiple representations for ratio of 3 to 4.

Procedural Fluency. Procedural fluency is sometimes confused with being able to do standard algorithms correctly and quickly, but it is much bigger than that. Look at the four descriptors of procedural fluency in Figure 2.11. Fluency includes having the ability to be flexible and to choose an appropriate strategy for a particular problem. Let's look at the problem 37 + 28. Younger students might be able to count all (see Figure 2.14(a)), or even start with the larger and count on to reach a total. Eventually skip counting can be used as a more efficient strategy, and students are able to count up by 10s and 1s (see Figure 2.14(b)). At a higher level of fluency, students are able to select a strategy that is efficient, for example, moving two from the 37 to the 28 to create a benchmark number or adding two on to 28 to add, and then taking it off again (see Figure 2.14(c)). Notice that to use these efficient and appropriate strategies requires a conceptual understanding of place value and addition.

The ineffective practice of teaching procedures in the absence of **conceptual understanding** (https://www.youtube.com/watch?v=FVKtQwARe6c) results in a lack of retention and increased errors, rigid approaches, and inefficient strategy use (Figure 2.14(d)).

Think about the following problem: 40,005−39,996 = _____. A student with rigid procedural skills may launch into the standard algorithm, regrouping across zeros (often with difficulty), rather than notice that the number 39,996 is just 4 away from 40,000, and therefore that the difference between the two numbers is 9.

Efficient Strategy: 39,996 $\xrightarrow{4}$ 40, 000 $\xrightarrow{5}$ 40,005

Developing conceptual understanding alongside procedural proficiency is crucial to becoming mathematically proficient (Baroody, Feil, & Johnson, 2007; Bransford et al., 2000; NCTM, 2014). **Click here** for an observation tool that focuses on evidence of mathematical proficiency in a classroom setting.

Perseverance and a Productive Disposition. As the Mathematical Practices and the Strands of Proficiency describe, being proficient at mathematics is not just what you know, but how you go about solving problems. Consider this short list of reflective prompts. Which ones might a proficient student say yes to often?

- When you read a problem you don't know how to solve, do you think, "Cool, something challenging. I can solve this. Let me now think how."?
- Do you consider several possible approaches before diving in to solve?
- Do you have a way of convincing yourself or your peer that it had to be correct?
- Do you recognize a wrong path and try something else?
- When you finish a problem, do you wonder whether it is right? If there are other answers?
- Do you look for patterns across examples and try to see a new shortcut or approach that might work?
- As you work, do you decide to draw a picture, use a calculator, or model the problem with a manipulative?

When students are in classrooms where they are able to do mathematics, these proficiencies develop and students build a stronger understanding of the mathematics they need to know, both the concepts and procedures, and are able to become mathematically proficient.

Complete Self-Check 2.3: What Does It Mean to Be Mathematically Proficient?

