Applications of Differentiation

2

Chapter Snapshot

What You'll Learn

2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

2.3 Graph Sketching: Asymptotes and Rational Functions

2.4 Using Derivatives to Find Absolute Maximum and Minimum Values

2.5 Maximum–Minimum Problems; Business and Economics Applications

2.6 Marginals and Differentials

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Why It's Important

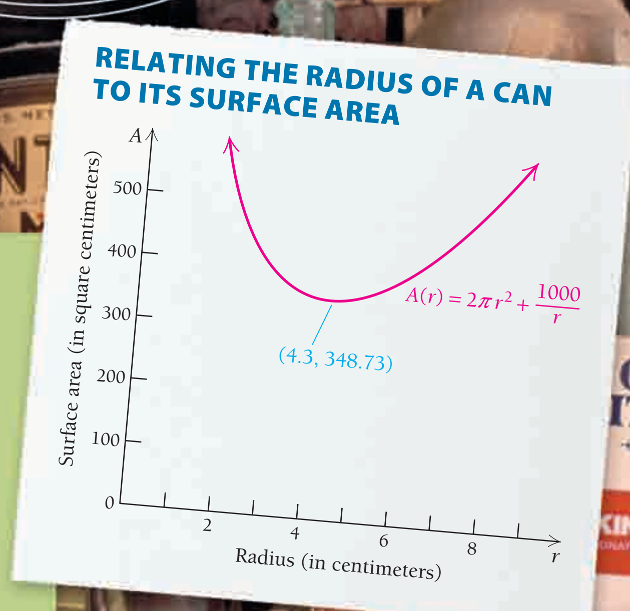
In this chapter, we explore many applications of differentiation. We learn to find maximum and minimum values of functions, and that skill allows us to solve many kinds of problems in which we need to find the largest and/or smallest value in a real-world situation. We also apply our differentiation skills to graphing and to approximating function values.

Where It's Used

MINIMIZING MATERIAL USED

Minimizing the amount of material used is a common goal in manufacturing, as it reduces overall costs as well as increases efficiency. For example, cylindrical food cans come in a variety of sizes. Suppose a can is to have a volume of 500 milliliters. Are there optimal dimensions for the can's height and radius that will minimize the material needed to produce each can? Can you see how minimizing the material used per can translates into minimized costs and conservation of resources?

This problem appears as Example 3 in Section 2.5.



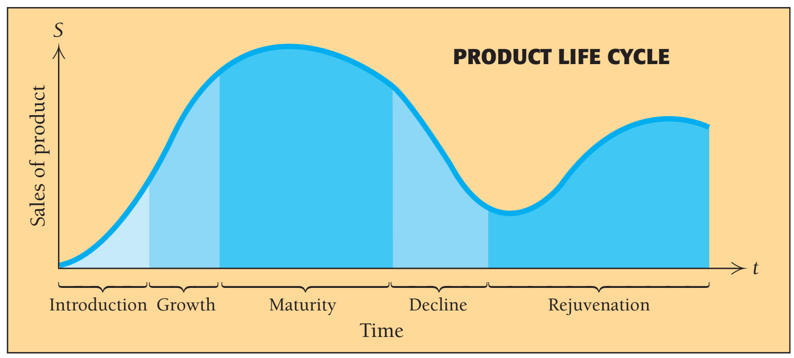
2.1

OBJECTIVES

- Find relative extrema of a continuous function using the First-Derivative Test.
- Sketch graphs of continuous functions.

Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

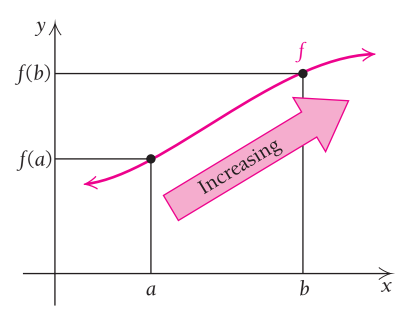
The graph below shows a typical life cycle of a retail product and is similar to graphs we will consider in this chapter. Note that the number of items sold varies with respect to time. Sales begin at a small level and increase to a point of maximum sales, after which they taper off to a low level, where the decline is probably due to the effect of new competitive products. The company then rejuvenates the product by making improvements. Think about versions of certain products: televisions can be traditional, flat-screen, or high-definition; music recordings have been produced as phonograph (vinyl) records, audiotapes, compact discs, and MP3 files. Where might each of these products be in a typical product life cycle? Does the curve seem appropriate for each product?



Finding the largest and smallest values of a function—that is, the maximum and minimum values—has extensive applications. The first and second derivatives of a function are calculus tools that provide information we can use in graphing functions and finding minimum and maximum values. Throughout this section we will assume, unless otherwise noted, that all functions are continuous. However, continuity of a function does not guarantee that its first and second derivatives are continuous.

Increasing and Decreasing Functions

If the graph of a function rises from left to right over an interval I, the function is said to be **increasing** on, or over, I.



f is an increasing function over I:
for all a, b in I, if $a < b$, then $f(a) < f(b)$.

TECHNOLOGY CONNECTION

Exploratory

Graph the function

$$y = -\tfrac{1}{3}x^3 + 6x^2 - 11x - 50$$

and its derivative

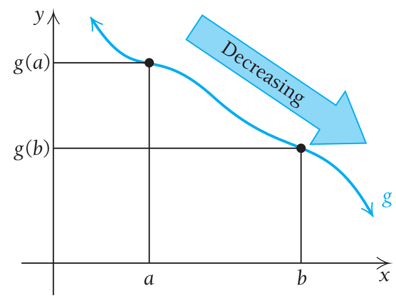
$$y' = -x^2 + 12x - 11$$

using the window $[-10, 25, -100, 150]$, with Xscl $= 5$ and Yscl $= 25$. Then TRACE from left to right along each graph. As you move the cursor from left to right, note that the x-coordinate always increases. If a function is increasing over an interval, the y-coordinate will increase as well. If a function is decreasing over an interval, the y-coordinate will decrease.

Over what intervals is the function increasing?
Over what intervals is the function decreasing?
Over what intervals is the derivative positive?
Over what intervals is the derivative negative?

What rules can you propose relating the sign of y' to the behavior of y?

If the graph drops from left to right, the function is said to be **decreasing** on, or over, I.



g is a decreasing function over I:
for all a, b in I, if $a < b$, then $g(a) > g(b)$.

We can describe these phenomena mathematically as follows.

DEFINITIONS

A function f is **increasing** over I if, for every a and b in I,

$$\text{if } a < b, \quad \text{then } f(a) < f(b).$$

(If the input a is less than the input b, then the output for a is less than the output for b.)

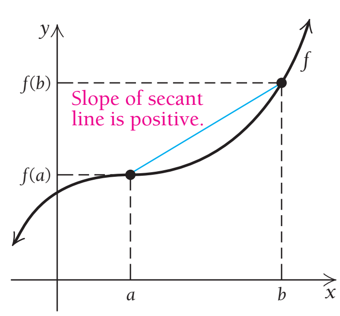
A function f is **decreasing** over I if, for every a and b in I,

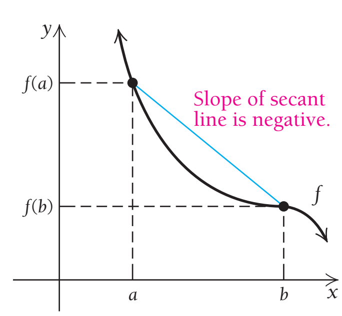
$$\text{if } a < b, \quad \text{then } f(a) > f(b).$$

(If the input a is less than the input b, then the output for a is greater than the output for b.)

The above definitions can be restated in terms of secant lines. If a graph is increasing over an interval I, then, for all a and b in I such that $a < b$, the slope of the secant line between $x = a$ and $x = b$ is positive. Similarly, if a graph is decreasing over an interval I, then, for all a and b in I such that $a < b$, the slope of the secant line between $x = a$ and $x = b$ is negative:

$$\text{Increasing: } \frac{f(b) - f(a)}{b - a} > 0. \qquad \text{Decreasing: } \frac{f(b) - f(a)}{b - a} < 0.$$



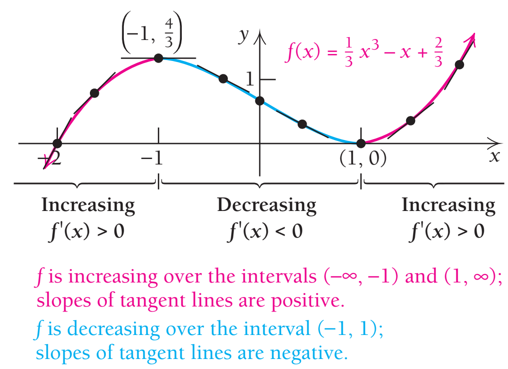


The following theorem shows how we can use the derivative (the slope of a tangent line) to determine whether a function is increasing or decreasing.

THEOREM 1

If $f'(x) > 0$ for all x in an open interval I, then f is increasing over I.
If $f'(x) < 0$ for all x in an open interval I, then f is decreasing over I.

Theorem 1 is illustrated in the following graph.



For determining increasing or decreasing behavior using a derivative, the interval I is an open interval; that is, it does not include its endpoints. Note how the intervals on which f is increasing and decreasing are written in the preceding graph: $x = -1$ and $x = 1$ are not included in any interval over which the function is increasing or decreasing. These values are examples of *critical values*.

Critical Values

Consider the graph of a continuous function f in Fig. 1.

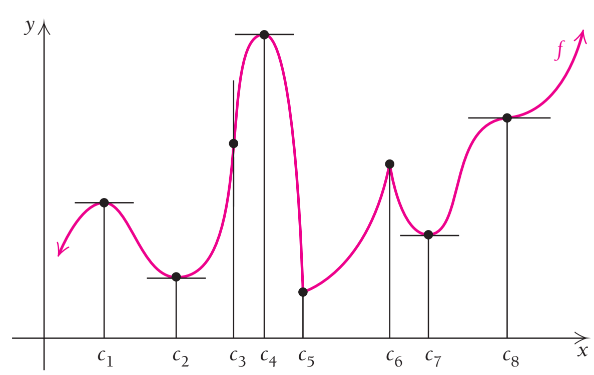


Figure 1

Note the following:

1. $f'(c) = 0$ at $x = c_1, c_2, c_4, c_7$, and c_8. That is, the tangent line to the graph is horizontal for these values.
2. $f'(c)$ does not exist at $x = c_3, c_5$, and c_6. The tangent line is vertical at c_3, and there are corner points at both c_5 and c_6. (See also the discussion at the end of Section 1.4.)

DEFINITION

A **critical value** of a function f is any number c in the domain of f for which the tangent line at $(c, f(c))$ is horizontal or for which the derivative does not exist. That is, c is a critical value if $f(c)$ exists and

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Thus, in the graph of f in Fig. 1:

1. c_1, c_2, c_4, c_7, and c_8 are critical values because $f'(c) = 0$ for each value.
2. c_3, c_5, and c_6 are critical values because $f'(c)$ does not exist for each value.

Also note that a continuous function can change from increasing to decreasing or from decreasing to increasing *only* at a critical value. In the graph in Fig. 1, c_1, c_2, c_4, c_5, c_6, and c_7 separate the intervals over which the function changes from increasing to decreasing or from decreasing to increasing. Although c_3 and c_8 are critical values, they do not separate intervals over which the function changes from increasing to decreasing or from decreasing to increasing.

Finding Relative Maximum and Minimum Values

Now consider the graph in Fig. 2. Note the "peaks" and "valleys" at the interior points c_1, c_2, and c_3.

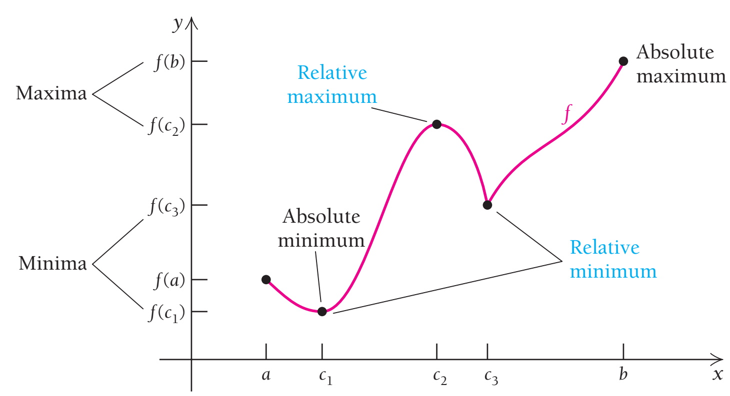


Figure 2

Here $f(c_2)$ is an example of a **relative maximum** (plural: **maxima**). Each of $f(c_1)$ and $f(c_3)$ is called a **relative minimum** (plural: **minima**). The terms **local maximum** and **local minimum** are also used.

DEFINITIONS

Let I be the domain of f.

$f(c)$ is a **relative minimum** if there exists within I an open interval I_1 containing c such that $f(c) \leq f(x)$, for all x in I_1;

and

$f(c)$ is a **relative maximum** if there exists within I an open interval I_2 containing c such that $f(c) \geq f(x)$, for all x in I_2.

A relative maximum can be thought of loosely as the second coordinate of a "peak" that may or may not be the highest point over all of I. Similarly, a relative minimum can be thought of as the second coordinate of a "valley" that may or may not be the lowest point on I. The second coordinates of the points that are the highest and the lowest on the interval are, respectively, the **absolute maximum** and the **absolute minimum.** For now, we focus on finding relative maximum or minimum values, collectively referred to as **relative extrema** (singular: **extremum**).

Look again at the graph in Fig. 2. The x-values at which a continuous function has relative extrema are those values for which the derivative is 0 or for which the derivative does not exist—the critical values.

THEOREM 2

If a function f has a relative extreme value $f(c)$ on an open interval, then c is a critical value, so

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

A *relative extreme point,* $(c, f(c))$, is higher or lower than all other points over some open interval containing c. A relative minimum point will have a y-value that is lower than that of points both to the left and to the right of it, and, similarly, a relative maximum point will have a y-value that is higher than that of points to the left and right of it. Thus, relative extrema cannot be located at the endpoints of a closed interval, since an endpoint lacks "both sides" with which to make the necessary comparisons. However, as we will see in Section 2.4, endpoints *can* be absolute extrema. Note that the right endpoint of the curve in Fig. 2 is the absolute maximum point.

Theorem 2 is very useful, but it is important to understand it precisely. What it says is that to find relative extrema, we need only consider those inputs for which the derivative is 0 or for which it does not exist. We can think of a critical value as a *candidate* for a value where a relative extremum *might* occur. That is, Theorem 2 does not say that every critical value will yield a relative maximum or minimum. Consider, for example, the graph of

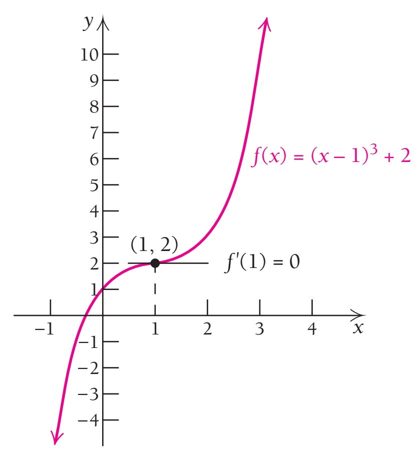
$$f(x) = (x - 1)^3 + 2,$$

shown at the right. Note that

$$f'(x) = 3(x - 1)^2,$$

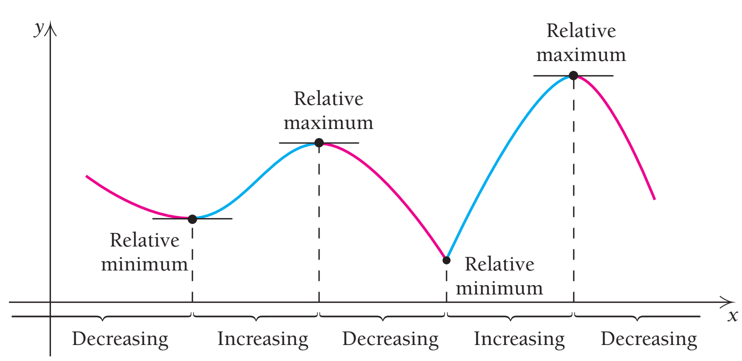
and

$$f'(1) = 3(1 - 1)^2 = 0.$$

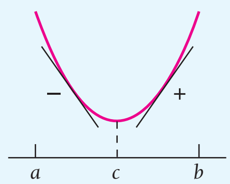
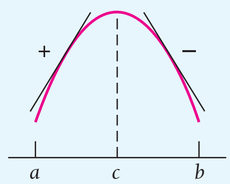
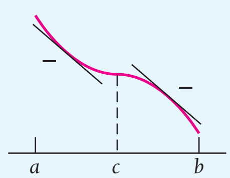
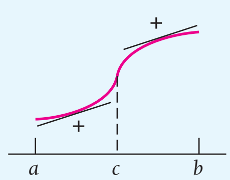


The function has $c = 1$ as a critical value, but has no relative maximum or minimum at that value.

Theorem 2 does say that if a relative maximum or minimum occurs, then the first coordinate of that extremum will be a critical value. How can we tell when the existence of a critical value leads us to a relative extremum? The following graph leads us to a test.



Note that at a critical value where there is a relative minimum, the function is decreasing to the left of the critical value and increasing to the right. At a critical value where there is a relative maximum, the function is increasing to the left of the critical value and decreasing to the right. In both cases, the derivative changes signs on either side of the critical value.

Graph over the interval (a, b)	$f(c)$	Sign of $f'(x)$ for x in (a, c)	Sign of $f'(x)$ for x in (c, b)	Increasing or decreasing
	Relative minimum	−	+	Decreasing on (a, c); increasing on (c, b)
	Relative maximum	+	−	Increasing on (a, c); decreasing on (c, b)
	No relative maxima or minima	−	−	Decreasing on (a, b)
	No relative maxima or minima	+	+	Increasing on (a, b)

Derivatives tell us when a function is increasing or decreasing. This leads us to the First-Derivative Test.

THEOREM 3 **The First-Derivative Test for Relative Extrema**

For any continuous function f that has exactly one critical value c in an open interval (a, b):

F1. f has a relative minimum at c if $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b). That is, f is decreasing to the left of c and increasing to the right of c.
F2. f has a relative maximum at c if $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b). That is, f is increasing to the left of c and decreasing to the right of c.
F3. f has neither a relative maximum nor a relative minimum at c if $f'(x)$ has the same sign on (a, c) as on (c, b).

Now let's see how we can use the First-Derivative Test to find relative extrema and create accurate graphs.

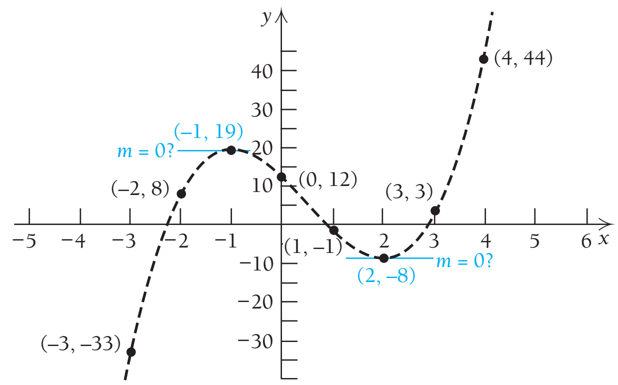
Exampl E 1 Graph the function f given by

$$f(x) = 2x^3 - 3x^2 - 12x + 12,$$

and find the relative extrema.

Solution Suppose that we are trying to graph this function but don't know any calculus. What can we do? We could plot several points to determine in which direction the graph seems to be turning. Let's pick some x-values and see what happens.

x	$f(x)$
−3	−33
−2	8
−1	19
0	12
1	−1
2	−8
3	3
4	44



We plot the points and use them to sketch a "best guess" of the graph, shown as the dashed line in the figure above. According to this rough sketch, it appears that the graph has a tangent line with slope 0 somewhere around $x = -1$ and $x = 2$. But how do we know for sure? We use calculus to support our observations. We begin by finding a general expression for the derivative:

$$f'(x) = 6x^2 - 6x - 12.$$

We next determine where $f'(x)$ does not exist or where $f'(x) = 0$. Since we can evaluate $f'(x) = 6x^2 - 6x - 12$ for any real number, there is no value for which $f'(x)$ does not exist. So the only possibilities for critical values are those where $f'(x) = 0$, locations at which there are horizontal tangents. To find such values, we solve $f'(x) = 0$:

$$6x^2 - 6x - 12 = 0$$

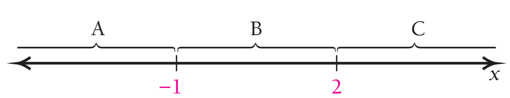
$$x^2 - x - 2 = 0 \qquad \text{Dividing both sides by 6}$$

$$(x + 1)(x - 2) = 0 \qquad \text{Factoring}$$

$$x + 1 = 0 \quad \text{or} \quad x - 2 = 0 \qquad \text{Using the Principle of Zero Products}$$

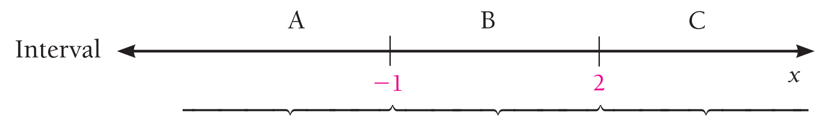
$$x = -1 \quad \text{or} \quad x = 2.$$

The critical values are -1 and 2. Since it is at these values that a relative maximum or minimum might exist, we examine the intervals on each side of the critical values: A is $(-\infty, -1)$, B is $(-1, 2)$, and C is $(2, \infty)$, as shown below.



Next, we analyze the sign of the derivative on each interval. If $f'(x)$ is positive for one value in the interval, then it will be positive for all values in the interval. Similarly, if it is negative for one value, it will be negative for all values in the interval. Thus, we choose a test value in each interval and make a substitution. The test values we choose are -2, 0, and 4.

A: Test -2, $f'(-2) = 6(-2)^2 - 6(-2) - 12$
$= 24 + 12 - 12 = 24 > 0$;
B: Test 0, $f'(0) = 6(0)^2 - 6(0) - 12 = -12 < 0$;
C: Test 4, $f'(4) = 6(4)^2 - 6(4) - 12$
$= 96 - 24 - 12 = 60 > 0$.



Test Value	$x = -2$	$x = 0$	$x = 4$
Sign of $f'(x)$	$f'(-2) > 0$	$f'(0) < 0$	$f'(4) > 0$
Result	f is increasing on $(-\infty, -1)$	f is decreasing on $(-1, 2)$	f is increasing on $(2, \infty)$

Change indicates a relative maximum.
Change indicates a relative minimum.

Therefore, by the First-Derivative Test,

f has a relative maximum at $x = -1$ given by

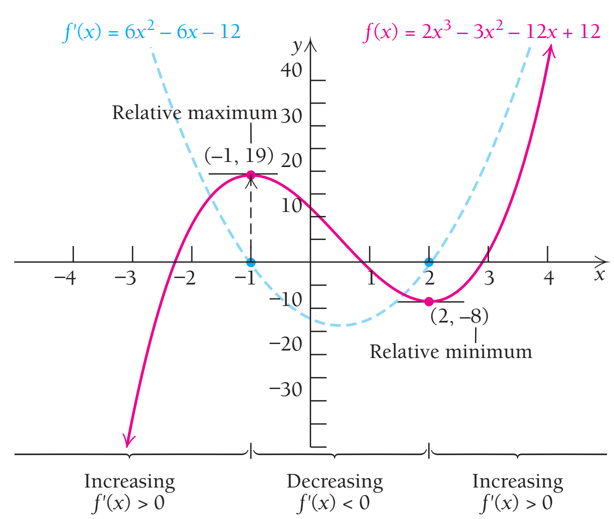
$$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 12 \quad \text{Substituting into the } \textit{original} \text{ function}$$
$$= 19 \quad \text{This is a relative maximum.}$$

and f has a relative minimum at $x = 2$ given by

$$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 12 = -8. \quad \text{This is a relative minimum.}$$

Thus, there is a relative maximum at $(-1, 19)$ and a relative minimum at $(2, -8)$, as we suspected from the sketch of the graph.

The information we have obtained from the first derivative can be very useful in sketching a graph of the function. We know that this polynomial is continuous, and we know where the function is increasing, where it is decreasing, and where it has relative extrema. We complete the graph by using a calculator to generate some additional function values. The graph of the function, shown below in red, has been scaled to clearly show its curving nature.



TECHNOLOGY CONNECTION

Exploratory

Consider the function f given by

$$f(x) = x^3 - 3x + 2.$$

Graph both f and f' using the same set of axes. Examine the graphs using the TABLE and TRACE features. Where do you think the relative extrema of $f(x)$ occur? Where is the derivative equal to 0? Where does $f(x)$ have critical values?

For reference, the graph of the derivative is shown in blue. Note that $f'(x) = 0$ where $f(x)$ has relative extrema. We summarize the behavior of this function as follows, by noting where it is increasing or decreasing, and by characterizing its critical points:

- The function f is increasing over the interval $(-\infty, -1)$.
- The function f has a relative maximum at the point $(-1, 19)$.
- The function f is decreasing over the interval $(-1, 2)$.
- The function f has a relative minimum at the point $(2, -8)$.
- The function f is increasing over the interval $(2, \infty)$.

❯ Quick Check 1

Graph the function g given by $g(x) = x^3 - 27x - 6$, and find the relative extrema.

❮ Quick Check 1

Interval notation and point notation look alike. Be clear when stating your answers whether you are identifying an interval or a point.

To use the first derivative for graphing a function f:

1. Find all critical values by determining where $f'(x)$ is 0 and where $f'(x)$ is undefined (but $f(x)$ is defined). Find $f(x)$ for each critical value.
2. Use the critical values to divide the x-axis into intervals and choose a test value in each interval.
3. Find the sign of $f'(x)$ for each test value chosen in step 2, and use this information to determine where $f(x)$ is increasing or decreasing and to classify any extrema as relative maxima or minima.
4. Plot some additional points and sketch the graph.

The *derivative* f' is used to find the critical values of f. The test values are substituted into the *derivative* f', and the function values are found using the *original* function f.

example 2 Find the relative extrema of the function f given by

$$f(x) = 2x^3 - x^4.$$

Then sketch the graph.

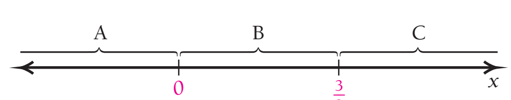
Solution First, we must determine the critical values. To do so, we find $f'(x)$:

$$f'(x) = 6x^2 - 4x^3.$$

Next, we find where $f'(x)$ does not exist or where $f'(x) = 0$. Since $f'(x) = 6x^2 - 4x^3$ is a polynomial, it exists for all real numbers x. Therefore, the only candidates for critical values are where $f'(x) = 0$, that is, where the tangent line is horizontal:

$$\begin{aligned} 6x^2 - 4x^3 &= 0 && \text{Setting } f'(x) \text{ equal to 0} \\ 2x^2(3 - 2x) &= 0 && \text{Factoring} \\ 2x^2 = 0 \quad &\text{or} \quad 3 - 2x = 0 \\ x^2 = 0 \quad &\text{or} \quad 3 = 2x \\ x = 0 \quad &\text{or} \quad x = \tfrac{3}{2}. \end{aligned}$$

The critical values are 0 and $\frac{3}{2}$. We use these values to divide the x-axis into three intervals as shown below: A is $(-\infty, 0)$; B is $(0, \frac{3}{2})$; and C is $(\frac{3}{2}, \infty)$.



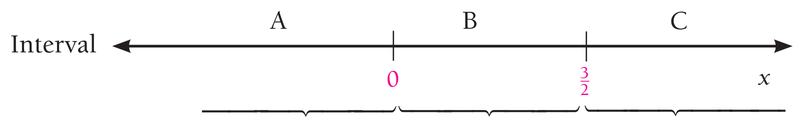
Note that $f(\frac{3}{2}) = 2(\frac{3}{2})^3 - (\frac{3}{2})^4 = \frac{27}{16}$ and $f(0) = 2 \cdot 0^3 - 0^4 = 0$ are possible extrema.

We now determine the sign of the derivative on each interval by choosing a test value in each interval and substituting. We generally choose test values for which it is easy to compute $f'(x)$.

A: Test -1, $f'(-1) = 6(-1)^2 - 4(-1)^3$
$= 6 + 4 = 10 > 0$;

B: Test 1, $f'(1) = 6(1)^2 - 4(1)^3$
$= 6 - 4 = 2 > 0$;

C: Test 2, $f'(2) = 6(2)^2 - 4(2)^3$
$= 24 - 32 = -8 < 0$.



Test Value	$x = -1$	$x = 1$	$x = 2$
Sign of $f'(x)$	$f'(-1) > 0$	$f'(1) > 0$	$f'(2) < 0$
Result	f is increasing on $(-\infty, 0)$	f is increasing on $\left(0, \frac{3}{2}\right)$	f is decreasing on $\left(\frac{3}{2}, \infty\right)$

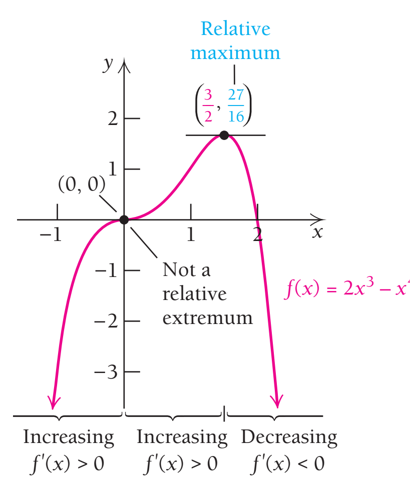
No change

Change indicates a relative maximum.

Therefore, by the First-Derivative Test, f has no extremum at $x = 0$ (since $f(x)$ is increasing on both sides of 0) and has a relative maximum at $x = \frac{3}{2}$. Thus, $f\left(\frac{3}{2}\right)$, or $\frac{27}{16}$, is a relative maximum.

We use the information obtained to sketch the graph below. Other function values are listed in the table.

x	$f(x)$, approximately
-1	-3
-0.5	-0.31
0	0
0.5	0.19
1	1
1.25	1.46
2	0



We summarize the behavior of f:

- The function f is increasing over the interval $(-\infty, 0)$.
- The function f has a critical point at $(0, 0)$, which is neither a minimum nor a maximum.

- The function f is increasing over the interval $\left(0, \frac{3}{2}\right)$.
- The function f has relative maximum at the point $\left(\frac{3}{2}, \frac{27}{16}\right)$.
- The function f is decreasing over the interval $\left(\frac{3}{2}, \infty\right)$.

Since f is increasing over the intervals $(-\infty, 0)$ and $\left(0, \frac{3}{2}\right)$, we can say that f is increasing over $\left(-\infty, \frac{3}{2}\right)$ despite the fact that $f'(0) = 0$ within this interval. In this case, we can observe that any secant line connecting two points within this interval will have a positive slope.

❮ Quick Check 2

> **❯ Quick Check 2**
>
> Find the relative extrema of the function h given by $h(x) = x^4 - \frac{8}{3}x^3$. Then sketch the graph.

example 3 Find the relative extrema of the function f given by

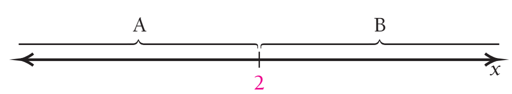
$$f(x) = (x-2)^{2/3} + 1.$$

Then sketch the graph.

Solution First, we determine the critical values. To do so, we find $f'(x)$:

$$f'(x) = \frac{2}{3}(x-2)^{-1/3}$$
$$= \frac{2}{3\sqrt[3]{x-2}}.$$

Next, we find where $f'(x)$ does not exist or where $f'(x) = 0$. Note that $f'(x)$ does not exist at 2, although $f(x)$ does. Thus, 2 is a critical value. Since the only way for a fraction to be 0 is if its numerator is 0, we see that $f'(x) = 0$ has no solution. Thus, 2 is the only critical value. We use 2 to divide the x-axis into the intervals A, which is $(-\infty, 2)$, and B, which is $(2, \infty)$. Note that $f(2) = (2-2)^{2/3} + 1 = 1$.



To determine the sign of the derivative, we choose a test value in each interval and substitute each value into the derivative. We choose test values 0 and 3. It is not necessary to find an exact value of the derivative; we need only determine the sign. Sometimes we can do this by just examining the formula for the derivative:

A: Test 0, $f'(0) = \dfrac{2}{3\sqrt[3]{0-2}} < 0;$

B: Test 3, $f'(3) = \dfrac{2}{3\sqrt[3]{3-2}} > 0.$

Test Value	$x = 0$	$x = 3$
Sign of $f'(x)$	$f'(0) < 0$	$f'(3) > 0$
Result	f is decreasing on $(-\infty, 2)$	f is increasing on $(2, \infty)$

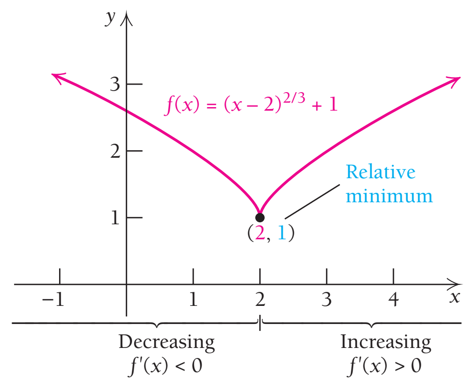
Change indicates a relative minimum.

> **TECHNOLOGY CONNECTION**
>
> **EXERCISES**
>
> In Exercises 1 and 2, consider the function f given by
>
> $$f(x) = 2 - (x-1)^{2/3}.$$
>
> **1.** Graph the function using the viewing window $[-4, 6, -2, 4]$.
>
> **2.** Graph the first derivative. What happens to the graph of the derivative at the critical values?

Since we have a change from decreasing to increasing, we conclude from the First-Derivative Test that a relative minimum occurs at $(2, f(2))$, or $(2, 1)$. The graph has *no* tangent line at $(2, 1)$ since $f'(2)$ does not exist.

We use the information obtained to sketch the graph. Other function values are listed in the table.

x	$f(x)$, approximately
−1	3.08
−0.5	2.84
0	2.59
0.5	2.31
1	2
1.5	1.63
2	1
2.5	1.63
3	2
3.5	2.31
4	2.59



We summarize the behavior of f:

- The function f is decreasing over the interval $(-\infty, 2)$.
- The function f has a relative minimum at the point $(2, 1)$.
- The function f is increasing over the interval $(2, \infty)$.

❯ **Quick Check 3**

Find the relative extrema of the function g given by $g(x) = 3 - x^{1/3}$. Then sketch the graph.

❮ Quick Check 3

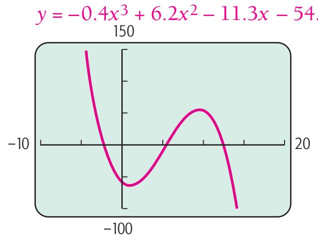
TECHNOLOGY CONNECTION

Finding Relative Extrema

To explore some methods for approximating relative extrema, let's find the relative extrema of

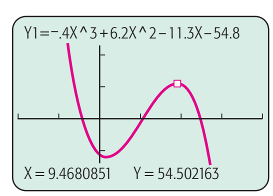
$$f(x) = -0.4x^3 + 6.2x^2 - 11.3x - 54.8.$$

We first graph the function, using a window that reveals the curvature.



Method 1: TRACE

Beginning with the window shown at left, we press TRACE and move the cursor along the curve, noting where relative extrema might occur.

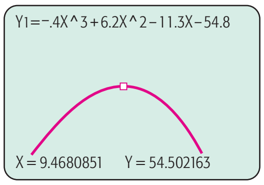


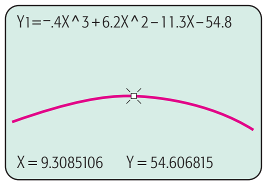
A relative maximum seems to be about $y = 54.5$ at $x = 9.47$. We can refine the approximation by zooming in to obtain the following window. We press TRACE and move

(continued)

Finding Relative Extrema (*continued*)

the cursor along the curve, again noting where the y-value is largest. The approximation is about $y = 54.61$ at $x = 9.31$.





We can continue in this manner until the desired accuracy is achieved.

Method 2: TABLE

We can also use the TABLE feature, adjusting starting points and step values to improve accuracy:

TblStart $= 9.3$ $\quad \Delta$Tbl $= .01$

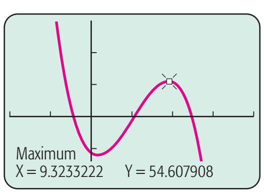
X	Y1	
9.3	54.605	
9.31	54.607	
9.32	54.608	
9.33	54.608	
9.34	54.607	
9.35	54.604	
9.36	54.601	

X = 9.32

The approximation seems to be nearly $y = 54.61$ at an x-value between 9.32 and 9.33. We could next set up a new table showing function values between $f(9.32)$ and $f(9.33)$ to refine the approximation.

Method 3: MAXIMUM, MINIMUM

Using the MAXIMUM option from the CALC menu, we find that a relative maximum of about 54.61 occurs at $x \approx 9.32$.



Method 4: fMax *or* fMin

This feature calculates a relative maximum or minimum value over any specified closed interval. We see from the initial graph that a relative maximum occurs in the interval $[-10, 20]$. Using the fMax option from the MATH menu, we see that a relative maximum occurs on $[-10, 20]$ when $x \approx 9.32$.

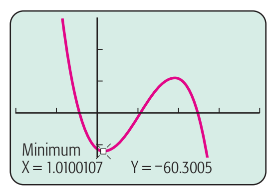
```
fMax(Y1,X,−10,20)
                9.323324165
■
```

To obtain the maximum value, we evaluate the function at the given x-value, obtaining the following.

```
fMax(Y1,X,−10,20)
                9.323324165
Y1(Ans)
                54.60790781
■
```

The approximation is about $y = 54.61$ at $x = 9.32$.

Using any of these methods, we find the relative minimum to be about $y = -60.30$ at $x = 1.01$.



EXERCISE

1. Using one of the methods just described, approximate the relative extrema of the function in Example 1.

TECHNOLOGY CONNECTION

Finding Relative Extrema with iPlot

We can use iPlot to graph a function and its derivative and then find relative extrema.

iPlot has the capability of graphing a function and its derivative on the same set of axes, though it does not give a formula for the derivative but merely draws the graph. As an example, let's consider the function given by $f(x) = x^3 - 3x + 4$.

To graph a function and its derivative, first open the iPlot app on your iPhone or iPad. You will get a screen like the one in Fig. 1. Notice the four icons at the bottom. The Functions icon is highlighted. Press [+] in the upper right; then enter $f(x) = x^3 - 3x + 4$ using the notation x^3−3*x+4. Press Done at the upper right and then Plot at the lower right (Fig. 2).

(*continued*)

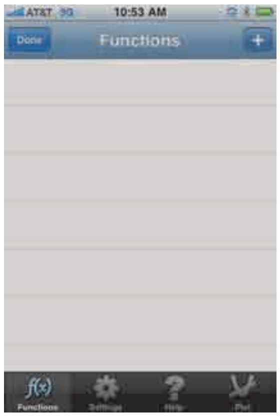


Figure 1

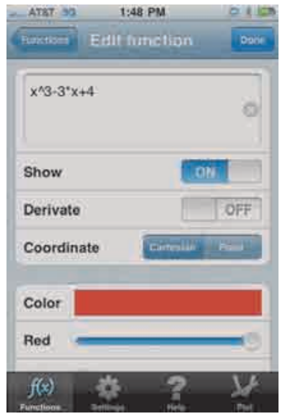


Figure 2

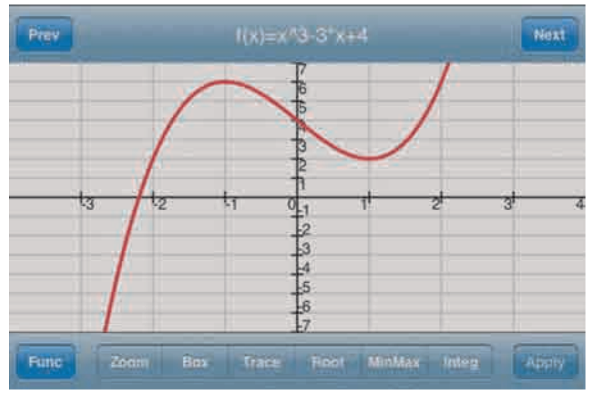


Figure 3

Figure 4

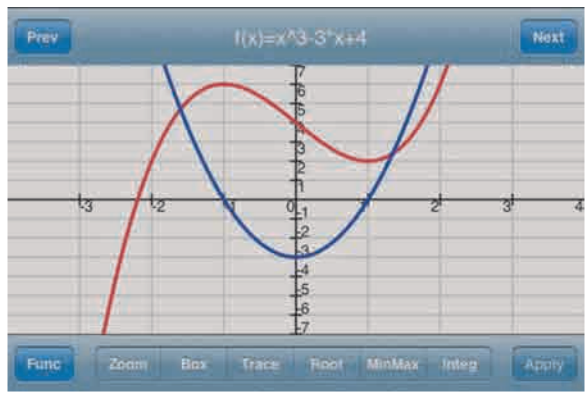


Figure 5

The graph of $f(x) = x^3 - 3x + 4$ is shown in red in Fig. 3. To graph the derivative of f, first click on the Functions icon again, and then press $\boxed{+}$. You will get the screen shown in Fig. 4.

Next, slide the Derivate button to the left. ("Derivate" means "Differentiate.") Then enter the same function as before, `x^3−3*x+4`, and press Done. `D(x^3−3*x+4)` will appear in the second line. Press Plot, and you will see both functions plotted, as shown in Fig. 5. Look over the two graphs, and use Trace to find various function values. Press Prev to jump between the function and its derivative. Look for x-values where the derivative is 0. What happens at these values of the original function? Examining the graphs in this way reveals that the graph of $f(x) = x^3 - 3x + 4$ has a relative maximum point at $(-1, 6)$ and a relative minimum point at $(1, 2)$.

iPlot has an additional feature that allows us to be more certain about these relative extrema. Go back to the original plot of $f(x) = x^3 - 3x + 4$ (Fig. 3), and press Settings. Change the window to $[-3, 3, 12, -10]$ to better see the graph. Press the MinMax button at the bottom. Touch the screen as closely as possible to what might be a relative extremum. See Figs. 6 and 7 for the relative maximum. The relative minimum can be found similarly.

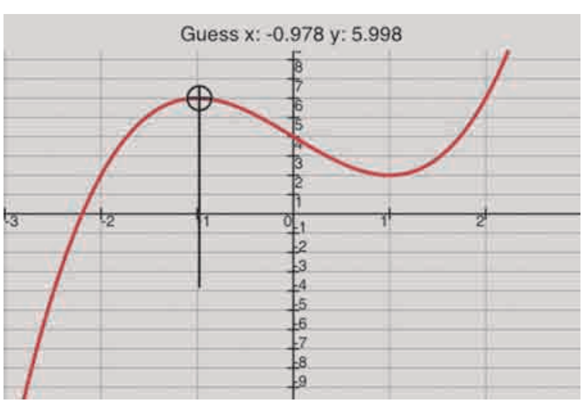


Figure 6

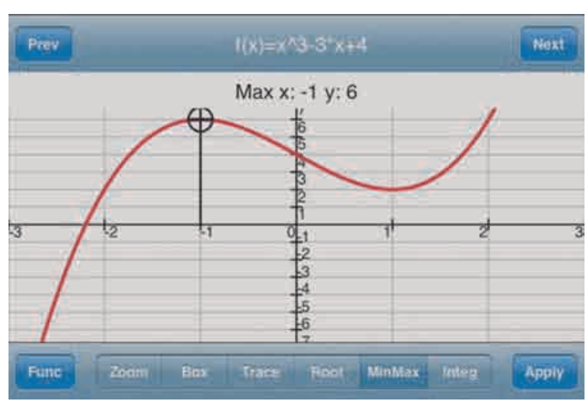


Figure 7

(continued)

EXERCISES

For each function, use iPlot to create the graph and find the derivative. Then explore each graph to look for possible relative extrema. Use `MinMax` to determine the relative extrema.

1. $f(x) = 2x^3 - x^4$

2. $f(x) = x(200 - x)$

3. $f(x) = x^3 - 6x^2$

4. $f(x) = -4.32 + 1.44x + 3x^2 - x^3$

5. $g(x) = x\sqrt{4 - x^2}$

6. $g(x) = \dfrac{4x}{x^2 + 1}$

7. $f(x) = \dfrac{x^2 - 3x}{x - 1}$

8. $f(x) = |x + 2| - 3$

Section Summary

- A function f is *increasing* over an interval I if, for all a and b in I such that $a < b$, $f(a) < f(b)$. Equivalently, the slope of the secant line connecting a and b is positive:
$$\frac{f(b) - f(a)}{b - a} > 0.$$
- A function f is *decreasing* over an interval I if, for all a and b in I such that $a < b$, $f(a) > f(b)$. Equivalently, the slope of the secant line connecting a and b is negative:
$$\frac{f(b) - f(a)}{b - a} < 0.$$
- Using the first derivative, a function is *increasing* over an open interval I if, for all x in I, the slope of the tangent line at x is positive; that is, $f'(x) > 0$. Similarly, a function is *decreasing* over an open interval I if, for all x in I, the slope of the tangent line is negative; that is, $f'(x) < 0$.
- A *critical value* is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist. The point $(c, f(c))$ is called a *critical point*.
- A relative maximum point is higher than all other points in some interval containing it. Similarly, a relative minimum point is lower than all other points in some interval containing it. The y-value of such a point is called a relative maximum (or minimum) *value* of the function.
- Minimum and maximum points are collectively called *extrema*.
- Critical values are candidates for possible relative extrema. The *First-Derivative Test* is used to classify a critical value as a relative minimum, a relative maximum, or neither.

EXERCISE SET 2.1

Find the relative extrema of each function, if they exist. List each extremum along with the x-value at which it occurs. Then sketch a graph of the function.

1. $f(x) = x^2 + 4x + 5$

2. $f(x) = x^2 + 6x - 3$

3. $f(x) = 5 - x - x^2$

4. $f(x) = 2 - 3x - 2x^2$

5. $g(x) = 1 + 6x + 3x^2$

6. $F(x) = 0.5x^2 + 2x - 11$

7. $G(x) = x^3 - x^2 - x + 2$

8. $g(x) = x^3 + \frac{1}{2}x^2 - 2x + 5$

9. $f(x) = x^3 - 3x + 6$

10. $f(x) = x^3 - 3x^2$

11. $f(x) = 3x^2 + 2x^3$

12. $f(x) = x^3 + 3x$

13. $g(x) = 2x^3 - 16$

14. $F(x) = 1 - x^3$

15. $G(x) = x^3 - 6x^2 + 10$

16. $f(x) = 12 + 9x - 3x^2 - x^3$

17. $g(x) = x^3 - x^4$

18. $f(x) = x^4 - 2x^3$

19. $f(x) = \frac{1}{3}x^3 - 2x^2 + 4x - 1$

20. $F(x) = -\frac{1}{3}x^3 + 3x^2 - 9x + 2$

21. $g(x) = 2x^4 - 20x^2 + 18$

22. $f(x) = 3x^4 - 15x^2 + 12$

23. $F(x) = \sqrt[3]{x - 1}$

24. $G(x) = \sqrt[3]{x + 2}$

25. $f(x) = 1 - x^{2/3}$

26. $f(x) = (x + 3)^{2/3} - 5$

27. $G(x) = \dfrac{-8}{x^2 + 1}$

28. $F(x) = \dfrac{5}{x^2 + 1}$

29. $g(x) = \dfrac{4x}{x^2 + 1}$

30. $g(x) = \dfrac{x^2}{x^2 + 1}$

31. $f(x) = \sqrt[3]{x}$

32. $f(x) = (x + 1)^{1/3}$

33. $g(x) = \sqrt{x^2 + 2x + 5}$

34. $F(x) = \dfrac{1}{\sqrt{x^2 + 1}}$

35–68. Check the results of Exercises 1–34 using a calculator.

For Exercises 69–84, draw a graph to match the description given. Answers will vary.

69. $f(x)$ is increasing over $(-\infty, 2)$ and decreasing over $(2, \infty)$.

70. $g(x)$ is decreasing over $(-\infty, -3)$ and increasing over $(-3, \infty)$.

71. $G(x)$ is decreasing over $(-\infty, 4)$ and $(9, \infty)$ and increasing over $(4, 9)$.

72. $F(x)$ is increasing over $(-\infty, 5)$ and $(12, \infty)$ and decreasing over $(5, 12)$.

73. $g(x)$ has a positive derivative over $(-\infty, -3)$ and a negative derivative over $(-3, \infty)$.

74. $f(x)$ has a negative derivative over $(-\infty, 1)$ and a positive derivative over $(1, \infty)$.

75. $F(x)$ has a negative derivative over $(-\infty, 2)$ and $(5, 9)$ and a positive derivative over $(2, 5)$ and $(9, \infty)$.

76. $G(x)$ has a positive derivative over $(-\infty, -2)$ and $(4, 7)$ and a negative derivative over $(-2, 4)$ and $(7, \infty)$.

77. $f(x)$ has a positive derivative over $(-\infty, 3)$ and $(3, 9)$, a negative derivative over $(9, \infty)$, and a derivative equal to 0 at $x = 3$.

78. $g(x)$ has a negative derivative over $(-\infty, 5)$ and $(5, 8)$, a positive derivative over $(8, \infty)$, and a derivative equal to 0 at $x = 5$.

79. $F(x)$ has a negative derivative over $(-\infty, -1)$ and a positive derivative over $(-1, \infty)$, and $F'(-1)$ does not exist.

80. $G(x)$ has a positive derivative over $(-\infty, 0)$ and $(3, \infty)$ and a negative derivative over $(0, 3)$, but neither $G'(0)$ nor $G'(3)$ exists.

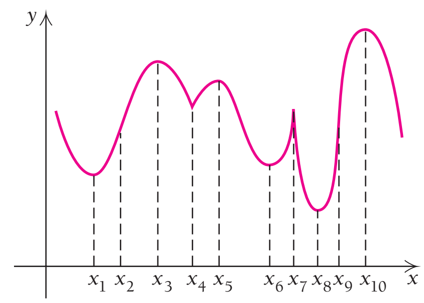
81. $f(x)$ has a negative derivative over $(-\infty, -2)$ and $(1, \infty)$ and a positive derivative over $(-2, 1)$, and $f'(-2) = 0$, but $f'(1)$ does not exist.

82. $g(x)$ has a positive derivative over $(-\infty, -3)$ and $(0, 3)$, a negative derivative over $(-3, 0)$ and $(3, \infty)$, and a derivative equal to 0 at $x = -3$ and $x = 3$, but $g'(0)$ does not exist.

83. $H(x)$ is increasing over $(-\infty, \infty)$, but the derivative does not exist at $x = 1$.

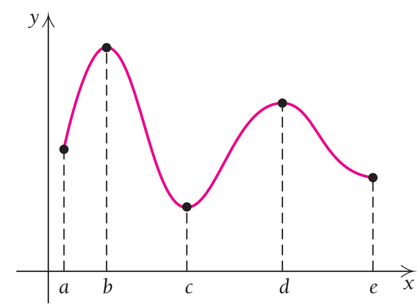
84. $K(x)$ is decreasing over $(-\infty, \infty)$, but the derivative does not exist at $x = 0$ and $x = 2$.

85. Consider this graph.



Explain the idea of a critical value. Then determine which x-values are critical values, and state why.

86. Consider this graph.



Using the graph and the intervals noted, explain how to relate the concept of the function being increasing or decreasing to the first derivative.

APPLICATIONS

Business and Economics

87. **Employment.** According to the U.S. Bureau of Labor Statistics, the number of professional services employees fluctuated during the period 2000–2009, as modeled by

$$E(t) = -28.31t^3 + 381.86t^2 - 1162.07t + 16{,}905.87,$$

where t is the number of years since 2000 ($t = 0$ corresponds to 2000) and E is thousands of employees. (*Source*: www.data.bls.gov.) Find the relative extrema of this function, and sketch the graph. Interpret the meaning of the relative extrema.

88. **Advertising.** Brody Electronics estimates that it will sell N units of a new toy after spending a thousands of dollars on advertising, where

$$N(a) = -a^2 + 300a + 6, \quad 0 \le a \le 300.$$

Find the relative extrema and sketch a graph of the function.

Life and Physical Sciences

89. **Temperature during an illness.** The temperature of a person during an intestinal illness is given by

$$T(t) = -0.1t^2 + 1.2t + 98.6, \quad 0 \le t \le 12,$$

where T is the temperature (°F) at time t, in days. Find the relative extrema and sketch a graph of the function.

90. **Solar eclipse.** On January 15, 2010, the longest annular solar eclipse until 3040 occurred over Africa and the Indian Ocean (in an annular eclipse, the sun is partially obscured by the moon and looks like a ring). The path of the full eclipse on the earth's surface is modeled by

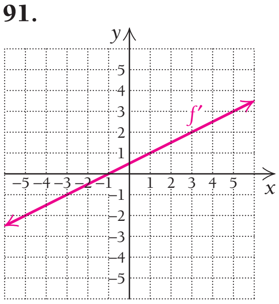
$$f(x) = 0.0125x^2 - 1.157x + 22.864, \quad 15 < x < 90,$$

where x is the number of degrees of longitude east of the prime meridian and $f(x)$ is the number of degrees of latitude north (positive) or south (negative) of the equator. (*Source*: NASA.) Find the longitude and latitude of the southernmost point at which the full eclipse could be viewed.