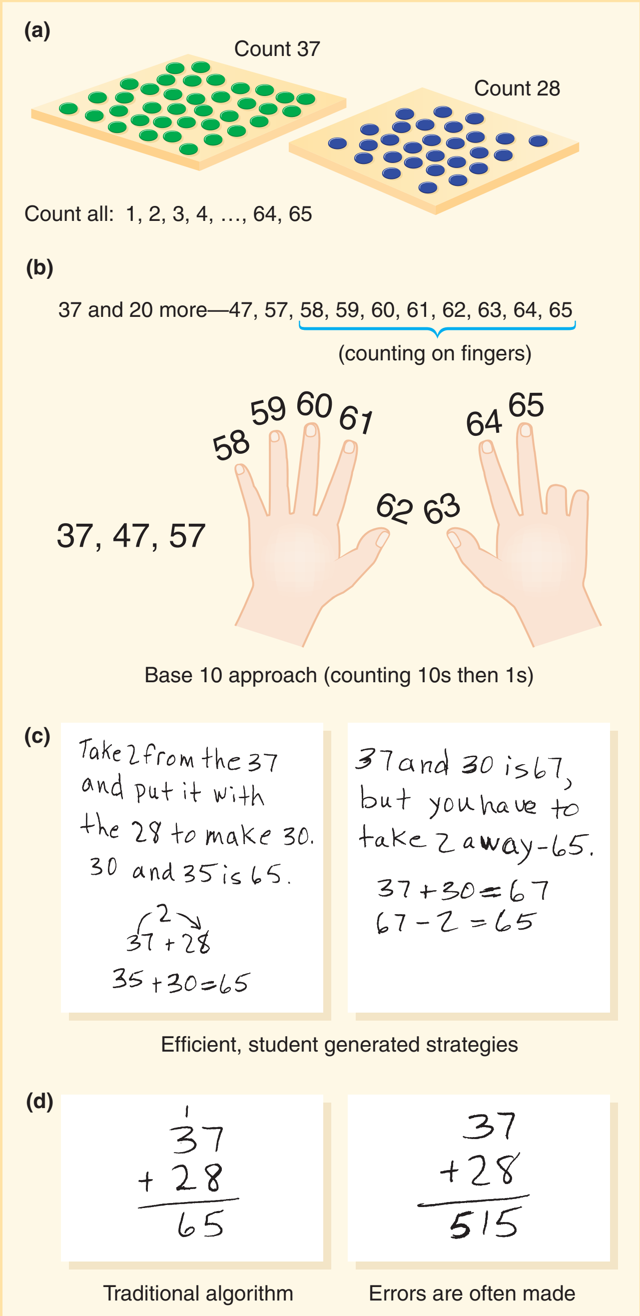


FIGURE 2.14 A range of levels of procedural fluency for 37+28.

How Do Students Learn Mathematics?

Now that you have had the chance to experience doing mathematics, you may have a series of questions: Can students solve such challenging tasks? Why take the time to solve these problems—isn't it better to do a lot of shorter problems? Why should students be doing problems like this especially if they are reluctant to do so? In other words, how does "doing mathematics" relate to student learning? The answer lies in learning theory and research on how people learn.

In mathematics education there is no consensus about what it means to know and understand mathematics. Theories, such as **behaviorism**, **cognitivism**, **constructivism**, and **sociocultural theory**, have influenced the way in which mathematics is taught. Even within these theories, there are different interpretations of what they mean and what the interpretation of that theory into classroom practice might look like. As a teacher, you rely on your own beliefs and theories as you decide what you think will most help your students learn. Your beliefs may be influenced by theorists and from your own pragmatic experiences. It is important for you to attend to your own beliefs and how they relate to your teaching practice (Davis & Sumara, 2012).

Learning theories have been developed through analysis of students and adults as they develop new understandings. They can be thought of as tools or lenses for interpreting how a person learns (Simon, 2009). Here we describe two theories, constructivism and sociocultural theory, that are commonly used by researchers to understand how students learn mathematics. These theories are not competing and are compatible (Norton & D'Ambrosio, 2008).

Constructivism

Constructivism is rooted in Jean Piaget's work which was developed in the 1930s and translated to English in the 1950s. At the heart of constructivism is the notion that learners are not blank slates but rather creators or constructors of their own learning. Integrated *networks*, or *cognitive schemas*, are both the product of constructing knowledge and the tools with which additional new knowledge can be constructed. As learning occurs the networks are rearranged, added to, or otherwise modified. This is an active endeavor on the part of the learner (Baroody, 1987; Cobb, 1988; Fosnot, 1996; von Glasersfeld, 1990, 1996).

All people construct or give meaning to things they perceive or think about. As you read these words you are giving meaning to them. Whether listening passively to a lecture or actively engaging in synthesizing findings in a project, your brain is applying prior knowledge your existing schemas to make sense of the new information.

Through *reflective thought*—the effort to connect existing ideas to new information—people modify their existing schemas to incorporate new ideas (Fosnot, 1996). This can happen through *assimilation* or *accommodation*. Assimilation occurs when a new concept "fits" with prior knowledge and the new information expands an existing network. Accommodation takes place when the new concept does not "fit" with the existing network, causing what Piaget called *disequilibrium*, so the brain revamps or replaces the existing schema.

Construction of Ideas. To construct or build something in the physical world requires tools, materials, and effort. The tools we use to build understanding are our existing ideas and knowledge. The materials we use may be things we see, hear, or touch, or our own thoughts and ideas. The effort required to connect new knowledge to old knowledge is reflective thought.

In Figure 2.15, blue and red dots are used as symbols for ideas. Consider the picture to be a small section of our cognitive makeup. The blue dots represent existing ideas. The lines joining the ideas represent our logical connections or relationships that have developed between and among ideas. The red dot is an emerging idea, one that is being constructed. Whatever existing ideas, blue dots, are used in the construction will be connected to the new idea, red dot, because those were the ideas that gave meaning to it. If a potentially relevant idea, blue dot, is not accessed by the learner when learning a new concept, red dot, then that potential connection will not be made. For more information on how constructivism applies to mathematics education, the Math Forum offers links to numerous sites and articles.