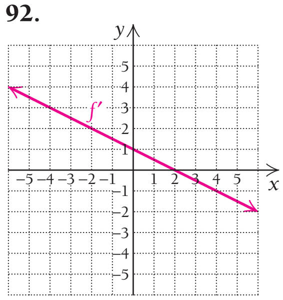
SYNTHESIS

In Exercises 91–96, the graph of a derivative f' is shown. Use the information in each graph to determine where f is increasing or decreasing and the x-values of any extrema. Then sketch a possible graph of f.

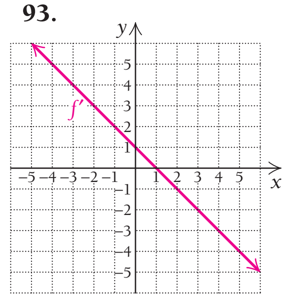
91.



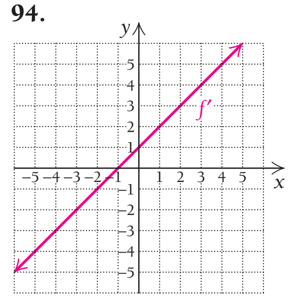
92.



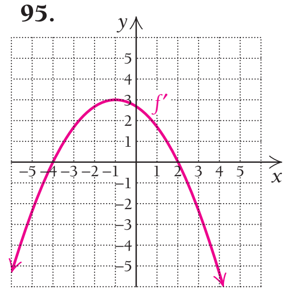
93.



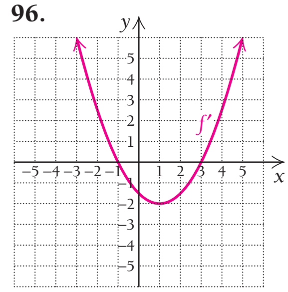
94.



95.



96.



TECHNOLOGY CONNECTION

Graph each function. Then estimate any relative extrema.

97. $f(x) = -x^6 - 4x^5 + 54x^4 + 160x^3 - 641x^2 - 828x + 1200$

98. $f(x) = x^4 + 4x^3 - 36x^2 - 160x + 400$

99. $f(x) = \sqrt[3]{|4 - x^2|} + 1$

100. $f(x) = x\sqrt{9 - x^2}$

Use your calculator's absolute-value feature to graph the following functions and determine relative extrema and intervals over which the function is increasing or decreasing. State the x-values at which the derivative does not exist.

101. $f(x) = |x - 2|$

102. $f(x) = |2x - 5|$

103. $f(x) = |x^2 - 1|$

104. $f(x) = |x^2 - 3x + 2|$

105. $f(x) = |9 - x^2|$

106. $f(x) = |-x^2 + 4x - 4|$

107. $f(x) = |x^3 - 1|$

108. $f(x) = |x^4 - 2x^2|$

Life science: caloric intake and life expectancy. *The data in the following table give, for various countries, daily caloric intake, projected life expectancy, and infant mortality. Use the data for Exercises 109 and 110.*

Country	Daily Caloric Intake	Life Expectancy at Birth (in years)	Infant Mortality (number of deaths before age 1 per 1000 births)
Argentina	3004	77	13
Australia	3057	82	5
Bolivia	2175	67	46
Canada	3557	81	5
Dominican Republic	2298	74	30
Germany	3491	79	4
Haiti	1835	61	62
Mexico	3265	76	17
United States	3826	78	6
Venezuela	2453	74	17

(Source: *U.N. FAO Statistical Yearbook*, 2009.)

109. **Life expectancy and daily caloric intake.**

a) Use the regression procedures of Section R.6 to fit a cubic function $y = f(x)$ to the data in the table, where x is daily caloric intake and y is life expectancy. Then fit a quartic function and decide which fits best. Explain.

b) What is the domain of the function?

c) Does the function have any relative extrema? Explain.

110. **Infant mortality and daily caloric intake.**

a) Use the regression procedures of Section R.6 to fit a cubic function $y = f(x)$ to the data in the table, where x is daily caloric intake and y is infant mortality. Then fit a quartic function and decide which fits best. Explain.

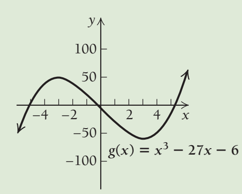
b) What is the domain of the function?

c) Does the function have any relative extrema? Explain.

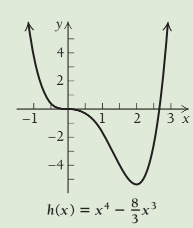
111. Describe a procedure that can be used to select an appropriate viewing window for the functions given in (a) Exercises 1–16 and (b) Exercises 97–100.

Answers to Quick Checks

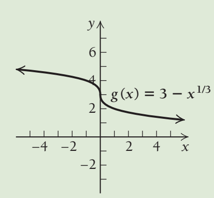
1. Relative maximum at $(-3, 48)$, relative minimum at $(3, -60)$



2. Relative minimum at $(2, -\frac{16}{3})$



3. There are no extrema.



2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

OBJECTIVES

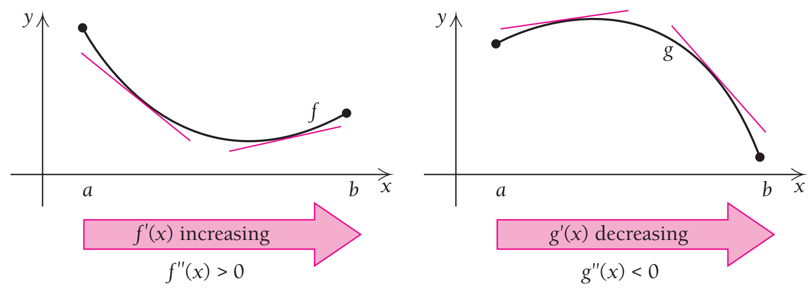
- Classify the relative extrema of a function using the Second-Derivative Test.
- Sketch the graph of a continuous function.

The "turning" behavior of a graph is called its *concavity*. The second derivative plays a pivotal role in analyzing the concavity of a function's graph.

Concavity: Increasing and Decreasing Derivatives

The graphs of two functions are shown below. The graph of f is turning up and the graph of g is turning down. Let's see if we can relate these observations to the functions' derivatives.

Consider first the graph of f. Take a ruler, or straightedge, and draw tangent lines as you move along the curve from left to right. What happens to the slopes of the tangent lines? Do the same for the graph of g. Look at the curvature and decide whether you see a pattern.



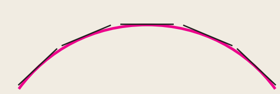
For the graph of f, the slopes of the tangent lines are increasing. That is, f' is increasing over the interval. This can be determined by noting that $f''(x)$ is positive, since the relationship between f' and f'' is like the relationship between f and f'. Note also that all the tangent lines for f are below the graph. For the graph of g, the slopes are decreasing. This can be determined by noting that g' is decreasing whenever $g''(x)$ is negative. For g, all tangent lines are above the graph.

DEFINITION

Suppose that f is a function whose derivative f' exists at every point in an open interval I. Then

f is **concave up** on I if f' is increasing over I.

f is **concave down** on I if f' is decreasing over I.

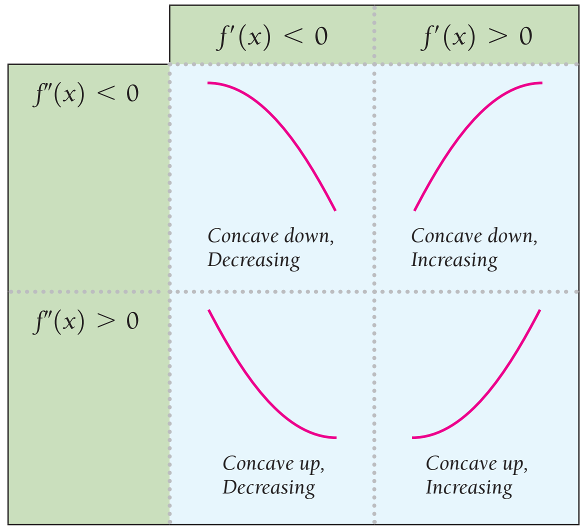


The following theorem states how the concavity of a function's graph and the second derivative of the function are related.

THEOREM 4 A Test for Concavity

1. If $f''(x) > 0$ on an interval I, then the graph of f is concave up. (f' is increasing, so f is turning up on I.)
2. If $f''(x) < 0$ on an interval I, then the graph of f is concave down. (f' is decreasing, so f is turning down on I.)

Keep in mind that a function can be decreasing and concave up, decreasing and concave down, increasing and concave up, or increasing and concave down. That is, *concavity* and *increasing/decreasing* are independent concepts. It is the increasing or decreasing aspect of the *derivative* that tells us about the function's concavity.



TECHNOLOGY CONNECTION

Exploratory

Graph the function

$$f(x) = -\tfrac{1}{3}x^3 + 6x^2 - 11x - 50$$

and its second derivative,

$$f''(x) = -2x + 12,$$

using the viewing window $[-10, 25, -100, 150]$, with Xscl $= 5$ and Yscl $= 25$.

Over what intervals is the graph of f concave up?
Over what intervals is the graph of f concave down?
Over what intervals is the graph of f'' positive?
Over what intervals is the graph of f'' negative?

What can you conjecture?

Now graph the first derivative

$$f'(x) = -x^2 + 12x - 11$$

and the second derivative

$$f''(x) = -2x + 12$$

using the viewing window $[-10, 25, -200, 50]$, with Xscl $= 5$ and Yscl $= 25$.

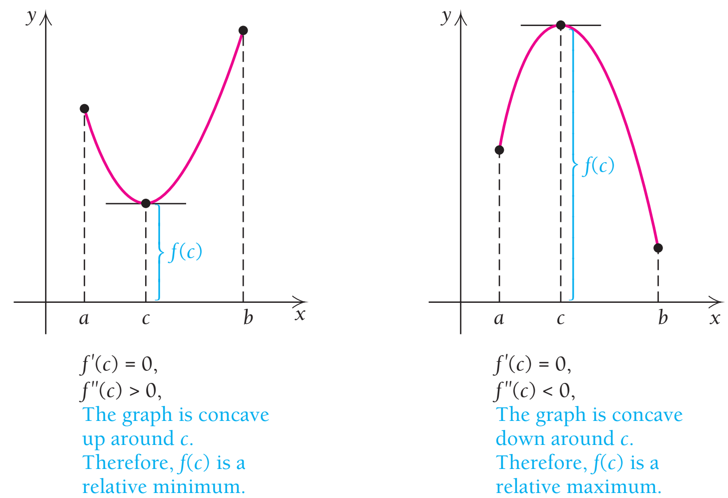
Over what intervals is the first derivative f' increasing?
Over what intervals is the first derivative f' decreasing?
Over what intervals is the graph of f'' positive?
Over what intervals is the graph of f'' negative?

What can you conjecture?

Classifying Relative Extrema Using Second Derivatives

Let's see how we can use second derivatives to determine whether a function has a relative extremum on an open interval.

The following graphs show both types of concavity at a critical value (where $f'(c) = 0$). When the second derivative is positive (graph is concave up) at the critical value, the critical point is a relative minimum point, and when the second derivative is negative (graph is concave down), the critical point is a relative maximum point.



This analysis is summarized in Theorem 5:

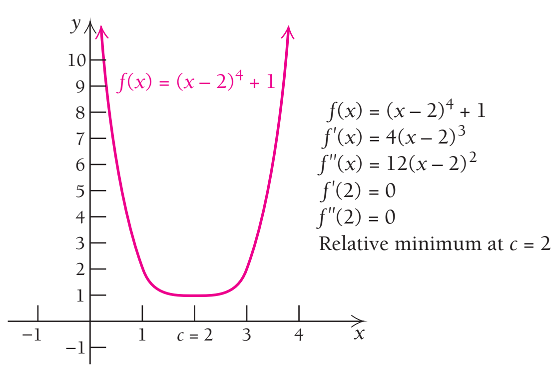
THEOREM 5 **The Second-Derivative Test for Relative Extrema**

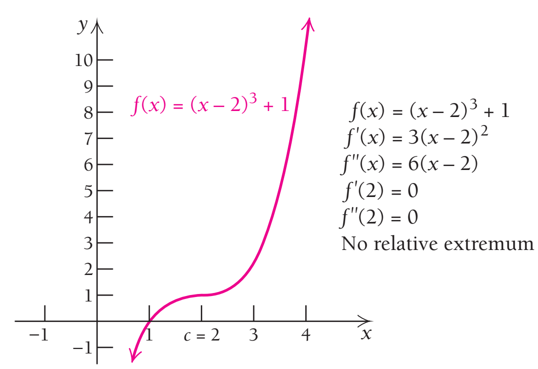
Suppose that f is differentiable for every x in an open interval (a, b) and that there is a critical value c in (a, b) for which $f'(c) = 0$. Then:

1. $f(c)$ is a relative minimum if $f''(c) > 0$.
2. $f(c)$ is a relative maximum if $f''(c) < 0$.

For $f''(c) = 0$, the First-Derivative Test can be used to determine whether $f(x)$ is a relative extremum.

Consider the following graphs. In each one, f' and f'' are both 0 at $c = 2$, but the first function has an extremum and the second function does not. When c is a critical value and $f''(c) = 0$, an extremum may or may not exist at c. Note too that if $f'(c)$ does not exist and c is a critical value, then $f''(c)$ also does not exist. Again, an approach other than the Second-Derivative Test must be used to determine whether $f(c)$ is an extremum.





The second derivative is used to help identify extrema and determine the overall behavior of a graph, as we see in the following examples.

example 1 Find the relative extrema of the function f given by

$$f(x) = x^3 + 3x^2 - 9x - 13,$$

and sketch the graph.

Solution To find any critical values, we determine $f'(x)$. To determine whether any critical values lead to extrema, we also find $f''(x)$:

$$f'(x) = 3x^2 + 6x - 9,$$
$$f''(x) = 6x + 6.$$

Then we solve $f'(x) = 0$:

$$3x^2 + 6x - 9 = 0$$
$$x^2 + 2x - 3 = 0 \qquad \text{Dividing both sides by 3}$$
$$(x + 3)(x - 1) = 0 \qquad \text{Factoring}$$
$$x + 3 = 0 \quad \text{or} \quad x - 1 = 0 \qquad \text{Using the Principle of Zero Products}$$
$$x = -3 \quad \text{or} \quad x = 1.$$

We next find second coordinates by substituting the critical values in the original function:

$$f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) - 13 = 14;$$
$$f(1) = (1)^3 + 3(1)^2 - 9(1) - 13 = -18.$$

Are the points $(-3, 14)$ and $(1, -18)$ relative extrema? Let's look at the second derivative. We use the Second-Derivative Test with the critical values -3 and 1:

$$f''(-3) = 6(-3) + 6 = -12 < 0; \longrightarrow \text{Relative maximum}$$
$$f''(1) = 6(1) + 6 = 12 > 0. \longrightarrow \text{Relative minimum}$$

Thus, $f(-3) = 14$ is a relative maximum and $f(1) = -18$ is a relative minimum. We plot both $(-3, 14)$ and $(1, -18)$, including short arcs at each point to indicate the graph's concavity. Then, by calculating and plotting a few more points, we can make a sketch, as shown below.

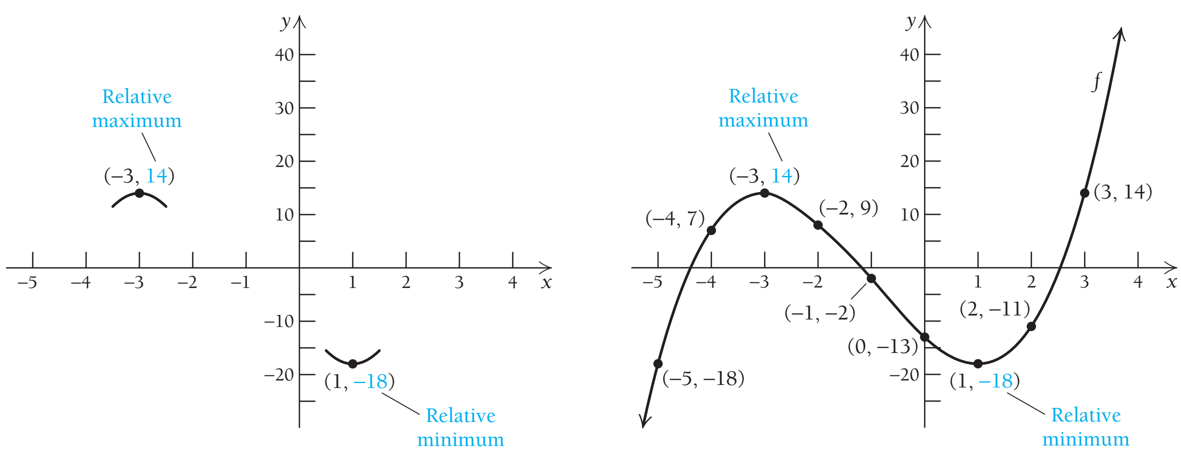
TECHNOLOGY CONNECTION

Exploratory

Consider the function given by

$$y = x^3 - 3x^2 - 9x - 1.$$

Use a graph to estimate the relative extrema. Then find the first and second derivatives. Graph both in the same window. Use the ZERO feature to determine where the first derivative is zero. Verify that relative extrema occur at those x-values by checking the sign of the second derivative. Then check your work using the approach of Example 1.



example 2 Find the relative extrema of the function f given by

$$f(x) = 3x^5 - 20x^3,$$

and sketch the graph.

Solution We find both the first and second derivatives:

$$f'(x) = 15x^4 - 60x^2,$$
$$f''(x) = 60x^3 - 120x.$$

Then we solve $f'(x) = 0$ to find any critical values:

$$15x^4 - 60x^2 = 0$$
$$15x^2(x^2 - 4) = 0$$
$$15x^2(x + 2)(x - 2) = 0 \quad \text{Factoring}$$
$$15x^2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{Using the Principle of Zero Products}$$
$$x = 0 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 2.$$

We next find second coordinates by substituting in the original function:

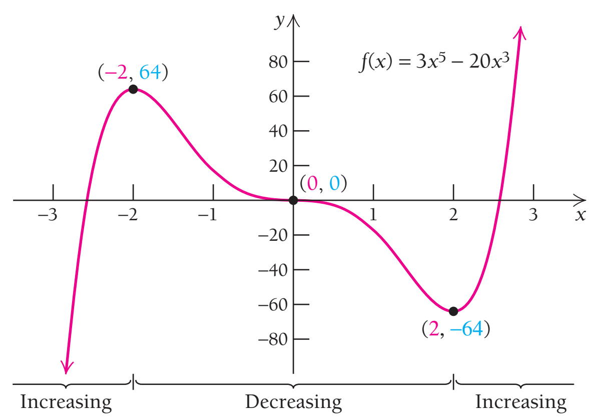
$$f(-2) = 3(-2)^5 - 20(-2)^3 = 64;$$
$$f(0) = 3(0)^5 - 20(0)^3 = 0.$$
$$f(2) = 3(2)^5 - 20(2)^3 = -64;$$

All three of these y-values are candidates for relative extrema.

We now use the Second-Derivative Test with the numbers -2, 2, and 0:

$$f''(-2) = 60(-2)^3 - 120(-2) = -240 < 0; \longrightarrow \text{Relative maximum}$$
$$f''(0) = 60(0)^3 - 120(0) = 0. \longrightarrow \text{The Second-Derivative Test fails. Use the First-Derivative Test.}$$
$$f''(2) = 60(2)^3 - 120(2) = 240 > 0; \longrightarrow \text{Relative minimum}$$

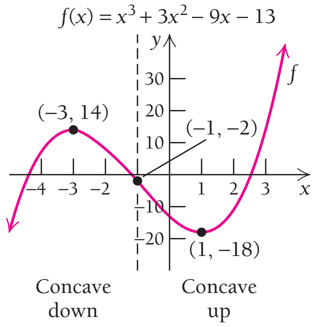
Thus, $f(-2) = 64$ is a relative maximum, and $f(2) = -64$ is a relative minimum. Since $f'(-1) < 0$ and $f'(1) < 0$, we know that f is decreasing on both $(-2, 0)$ and $(0, 2)$. Thus, we know by the First-Derivative Test that f has no relative extremum at $(0, 0)$. We complete the graph, plotting other points as needed. The extrema are shown in the graph at right.



❮ Quick Check 1

❯ **Quick Check 1**

Find the relative extrema of the function g given by $g(x) = 10x^3 - 6x^5$, and sketch the graph.



Points of Inflection

Look again at the graphs in Examples 1 and 2. The concavity changes from down to up at the point $(-1, -2)$ in Example 1, and the concavity changes from up to down at the point $(0, 0)$ in Example 2. (In fact, the graph in Example 2 has other points where the concavity changes direction. This is addressed in Example 3.)

A **point of inflection**, or an **inflection point**, is a point across which the direction of concavity changes. For example, in Figs. 1–3, point P is an inflection point. The figures display the sign of $f''(x)$ to indicate the concavity on either side of P.

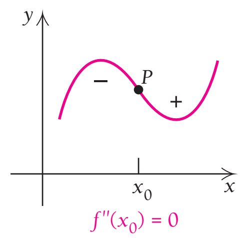


Figure 1

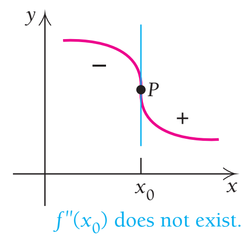


Figure 2

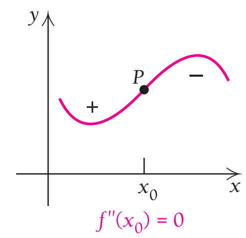


Figure 3

As we move to the right along each curve, the concavity changes at P. Since, as we move through P, the sign of $f''(x)$ changes, either the value of $f''(x_0)$ at P must be 0, as in Figs. 1 and 3, or $f''(x_0)$ must not exist, as in Fig. 2.

THEOREM 6 **Finding Points of Inflection**

If a function f has a point of inflection, it must occur at a point x_0, where

$$f''(x_0) = 0 \quad \text{or} \quad f''(x_0) \text{ does not exist.}$$

The converse of Theorem 6 is not necessarily true. That is, if $f''(x_0)$ is 0 or does not exist, then there is not necessarily a point of inflection at x_0. There must be a change in the direction of concavity on either side of x_0 for $(x_0, f(x_0))$ to be a point of inflection. For example, for $f(x) = (x - 2)^4 + 1$ (see the graph on p. 218), we have $f''(2) = 0$, but $(2, f(2))$ is not a point of inflection since the graph of f is concave up to the left and to the right of $x = 2$.

To find candidates for points of inflection, we look for numbers x_0 for which $f''(x_0) = 0$ or for which $f''(x_0)$ does not exist. Then, if $f''(x)$ changes sign as x moves through x_0 (see Figs. 1–3), we have a point of inflection at x_0.

Theorem 6, about points of inflection, is completely analogous to Theorem 2 about relative extrema. Theorem 2 tells us that relative extrema occur when $f'(x) = 0$ or $f'(x)$ does not exist. Theorem 6 tells us that points of inflection occur when $f''(x) = 0$ or $f''(x)$ does not exist.

example 3 Use the second derivative to determine the point(s) of inflection for the function in Example 2.

Solution The function is $f(x) = 3x^5 - 20x^3$, and its second derivative is $f''(x) = 60x^3 - 120x$. We set the second derivative equal to 0 and solve for x:

$$60x^3 - 120x = 0$$

$$60x(x^2 - 2) = 0 \qquad \text{Factoring out } 60x$$

$$60x(x + \sqrt{2})(x - \sqrt{2}) = 0 \qquad \text{Factoring } x^2 - 2 \text{ as a difference of squares}$$

$$x = 0 \quad \text{or} \quad x = -\sqrt{2} \quad \text{or} \quad x = \sqrt{2}. \qquad \text{Using the Principle of Zero Products}$$

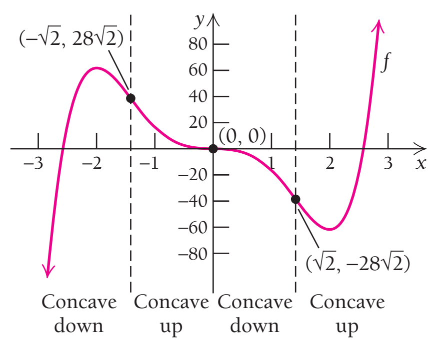
Next, we check the sign of $f''(x)$ over the intervals bounded by these three x-values. We are looking for a change in sign from one interval to the next:

Test Value	$x = -2$	$x = -1$	$x = 1$	$x = 2$
Sign of $f''(x)$	$f''(-2) < 0$	$f''(-1) > 0$	$f''(1) < 0$	$f''(2) > 0$
Result	Concave down	Concave up	Concave down	Concave up

The graph changes from concave down to concave up at $x = -\sqrt{2}$, from concave up to concave down at $x = 0$, and from concave down to concave up at $x = \sqrt{2}$. Therefore, $(-\sqrt{2}, f(-\sqrt{2}))$, $(0, f(0))$, and $(\sqrt{2}, f(\sqrt{2}))$ are points of inflection. Since

$$f(-\sqrt{2}) = 3(-\sqrt{2})^5 - 20(-\sqrt{2})^3 = 28\sqrt{2},$$
$$f(0) = 3(0)^5 - 20(0)^3 = 0,$$
and
$$f(\sqrt{2}) = 3(\sqrt{2})^5 - 20(\sqrt{2})^3 = -28\sqrt{2},$$

these points are $(-\sqrt{2}, 28\sqrt{2})$, $(0, 0)$, and $(\sqrt{2}, -28\sqrt{2})$, shown in the graph below.



> **Quick Check 2**
>
> Determine the points of inflection for the function given by $g(x) = 10x^3 - 6x^5$.

Quick Check 2

Curve Sketching

The first and second derivatives enhance our ability to sketch curves. We use the following strategy:

Strategy for Sketching Graphs*

a) *Derivatives and domain.* Find $f'(x)$ and $f''(x)$. Note the domain of f.

b) *Critical values of f.* Find the critical values by solving $f'(x) = 0$ and finding where $f'(x)$ does not exist. These numbers yield candidates for relative maxima or minima. Find the function values at these points.

c) *Increasing and/or decreasing; relative extrema.* Substitute each critical value, x_0, from step (b) into $f''(x)$. If $f''(x_0) < 0$, then $f(x_0)$ is a relative maximum and f is increasing to the left of x_0 and decreasing to the right. If $f''(x_0) > 0$, then $f(x_0)$ is a relative minimum and f is decreasing to the left of x_0 and increasing to the right.

d) *Inflection points.* Determine candidates for inflection points by finding where $f''(x) = 0$ or where $f''(x)$ does not exist. Find the function values at these points.

e) *Concavity.* Use the candidates for inflection points from step (d) to define intervals. Substitute test values into $f''(x)$ to determine where the graph is concave up ($f''(x) > 0$) and where it is concave down ($f''(x) < 0$).

f) *Sketch the graph.* Sketch the graph using the information from steps (a)–(e), calculating and plotting extra points as needed.

*This strategy is refined further, for rational functions, in Section 2.3.

The examples that follow apply this step-by-step strategy to sketch the graphs of several functions.

example 4 Find the relative extrema of the function f given by

$$f(x) = x^3 - 3x + 2,$$

and sketch the graph.

Solution

a) *Derivatives and domain.* Find $f'(x)$ and $f''(x)$:

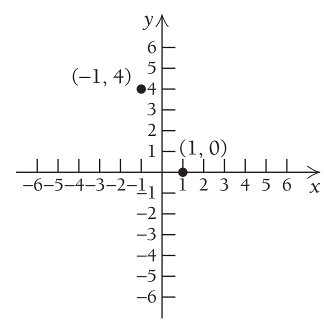
$$f'(x) = 3x^2 - 3,$$
$$f''(x) = 6x.$$

The domain of f (and of any polynomial function) is $(-\infty, \infty)$, or the set of all real numbers, which is also written as $\mathbb{R}$.

b) *Critical values of f.* Find the critical values by determining where $f'(x)$ does not exist and by solving $f'(x) = 0$. We know that $f'(x) = 3x^2 - 3$ exists for all values of x, so the only critical values are where $f'(x)$ is 0:

$$3x^2 - 3 = 0 \qquad \text{Setting } f'(x) \text{ equal to 0}$$
$$3x^2 = 3$$
$$x^2 = 1$$
$$x = \pm 1.$$

We have $f(-1) = 4$ and $f(1) = 0$, so $(-1, 4)$ and $(1, 0)$ are on the graph.



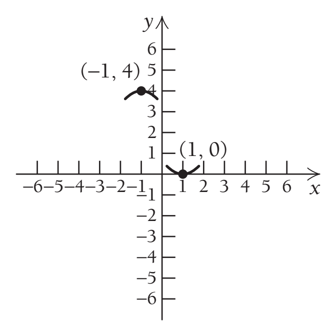
c) *Increasing and/or decreasing; relative extrema.* Substitute the critical values into $f''(x)$:

$$f''(-1) = 6(-1) = -6 < 0,$$

so $f(-1) = 4$ is a relative maximum, with f increasing on $(-\infty, -1)$ and decreasing on $(-1, 1)$.

$$f''(1) = 6 \cdot 1 = 6 > 0,$$

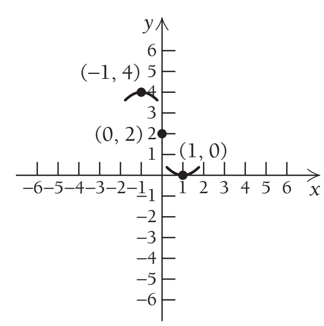
so $f(1) = 0$ is a relative minimum, with f decreasing on $(-1, 1)$ and increasing on $(1, \infty)$.



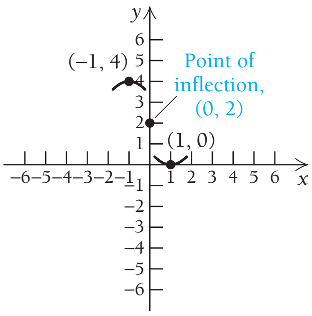
d) *Inflection points.* Find possible inflection points by finding where $f''(x)$ does not exist and by solving $f''(x) = 0$. We know that $f''(x) = 6x$ exists for all values of x, so we try to solve $f''(x) = 0$:

$$6x = 0 \qquad \text{Setting } f''(x) \text{ equal to 0}$$
$$x = 0. \qquad \text{Dividing both sides by 6}$$

We have $f(0) = 2$, which gives us another point, $(0, 2)$, that lies on the graph.



e) *Concavity*. Find the intervals on which f is concave up or concave down, using the point $(0, 2)$ from step (d). From step (c), we can conclude that f is concave down over the interval $(-\infty, 0)$ and concave up over $(0, \infty)$.

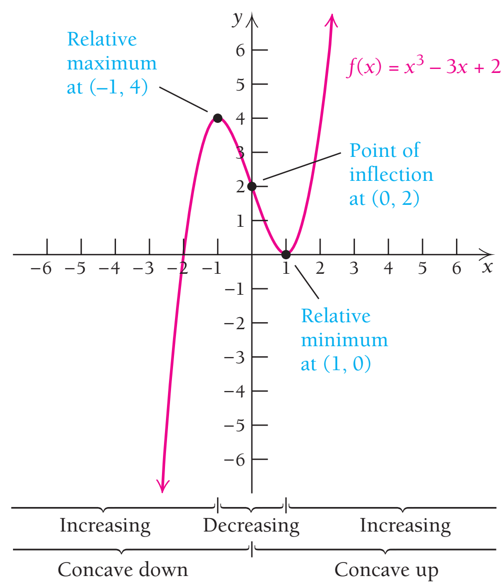


Test Value	$x = -1$	$x = 1$
Sign of $f''(x)$	$f''(-1) < 0$	$f''(1) > 0$
Result	f' is decreasing; f is concave down.	f' is increasing; f is concave up.

Change indicates a point of inflection.

f) *Sketch the graph*. Sketch the graph using the information in steps (a)–(e). Calculate some extra function values if desired. The graph follows.

x	$f(x)$
−3	−16
−2	0
−1	4
0	2
1	0
2	4
3	20



TECHNOLOGY CONNECTION

Check the results of Example 4 using a calculator.

example 5 Find the relative maxima and minima of the function f given by

$$f(x) = x^4 - 2x^2,$$

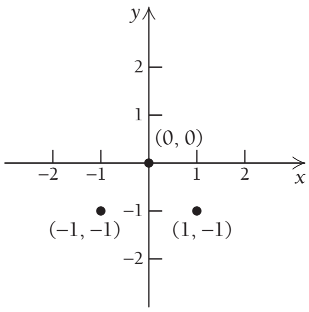
and sketch the graph.

Solution

a) *Derivatives and domain*. Find $f'(x)$ and $f''(x)$:

$$f'(x) = 4x^3 - 4x,$$
$$f''(x) = 12x^2 - 4.$$

The domain of f is $\mathbb{R}$.



b) *Critical values.* Since $f'(x) = 4x^3 - 4x$ exists for all values of x, the only critical values are where $f'(x) = 0$:

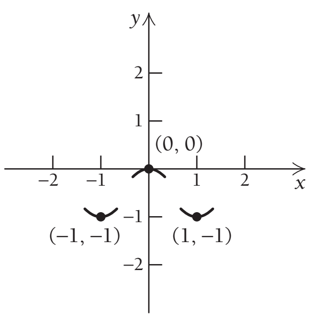
$$\begin{aligned} 4x^3 - 4x &= 0 \qquad \text{Setting } f'(x) \text{ equal to } 0 \\ 4x(x^2 - 1) &= 0 \\ 4x = 0 \quad &\text{or} \quad x^2 - 1 = 0 \\ x = 0 \quad &\text{or} \quad x^2 = 1 \\ & \quad x = \pm 1. \end{aligned}$$

We have $f(0) = 0, f(-1) = -1$, and $f(1) = -1$, which gives the points $(0, 0)$, $(-1, -1)$, and $(1, -1)$ on the graph.

c) *Increasing and/or decreasing; relative extrema.* Substitute the critical values into $f''(x)$:

$$f''(0) = 12 \cdot 0^2 - 4 = -4 < 0,$$

so $f(0) = 0$ is a relative maximum, with f increasing on $(-1, 0)$ and decreasing on $(0, 1)$.



$$f''(-1) = 12(-1)^2 - 4 = 8 > 0,$$

so $f(-1) = -1$ is a relative minimum, with f decreasing on $(-\infty, -1)$ and increasing on $(-1, 0)$.

$$f''(1) = 12 \cdot 1^2 - 4 = 8 > 0,$$

so $f(1) = -1$ is also a relative minimum, with f decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

d) *Inflection points.* Find where $f''(x)$ does not exist and where $f''(x) = 0$. Since $f''(x)$ exists for all real numbers, we just solve $f''(x) = 0$:

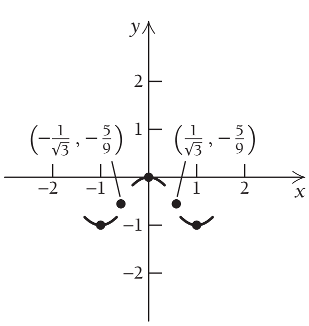
$$\begin{aligned} 12x^2 - 4 &= 0 \qquad \text{Setting } f''(x) \text{ equal to } 0 \\ 4(3x^2 - 1) &= 0 \\ 3x^2 - 1 &= 0 \\ 3x^2 &= 1 \\ x^2 &= \frac{1}{3} \\ x &= \pm\sqrt{\frac{1}{3}} \\ &= \pm\frac{1}{\sqrt{3}}. \end{aligned}$$

We have

$$f\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right)^4 - 2\left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{9} - \frac{2}{3} = -\frac{5}{9}$$

and

$$f\left(-\frac{1}{\sqrt{3}}\right) = -\frac{5}{9}.$$

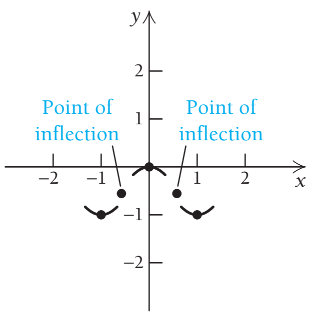


These values give

$$\left(-\frac{1}{\sqrt{3}}, -\frac{5}{9}\right) \text{ and } \left(\frac{1}{\sqrt{3}}, -\frac{5}{9}\right)$$

as possible inflection points.

e) *Concavity.* Find the intervals on which f is concave up or concave down, using the points $\left(-\frac{1}{\sqrt{3}}, -\frac{5}{9}\right)$ and $\left(\frac{1}{\sqrt{3}}, -\frac{5}{9}\right)$, from step (d). From step (c), we can conclude that f is concave up over the intervals $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$ and concave down over the interval $(-1/\sqrt{3}, 1/\sqrt{3})$.



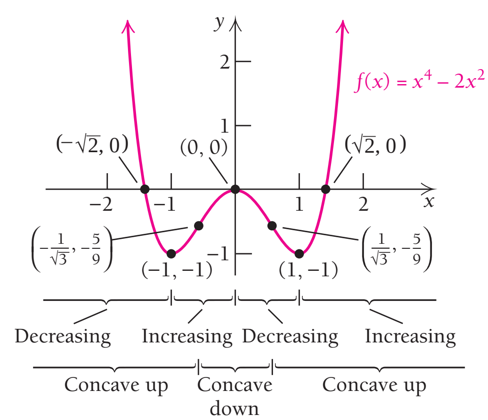
Interval ← $-1/\sqrt{3}$ — $-1/\sqrt{3}$ → x

Test Value	$x = -1$	$x = 0$	$x = 1$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(0) < 0$	$f''(1) > 0$
Result	f' is increasing; f is concave up.	f' is decreasing; f is concave down.	f' is increasing; f is concave up.

Change indicates a point of inflection. Change indicates a point of inflection.

f) *Sketch the graph.* Sketch the graph using the information in steps (a)–(e). By solving $x^4 - 2x^2 = 0$, we can find the x-intercepts easily. They are $(-\sqrt{2}, 0)$, $(0, 0)$, and $(\sqrt{2}, 0)$. This also aids with graphing. Extra function values can be calculated if desired. The graph is shown below.

x	$f(x)$, approximately
−2	8
−1.5	0.56
−1	−1
−0.5	−0.44
0	0
0.5	−0.44
1	−1
1.5	0.56
2	8



TECHNOLOGY CONNECTION

EXERCISE

1. Consider

$$f(x) = x^3(x-2)^3.$$

How many relative extrema do you anticipate finding? Where do you think they will be?

Graph f, f', and f'' using $[-1, 3, -2, 6]$ as a viewing window. Estimate the relative extrema and the inflection points of f. Then check your work using the analytic methods of Examples 4 and 5.

❯ Quick Check 3

Find the relative maxima and minima of the function f given by $f(x) = 1 + 8x^2 - x^4$, and sketch the graph.

❮ Quick Check 3

Exampl E 6 Graph the function f given by

$$f(x) = (2x-5)^{1/3} + 1.$$

List the coordinates of any extrema and points of inflection. State where the function is increasing or decreasing, as well as where it is concave up or concave down.

Solution

a) *Derivatives and domain*. Find $f'(x)$ and $f''(x)$:

$$f'(x) = \tfrac{1}{3}(2x-5)^{-2/3}\cdot 2 = \tfrac{2}{3}(2x-5)^{-2/3}, \text{ or } \frac{2}{3(2x-5)^{2/3}};$$

$$f''(x) = -\tfrac{4}{9}(2x-5)^{-5/3}\cdot 2 = -\tfrac{8}{9}(2x-5)^{-5/3}, \text{ or } \frac{-8}{9(2x-5)^{5/3}}.$$

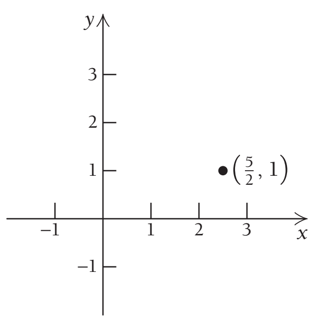
The domain of f is $\mathbb{R}$.

b) *Critical values*. Since

$$f'(x) = \frac{2}{3(2x-5)^{2/3}}$$

is never 0 (a fraction equals 0 only when its numerator is 0), the only critical value is when $f'(x)$ does not exist. The only time $f'(x)$ does not exist is when its denominator is 0:

$$3(2x-5)^{2/3} = 0$$
$$(2x-5)^{2/3} = 0 \qquad \text{Dividing both sides by 3}$$
$$(2x-5)^2 = 0 \qquad \text{Cubing both sides}$$
$$2x-5 = 0$$
$$2x = 5$$
$$x = \tfrac{5}{2}$$



We now have $f\left(\frac{5}{2}\right) = \left(2\cdot\frac{5}{2} - 5\right)^{1/3} + 1 = 0 + 1 = 1$, so the point $\left(\frac{5}{2}, 1\right)$ is on the graph.

c) *Increasing and/or decreasing; relative extrema*. Substitute the critical value into $f''(x)$:

$$f''\left(\tfrac{5}{2}\right) = \frac{-8}{9\left(2\cdot\frac{5}{2} - 5\right)^{5/3}} = \frac{-8}{9\cdot 0} = \frac{-8}{0}.$$

Since $f''\left(\frac{5}{2}\right)$ does not exist, the Second-Derivative Test cannot be used at $x = \frac{5}{2}$. Instead, we use the First-Derivative Test, selecting 2 and 3 as test values on either side of $\frac{5}{2}$:

$$f'(2) = \frac{2}{3(2\cdot 2 - 5)^{2/3}} = \frac{2}{3(-1)^{2/3}} = \frac{2}{3\cdot 1} = \frac{2}{3},$$

$$\text{and } f'(3) = \frac{2}{3(2\cdot 3 - 5)^{2/3}} = \frac{2}{3\cdot 1^{2/3}} = \frac{2}{3\cdot 1} = \frac{2}{3}.$$

Since $f'(x) > 0$ on either side of $x = \frac{5}{2}$, we know that f is increasing on both $\left(-\infty, \frac{5}{2}\right)$ and $\left(\frac{5}{2}, \infty\right)$; thus, $f\left(\frac{5}{2}\right) = 1$ is not an extremum.

d) *Inflection points*. Find where $f''(x)$ does not exist and where $f''(x) = 0$. Since $f''(x)$ is never 0 (why?), we only need to find where $f''(x)$ does not exist. Since $f''(x)$ cannot exist where $f'(x)$ does not exist, we know from step (b) that a possible inflection point is $\left(\frac{5}{2}, 1\right)$.

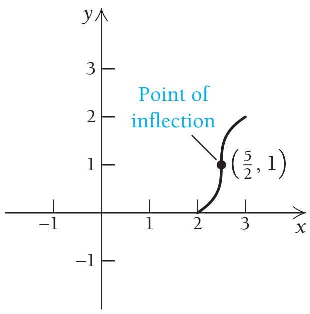
e) *Concavity*. We check the concavity on either side of $x = \frac{5}{2}$. We choose $x = 2$ and $x = 3$ as our test values.

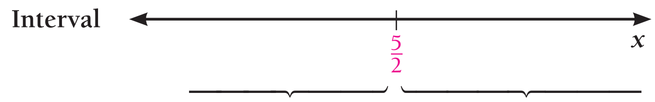
$$f''(2) = \frac{-8}{9(2\cdot 2 - 5)^{5/3}} = \frac{-8}{9(-1)} > 0,$$

so f is concave up on $\left(-\infty, \frac{5}{2}\right)$.

$$f''(3) = \frac{-8}{9(2\cdot 3 - 5)^{5/3}} = \frac{-8}{9\cdot 1} < 0,$$

so f is concave down on $\left(\frac{5}{2}, \infty\right)$.



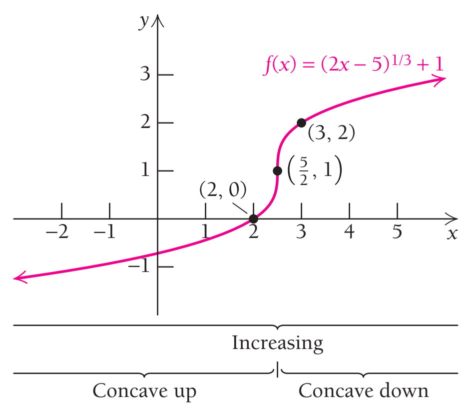


Test Value	$x = 2$	$x = 3$
Sign of $f''(x)$	$f''(2) > 0$	$f''(3) < 0$
Result	f' is increasing; f is concave up.	f' is decreasing; f is concave down.

Change indicates a point of inflection.

f) *Sketch the graph*. Sketch the graph using the information in steps (a)–(e). By solving $(2x - 5)^{1/3} + 1 = 0$, we can find the x-intercept—it is $(2, 0)$. Extra function values can be calculated, if desired. The graph is shown below.

x	$f(x)$, approximately
0	−0.71
1	−0.44
2	0
$\frac{5}{2}$	1
3	2
4	2.44
5	2.71



The following figures illustrate some information concerning the function in Example 1 that can be found from the first and second derivatives of f. The relative extrema are shown in Figs. 4 and 5. In Fig. 5, we see that the x-coordinates of the x-intercepts of f' are the critical values of f. Note that the intervals over which f is increasing or decreasing are those intervals for which f' is positive or negative, respectively.

Not for Sale

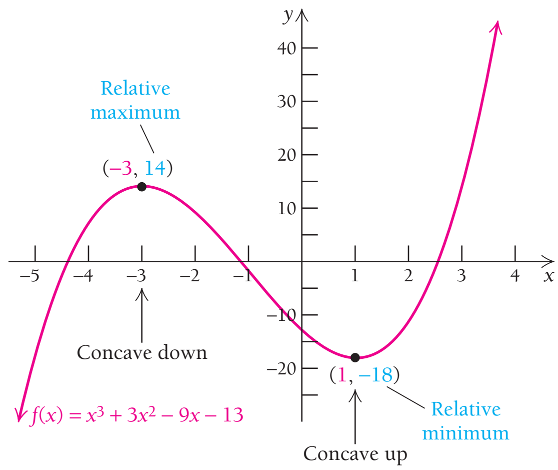


Figure 4

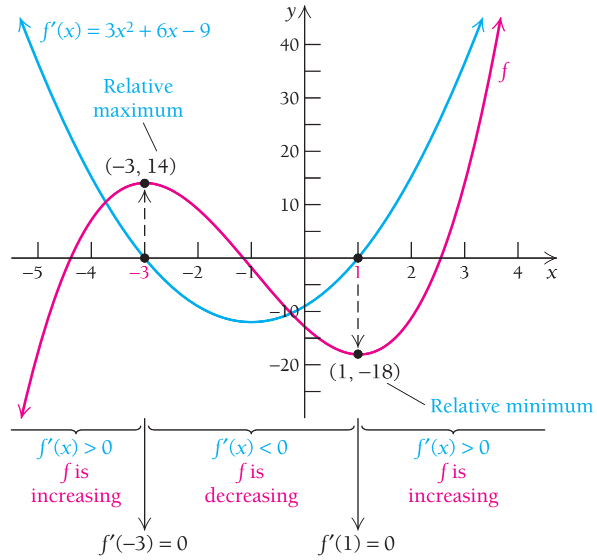


Figure 5

In Fig. 6, the intervals over which f' is increasing or decreasing are, respectively, those intervals over which f'' is positive or negative. And finally, in Fig. 7, we note that when $f''(x) < 0$, the graph of f is concave down, and when $f''(x) > 0$, the graph of f is concave up.

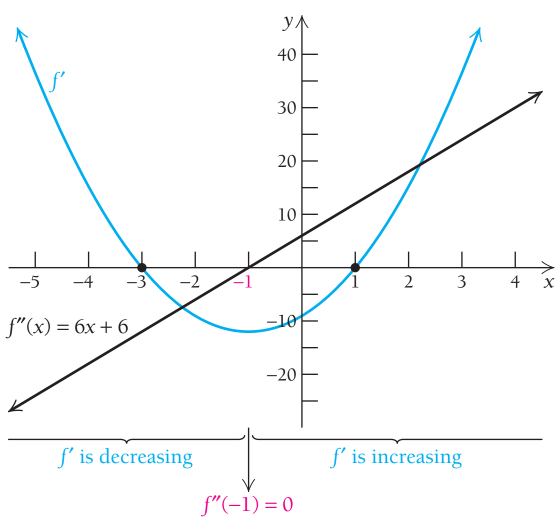


Figure 6

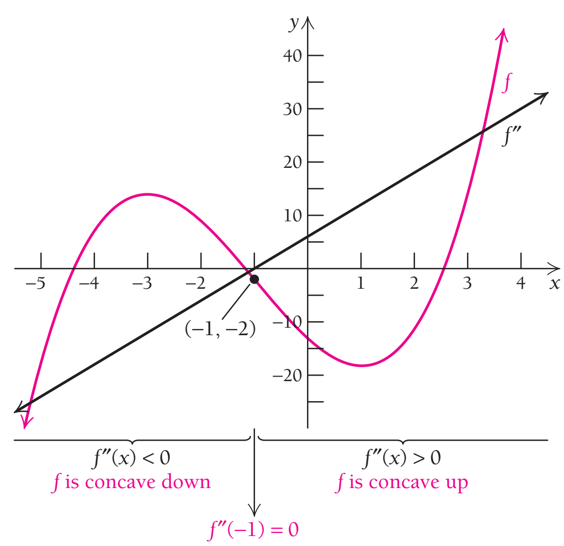


Figure 7

TECHNOLOGY CONNECTION

Using Graphicus to Find Roots, Extrema, and Inflection Points

Graphicus has the capability of finding roots, relative extrema, and points of inflection. Let's consider the function of Example 5, given by $f(x) = x^4 - 2x^2$.

Graphing a Function and Finding Its Roots, Relative Extrema, and Points of Inflection

After opening Graphicus, touch the blank rectangle at the top of the screen and enter the function as y(x)=x^4-2x^2. Press [+] in the upper right. You will see the graph (Fig. 1).

(continued)

Graphing a Function and Finding Its Roots, Relative Extrema, and Points of Inflection (continued)

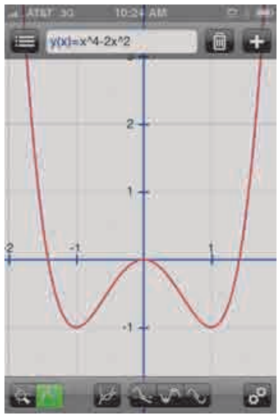


Figure 1

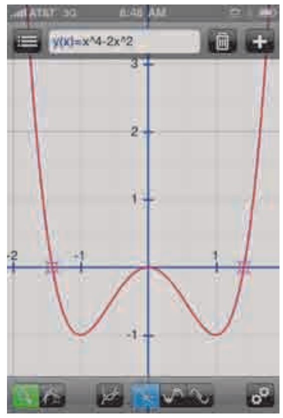


Figure 2

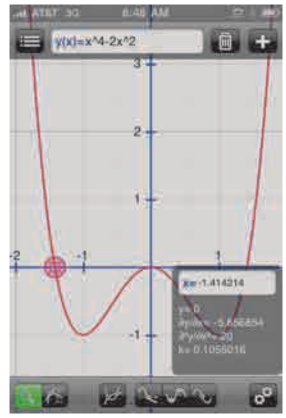


Figure 3

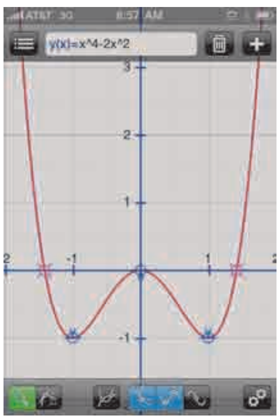


Figure 4

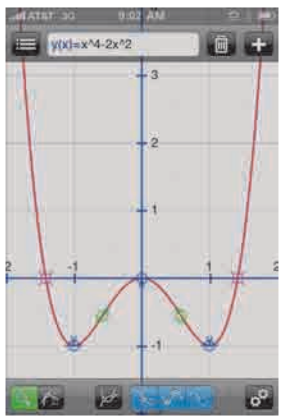


Figure 5

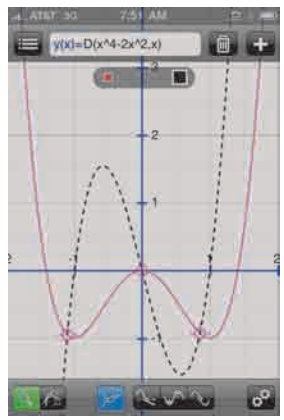


Figure 6

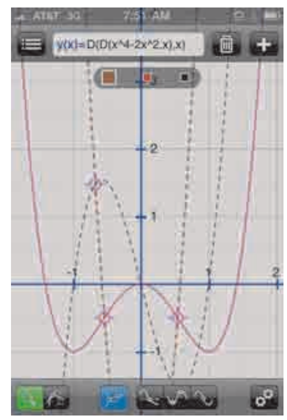


Figure 7

Then touch the fourth icon from the left at the bottom, and you will see the roots highlighted on the graph (Fig. 2). Touch the left-hand root symbol, and a value for one root, in this case, an approximation, -1.414214, is displayed (Fig. 3).

To find the relative extrema, touch the fifth icon from the left; the relative extrema will be highlighted on the graph in a different color. Touch each point to identify the relative minima at $(-1, -1)$, and $(1, -1)$, and a relative maximum at $(0, 0)$ (Fig. 4).

To find points of inflection, touch the sixth icon from the left, and the points of inflection will be highlighted as shown in Fig. 5. Touch these points and the approximations $(-0.577, -0.556)$ and $(0.577, -0.556)$ will be displayed.

Graphing a Function and Its Derivatives

Graphicus can graph derivatives. Go back to the original graph of f (Fig. 1). Touch [+]; then press `Add derivative`, and you will see the original graph and the graph of the first derivative as a dashed line of a different color (Fig. 6). Touch [+] and `Add derivative` again, and you will see the graph of the original function, with dashed graphs of the first and second derivatives in different colors (Fig. 7). You can toggle between the derivatives by pressing the colored squares above the graphs.