Sociocultural Theory

In the 1920s and 1930s, Lev Vygotsky, a Russian psychologist, began developing what is now called sociocultural theory. There are many theorical ideas that sociocultural theory shares with constructivism (for example, the learning process as active meaning-seeking on the part of the learner), but sociocultural theory has several unique features. One is that mental processes exist between and among people in social learning settings, and that from these social settings the learner moves ideas into his or her own psychological realm (Forman, 2003).

An important aspect of sociocultural theory is that the way in which information is internalized, or learned, depends on whether it was within a learner's zone of proximal development (ZPD) (Vygotsky, 1978). Simply put, the ZPD refers to a range of knowledge that may be out of reach for a person to learn on his or her own, but is accessible if the learner has support from peers or more knowledgeable others. Researchers Cobb (1994) and Goos (2004) suggest that in a true mathematical community of learners there is something of a common ZPD that emerges across learners and there are also the ZPDs of individual learners.

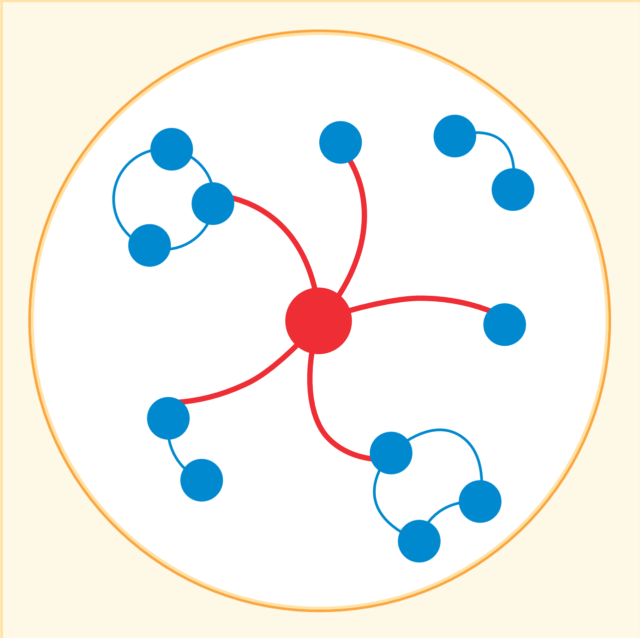


FIGURE 2.15 We use the ideas we already have (blue dots) to construct a new idea (red dot), in the process developing a network of connections between ideas. The more ideas used and the more connections made, the better we understand.

Another major component in sociocultural theory is *semiotic mediation*. Semiotic refers to the use of language and other tools, such as diagrams, pictures, and actions, to convey cultural practices. Mediation means that these semiotics are exchanged between and among people. So, semiotic mediation is the way in which an individual's beliefs, attitudes, and goals affect and are affected by sociocultural practices (Forman & McPhail, 1993). In mathematics, semiotics include mathematical symbols (e.g., the equal sign) and it is through classroom interactions and activities that the meaning of these symbols are developed.

Social interaction is essential for learning to occur. The nature of the community of learners is affected by not just the culture the teacher creates, but the broader social and historical culture of the members of the classroom (Forman, 2003). In summary, from a sociocultural perspective, learning is dependent on the new knowledge falling within the ZPD of the learner who must have access to the assistance, and occurs through interactions that are influenced by tools of mediation and the culture within and beyond the classroom.

Implications for Teaching Mathematics

It is not necessary to choose between a social constructivist theory that favors the views of Vygotsky and a cognitive constructivism built on the theories of Piaget (Cobb, 1994; Simon, 2009). In fact, when considering classroom practices that maximize opportunities to construct ideas, or to provide tools to promote mediation, they are quite similar. Classroom discussion based on students' own ideas and solutions to problems is essential to learning (Wood & Turner-Vorbeck, 2001).

Remember that learning theory is not a teaching strategy—theory *informs* teaching. This section outlines teaching strategies that are informed by constructivist and sociocultural perspectives. You will see these strategies revisited in more detail in Chapters 3 and 4, where a problem-based model for instruction is discussed, and in Section II of this book, where you learn how to apply these ideas to specific areas of mathematics.

Importantly, if these strategies are grounded in how people learn, it means *all* people learn this way—students with special needs, English language learners, students who struggle, and students who are gifted. Too often, when teachers make adaptations and modifications for particular learners, they trade in strategies that align with learning theories and research for methods that seem "easier" for students. These strategies, however, provide fewer opportunities for students to connect ideas and build knowledge—thereby impeding, not supporting, learning.