EXERCISES

Use Graphicus to graph each function and its first and second derivatives. Then find approximations for roots, relative extrema, and points of inflection.

1. $f(x) = 2x^3 - x^4$

2. $f(x) = x(200 - x)$

3. $f(x) = x^3 - 6x^2$

4. $f(x) = -4.32 + 1.44x + 3x^2 - x^3$

5. $g(x) = x\sqrt{4 - x^2}$

6. $g(x) = \dfrac{4x}{x^2 + 1}$

7. $f(x) = \dfrac{x^2 - 3x}{x - 1}$

8. $f(x) = |x + 2| - 3$

TECHNOLOGY CONNECTION

EXERCISES

Graph the following:

$$f(x) = 3x^5 - 5x^3, \quad f'(x) = 15x^4 - 15x^2,$$

and

$$f''(x) = 60x^3 - 30x,$$

using the window $[-3, 3, -10, 10]$.

1. From the graph of f', estimate the critical values of f.
2. From the graph of f'', estimate the x-values of any inflection points of f.

Section Summary

- The second derivative f'' determines the *concavity* of the graph of function f.
- If $f''(x) > 0$ for all x in an open interval I, then the graph of f is *concave up* over I.
- If $f''(x) < 0$ for all x in an open interval I, then the graph of f is *concave down* over I.
- If c is a critical value and $f''(c) > 0$, then $f(c)$ is a relative minimum.
- If c is a critical value and $f''(c) < 0$, then $f(c)$ is a relative maximum.
- If c is a critical value and $f''(c) = 0$, the First-Derivative Test must be used to classify $f(c)$.
- If $f''(x_0) = 0$ or $f''(x_0)$ does not exist, and there is a change in concavity to the left and to the right of x_0, then the point $(x_0, f(x_0))$ is called a *point of inflection*.
- Finding relative extrema, intervals over which a function is increasing or decreasing, intervals of upward or downward concavity, and points of inflection is all part of a strategy for accurate curve sketching.

EXERCISE SET 2.2

For each function, find all relative extrema and classify each as a maximum or minimum. Use the Second-Derivative Test where possible.

1. $f(x) = 5 - x^2$
2. $f(x) = 4 - x^2$
3. $f(x) = x^2 - x$
4. $f(x) = x^2 + x - 1$
5. $f(x) = -5x^2 + 8x - 7$
6. $f(x) = -4x^2 + 3x - 1$
7. $f(x) = 8x^3 - 6x + 1$
8. $f(x) = x^3 - 12x - 1$

Sketch the graph of each function. List the coordinates of where extrema or points of inflection occur. State where the function is increasing or decreasing, as well as where it is concave up or concave down.

9. $f(x) = x^3 - 12x$
10. $f(x) = x^3 - 27x$
11. $f(x) = 3x^3 - 36x - 3$
12. $f(x) = 2x^3 - 3x^2 - 36x + 28$
13. $f(x) = \frac{8}{3}x^3 - 2x + \frac{1}{3}$
14. $f(x) = 80 - 9x^2 - x^3$
15. $f(x) = -x^3 + 3x^2 - 4$
16. $f(x) = -x^3 + 3x - 2$
17. $f(x) = 3x^4 - 16x^3 + 18x^2$
    (Round results to three decimal places.)
18. $f(x) = 3x^4 + 4x^3 - 12x^2 + 5$
    (Round results to three decimal places.)
19. $f(x) = x^4 - 6x^2$
20. $f(x) = 2x^2 - x^4$
21. $f(x) = x^3 - 2x^2 - 4x + 3$
22. $f(x) = x^3 - 6x^2 + 9x + 1$
23. $f(x) = 3x^4 + 4x^3$
24. $f(x) = x^4 - 2x^3$
25. $f(x) = x^3 - 6x^2 - 135x$
26. $f(x) = x^3 - 3x^2 - 144x - 140$
27. $f(x) = x^4 - 4x^3 + 10$
28. $f(x) = \frac{4}{3}x^3 - 2x^2 + x$
29. $f(x) = x^3 - 6x^2 + 12x - 6$

30. $f(x) = x^3 + 3x + 1$

31. $f(x) = 5x^3 - 3x^5$

32. $f(x) = 20x^3 - 3x^5$
(Round results to three decimal places.)

33. $f(x) = x^2(3 - x)^2$
(Round results to three decimal places.)

34. $f(x) = x^2(1 - x)^2$
(Round results to three decimal places.)

35. $f(x) = (x + 1)^{2/3}$

36. $f(x) = (x - 1)^{2/3}$

37. $f(x) = (x - 3)^{1/3} - 1$

38. $f(x) = (x - 2)^{1/3} + 3$

39. $f(x) = -2(x - 4)^{2/3} + 5$

40. $f(x) = -3(x - 2)^{2/3} + 3$

41. $f(x) = x\sqrt{4 - x^2}$

42. $f(x) = -x\sqrt{1 - x^2}$

43. $f(x) = \dfrac{x}{x^2 + 1}$

44. $f(x) = \dfrac{8x}{x^2 + 1}$

45. $f(x) = \dfrac{3}{x^2 + 1}$

46. $f(x) = \dfrac{-4}{x^2 + 1}$

For Exercises 47–56, sketch a graph that possesses the characteristics listed. Answers may vary.

47. f is increasing and concave up on $(-\infty, 4)$,
f is increasing and concave down on $(4, \infty)$.

48. f is decreasing and concave up on $(-\infty, 2)$,
f is decreasing and concave down on$(2, \infty)$.

49. f is increasing and concave down on $(-\infty, 1)$,
f is increasing and concave up on $(1, \infty)$.

50. f is decreasing and concave down on $(-\infty, 3)$,
f is decreasing and concave up on $(3, \infty)$.

51. f is concave down at $(1, 5)$, concave up at $(7, -2)$, and has an inflection point at $(4, 1)$.

52. f is concave up at $(1, -3)$, concave down at $(8, 7)$, and has an inflection point at $(5, 4)$.

53. $f'(-1) = 0, f''(-1) > 0, f(-1) = -5;$
$f'(7) = 0, f''(7) < 0, f(7) = 10; f''(3) = 0$, and
$f(3) = 2$

54. $f'(-3) = 0, f''(-3) < 0, f(-3) = 8; f'(9) = 0,$
$f''(9) > 0, f(9) = -6; f''(2) = 0$, and $f(2) = 1$

55. $f'(-1) = 0, f''(-1) > 0, f(-1) = -2;$
$f'(1) = 0, f''(1) > 0, f(1) = -2; f'(0) = 0,$
$f''(0) < 0$, and $f(0) = 0$

56. $f'(0) = 0, f''(0) < 0, f(0) = 5; f'(2) = 0, f''(2) > 0,$
$f(2) = 2; f'(4) = 0, f''(4) < 0$, and $f(4) = 3$

57–102. Check the results of Exercises 1–46 with a graphing calculator.

APPLICATIONS

Business and Economics

Total revenue, cost, and profit. *Using the same set of axes, sketch the graphs of the total-revenue, total-cost, and total-profit functions.*

103. $R(x) = 50x - 0.5x^2, \quad C(x) = 4x + 10$

104. $R(x) = 50x - 0.5x^2, \quad C(x) = 10x + 3$

105. **Small business.** The percentage of the U.S. national income generated by nonfarm proprietors may be modeled by the function

$$p(x) = \frac{13x^3 - 240x^2 - 2460x + 585{,}000}{75{,}000},$$

where x is the number of years since 1970. Sketch the graph of this function for $0 \le x \le 40$.

106. **Labor force.** The percentage of the U.S. civilian labor force aged 45–54 may be modeled by the function

$$f(x) = 0.025x^2 - 0.71x + 20.44,$$

where x is the number of years after 1970. Sketch the graph of this function for $0 \le x \le 30$.

Life and Physical Sciences

107. **Coughing velocity.** A person coughs when a foreign object is in the windpipe. The velocity of the cough depends on the size of the object. Suppose a person has a windpipe with a 20-mm radius. If a foreign object has a radius r, in millimeters, then the velocity V, in millimeters per second, needed to remove the object by a cough is given by

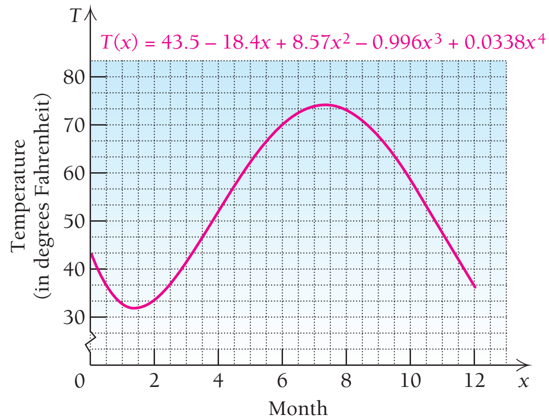
$$V(r) = k(20r^2 - r^3), \quad 0 \le r \le 20,$$

where k is some positive constant. For what size object is the maximum velocity required to remove the object?

108. **New York temperatures.** The average temperature in New York can be approximated by the function

$$T(x) = 43.5 - 18.4x + 8.57x^2 - 0.996x^3 + 0.0338x^4,$$

where T represents the temperature, in degrees Fahrenheit, $x = 1$ represents the middle of January, $x = 2$ represents the middle of February, and so on. (*Source:* www.WorldClimate.com.)

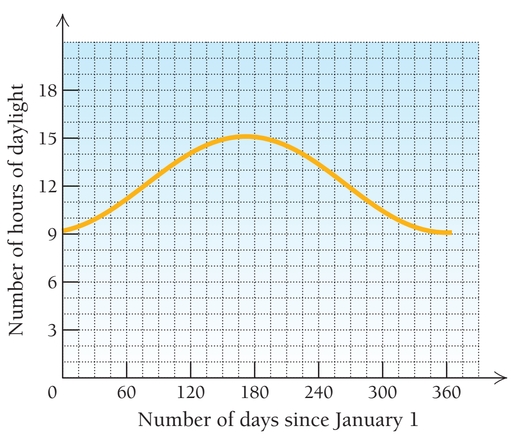


a) Based on the graph, when would you expect the highest temperature to occur in New York?

b) Based on the graph, when would you expect the lowest temperature to occur?

c) Use the Second-Derivative Test to estimate the points of inflection for the function $T(x)$. What is the significance of these points?

109. Hours of daylight. The number of hours of daylight in Chicago is represented in the graph below. On what dates is the number of hours of daylight changing most rapidly? How can you tell?

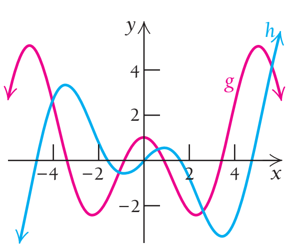


(*Source*: Astronomical Applications Dept., U.S. Naval Observatory.)

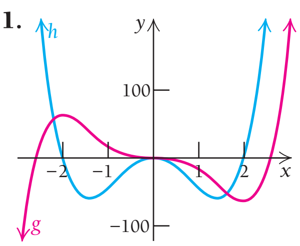
SYNTHESIS

In each of Exercises 110 and 111, determine which graph is the derivative of the other and explain why.

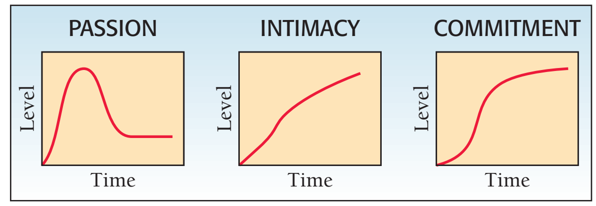
110.



111.



112. Social sciences: three aspects of love. Researchers at Yale University have suggested that the following graphs may represent three different aspects of love.



(*Source*: From "A Triangular Theory of Love," by R. J. Sternberg, 1986, *Psychological Review*, 93(2), 119–135. Copyright 1986 by the American Psychological Association, Inc. Reprinted by permission.)

Analyze each of these graphs in terms of the concepts you have learned: relative extrema, concavity, increasing, decreasing, and so on. Do you agree with the researchers regarding the shapes of these graphs? Why or why not?

113. Use calculus to prove that the relative minimum or maximum for any function f for which

$$f(x) = ax^2 + bx + c, a \neq 0,$$

occurs at $x = -b/(2a)$.

114. Use calculus to prove that the point of inflection for any function g given by

$$g(x) = ax^3 + bx^2 + cx + d, \qquad a \neq 0,$$

occurs at $x = -b/(3a)$.

For Exercises 115–121, assume the function f is differentiable over the interval $(-\infty, \infty)$; that is, it is smooth and continuous for all real numbers x and has no corners or vertical tangents. Classify each of the following statements as either true or false. If you choose false, explain why.

115. If f has exactly two critical values at $x = a$ and $x = b$, where $a < b$, then there must exist exactly one point of inflection at $x = c$ such that $a < c < b$. In other words, exactly one point of inflection must exist between any two critical points.

116. If f has exactly two critical values at $x = a$ and $x = b$, where $a < b$, then there must exist at least one point of inflection at $x = c$ such that $a < c < b$. In other words, at least one point of inflection must exist between any two critical points.

117. The function f can have no extrema but can have at least one point of inflection.

118. If the function f has two points of inflection, then there must be a critical value located between those points of inflection.

119. The function f can have a point of inflection at a critical value.

120. The function f can have a point of inflection at an extreme value.

121. The function f can have exactly one extreme value but no points of inflection.

TECHNOLOGY CONNECTION

Graph each function. Then estimate any relative extrema. Where appropriate, round to three decimal places.

122. $f(x) = 3x^{2/3} - 2x$

123. $f(x) = 4x - 6x^{2/3}$

124. $f(x) = x^2(x - 2)^3$

125. $f(x) = x^2(1 - x)^3$

126. $f(x) = x - \sqrt{x}$

127. $f(x) = (x - 1)^{2/3} - (x + 1)^{2/3}$

128. Social sciences: time spent on home computer. The following data relate the average number of minutes spent per month on a home computer to a person's age.

Age (in years)	Average Use (in minutes per month)
6.5	363
14.5	645
21	1377
29.5	1727
39.5	1696
49.5	2052
55 and up	2299

(Source: *Media Matrix; The PC Meter Company.*)

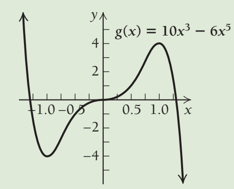
a) Use the regression procedures of Section R.6 to fit linear, cubic, and quartic functions $y = f(x)$ to the data, where x is age and y is average use per month. Decide which function best fits the data. Explain.

b) What is the domain of the function?

c) Does the function have any relative extrema? Explain.

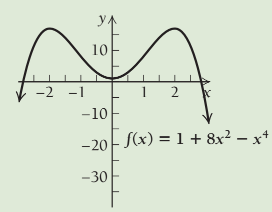
Answers to Quick Checks

1. Relative minimum: -4 at $x = -1$; relative maximum: 4 at $x = 1$



2. $(0, 0)$, $\left(\frac{\sqrt{2}}{2}, 2.475\right)$, and $\left(-\frac{\sqrt{2}}{2}, -2.475\right)$

3. Relative maxima: 17 at $x = -2$ and $x = 2$; relative minimum: 1 at $x = 0$



2.3 Graph Sketching: Asymptotes and Rational Functions

OBJECTIVES

- Find limits involving infinity.
- Determine the asymptotes of a function's graph.
- Graph rational functions.

Thus far we have considered a strategy for graphing a continuous function using the tools of calculus. We now want to consider some discontinuous functions, most of which are rational functions. Our graphing skills must now allow for discontinuities as well as certain lines called *asymptotes*.

Let's review the definition of a rational function.

Rational Functions

DEFINITION

A **rational function** is a function f that can be described by

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials, with $Q(x)$ not the zero polynomial. The domain of f consists of all inputs x for which $Q(x) \neq 0$.

Polynomials are themselves a special kind of rational function, since $Q(x)$ can be 1. Here we are considering graphs of rational functions in which the denominator is not a constant. Before we do so, however, we need to reconsider limits.

Vertical and Horizontal Asymptotes

Figure 1 shows the graph of the rational function

$$f(x) = \frac{x^2 - 1}{x^2 + x - 6} = \frac{(x - 1)(x + 1)}{(x - 2)(x + 3)}.$$

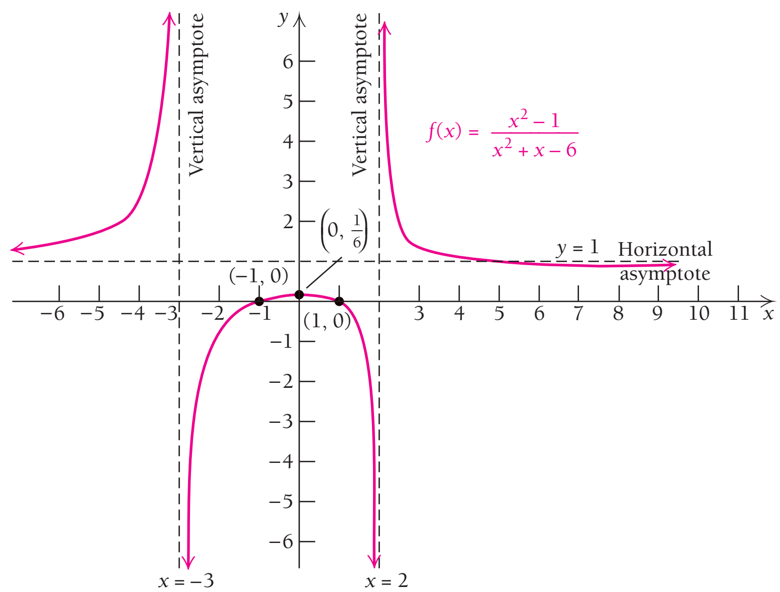


Figure 1

Note that as x gets closer to 2 from the left, the function values get smaller and smaller negatively, approaching $-\infty$. As x gets closer to 2 from the right, the function values get larger and larger positively. Thus,

$$\lim_{x \to 2^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to 2^+} f(x) = \infty.$$

For this graph, we can think of the line $x = 2$ as a "limiting line" called a *vertical asymptote*. Similarly, the line $x = -3$ is another vertical asymptote.

DEFINITION

The line $x = a$ is a **vertical asymptote** if any of the following limit statements is true:

$$\lim_{x \to a^-} f(x) = \infty, \quad \lim_{x \to a^-} f(x) = -\infty, \quad \lim_{x \to a^+} f(x) = \infty, \quad \text{or} \quad \lim_{x \to a^+} f(x) = -\infty.$$

The graph of a rational function *never* crosses a vertical asymptote. If the expression that defines the rational function f is simplified, meaning that it has no common factor other than -1 or 1, then if a is an input that makes the denominator 0, the line $x = a$ is a vertical asymptote.

For example,

$$f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3}$$

does not have a vertical asymptote at $x = 3$, even though 3 is an input that makes the denominator 0. This is because when $(x^2 - 9)/(x - 3)$ is simplified, it has $x - 3$ as a common factor of the numerator and the denominator. In contrast,

$$g(x) = \frac{x^2 - 4}{x^2 + x - 12} = \frac{(x + 2)(x - 2)}{(x - 3)(x + 4)}$$

is simplified and has $x = 3$ and $x = -4$ as vertical asymptotes.

Figure 2 shows the four ways in which a vertical asymptote can occur. The dashed lines represent the asymptotes. They are sketched in for visual assistance only; they are not part of the graphs of the functions.

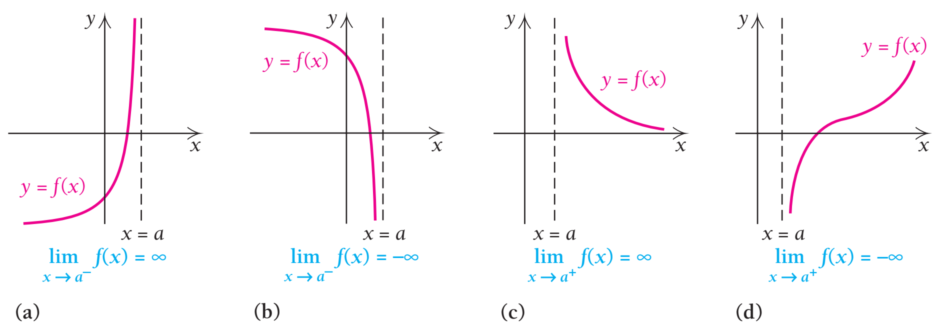


Figure 2

❯ Quick Check 1

Determine the vertical asymptotes:

$$f(x) = \frac{1}{x(x^2 - 16)}.$$

■ **example 1** Determine the vertical asymptotes: $f(x) = \dfrac{3x - 2}{x(x - 5)(x + 3)}$.

Solution The expression is in simplified form. The vertical asymptotes are the lines $x = 0$, $x = 5$, and $x = -3$.

❮ Quick Check 1

■ **example 2** Determine the vertical asymptotes of the function given by

$$f(x) = \frac{x^2 - 2x}{x^3 - x}.$$

Solution We write the expression in simplified form:

$$f(x) = \frac{x^2 - 2x}{x^3 - x} = \frac{x(x - 2)}{x(x - 1)(x + 1)}$$

$$= \frac{x - 2}{(x - 1)(x + 1)}, \quad x \neq 0.$$

The expression is now in simplified form. The vertical asymptotes are the lines $x = -1$ and $x = 1$.

❮ Quick Check 2

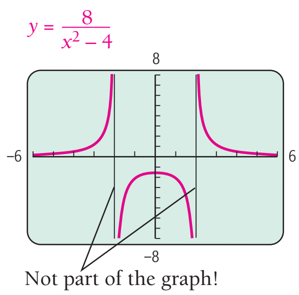
❯ Quick Check 2

For the function in Example 2, explain why $x = 0$ does not correspond to a vertical asymptote. What kind of discontinuity occurs at $x = 0$?

TECHNOLOGY CONNECTION

Asymptotes

Our discussion here allows us to attach the term "vertical asymptote" to those mysterious vertical lines that appear when graphing rational functions in the CONNECTED mode. For example, consider the graph of $f(x) = 8/(x^2 - 4)$, using the window $[-6, 6, -8, 8]$. Vertical asymptotes occur at $x = -2$ and $x = 2$. These lines are not part of the graph.



EXERCISES

Graph each of the following in both DOT and CONNECTED modes. Try to locate the vertical asymptotes visually. Then verify your results using the method of Examples 1 and 2. You may need to try different viewing windows.

1. $f(x) = \dfrac{x^2 + 7x + 10}{x^2 + 3x - 28}$

2. $f(x) = \dfrac{x^2 + 5}{x^3 - x^2 - 6x}$

Look again at the graph in Fig. 1. Note that function values get closer and closer to 1 as x approaches $-\infty$, meaning that $f(x) \to 1$ as $x \to -\infty$. Also, function values get closer and closer to 1 as x approaches ∞, meaning that $f(x) \to 1$ as $x \to \infty$. Thus,

$$\lim_{x \to -\infty} f(x) = 1 \quad \text{and} \quad \lim_{x \to \infty} f(x) = 1.$$

The line $y = 1$ is called a *horizontal asymptote*.

DEFINITION

The line $y = b$ is a **horizontal asymptote** if either or both of the following limit statements is true:

$$\lim_{x \to -\infty} f(x) = b \quad \text{or} \quad \lim_{x \to \infty} f(x) = b.$$

The graph of a rational function may or may not cross a horizontal asymptote. Horizontal asymptotes occur when the degree of the numerator is less than or equal to the degree of the denominator. (The degree of a polynomial in one variable is the highest power of that variable.)

In Figs. 3–5, we see three ways in which horizontal asymptotes can occur.

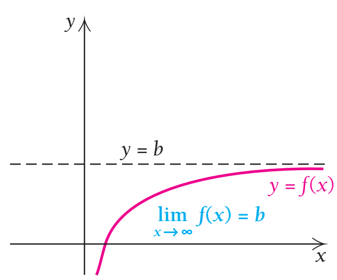


Figure 3

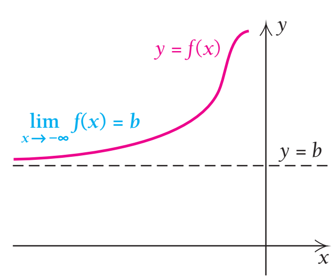


Figure 4

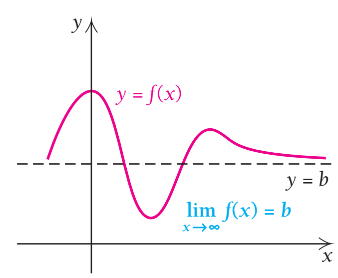


Figure 5

Horizontal asymptotes are found by determining the limit of a rational function as inputs approach $-\infty$ or ∞.

example 3 Determine the horizontal asymptote of the function given by

$$f(x) = \frac{3x - 4}{x}.$$

Solution To find the horizontal asymptote, we consider

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{3x - 4}{x}.$$

One way to find such a limit is to use an input–output table, as follows, using progressively larger x-values.

Inputs, x	1	10	50	100	2000
Outputs, $\dfrac{3x - 4}{x}$	−1	2.6	2. 92	2.96	2.998

TECHNOLOGY CONNECTION

EXERCISES

1. Verify the limit

$$\lim_{x \to \infty} \frac{3x - 4}{x} = 3$$

by using the TABLE feature with larger and larger x-values.

X	Y1
50	2.92
150	2.9733
250	2.984
350	2.9886
450	2.9911
550	2.9927
650	2.9938

X = 50

X	Y1
500	2.992
1500	2.9973
2500	2.9984
3500	2.9989
4500	2.9991
5500	2.9993
6500	2.9994

X = 500

2. Graph the function

$$f(x) = \frac{3x - 4}{x}$$

in DOT mode. Then use TRACE, moving the cursor along the graph from left to right, and observe the behavior of the y-coordinates.

For Exercises 3 and 4, consider $\lim_{x \to \infty} \frac{2x + 5}{x}$.

3. Use the TABLE feature to find the limit.

4. Graph the function in DOT mode, and use TRACE to find the limit.

As the inputs get larger and larger without bound, the outputs get closer and closer to 3. Thus,

$$\lim_{x \to \infty} \frac{3x - 4}{x} = 3.$$

Another way to find this limit is to use some algebra and the fact that

$$\text{as } x \to \infty, \text{ we have } \frac{1}{x} \to 0, \text{ and more generally, } \frac{b}{ax^n} \to 0,$$

for any positive integer n and any constants a and b, $a \neq 0$. We multiply by 1, using $(1/x) \div (1/x)$. This amounts to dividing both the numerator and the denominator by x:

$$\begin{aligned}
\lim_{x \to \infty} \frac{3x - 4}{x} &= \lim_{x \to \infty} \frac{3x - 4}{x} \cdot \frac{(1/x)}{(1/x)} \\
&= \lim_{x \to \infty} \frac{\frac{3x}{x} - \frac{4}{x}}{\frac{x}{x}} \\
&= \lim_{x \to \infty} \frac{3 - \frac{4}{x}}{1} \\
&= \lim_{x \to \infty} \left(3 - \frac{4}{x}\right) \\
&= 3 - 0 = 3.
\end{aligned}$$

In a similar manner, it can be shown that

$$\lim_{x \to -\infty} f(x) = 3.$$

The horizontal asymptote is the line $y = 3$.

example 4

Determine the horizontal asymptote of the function given by

$$f(x) = \frac{3x^2 + 2x - 4}{2x^2 - x + 1}.$$

Solution As in Example 3, the degree of the numerator is the same as the degree of the denominator. Let's adapt the algebraic approach used in that example.