Build New Knowledge from Prior Knowledge. If you are teaching a new concept, like division, it must be developed using what students already know about equal subtraction and sharing. Consider the following task.

Goodies Toy Store is creating bags with 3 squishy balls in each. If they have 24 squishy balls, how many bags will they be able to make

---

Here, consider how you might introduce division to third graders and what your expectations might be for this problem as a teacher grounding your work in constructivist or sociocultural learning theory.

From a constructivist and sociocultural perspective, this classroom culture allows students to access their prior knowledge, use cultural tools, and build new knowledge. You might ask students to use manipulatives or to draw pictures to solve this problem. As they work, they might have different ways of thinking about the problem (e.g., skip counting up by 3s, or skip counting down by 3s). These ideas become part of a classroom discussion, connecting what they know about equal subtraction and addition, and connecting that to multiplication and division. Interestingly, this practice of connecting ideas is not only grounded in learning theory, but has been established through research studies.

Recall that making mathematical relationships explicit is connected with improving student conceptual understanding (Hiebert & Grouws, 2007). The teacher's role in making mathematical relationships explicit is to be sure that students are making the connections that are implied in a task. For example, asking students to relate today's topic to one they investigated last week, or by asking "How is Lisa's strategy like Marco's strategy?" when the two students have picked different ways to solve a problem, are ways to be "explicit" about mathematical relationships. Students apply their prior knowledge, test ideas, make connections, compare, and make conjectures. The more students see the connections among problems and among mathematical concepts, the more deeply they understand mathematics.

Provide Opportunities to Communicate about Mathematics. Learning is enhanced when the learner is engaged with others working on the same ideas. The rich interaction in such a classroom allows students to engage in reflective thinking and to internalize concepts that may be out of reach without the interaction and input from peers and their teacher. In discussions with peers, students will be adapting and expanding on their existing networks of concepts.

Create Opportunities for Reflective Thought. Classrooms need to provide structures and supports to help students make sense of mathematics in light of what they know. For a new idea you are teaching to be interconnected in a rich web of interrelated ideas, children must be mentally engaged. They must find the relevant ideas they possess and bring them to bear on the development of the new idea. In terms of the dots in Figure 2.15 we want to activate every blue dot students have that is related to the new red dot we want them to learn. It is through thinking, talking, and writing, that we can help students reflect on how mathematical ideas are connected to each other.

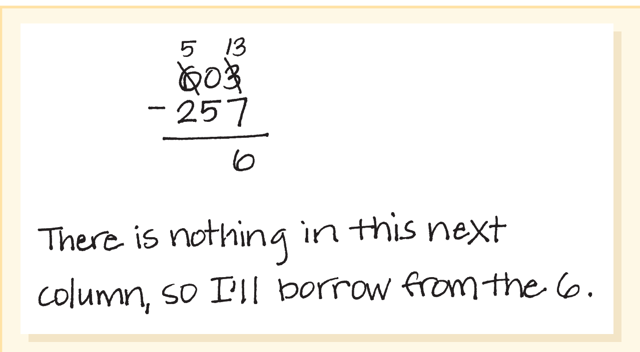


FIGURE 2.16 This student's work indicates that she has a misconception about place value and regrouping.

Encourage Multiple Approaches. Encourage students to use strategies that make sense to them. The student whose work is presented in Figure 2.16 may not understand the algorithm she used. If instead she were asked to use her own approach to find the difference, she might be able to get to a correct solution and build on her understanding of place value and subtraction.

Even learning a basic fact, like 7×8, can have better results if a teacher promotes multiple strategies. Imagine a class where students discuss and share clever ways to figure out the product. One student might think of 5 eights (40) and then 2 more eights (16) to equal 56. Another may have learned 7×7 (49) and added on 7 more to get 56. Still another might think "8 sevens" and take half of the sevens (4×7) to get 28 and double 28 to get 56. A class discussion sharing these ideas brings to the fore a wide range of useful mathematical "dots" relating addition and multiplication concepts.

Engage Students in Productive Struggle. Have you ever just wanted to think through something yourself without being interrupted or told how to do it? Yet, how often in mathematics class does this happen? As soon as a student pauses in solving a problem the teacher steps in to show or explain. While this may initially get the student to an answer faster, it does not help the student learn mathematics—engaging in productive struggle is what helps students learn mathematics. As Piaget describes, learners are going to experience disequilibrium in developing new ideas. Let students know this disequilibrium is part of the process. This is also one of the findings mentioned earlier as key to developing conceptual understanding (Hiebert & Grouws, 2007).