To do so, we divide the numerator and the denominator by x^2 and find the limit as $|x|$ gets larger and larger:

$$f(x) = \frac{3x^2 + 2x - 4}{2x^2 - x + 1} = \frac{3 + \frac{2}{x} - \frac{4}{x^2}}{2 - \frac{1}{x} + \frac{1}{x^2}}.$$

As $|x|$ gets very large, the numerator approaches 3 and the denominator approaches 2. Therefore, the value of the function gets very close to $\frac{3}{2}$. Thus,

$$\lim_{x \to -\infty} f(x) = \frac{3}{2} \quad \text{and} \quad \lim_{x \to \infty} f(x) = \frac{3}{2}.$$

The line $y = \frac{3}{2}$ is a horizontal asymptote.

❯ Quick Check 3

Determine the horizontal asymptote of the function given by

$$f(x) = \frac{(2x - 1)(x + 1)}{(3x + 2)(5x + 6)}.$$

❮ Quick Check 3

TECHNOLOGY CONNECTION

To see why a horizontal asymptote is determined by the leading terms in the expression's numerator and denominator, consider the terms $3x^2$, $2x$, and -4 in Example 4. If $y_1 = 3x^2$ and $y_2 = 3x^2 + 2x - 4$, a table reveals that as x gets larger, the difference between y_2 and y_1 becomes less significant. This is because $3x^2$ grows far more rapidly than does $2x - 4$.

X	Y1	Y2
20	1200	1236
120	43200	43436
220	145200	145636
320	307200	307836
420	529200	530036
520	811200	812236
620	1.15 E6	1.15 E6

X = 20

EXERCISES

1. Let $y_1 = 2x^2$ and $y_2 = 2x^2 - x + 1$. Use a table to show that as x gets large, $y_1 \approx y_2$.

2. Let $y_1 = (3x^2)/(2x^2)$ and $y_2 = (3x^2 + 2x - 1)/(2x^2 - x + 1)$. Show that for large x, we have $y_1 \approx y_2$.

TECHNOLOGY CONNECTION

EXERCISES

Graph each of the following. Try to locate the horizontal asymptotes using the TABLE and TRACE features. Verify your results using the methods of Examples 3–5.

1. $f(x) = \dfrac{x^2 + 5}{x^3 - x^2 - 6x}$

2. $f(x) = \dfrac{9x^4 - 7x^2 - 9}{3x^4 + 7x^2 + 9}$

3. $f(x) = \dfrac{135x^5 - x^2}{x^7}$

4. $f(x) = \dfrac{3x^2 - 4x + 3}{6x^2 + 2x - 5}$

Examples 3 and 4 lead to the following result.

When the degree of the numerator is the same as the degree of the denominator, the line $y = a/b$ is a horizontal asymptote, where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator.

example 5 Determine the horizontal asymptote:

$$f(x) = \frac{2x + 3}{x^3 - 2x^2 + 4}.$$

Solution Since the degree of the numerator is less than the degree of the denominator, there is a horizontal asymptote. To identify that asymptote, we divide both the numerator and denominator by the highest power of x in the denominator, just as in Examples 3 and 4, and find the limits as $|x| \to \infty$:

$$f(x) = \frac{2x + 3}{x^3 - 2x^2 + 4} = \frac{\dfrac{2}{x^2} + \dfrac{3}{x^3}}{1 - \dfrac{2}{x} + \dfrac{4}{x^3}}.$$

As x gets smaller and smaller negatively, $|x|$ gets larger and larger. Similarly, as x gets larger and larger positively, $|x|$ gets larger and larger. Thus, as $|x|$ becomes very large, every expression with a denominator that is a power of x gets ever closer to 0. Thus, the numerator of $f(x)$ approaches 0 as its denominator approaches 1; hence, the entire expression takes on values ever closer to 0. That is, for $x \to -\infty$ or $x \to \infty$, we have

$$f(x) \approx \frac{0 + 0}{1 - 0 + 0},$$

so $$\lim_{x\to-\infty} f(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} f(x) = 0,$$

and the x-axis, the line $y = 0$, is a horizontal asymptote.

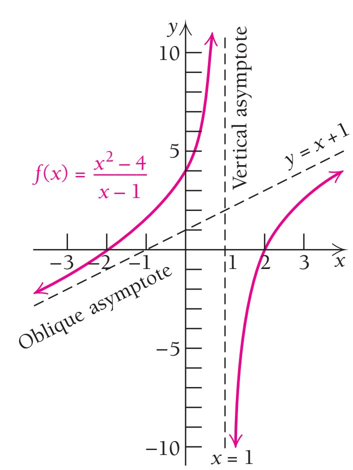
When the degree of the numerator is less than the degree of the denominator, the x-axis, or the line $y = 0$, is a horizontal asymptote.

Slant Asymptotes

Some asymptotes are neither vertical nor horizontal. For example, in the graph of

$$f(x) = \frac{x^2 - 4}{x - 1},$$

shown at right, as $|x|$ gets larger and larger, the curve gets closer and closer to $y = x + 1$. The line $y = x + 1$ is called a *slant asymptote*, or *oblique asymptote*. In Example 6, we will see how the line $y = x + 1$ was determined.



DEFINITION

A linear asymptote that is neither vertical nor horizontal is called a **slant asymptote**, or an **oblique asymptote.** For any rational function of the form $f(x) = p(x)/q(x)$, a slant asymptote occurs when the degree of $p(x)$ is exactly 1 more than the degree of $q(x)$. A graph can cross a slant asymptote.

How can we find a slant asymptote? One way is by division.

TECHNOLOGY CONNECTION

EXERCISES

Graph each of the following. Try to visually locate the slant asymptotes. Then use the method in Example 6 to find each slant asymptote and graph it along with the original function.

1. $f(x) = \dfrac{3x^2 - 7x + 8}{x - 2}$

2. $f(x) = \dfrac{5x^3 + 2x + 1}{x^2 - 4}$

example 6 Find the slant asymptote:

$$f(x) = \frac{x^2 - 4}{x - 1}.$$

Solution When we divide the numerator by the denominator, we obtain a quotient of $x + 1$ and a remainder of -3:

$$\begin{array}{r|l} & x + 1 \\ \hline x - 1 & x^2 \qquad - 4 \\ & \underline{x^2 - x} \\ & x - 4 \\ & \underline{x - 1} \\ & -3 \end{array}$$

$$f(x) = \frac{x^2 - 4}{x - 1} = (x + 1) + \frac{-3}{x - 1}.$$

Now we can see that when $|x|$ gets very large, $-3/(x - 1)$ approaches 0. Thus, for very large $|x|$, the expression $x + 1$ is the dominant part of

$$(x + 1) + \frac{-3}{x - 1}.$$

Thus, $y = x + 1$ is a slant asymptote.

❮ Quick Check 4

❯ Quick Check 4

Find the slant asymptote:

$$g(x) = \frac{2x^2 + x - 1}{x - 3}.$$

Intercepts

If they exist, the **x-intercepts** of a function occur at those values of x for which $y = f(x) = 0$, and they give us points at which the graph crosses the x-axis. If it exists, the **y-intercept** of a function occurs at the value of y for which $x = 0$, and it gives us the point at which the graph crosses the y-axis.

example 7 Find the intercepts of the function given by

$$f(x) = \frac{x^3 - x^2 - 6x}{x^2 - 3x + 2}.$$

Solution We factor the numerator and the denominator:

$$f(x) = \frac{x(x + 2)(x - 3)}{(x - 1)(x - 2)}.$$

To find the x-intercepts, we solve the equation $f(x) = 0$. Such values occur when the numerator is 0 and the denominator is not. Thus, we solve the equation

$$x(x + 2)(x - 3) = 0.$$

The x-values that make the numerator 0 are 0, -2, and 3. Since none of these make the denominator 0, they yield the x-intercepts $(0, 0)$, $(-2, 0)$, and $(3, 0)$.

TECHNOLOGY CONNECTION

EXERCISES

Graph each of the following. Use the ZERO feature and a table in ASK mode to find the x- and y-intercepts.

1. $f(x) = \dfrac{x(x - 3)(x + 5)}{(x + 2)(x - 4)}$

2. $f(x) = \dfrac{x^3 + 2x^2 - 3x}{x^2 + 5}$

To find the y-intercept, we let $x = 0$:

$$f(0) = \frac{0^3 - 0^2 - 6(0)}{0^2 - 3(0) + 2} = 0.$$

In this case, the y-intercept is also an x-intercept, $(0, 0)$.

❮ Quick Check 5

❯ **Quick Check 5**

Find the intercepts of the function given by

$$h(x) = \frac{x^3 - x}{x^2 - 4}.$$

Sketching Graphs

We can now refine our analytic strategy for graphing.

Strategy for Sketching Graphs

a) *Intercepts.* Find the x-intercept(s) and the y-intercept of the graph.
b) *Asymptotes.* Find any vertical, horizontal, or slant asymptotes.
c) *Derivatives and domain.* Find $f'(x)$ and $f''(x)$. Find the domain of f.
d) *Critical values of f.* Find any inputs for which $f'(x)$ is not defined or for which $f'(x) = 0$.
e) *Increasing and/or decreasing; relative extrema.* Substitute each critical value, x_0, from step (d) into $f''(x)$. If $f''(x_0) < 0$, then x_0 yields a relative maximum and f is increasing to the left of x_0 and decreasing to the right. If $f''(x_0) > 0$, then x_0 yields a relative minimum and f is decreasing to the left of x_0 and increasing to the right. On intervals where no critical value exists, use f' and test values to find where f is increasing or decreasing.
f) *Inflection points.* Determine candidates for inflection points by finding x-values for which $f''(x)$ does not exist or for which $f''(x) = 0$. Find the function values at these points. If a function value $f(x)$ does not exist, then the function does not have an inflection point at x.
g) *Concavity.* Use the values from step (f) as endpoints of intervals. Determine the concavity over each interval by checking to see where f' is increasing—that is, where $f''(x) > 0$—and where f' is decreasing—that is, where $f''(x) < 0$. Do this by substituting a test value from each interval into $f''(x)$. Use the results of step (d).
h) *Sketch the graph.* Use the information from steps (a)–(g) to sketch the graph, plotting extra points as needed.

example 8 Sketch the graph of $f(x) = \dfrac{8}{x^2 - 4}$.

Solution

a) *Intercepts.* The x-intercepts occur at values for which the numerator is 0 but the denominator is not. Since in this case the numerator is the constant 8, there are no x-intercepts. To find the y-intercept, we compute $f(0)$:

$$f(0) = \frac{8}{0^2 - 4} = \frac{8}{-4} = -2.$$

This gives us one point on the graph, $(0, -2)$.

b) *Asymptotes.*

Vertical: The denominator, $x^2 - 4 = (x + 2)(x - 2)$, is 0 for x-values of -2 and 2. Thus, the graph has the lines $x = -2$ and $x = 2$ as vertical asymptotes. We draw them using dashed lines (they are *not* part of the actual graph, just guidelines).

Horizontal: The degree of the numerator is less than the degree of the denominator, so the x-axis, $y = 0$, is the horizontal asymptote.

Slant: There is no slant asymptote since the degree of the numerator is not 1 more than the degree of the denominator.

c) *Derivatives and domain.* We find $f'(x)$ and $f''(x)$ using the Quotient Rule:

$$f'(x) = \frac{-16x}{(x^2 - 4)^2} \quad \text{and} \quad f''(x) = \frac{16(3x^2 + 4)}{(x^2 - 4)^3}.$$

The domain of f is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ as determined in step (b).

d) *Critical values of f.* We look for values of x for which $f'(x) = 0$ or for which $f'(x)$ does not exist. From step (c), we see that $f'(x) = 0$ for values of x for which $-16x = 0$, but the denominator is not 0. The only such number is 0 itself. The derivative $f'(x)$ does not exist at -2 and 2, but neither value is in the domain of f. Thus, the only critical value is 0.

e) *Increasing and/or decreasing; relative extrema.* We use the undefined values and the critical values to determine the intervals over which f is increasing and the intervals over which f is decreasing. The values to consider are -2, 0, and 2.

Since

$$f''(0) = \frac{16(3 \cdot 0^2 + 4)}{(0^2 - 4)^3} = \frac{64}{-64} < 0,$$

we know that a relative maximum exists at $(0, f(0))$, or $(0, -2)$. Thus, f is increasing on the interval $(-2, 0)$ and decreasing on $(0, 2)$.

Since $f''(x)$ does not exist for the x-values -2 and 2, we use $f'(x)$ and test values to see if f is increasing or decreasing on $(-\infty, -2)$ and $(2, \infty)$:

$$\text{Test } -3, \quad f'(-3) = \frac{-16(-3)}{[(-3)^2 - 4]^2} = \frac{48}{25} > 0, \text{ so } f \text{ is increasing on } (-\infty, -2);$$

$$\text{Test } 3, \quad f'(3) = \frac{-16(3)}{[(3)^2 - 4]^2} = \frac{-48}{25} < 0, \text{ so } f \text{ is decreasing on } (2, \infty).$$

f) *Inflection points.* We determine candidates for inflection points by finding where $f''(x)$ does not exist and where $f''(x) = 0$. The only values for which $f''(x)$ does not exist are where $x^2 - 4 = 0$, or -2 and 2. Neither value is in the domain of f, so we focus solely on where $f''(x) = 0$, or

$$16(3x^2 + 4) = 0.$$

Since $16(3x^2 + 4) > 0$ for all real numbers x, there are no points of inflection.

g) *Concavity.* Since no values were found in step (f), the only place where concavity could change is on either side of the vertical asymptotes, $x = -2$ and $x = 2$. To determine the concavity, we check to see where $f''(x)$ is positive or negative. The numbers -2 and 2 divide the x-axis into three intervals. We choose test values in each interval and make a substitution into f'':

$$\text{Test } -3, \quad f''(-3) = \frac{16[3(-3)^2 + 4]}{[(-3)^2 - 4]^3} > 0;$$

$$\text{Test } 0, \quad f''(0) = \frac{16[3(0)^2 + 4]}{[(0)^2 - 4]^3} < 0;$$

We already knew this from step (e).

$$\text{Test } 3, \quad f''(3) = \frac{16[3(3)^2 + 4]}{[(3)^2 - 4]^3} > 0.$$

Interval: -2, 2, x

Test Value	$x = -3$	$x = 0$	$x = 3$
Sign of $f''(x)$	$f''(-3) > 0$	$f''(0) < 0$	$f''(3) > 0$
Result	f' is increasing; f is concave up.	f' is decreasing; f is concave down.	f' is increasing; f is concave up.

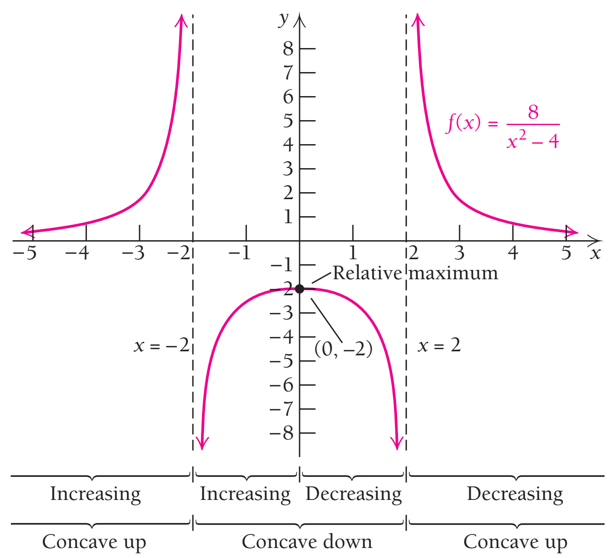
Change does not indicate a point of inflection since $f(-2)$ does not exist.

Change does not indicate a point of inflection since $f(2)$ does not exist.

The function is concave up over the intervals $(-\infty, -2)$ and $(2, \infty)$. The function is concave down over the interval $(-2, 2)$.

h) *Sketch the graph.* We sketch the graph using the information in the following table, plotting extra points as needed. The graph is shown below.

x	$f(x)$ approximately
-5	0.38
-4	0.67
-3	1.6
-1	-2.67
0	-2
1	-2.67
3	1.6
4	0.67
5	0.38



Exampl E 9

Sketch the graph of the function given by $f(x) = \dfrac{x^2 + 4}{x}$.

Solution

a) *Intercepts.* The equation $f(x) = 0$ has no real-number solution. Thus, there are no x-intercepts. The number 0 is not in the domain of the function. Thus, there is no y-intercept.

b) *Asymptotes.*

Vertical: Since replacing x with 0 makes the denominator 0, the line $x = 0$ is a vertical asymptote.

Horizontal: The degree of the numerator is greater than the degree of the denominator, so there is no horizontal asymptote.

Slant: The degree of the numerator is 1 greater than the degree of the denominator, so there is a slant asymptote. We do the division

$$\begin{array}{r} x \\ x\overline{)x^2 + 4} \\ \underline{x^2 \quad\;\;} \\ 4 \end{array}$$

and express the function in the form

$$f(x) = x + \frac{4}{x}.$$

As $|x|$ gets larger, $4/x$ approaches 0, so the line $y = x$ is a slant asymptote.

c) *Derivatives and domain.* We find $f'(x)$ and $f''(x)$:

$$f'(x) = 1 - 4x^{-2} = 1 - \frac{4}{x^2};$$

$$f''(x) = 8x^{-3} = \frac{8}{x^3}.$$

The domain of f is $(-\infty, 0) \cup (0, \infty)$, or all real numbers except 0.

d) *Critical values of f.* We see from step (c) that $f'(x)$ is undefined at $x = 0$, but 0 is not in the domain of f. Thus, to find critical values, we solve $f'(x) = 0$, looking for solutions other than 0:

$$1 - \frac{4}{x^2} = 0 \qquad \text{Setting } f'(x) \text{ equal to 0}$$

$$1 = \frac{4}{x^2}$$

$$x^2 = 4 \qquad \text{Multiplying both sides by } x^2$$

$$x = \pm 2.$$

Thus, -2 and 2 are critical values.

e) *Increasing and/or decreasing; relative extrema.* We use the points found in step (d) to find intervals over which f is increasing and intervals over which f is decreasing. The points to consider are -2, 0, and 2.

Since

$$f''(-2) = \frac{8}{(-2)^3} = -1 < 0,$$

we know that a relative maximum exists at $(-2, f(-2))$, or $(-2, -4)$. Thus, f is increasing on $(-\infty, -2)$ and decreasing on $(-2, 0)$.

Since

$$f''(2) = \frac{8}{(2)^3} = 1 > 0,$$

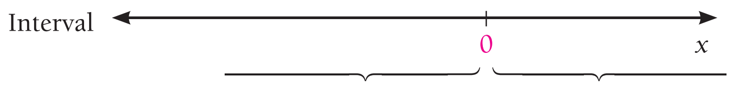
we know that a relative minimum exists at $(2, f(2))$, or $(2, 4)$. Thus, f is decreasing on $(0, 2)$ and increasing on $(2, \infty)$.

f) *Inflection points.* We determine candidates for inflection points by finding where $f''(x)$ does not exist or where $f''(x) = 0$. The only value for which $f''(x)$ does not exist is 0, but 0 is not in the domain of f. Thus, the only place an inflection point could occur is where $f''(x) = 0$:

$$\frac{8}{x^3} = 0.$$

But this equation has no solution. Thus, there are no points of inflection.

g) *Concavity*. Since no values were found in step (f), the only place where concavity could change would be on either side of the vertical asymptote $x = 0$. In step (e), we used the Second-Derivative Test to determine relative extrema. From that work, we know that f is concave down over the interval $(-\infty, 0)$ and concave up over $(0, \infty)$.

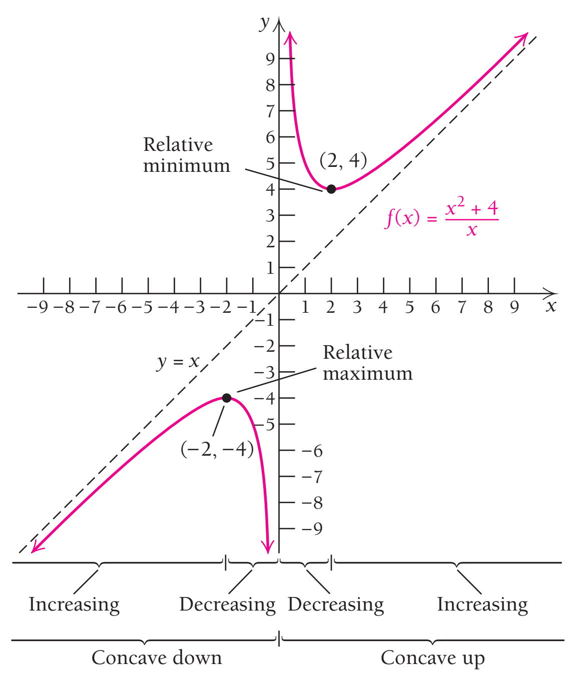


Test Value	$x = -2$	$x = 2$
Sign of $f''(x)$	$f''(-2) < 0$	$f''(2) > 0$
Result	f' is decreasing; f is concave down.	f' is increasing; f is concave up.

Change does not indicate a point of inflection since $f(0)$ does not exist.

x	$f(x)$, approximately
-6	-6.67
-5	-5.8
-4	-5
-3	-4.3
-2	-4
-1	-5
-0.5	-8.5
0.5	8.5
1	5
2	4
3	4.3
4	5
5	5.8
6	6.67

h) *Sketch the graph*. We sketch the graph using the preceding information and additional computed values of f, as needed. The graph follows.



❯ Quick Check 6

Sketch the graph of the function given by

$$f(x) = \frac{x^2 - 9}{x - 1}.$$

❮ Quick Check 6

We can apply our analytic strategy for graphing to "building" a rational function that meets certain initial conditions.

Exampl E 10 Determine a rational function f (in lowest terms) whose graph has vertical asymptotes at $x = -5$ and $x = 2$ and a horizontal asymptote at $y = 2$ and for which $f(1) = 3$.

Solution We know that the graph of f has vertical asymptotes at $x = -5$ and $x = 2$, so we can conclude that the denominator must contain the factors $x + 5$ and $x - 2$. Writing these as a product, we see that the denominator will have degree 2 (although we do not carry out the multiplication). Since the graph has a horizontal asymptote, the function must have a polynomial of degree 2 in the numerator, and the leading coefficients must form a ratio of 2. Therefore, a reasonable first guess for f is given by

$$f(x) = \frac{2x^2}{(x+5)(x-2)}. \quad \text{This is a first guess.}$$

However, this does not satisfy the requirement that $f(1) = 3$. We can add a constant in the numerator and then use the fact that $f(1) = 3$ to solve for this constant:

$$f(x) = \frac{2x^2 + B}{(x+5)(x-2)}$$

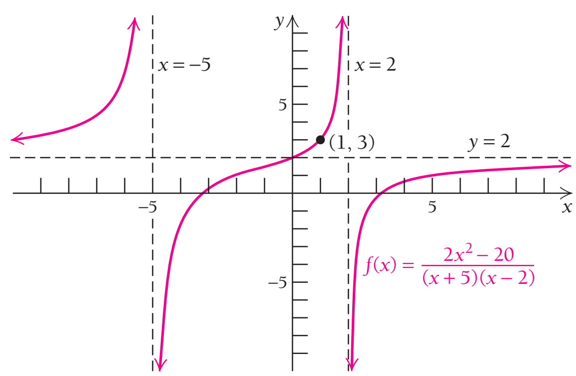
$$3 = \frac{2(1)^2 + B}{(1+5)(1-2)} \quad \text{Setting } x = 1 \text{ and } f(x) = 3$$

$$3 = \frac{2 + B}{-6}$$

Multiplying both sides by -6, we have $-18 = 2 + B$. Therefore, $B = -20$. The rational function is given by

$$f(x) = \frac{2x^2 - 20}{(x+5)(x-2)}.$$

A sketch of its graph serves as a visual check that all of the initial conditions are met.



› Quick Check 7

Determine a rational function g that has vertical asymptotes at $x = -2$ and $x = 2$ and a horizontal asymptote at $y = 3$, and for which $g(1) = -3$.

‹ Quick Check 7

Section Summary

- A line $x = a$ is a *vertical asymptote* if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.
- A line $y = b$ is a *horizontal asymptote* if $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.
- A graph may cross a horizontal asymptote but never a vertical asymptote.
- A *slant asymptote* occurs when the degree of the numerator is 1 greater than the degree of the denominator. Long division of polynomials can be used to determine the equation of the slant asymptote.
- Vertical, horizontal, and slant asymptotes can be used as guides for accurate curve sketching. Asymptotes are not a part of a graph but are visual guides only.

EXERCISE SET 2.3

Determine the vertical asymptote(s) of each function. If none exists, state that fact.

1. $f(x) = \frac{2x - 3}{x - 5}$
2. $f(x) = \frac{x + 4}{x - 2}$
3. $f(x) = \frac{3x}{x^2 - 9}$
4. $f(x) = \frac{5x}{x^2 - 25}$
5. $f(x) = \frac{x + 2}{x^3 - 6x^2 + 8x}$
6. $f(x) = \frac{x + 3}{x^3 - x}$
7. $f(x) = \frac{x + 6}{x^2 + 7x + 6}$
8. $f(x) = \frac{x + 2}{x^2 + 6x + 8}$
9. $f(x) = \frac{6}{x^2 + 36}$
10. $f(x) = \frac{7}{x^2 + 49}$

Determine the horizontal asymptote of each function. If none exists, state that fact.

11. $f(x) = \frac{6x}{8x + 3}$
12. $f(x) = \frac{3x^2}{6x^2 + x}$
13. $f(x) = \frac{4x}{x^2 - 3x}$
14. $f(x) = \frac{2x}{3x^3 - x^2}$
15. $f(x) = 5 - \frac{3}{x}$
16. $f(x) = 4 + \frac{2}{x}$
17. $f(x) = \frac{8x^4 - 5x^2}{2x^3 + x^2}$
18. $f(x) = \frac{6x^3 + 4x}{3x^2 - x}$
19. $f(x) = \frac{6x^4 + 4x^2 - 7}{2x^5 - x + 3}$
20. $f(x) = \frac{4x^3 - 3x + 2}{x^3 + 2x - 4}$
21. $f(x) = \frac{2x^3 - 4x + 1}{4x^3 + 2x - 3}$
22. $f(x) = \frac{5x^4 - 2x^3 + x}{x^5 - x^3 + 8}$

Sketch the graph of each function. Indicate where each function is increasing or decreasing, where any relative extrema occur, where asymptotes occur, where the graph is concave up or concave down, where any points of inflection occur, and where any intercepts occur.