Notice the importance of both words in "productive struggle." Students must have the tools and prior knowledge to solve a problem, and not be given a problem that is out of reach, or they will struggle without being productive; yet students should not be given tasks that are straightforward and easy or they will not be struggling with mathematical ideas. When students. even very young students, know that struggle is expected as part of the process of doing mathematics, they embrace the struggle and feel success when they reach a solution (Carter, 2008).

This means redefining what it means to "help" students! Rather than showing students how to do something, your role is to ask probing questions that keep students engaged in the productive struggle until they reach a solution. This communicates high expectations and maximizes students' opportunities to learn with understanding.

Treat Errors as Opportunities for Learning. When students make errors, it can mean a misapplication of their prior knowledge in the new situation. Remember that from a constructivist perspective, the mind is sifting through what it knows in order to find useful approaches for the new situation. Students rarely give random responses, so their errors are insight into misconceptions they might have. For example, students comparing decimals may incorrectly apply "rules" of whole numbers, such as "the more digits, the bigger the number" (Martinie, 2014). Often one student's misconception is shared by others in the class and discussing the problem publicly can help other students understand (Hoffman, Breyfogle, & Dressler, 2009). You can introduce errors and ask students to imagine what might have led to that answer (Rathouz, 2011). This public negotiation of meaning allows students to construct deeper meaning for the mathematics they are learning.

Scaffold New Content. The practice of *scaffolding*, often associated with sociocultural theory, is based on the idea that a task otherwise outside of a student's ZPD can become accessible if it is carefully structured. For concepts completely new to students, the learning requires more structure or assistance, including the use of tools like manipulatives or more assistance from peers. As students become more comfortable with the content, the scaffolds are removed and the student becomes more independent. Scaffolding can provide support for those students who may not have a robust collection of "blue dots."

Honor Diversity. Finally, and importantly, these theories emphasize that each learner is unique, with a different collection of prior knowledge and cultural experiences. Since new knowledge is built on existing knowledge and experience, effective teaching incorporates and builds on what the students bring to the classroom, honoring those experiences. Thus, lessons begin with eliciting prior experiences, and understandings and contexts for the lessons are selected based on students' knowledge and experiences. Some students will not have all the "blue dots" they need—it is your job to provide experiences where those blue dots are developed and then connected to the concept being learned.

Classroom culture influences the individual learning of your students. As stated previously, you should support a range of approaches and strategies for doing mathematics. Students' ideas should be valued and included in classroom discussion of the mathematics. This shift in practice, away from the teacher telling one way to do the problem, establishes a classroom culture where ideas are valued. This approach values the uniqueness of each individual.

Create a Classroom Environment for Doing Mathematics. Classrooms where students are making sense of mathematics do not happen by accident—they happen because the teacher establishes practices and expectations that encourage risk taking, reasoning, sharing, and so on. The list below provides expectations that are often cited as ones that support students in doing mathematics (Clarke & Clarke, 2004; CCSSO, 2010; Hiebert et al., 1997; NCTM, 2007).

1. *Persistence, effort, and concentration are important in learning mathematics.* Engaging in productive struggle is important in learning! The more a student stays with a problem, the more likely they are to get it right. Getting a tough problem right leads to a stronger sense of accomplishment than getting a quick, easy problem correct.
2. *Students share their ideas.* Everyone's ideas are important, and hearing different ideas helps students to become strategic in selecting good strategies.
3. *Students listen to each other.* All students have something to contribute and these ideas should be considered and evaluated for whether they will work in that situation.
4. *Errors or strategies that didn't work are opportunities for learning.* Mistakes are opportunities for learning—why did that approach not work? Could it be adapted and work or is a completely different approach needed? Doing mathematics involves monitoring and reflecting on the process—catching and adjusting errors along the way.
5. *Students look for and discuss connections.* Students should see connections between different strategies to solve a particular problem, as well as connections to other mathematics concepts and to real contexts and situations. When students look for and discuss connections, they see mathematics as worthwhile and important, rather than an isolated collection of facts.

These five features are evident in what teachers do and what students do. You can observe video, such as **Cathy's Class** to look for these classroom practices, as well as to see the extent to which students are becoming mathematically proficient. You might also visit classrooms or record your own teaching and use the **Observation Tool: Classroom Environment for Doing Mathematics**, as a way to see how you can establish such an environment in your own classroom.

Complete Self-Check 2.4: How Do Students Learn Mathematics?

Connecting the Dots

It seems appropriate to close this chapter by connecting some dots, especially because the ideas represented here are the foundation for the approach to each topic in the content chapters. This chapter began with discussing what *doing* mathematics is and challenging you to do some mathematics. Each of these tasks offered opportunities to make connections between mathematics concepts—connecting the blue dots.

Second, you read about what is important to know about mathematics—that having relational knowledge (knowledge in which blue dots are well connected) requires conceptual and procedural understanding as well as other proficiencies. The problems that you solved in the first section emphasized concepts and procedures while placing you in a position to use strategic competence, adaptive reasoning, and a productive disposition.

Finally, you read how learning theory—the importance of having opportunities to connect the dots—connects to mathematics learning. The best learning opportunities, according to constructivism and sociocultural theories, are those that engage learners in using their own knowledge and experience to solve problems through social interactions and reflection. This is what you were asked to do in the four tasks. Did you learn something new about mathematics? Did you connect an idea that you had not previously connected?

This chapter focused on connecting the dots between theory and practice—building a case that your teaching must focus on opportunities for students to develop their own networks of blue dots. As you plan and design instruction, you should constantly reflect on how to elicit prior knowledge by designing tasks that reflect the social and cultural backgrounds of students, to challenge students to think critically and creatively, and to include a comprehensive treatment of mathematics.

Complete Self-Check 2.5: Connecting the Dots

REFLECTIONS ON CHAPTER 2

WRITING TO LEARN

Click here to assess your understanding and application of chapter content.

1. How would you describe what it means to "do mathematics"?
2. Select three of the verbs for doing mathematics. For each, think about what it looks like when a student is "doing" it, then explain or draw a picture of what it might look like.
3. What is important to know about relational understanding?
4. Using the task One Up, One Down as the example, describe how to implement it with students in a way that reflects constructivist and/or sociocultural learning theory.

FOR DISCUSSION AND EXPLORATION

- Consider the following task and respond to these three questions.
- Some people say that to add four consecutive numbers, you add the first and the last numbers and multiply by.
- Is this always true? How do you know? (Stoessiger & Edmunds, 1992)

  a. What features of "doing mathematics" does it have?
  b. What web of ideas do you need to draw on to make sense of the problem?
  c. To what extent does it task have the potential to develop mathematical proficiency?

- Not every educator believes in the constructivist-oriented approach to teaching mathematics. Some of their reasons include the following: There is not enough time to let kids discover everything. Basic facts and ideas are better taught through quality explanations. Students should not have to "reinvent the wheel." How would you respond to these arguments?

RESOURCES FOR CHAPTER 2

RECOMMENDED READINGS

Articles

Berkman, R. M. (2006). One, some, or none: Finding beauty in ambiguity. *Mathematics Teaching in the Middle School*, *11*(7), 324–327.

This article offers a great teaching strategy for nurturing relational thinking. Examples of the engaging "one, some, or none" activity are given for geometry, number, and algebra activities.

Carter, S. (2008). Disequilibrium & questioning in the primary classroom: Establishing routines that help students learn. *Teaching Children Mathematics*, *15*(3), 134–137.

This is a wonderful teacher's story of how she infused the constructivist notion of disequilibrium and the related idea of productive struggle to support learning in her first-grade class.

Hedges, M., Huinker, D., & Steinmeyer, M. (2005). Unpacking division to build teachers' mathematical knowledge. *Teaching Children Mathematics*, *11*(9), 478–483.

This article describes the many concepts related to division.

Suh, J. (2007). Tying it all together: Classroom practices that promote mathematical proficiency for all students. *Teaching Children Mathematics*, *14*(3), 163–169.

As the title implies, this is a great resource for connecting the NRC's Mathematics Proficiencies (National Research Council, 2001) to teaching.

Books

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Lampert reflects on her personal experiences in teaching fifth grade and shares with us her perspectives on the many issues and complexities of teaching. It is wonderfully written and easily accessed at any point in the book.

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically* (2nd ed.). Harlow, England: Pearson Education.

This classic book is about doing mathematics. There are excellent problems to explore along the way, with strategy suggestions. It is an engaging book that will help you learn more about your own problem solving and become a better teacher of mathematics.