23. $f(x) = -\frac{5}{x}$
24. $f(x) = \frac{4}{x}$
25. $f(x) = \frac{1}{x - 5}$
26. $f(x) = \frac{-2}{x - 5}$
27. $f(x) = \frac{1}{x + 2}$
28. $f(x) = \frac{1}{x - 3}$
29. $f(x) = \frac{-3}{x - 3}$
30. $f(x) = \frac{-2}{x + 5}$
31. $f(x) = \frac{3x - 1}{x}$
32. $f(x) = \frac{2x + 1}{x}$
33. $f(x) = x + \frac{2}{x}$
34. $f(x) = x + \frac{9}{x}$
35. $f(x) = \frac{-1}{x^2}$
36. $f(x) = \frac{2}{x^2}$
37. $f(x) = \frac{x}{x + 2}$
38. $f(x) = \frac{x}{x - 3}$
39. $f(x) = \frac{-1}{x^2 + 2}$
40. $f(x) = \frac{1}{x^2 + 3}$
41. $f(x) = \frac{x + 3}{x^2 - 9}$ (*Hint*: Simplify.)
42. $f(x) = \frac{x - 1}{x^2 - 1}$
43. $f(x) = \frac{x - 1}{x + 2}$
44. $f(x) = \frac{x - 2}{x + 1}$
45. $f(x) = \frac{x^2 - 4}{x + 3}$
46. $f(x) = \frac{x^2 - 9}{x + 1}$
47. $f(x) = \frac{x + 1}{x^2 - 2x - 3}$
48. $f(x) = \frac{x - 3}{x^2 + 2x - 15}$
49. $f(x) = \frac{2x^2}{x^2 - 16}$
50. $f(x) = \frac{x^2 + x - 2}{2x^2 - 2}$
51. $f(x) = \frac{1}{x^2 - 1}$
52. $f(x) = \frac{10}{x^2 + 4}$
53. $f(x) = \frac{x^2 + 1}{x}$
54. $f(x) = \frac{x^3}{x^2 - 1}$
55. $f(x) = \frac{x^2 - 9}{x - 3}$
56. $f(x) = \frac{x^2 - 16}{x + 4}$

In Exercises 57–62, determine a rational function that meets the given conditions, and sketch its graph.

57. The function f has a vertical asymptote at $x = 2$, a horizontal asymptote at $y = -2$, and $f(0) = 0$.
58. The function f has a vertical asymptote at $x = 0$, a horizontal asymptote at $y = 3$, and $f(1) = 2$.
59. The function g has vertical asymptotes at $x = -1$ and $x = 1$, a horizontal asymptote at $y = 1$, and $g(0) = 2$.
60. The function g has vertical asymptotes at $x = -2$ and $x = 0$, a horizontal asymptote at $y = -3$, and $g(1) = 4$.
61. The function h has vertical asymptotes at $x = -3$ and $x = 2$, a horizontal asymptote at $y = 0$, and $h(1) = 2$.
62. The function h has vertical asymptotes at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$, a horizontal asymptote at $y = 0$, and $h(0) = -3$.

APPLICATIONS

Business and Economics

63. Depreciation. Suppose that the value V of the inventory at Fido's Pet Supply decreases, or depreciates, with time t, in months, where

$$V(t) = 50 - \frac{25t^2}{(t+2)^2}.$$

a) Find $V(0)$, $V(5)$, $V(10)$, and $V(70)$.
b) Find the maximum value of the inventory over the interval $[0, \infty)$.
c) Sketch a graph of V.
d) Does there seem to be a value below which $V(t)$ will never fall? Explain.

64. Average cost. The total-cost function for Acme, Inc., to produce x units of a product is given by

$$C(x) = 3x^2 + 80.$$

a) The *average cost* is given by $A(x) = C(x)/x$. Find $A(x)$.
b) Graph the average cost.
c) Find the slant asymptote for the graph of $y = A(x)$, and interpret its significance.

65. Cost of pollution control. Cities and companies find that the cost of pollution control increases along with the percentage of pollutants to be removed in a situation. Suppose that the cost C, in dollars, of removing p% of the pollutants from a chemical spill is given by

$$C(p) = \frac{48{,}000}{100 - p}.$$

a) Find $C(0)$, $C(20)$, $C(80)$, and $C(90)$.
b) Find the domain of C.
c) Sketch a graph of C.
d) Can the company or city afford to remove 100% of the pollutants due to this spill? Explain.

66. Total cost and revenue. The total cost and total revenue, in dollars, from producing x couches are given by

$$C(x) = 5000 + 600x \quad \text{and} \quad R(x) = -\tfrac{1}{2}x^2 + 1000x.$$

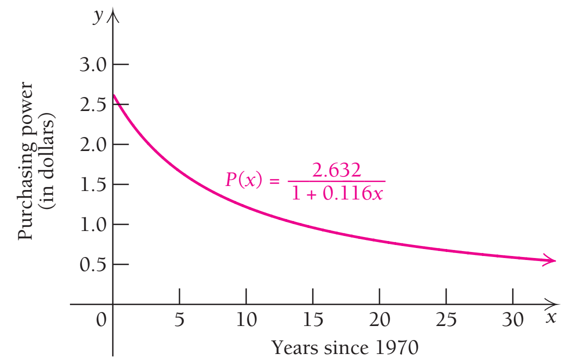
a) Find the total-profit function, $P(x)$.
b) The *average profit* is given by $A(x) = P(x)/x$. Find $A(x)$.
c) Graph the average profit.
d) Find the slant asymptote for the graph of $y = A(x)$.

67. Purchasing power. Since 1970, the purchasing power of the dollar, as measured by consumer prices, can be modeled by the function

$$P(x) = \frac{2.632}{1 + 0.116x},$$

where x is the number of years since 1970. (*Source*: U.S. Bureau of Economic Analysis.)

a) Find $P(10)$, $P(20)$, and $P(40)$.
b) When was the purchasing power \$0.50?
c) Find $\lim_{x \to \infty} P(x)$.



Life and Physical Sciences

68. Medication in the bloodstream. After an injection, the amount of a medication A, in cubic centimeters (cc), in the bloodstream decreases with time t, in hours. Suppose that under certain conditions A is given by

$$A(t) = \frac{A_0}{t^2 + 1},$$

where A_0 is the initial amount of the medication. Assume that an initial amount of 100 cc is injected.

a) Find $A(0)$, $A(1)$, $A(2)$, $A(7)$, and $A(10)$.
b) Find the maximum amount of medication in the bloodstream over the interval $[0, \infty)$.
c) Sketch a graph of the function.
d) According to this function, does the medication ever completely leave the bloodstream? Explain your answer.

General Interest

69. Baseball: earned-run average. A pitcher's *earned-run average* (the average number of runs given up every 9 innings, or 1 game) is given by

$$E = 9 \cdot \frac{r}{n},$$

where r is the number of earned runs allowed in n innings. Suppose that we fix the number of earned runs allowed at 4 and let n vary. We get a function given by

$$E(n) = 9 \cdot \frac{4}{n}.$$

a) Complete the following table, rounding to two decimal places:

Innings Pitched, n	9	6	3	1	$\frac{2}{3}$	$\frac{1}{3}$
Earned-Run Average, E						

b) The number of innings pitched n is equivalent to the number of outs that a pitcher is able to get while pitching, divided by 3. For example, if the pitcher gets just 1 out, he is credited with pitching $\frac{1}{3}$ of an inning. Find $\lim_{n\to 0} E(n)$. Under what circumstances might this limit be plausible?

c) Suppose a pitcher gives up 4 earned runs over two complete games, or 18 innings. Calculate the pitcher's earned-run average, and interpret this result.

While pitching for the St. Louis Cardinals in 1968, Bob Gibson had an earned-run average of 1.12, a record low.

SYNTHESIS

70. Explain why a vertical asymptote is only a guide and is not part of the graph of a function.

71. Using graphs and limits, explain the idea of an asymptote to the graph of a function. Describe three types of asymptotes.

Find each limit, if it exists.

72. $\lim_{x\to -\infty} \frac{-3x^2 + 5}{2 - x}$

73. $\lim_{x\to 0} \frac{|x|}{x}$

74. $\lim_{x\to -2} \frac{x^3 + 8}{x^2 - 4}$

75. $\lim_{x\to \infty} \frac{-6x^3 + 7x}{2x^2 - 3x - 10}$

76. $\lim_{x\to -\infty} \frac{-6x^3 + 7x}{2x^2 - 3x - 10}$

77. $\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1}$

78. $\lim_{x\to -\infty} \frac{7x^5 + x - 9}{6x + x^3}$

79. $\lim_{x\to -\infty} \frac{2x^4 + x}{x + 1}$

TECHNOLOGY CONNECTION

Graph each function using a calculator, iPlot, or Graphicus.

80. $f(x) = x^2 + \frac{1}{x^2}$

81. $f(x) = \frac{x}{\sqrt{x^2 + 1}}$

82. $f(x) = \frac{x^3 + 4x^2 + x - 6}{x^2 - x - 2}$

83. $f(x) = \frac{x^3 + 2x^2 - 15x}{x^2 - 5x - 14}$

84. $f(x) = \frac{x^3 + 2x^2 - 3x}{x^2 - 25}$

85. $f(x) = \left|\frac{1}{x} - 2\right|$

86. Graph the function

$$f(x) = \frac{x^2 - 3}{2x - 4}.$$

a) Find all the x-intercepts.
b) Find the y-intercept.
c) Find all the asymptotes.

87. Graph the function given by

$$f(x) = \frac{\sqrt{x^2 + 3x + 2}}{x - 3}.$$

a) Estimate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$ using the graph and input–output tables as needed to refine your estimates.
b) Describe the outputs of the function over the interval $(-2, -1)$.
c) What appears to be the domain of the function? Explain.
d) Find $\lim_{x\to -2^-} f(x)$ and $\lim_{x\to -1^+} f(x)$.

88. Not all asymptotes are linear. Use long division to find an equation for the nonlinear asymptote that is approached by the graph of

$$f(x) = \frac{x^5 + x - 9}{x^3 + 6x}.$$

Then graph the function and its asymptote.

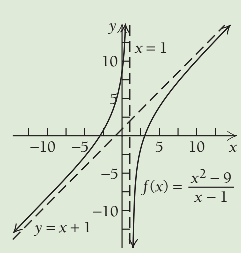
89. Refer to Fig. 1 on p. 235. The function is given by

$$f(x) = \frac{x^2 - 1}{x^2 + x - 6}.$$

a) Inspect the graph and estimate the coordinates of any extrema.
b) Find f' and use it to determine the critical values. (*Hint*: you will need the quadratic formula.) Round the x-values to the nearest hundredth.
c) Graph this function in the window $[0, 0.2, 0.16, 0.17]$. Use TRACE or MAXIMUM to confirm your results from part (b).
d) Graph this function in the window $[9.8, 10, 0.9519, 0.95195]$. Use TRACE or MINIMUM to confirm your results from part (b).
e) How close were your estimates of part (a)? Would you have been able to identify the relative minimum point without calculus techniques?

Answers to Quick Checks

1. The lines $x = 0$, $x = 4$, and $x = -4$ are vertical asymptotes.
2. The line $x = 0$ is not a vertical asymptote because $\lim_{x\to 0} f(x) = 2$. There is a deleted point discontinuity at $x = 0$.
3. The line $y = \frac{2}{15}$ is a horizontal asymptote.
4. The line $y = 2x + 7$ is a slant asymptote.
5. x-intercepts: $(1, 0)$, $(-1, 0)$, $(0, 0)$; y-intercept: $(0, 0)$
6.



7. $g(x) = \frac{3x^2 + 6}{x^2 - 4}$

2.4 Using Derivatives to Find Absolute Maximum and Minimum Values

OBJECTIVES

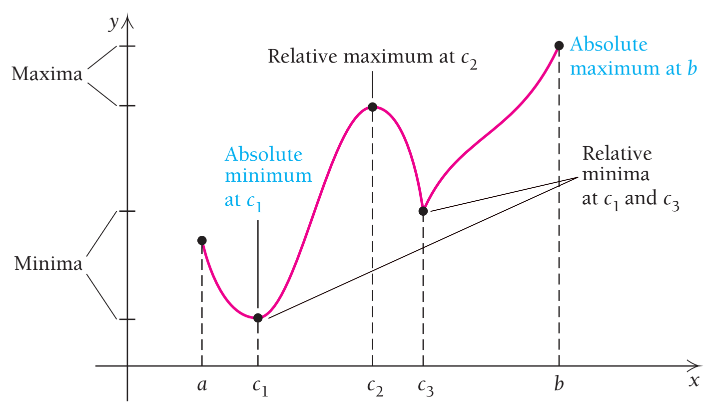
- Find absolute extrema using Maximum–Minimum Principle 1.
- Find absolute extrema using Maximum–Minimum Principle 2.

An extremum may be at the highest or lowest point for a function's entire graph, in which case it is called an *absolute extremum*. For example, the parabola given by $f(x) = x^2$ has a relative minimum at $(0, 0)$. This is also the lowest point for the *entire* graph of f, so it is also called the *absolute minimum*. Relative extrema are useful for graph sketching and understanding the behavior of a function. In many applications, however, we are more concerned with absolute extrema.

Absolute Maximum and Minimum Values

A relative minimum may or may not be an absolute minimum, meaning the smallest value of the function over its entire domain. Similarly, a relative maximum may or may not be an absolute maximum, meaning the greatest value of a function over its entire domain.

The function in the following graph has relative minima at interior points c_1 and c_3 of the closed interval $[a, b]$.



The relative minimum at c_1 is also the absolute minimum. On the other hand, the relative maximum at c_2 is *not* the absolute maximum. The absolute maximum occurs at the endpoint b.

DEFINITION

Suppose that f is a function with domain I.

$f(c)$ is an **absolute minimum** if $f(c) \leq f(x)$ for all x in I.

$f(c)$ is an **absolute maximum** if $f(c) \geq f(x)$ for all x in I.

Finding Absolute Maximum and Minimum Values over Closed Intervals

We first consider a continuous function for which the domain is a closed interval. Look at the graphs in Figs. 1 and 2 and try to determine where the absolute maxima and minima (extrema) occur for each interval.

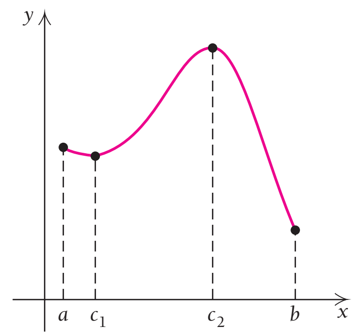


Figure 1

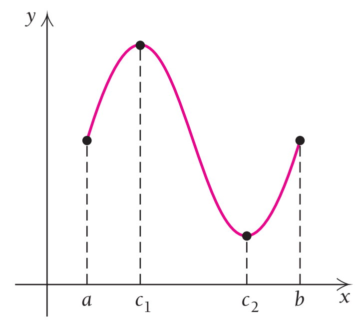


Figure 2

Note that each of the functions does indeed have an absolute maximum value and an absolute minimum value. This leads us to the following theorem.

THEOREM **The Extreme-Value Theorem**

A continuous function f defined over a closed interval $[a, b]$ must have an absolute maximum value and an absolute minimum value over $[a, b]$.

Look again at the graphs in Figs. 1 and 2 and consider the critical values and the endpoints. In Fig. 1, the graph starts at $f(a)$ and falls to $f(c_1)$. Then it rises from $f(c_1)$ to $f(c_2)$. From there, it falls to $f(b)$. In Fig. 2, the graph starts at $f(a)$ and rises to $f(c_1)$. Then it falls from $f(c_1)$ to $f(c_2)$. From there, it rises to $f(b)$. It seems reasonable that whatever the maximum and minimum values are, they occur among the function values $f(a)$, $f(c_1)$, $f(c_2)$, and $f(b)$. This leads us to a procedure for determining *absolute extrema*.

THEOREM 8 **Maximum–Minimum Principle 1**

Suppose that f is a continuous function defined over a closed interval $[a, b]$. To find the absolute maximum and minimum values over $[a, b]$:

a) First find $f'(x)$.
b) Then determine all critical values in $[a, b]$. That is, find all c in $[a, b]$ for which

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

c) List the values from step (b) and the endpoints of the interval:

$$a, c_1, c_2, \ldots, c_n, b.$$

d) Evaluate $f(x)$ for each value in step (c):

$$f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b).$$

The largest of these is the **absolute maximum of f over $[a, b]$**. The smallest of these is the **absolute minimum of f over $[a, b]$**.

A reminder: endpoints of a closed interval can be absolute extrema but *not* relative extrema.

Exampl E 1 Find the absolute maximum and minimum values of

$$f(x) = x^3 - 3x + 2$$

over the interval $\left[-2, \frac{3}{2}\right]$.

Solution Keep in mind that we are considering only the interval $\left[-2, \frac{3}{2}\right]$.

a) Find $f'(x)$: $f'(x) = 3x^2 - 3$.

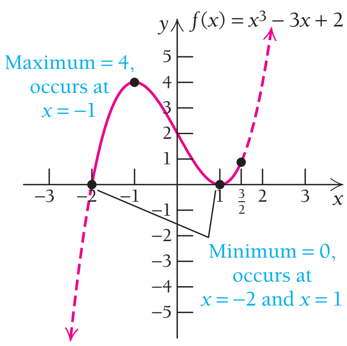
b) Find the critical values. The derivative exists for all real numbers. Thus, we merely solve $f'(x) = 0$:

$$\begin{aligned} 3x^2 - 3 &= 0 \\ 3x^2 &= 3 \\ x^2 &= 1 \\ x &= \pm 1. \end{aligned}$$

c) List the critical values and the endpoints: $-2, -1, 1$, and $\frac{3}{2}$.

d) Evaluate f for each value in step (c):

$$\begin{aligned} f(-2) &= (-2)^3 - 3(-2) + 2 = -8 + 6 + 2 = 0; \longrightarrow \text{Minimum} \\ f(-1) &= (-1)^3 - 3(-1) + 2 = -1 + 3 + 2 = 4; \longrightarrow \text{Maximum} \\ f(1) &= (1)^3 - 3(1) + 2 = 1 - 3 + 2 = 0; \longrightarrow \text{Minimum} \\ f\left(\tfrac{3}{2}\right) &= \left(\tfrac{3}{2}\right)^3 - 3\left(\tfrac{3}{2}\right) + 2 = \tfrac{27}{8} - \tfrac{9}{2} + 2 = \tfrac{7}{8} \end{aligned}$$



A visualization of Example 1

The largest of these values, 4, is the maximum. It occurs at $x = -1$. The smallest of these values is 0. It occurs twice: at $x = -2$ and $x = 1$. Thus, over the interval $\left[-2, \frac{3}{2}\right]$, the

absolute maximum $= 4$ at $x = -1$

and the

absolute minimum $= 0$ at $x = -2$ and $x = 1$.

Note that an absolute maximum or minimum value can occur at more than one point.

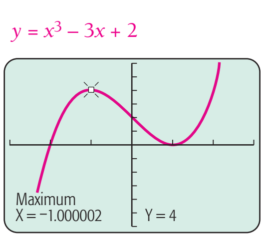
TECHNOLOGY CONNECTION

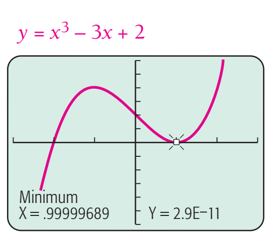
Finding Absolute Extrema

To find the absolute extrema of Example 1, we can use any of the methods described in the Technology Connection on pp. 209–210. In this case, we adapt Methods 3 and 4.

Method 3

Method 3 is selected because there are relative extrema in the interval $\left[-2, \frac{3}{2}\right]$. This method gives us approximations for the relative extrema.





Next, we check function values at these x-values and at the endpoints, using Maximum–Minimum Principle 1 to determine the absolute maximum and minimum values over $\left[-2, \frac{3}{2}\right]$.

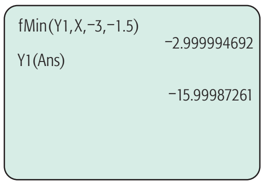
X	Y1	
−2	0	Min
−1	4	Max
1	0	Min
1.5	.875	

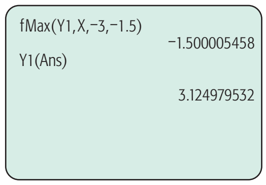
X =

Method 4

Example 2 considers the same function as in Example 1, but over a different interval. Because there are no relative extrema, we can use fMax and fMin features from the MATH menu. The minimum and maximum values occur at the endpoints, as the following graphs show.

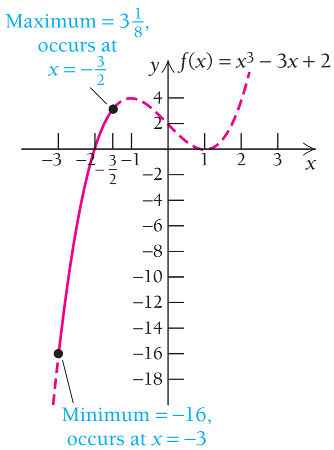
(continued)





EXERCISE

1. Use a graph to estimate the absolute maximum and minimum values of $f(x) = x^3 - x^2 - x + 2$, first over the interval $[-2, 1]$ and then over the interval $[-1, 2]$. Then check your work using the methods of Examples 1 and 2.



A visualization of Example 2

Exampl E 2

Find the absolute maximum and minimum values of

$$f(x) = x^3 - 3x + 2$$

over the interval $\left[-3, -\frac{3}{2}\right]$.

Solution As in Example 1, the derivative is 0 at -1 and 1. But neither -1 nor 1 is in the interval $\left[-3, -\frac{3}{2}\right]$, so there are no critical values in this interval. Thus, the maximum and minimum values occur at the endpoints:

$$f(-3) = (-3)^3 - 3(-3) + 2$$
$$= -27 + 9 + 2 = -16; \longrightarrow \text{Minimum}$$
$$f\left(-\frac{3}{2}\right) = \left(-\frac{3}{2}\right)^3 - 3\left(-\frac{3}{2}\right) + 2$$
$$= -\frac{27}{8} + \frac{9}{2} + 2 = \frac{25}{8} = 3\frac{1}{8}. \longrightarrow \text{Maximum}$$

Thus, the absolute maximum over the interval $\left[-3, -\frac{3}{2}\right]$, is $3\frac{1}{8}$, which occurs at $x = -\frac{3}{2}$, and the absolute minimum over $\left[-3, -\frac{3}{2}\right]$ is -16, which occurs at $x = -3$.

Quick Check 1

Find the absolute maximum and minimum values of the function given in Example 2 over the interval $[0, 3]$.

❮ Quick Check 1

Finding Absolute Maximum and Minimum Values over Other Intervals

When there is only one critical value c in I, we may not need to check endpoint values to determine whether the function has an absolute maximum or minimum value at that point.

THEOREM 9 **Maximum–Minimum Principle 2**

Suppose that f is a function such that $f'(x)$ exists for every x in an interval I and that there is *exactly one* (critical) value c in I, for which $f'(c) = 0$. Then

$f(c)$ is the absolute maximum value over I if $f''(c) < 0$

or

$f(c)$ is the absolute minimum value over I if $f''(c) > 0$.

Theorem 9 holds no matter what the interval I is—whether open, closed, or infinite in length. If $f''(c) = 0$, either we must use Maximum–Minimum Principle 1 or we must know more about the behavior of the function over the given interval.

TECHNOLOGY CONNECTION

Finding Absolute Extrema

Let's do Example 3 graphically, by adapting Methods 1 and 2 of the Technology Connection on pp. 209–210. Strictly speaking, we cannot use the fMin or fMax options of the MATH menu or the MAXIMUM or MINIMUM options from the CALC menu since we do not have a closed interval.

Methods 1 and 2

We create a graph, examine its shape, and use TRACE and/or TABLE. This procedure leads us to see that there is indeed no absolute minimum. We do find an absolute maximum: $f(x) = 4$ at $x = 2$.

EXERCISE

1. Use a graph to estimate the absolute maximum and minimum values of $f(x) = x^2 - 4x$. Then check your work using the method of Example 3.

Exampl E 3

Find the absolute maximum and minimum values of

$$f(x) = 4x - x^2.$$

Solution When no interval is specified, we consider the entire domain of the function. In this case, the domain is the set of all real numbers.

a) Find $f'(x)$:

$$f'(x) = 4 - 2x.$$

b) Find the critical values. The derivative exists for all real numbers. Thus, we merely solve $f'(x) = 0$:

$$4 - 2x = 0$$
$$-2x = -4$$
$$x = 2.$$

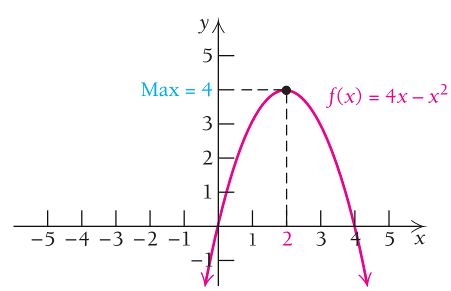
c) Since there is only one critical value, we can apply Maximum–Minimum Principle 2 using the second derivative:

$$f''(x) = -2.$$

The second derivative is constant. Thus, $f''(2) = -2$, and since this is negative, we have the absolute maximum:

$$f(2) = 4 \cdot 2 - 2^2,$$
$$= 8 - 4 = 4 \text{ at } x = 2.$$

The function has no minimum, as the graph, shown below, indicates.



Exampl E 4

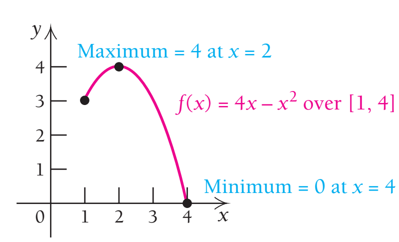
Find the absolute maximum and minimum values of $f(x) = 4x - x^2$ over the interval $[1, 4]$.

Solution By the reasoning in Example 3, we know that the absolute maximum of f on $(-\infty, \infty)$ is $f(2)$, or 4. Since 2 is in the interval $[1, 4]$, we know that the absolute maximum of f over $[1, 4]$ will occur at 2. To find the absolute minimum, we need to check the endpoints:

$$f(1) = 4 \cdot 1 - 1^2 = 3$$

and

$$f(4) = 4 \cdot 4 - 4^2 = 0.$$



We see from the graph that the minimum is 0. It occurs at $x = 4$. Thus, the

absolute maximum $= 4$ at $x = 2$,

and the

absolute minimum $= 0$ at $x = 4$.

❮ Quick Check 2

> **❯ Quick Check 2**
>
> Find the absolute maximum and minimum values of $f(x) = x^2 - 10x$ over each interval:
>
> **a)** $[0, 6]$; **b)** $[4, 10]$.

A Strategy for Finding Absolute Maximum and Minimum Values

The following general strategy can be used when finding absolute maximum and minimum values of continuous functions.

A Strategy for Finding Absolute Maximum and Minimum Values

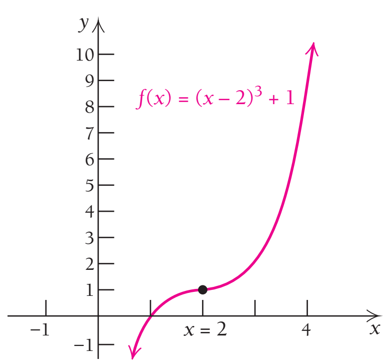
To find absolute maximum and minimum values of a continuous function over an interval:

a) Find $f'(x)$.
b) Find the critical values.
c) If the interval is closed and there is more than one critical value, use Maximum–Minimum Principle 1.
d) If the interval is closed and there is exactly one critical value, use either Maximum–Minimum Principle 1 or Maximum–Minimum Principle 2. If it is easy to find $f''(x)$, use Maximum–Minimum Principle 2.
e) If the interval is not closed, such as $(-\infty, \infty)$, $(0, \infty)$, or (a, b), and the function has only one critical value, use Maximum–Minimum Principle 2. In such a case, if the function has a maximum, it will have no minimum; and if it has a minimum, it will have no maximum.

Finding absolute maximum and minimum values when more than one critical value occurs in an interval that is not closed, such as any of those listed in step (e) above, requires a detailed graph or techniques beyond the scope of this book.

Exampl E 5 Find the absolute maximum and minimum values of

$$f(x) = (x - 2)^3 + 1.$$



A visualization of Example 5

Solution

a) Find $f'(x)$.

$$f'(x) = 3(x - 2)^2.$$

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $f'(x) = 0$:

$$3(x - 2)^2 = 0$$
$$(x - 2)^2 = 0$$
$$x - 2 = 0$$
$$x = 2.$$

c) Since there is only one critical value and there are no endpoints, we can try to apply Maximum–Minimum Principle 2 using the second derivative:

$$f''(x) = 6(x - 2).$$

We have

$$f''(2) = 6(2 - 2) = 0,$$

so Maximum–Minimum Principle 2 does not apply. We cannot use Maximum–Minimum Principle 1 because there are no endpoints. But note that $f'(x) = 3(x - 2)^2$ is never negative. Thus, $f(x)$ is increasing everywhere except at $x = 2$, so there is no maximum and no minimum. For $x < 2$, say $x = 1$, we have $f''(1) = -6 < 0$. For $x > 2$, say $x = 3$, we have $f''(3) = 6 > 0$. Thus, at $x = 2$, the function has a *point of inflection*.

Quick Check 3

Let $f(x) = x^n$, where n is a positive odd integer. Explain why functions of this form never have an absolute minimum or maximum.

Quick Check 3

example 6 Find the absolute maximum and minimum values of

$$f(x) = 5x + \frac{35}{x}$$

over the interval $(0, \infty)$.

Solution

a) Find $f'(x)$. We first express $f(x)$ as

$$f(x) = 5x + 35x^{-1}.$$

Then

$$f'(x) = 5 - 35x^{-2}$$
$$= 5 - \frac{35}{x^2}.$$

b) Find the critical values. Since $f'(x)$ exists for all values of x in $(0, \infty)$, the only critical values are those for which $f'(x) = 0$:

$$5 - \frac{35}{x^2} = 0$$
$$5 = \frac{35}{x^2}$$
$$5x^2 = 35 \quad \text{Multiplying both sides by } x^2\text{, since } x \neq 0$$
$$x^2 = 7$$
$$x = \pm\sqrt{7} \approx \pm 2.646.$$

c) The interval is not closed and is $(0, \infty)$. The only critical value is $\sqrt{7}$. Therefore, we can apply Maximum–Minimum Principle 2 using the second derivative,

$$f''(x) = 70x^{-3} = \frac{70}{x^3},$$

to determine whether we have a maximum or a minimum. Since

$$f''(\sqrt{7}) = \frac{70}{(\sqrt{7})^3} > 0,$$

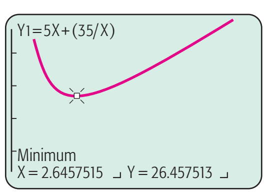
an absolute minimum occurs at $x = \sqrt{7}$:

$$\text{Absolute minimum} = f(\sqrt{7})$$
$$= 5 \cdot \sqrt{7} + \frac{35}{\sqrt{7}}$$
$$= 5\sqrt{7} + \frac{35}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}$$
$$= 5\sqrt{7} + \frac{35\sqrt{7}}{7}$$
$$= 5\sqrt{7} + 5\sqrt{7}$$
$$= 10\sqrt{7} \approx 26.458 \quad \text{at } x = \sqrt{7}.$$

TECHNOLOGY CONNECTION

Finding Absolute Extrema

Let's do Example 6 using `MAXIMUM` and `MINIMUM` from the `CALC` menu. The shape of the graph leads us to see that there is no absolute maximum, but there is an absolute minimum.



[0, 10, 0, 50]

Note that

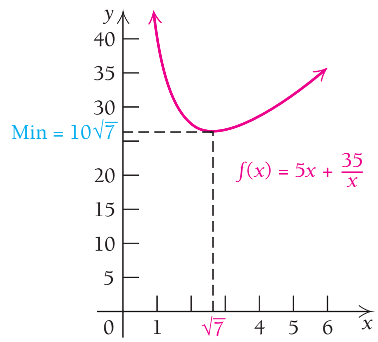
$$\sqrt{7} \approx 2.6458 \quad \text{and} \quad 10\sqrt{7} \approx 26.458,$$

which confirms the analytic solution.

EXERCISE

1. Use a graph to estimate the absolute maximum and minimum values of $f(x) = 10x + 1/x$ over the interval $(0, \infty)$. Then check your work using the analytic method of Example 6.

The function has no maximum value, which can happen since the interval $(0, \infty)$ is *not* closed.



❯ Quick Check 4

Find the absolute maximum and minimum values of

$$g(x) = \frac{2x^2 + 18}{x}$$

over the interval $(0, \infty)$.

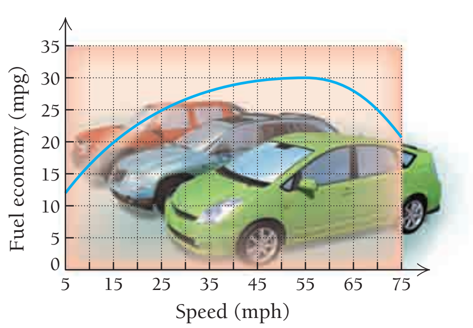
❮ Quick Check 4

Section Summary

- An *absolute minimum* of a function f is a value $f(c)$ such that $f(c) \leq f(x)$ for all x in the domain of f.
- An *absolute maximum* of a function f is a value $f(c)$ such that $f(c) \geq f(x)$ for all x in the domain of f.
- If the domain of f is a closed interval and f is continuous over that domain, then the *Extreme–Value Theorem* guarantees the existence of both an absolute minimum and an absolute maximum.
- Endpoints of a closed interval may be absolute extrema, but not relative extrema.
- If there is exactly one critical value c such that $f'(c) = 0$ in the domain of f, then *Maximum–Minimum Principle 2* may be used. Otherwise, *Maximum–Minimum Principle 1* has to be used.

EXERCISE SET 2.4

1. Fuel economy. According to the U.S. Department of Energy, a vehicle's fuel economy, in miles per gallon (mpg), decreases rapidly for speeds over 60 mph.



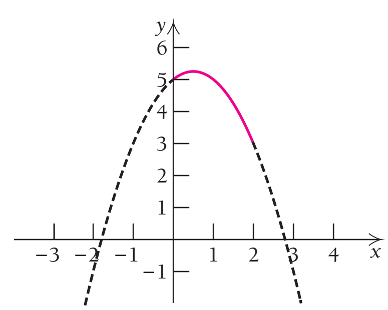
(*Sources*: U.S. Dept. of Energy and a study by West, B.H., McGill, R.N., Hodgson, J.W., Sluder, S.S., and Smith, D.E., Oak Ridge National Laboratory, 1999.)

a) Estimate the speed at which the absolute maximum gasoline mileage is obtained.
b) Estimate the speed at which the absolute minimum gasoline mileage is obtained.
c) What is the mileage obtained at 70 mph?

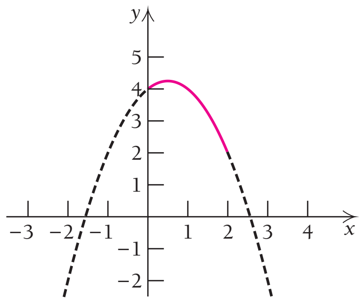
2. Fuel economy. Using the graph in Exercise 1, estimate the absolute maximum and the absolute minimum fuel economy over the interval $[30, 70]$.

Find the absolute maximum and minimum values of each function over the indicated interval, and indicate the x-values at which they occur.

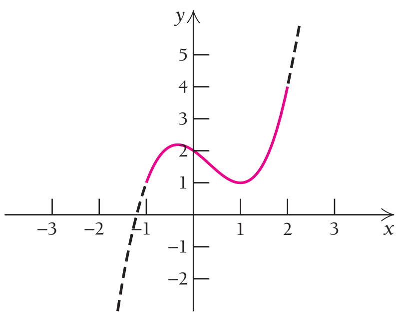
3. $f(x) = 5 + x - x^2$; $[0, 2]$



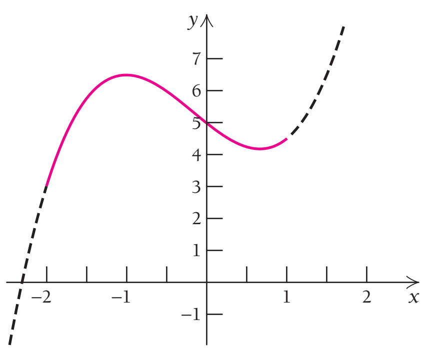
4. $f(x) = 4 + x - x^2$; $[0, 2]$



5. $f(x) = x^3 - x^2 - x + 2$; $[-1, 2]$



6. $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 5$; $[-2, 1]$



7. $f(x) = x^3 - x^2 - x + 3$; $[-1, 0]$

8. $f(x) = x^3 + \frac{1}{2}x^2 - 2x + 4$; $[-2, 0]$

9. $f(x) = 5x - 7$; $[-2, 3]$

10. $f(x) = 2x + 4$; $[-1, 1]$

11. $f(x) = 7 - 4x$; $[-2, 5]$

12. $f(x) = -2 - 3x$; $[-10, 10]$

13. $f(x) = -5$; $[-1, 1]$

14. $g(x) = 24$; $[4, 13]$

15. $f(x) = x^2 - 6x - 3$; $[-1, 5]$

16. $f(x) = x^2 - 4x + 5$; $[-1, 3]$

17. $f(x) = 3 - 2x - 5x^2$; $[-3, 3]$

18. $f(x) = 1 + 6x - 3x^2$; $[0, 4]$

19. $f(x) = x^3 - 3x^2$; $[0, 5]$

20. $f(x) = x^3 - 3x + 6$; $[-1, 3]$

21. $f(x) = x^3 - 3x$; $[-5, 1]$

22. $f(x) = 3x^2 - 2x^3$; $[-5, 1]$

23. $f(x) = 1 - x^3$; $[-8, 8]$

24. $f(x) = 2x^3$; $[-10, 10]$

25. $f(x) = 12 + 9x - 3x^2 - x^3$; $[-3, 1]$

26. $f(x) = x^3 - 6x^2 + 10$; $[0, 4]$

27. $f(x) = x^4 - 2x^3$; $[-2, 2]$

28. $f(x) = x^3 - x^4$; $[-1, 1]$

29. $f(x) = x^4 - 2x^2 + 5$; $[-2, 2]$

30. $f(x) = x^4 - 8x^2 + 3$; $[-3, 3]$

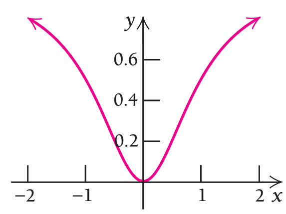
31. $f(x) = (x + 3)^{2/3} - 5$; $[-4, 5]$

32. $f(x) = 1 - x^{2/3}$; $[-8, 8]$

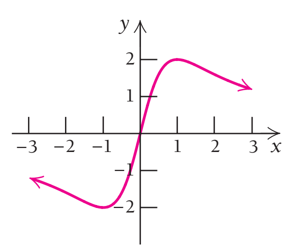
33. $f(x) = x + \dfrac{1}{x}$; $[1, 20]$

34. $f(x) = x + \dfrac{4}{x}$; $[-8, -1]$

35. $f(x) = \dfrac{x^2}{x^2 + 1}$; $[-2, 2]$



36. $f(x) = \dfrac{4x}{x^2 + 1}$, $[-3, 3]$



37. $f(x) = (x + 1)^{1/3}$; $[-2, 26]$

38. $f(x) = \sqrt[3]{x}$; $[8, 64]$

39–48. Check Exercises 3, 5, 9, 13, 19, 23, 33, 35, 37, and 38 with a graphing calculator.

Find the absolute maximum and minimum values of each function, if they exist, over the indicated interval. Also indicate the x-value at which each extremum occurs. When no interval is specified, use the real line, $(-\infty, \infty)$.

49. $f(x) = 12x - x^2$

50. $f(x) = 30x - x^2$

51. $f(x) = 2x^2 - 40x + 270$

52. $f(x) = 2x^2 - 20x + 340$

53. $f(x) = x - \frac{4}{3}x^3; \quad (0, \infty)$

54. $f(x) = 16x - \frac{4}{3}x^3; \quad (0, \infty)$

55. $f(x) = x(60 - x)$

56. $f(x) = x(25 - x)$

57. $f(x) = \frac{1}{3}x^3 - 3x; \quad [-2, 2]$

58. $f(x) = \frac{1}{3}x^3 - 5x; \quad [-3, 3]$

59. $f(x) = -0.001x^2 + 4.8x - 60$

60. $f(x) = -0.01x^2 + 1.4x - 30$

61. $f(x) = -\frac{1}{3}x^3 + 6x^2 - 11x - 50; \quad (0, 3)$

62. $f(x) = -x^3 + x^2 + 5x - 1; \quad (0, \infty)$

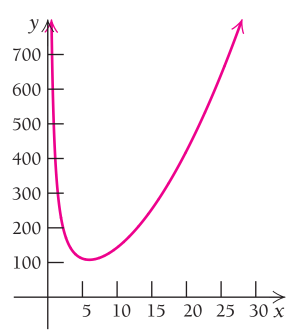
63. $f(x) = 15x^2 - \frac{1}{2}x^3; \quad [0, 30]$

64. $f(x) = 4x^2 - \frac{1}{2}x^3; \quad [0, 8]$

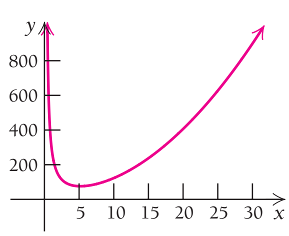
65. $f(x) = 2x + \dfrac{72}{x}; \quad (0, \infty)$

66. $f(x) = x + \dfrac{3600}{x}; \quad (0, \infty)$

67. $f(x) = x^2 + \dfrac{432}{x}; \quad (0, \infty)$



68. $f(x) = x^2 + \dfrac{250}{x}; \quad (0, \infty)$



69. $f(x) = 2x^4 - x; \quad [-1, 1]$

70. $f(x) = 2x^4 + x; \quad [-1, 1]$

71. $f(x) = \sqrt[3]{x}; \quad [0, 8]$

72. $f(x) = \sqrt{x}; \quad [0, 4]$

73. $f(x) = (x + 1)^3$

74. $f(x) = (x - 1)^3$

75. $f(x) = 2x - 3; \quad [-1, 1]$

76. $f(x) = 9 - 5x; \quad [-10, 10]$

77. $f(x) = 2x - 3; \quad [-1, 5)$

78. $f(x) = 9 - 5x; \quad [-2, 3)$

79. $f(x) = x^{2/3}; \quad [-1, 1]$

80. $g(x) = x^{2/3}$

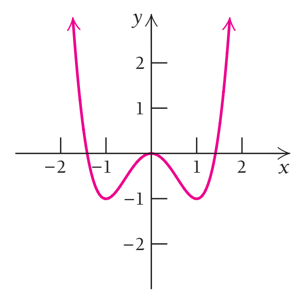
81. $f(x) = \frac{1}{3}x^3 - x + \frac{2}{3}$

82. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 1$

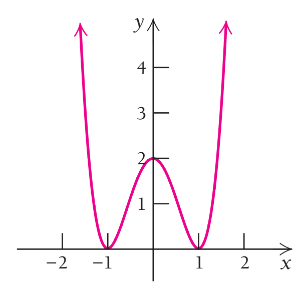
83. $f(x) = \frac{1}{3}x^3 - 2x^2 + x; \quad [0, 4]$

84. $g(x) = \frac{1}{3}x^3 + 2x^2 + x; \quad [-4, 0]$

85. $t(x) = x^4 - 2x^2$



86. $f(x) = 2x^4 - 4x^2 + 2$



87–96. Check Exercises 49, 51, 53, 57, 61, 65, 67, 69, 73, and 85 with a graphing calculator.

APPLICATIONS

Business and Economics

97. Monthly productivity. An employee's monthly productivity M, in number of units produced, is found to be a function of t, the number of years of service. For a certain product, a productivity function is given by

$$M(t) = -2t^2 + 100t + 180, \quad 0 \le t \le 40.$$

Find the maximum productivity and the year in which it is achieved.

98. **Advertising.** Sound Software estimates that it will sell N units of a program after spending a dollars on advertising, where

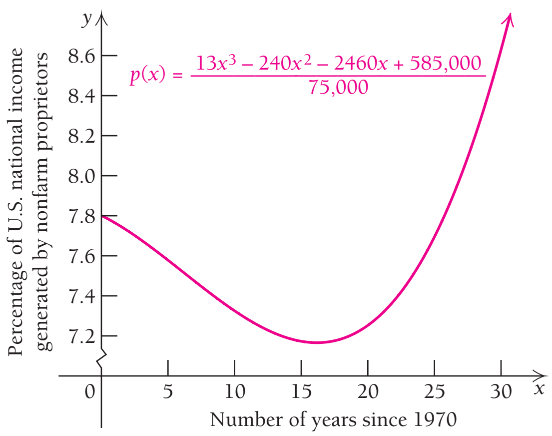
$$N(a) = -a^2 + 300a + 6, \quad 0 \le a \le 300,$$

and a is in thousands of dollars. Find the maximum number of units that can be sold and the amount that must be spent on advertising in order to achieve that maximum.

99. **Small business.** The percentage of the U.S. national income generated by nonfarm proprietors may be modeled by

$$p(x) = \frac{13x^3 - 240x^2 - 2460x + 585{,}000}{75{,}000},$$

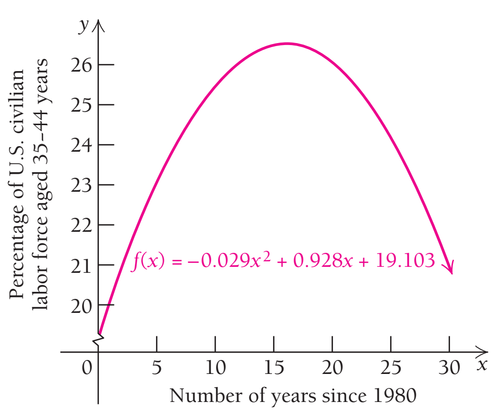
where x is the number of years since 1970. (*Source*: U.S. Census Bureau.) According to this model, in what year from 1970 through 2000 was this percentage a minimum? Calculate the answer, and then check it on the graph.



100. The percentage of the U.S. civilian labor force aged 35–44 may be modeled by

$$f(x) = -0.029x^2 + 0.928x + 19.103,$$

where x is the number of years since 1980. (*Source*: U.S. Census Bureau.) According to this model, in what year from 1980 through 2010 was this percentage a maximum? Calculate the answer, and then check it on the graph.



101. **Worldwide oil production.** One model of worldwide oil production is the function given by

$$P(t) = 0.000008533t^4 - 0.001685t^3 + 0.090t^2 - 0.687t + 4.00, \quad 0 \le t \le 90,$$

where $P(t)$ is the number of barrels, in billions, produced in a year, t years after 1950. (*Source*: *Beyond Oil*, by Kenneth S. Deffeyes, p. xii, Hill and Wang, New York, 2005.) According to this model, in what year did worldwide oil production achieve an absolute maximum? What was that maximum? (*Hint*: Do not solve $P'(t) = 0$ algebraically.)

102. **Maximizing profit.** Corner Stone Electronics determines that its total weekly profit, in dollars, from the production and sale of x amplifiers is given by

$$P(x) = \frac{1500}{x^2 - 6x + 10}.$$

Find the number of amplifiers, x, for which the total weekly profit is a maximum.

Maximizing profit. *The total-cost and total-revenue functions for producing x items are*

$$C(x) = 5000 + 600x \quad \text{and} \quad R(x) = -\frac{1}{2}x^2 + 1000x,$$

where $0 \le x \le 600$. *Use these functions for Exercises 103 and 104.*

103. **a)** Find the total-profit function $P(x)$.
b) Find the number of items, x, for which the total profit is a maximum.

104. **a)** The *average profit* is given by $A(x) = P(x)/x$. Find $A(x)$.
b) Find the number of items, x, for which the average profit is a maximum.

Life and Physical Sciences

105. **Blood pressure.** For a dosage of x cubic centimeters (cc) of a certain drug, the resulting blood pressure B is approximated by

$$B(x) = 305x^2 - 1830x^3, \quad 0 \le x \le 0.16.$$

Find the maximum blood pressure and the dosage at which it occurs.

SYNTHESIS

106. Explain the usefulness of the second derivative in finding the absolute extrema of a function.

For Exercises 107–110, find the absolute maximum and minimum values of each function, and sketch the graph.

107. $f(x) = \begin{cases} 2x + 1 & \text{for } -3 \le x \le 1, \\ 4 - x^2, & \text{for } 1 < x \le 2 \end{cases}$

108. $g(x) = \begin{cases} x^2, & \text{for } -2 \le x \le 0, \\ 5x, & \text{for } 0 < x \le 2 \end{cases}$

109. $h(x) = \begin{cases} 1 - x^2, & \text{for } -4 \le x < 0, \\ 1 - x, & \text{for } 0 \le x < 1, \\ x - 1, & \text{for } 1 \le x \le 2 \end{cases}$

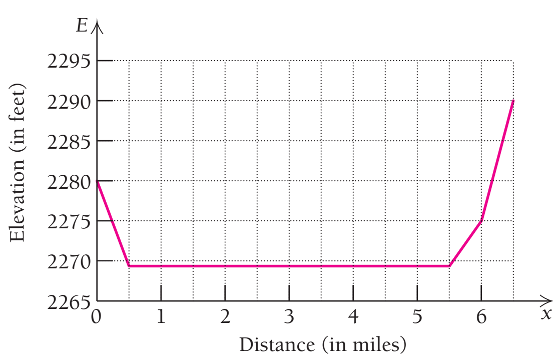
110. $F(x) = \begin{cases} x^2 + 4, & \text{for } -2 \le x < 0, \\ 4 - x, & \text{for } 0 \le x < 3, \\ \sqrt{x - 2}, & \text{for } 3 \le x \le 67 \end{cases}$

111. Consider the piecewise-defined function f defined by:

$$f(x) = \begin{cases} x^2 + 2, & \text{for } -2 \le x \le 0, \\ 2, & \text{for } 0 < x < 4, \\ x - 2, & \text{for } 4 \le x \le 6. \end{cases}$$

a) Sketch its graph.
b) Identify the absolute maximum.
c) How would you describe the absolute minimum?

112. Physical science: dry lake elevation. Dry lakes are common in the Western deserts of the United States. These beds of ancient lakes are notable for having perfectly flat terrain. Rogers Dry Lake in California has been used as a landing site for space shuttle missions in recent years. The graph shows the elevation E, in feet, as a function of the distance x, in miles, from a point west ($x = 0$) of Rogers Dry Lake to a point east of the dry lake. (*Source*: www.mytopo.com.)



a) What is the maximum elevation?
b) How would you describe the minimum elevation?

Find the absolute maximum and minimum values of the function, if they exist, over the indicated interval.

113. $g(x) = x\sqrt{x + 3}$; $[-3, 3]$

114. $h(x) = x\sqrt{1 - x}$; $[0, 1]$

115. Business: total cost. Certain costs in a business environment can be separated into two components: those that increase with volume and those that decrease with volume. For example, customer service becomes more expensive as its quality increases, but part of the increased cost is offset by fewer customer complaints. A firm has determined that its cost of service, $C(x)$, in thousands of dollars, is modeled by

$$C(x) = (2x + 4) + \left(\frac{2}{x - 6}\right), \quad x > 6,$$

where x represents the number of "quality units." Find the number of "quality units" that the firm should use in order to minimize its total cost of service.

116. Let

$$y = (x - a)^2 + (x - b)^2.$$

For what value of x is y a minimum?

117. Explain the usefulness of the first derivative in finding the absolute extrema of a function.

TECHNOLOGY CONNECTION

118. Business: worldwide oil production. Refer to Exercise 101. In what year was worldwide oil production increasing most rapidly and at what rate was it increasing?

119. Business: U.S. oil production. One model of oil production in the United States is given by

$$P(t) = 0.0000000219t^4 - 0.0000167t^3 + 0.00155t^2 + 0.002t + 0.22, \quad 0 \le t \le 110,$$

where $P(t)$ is the number of barrels of oil, in billions, produced in a year, t years after 1910. (*Source*: *Beyond Oil*, by Kenneth S. Deffeyes, p. 41, Hill and Wang, New York, 2005.)

a) According to this model, what is the absolute maximum amount of oil produced in the United States and in what year did that production occur?
b) According to this model, at what rate was United States oil production declining in 2004 and in 2010?

Graph each function over the given interval. Visually estimate where absolute maximum and minimum values occur. Then use the TABLE *feature to refine your estimate.*

120. $f(x) = x^{2/3}(x - 5)$; $[1, 4]$

121. $f(x) = \frac{3}{4}(x^2 - 1)^{2/3}$; $\left[\frac{1}{2}, \infty\right)$

122. $f(x) = x\left(\frac{x}{2} - 5\right)^4$; $\mathbb{R}$

123. Life and physical sciences: contractions during pregnancy. The following table and graph give the pressure of a pregnant woman's contractions as a function of time.

Time, t (in minutes)	Pressure (in millimeters of mercury)
0	10
1	8
2	9.5
3	15
4	12
5	14
6	14.5

Use a calculator that has the REGRESSION option.

a) Fit a linear equation to the data. Predict the pressure of the contractions after 7 min.

b) Fit a quartic polynomial to the data. Predict the pressure of the contractions after 7 min. Find the smallest contraction over the interval $[0, 10]$.

Answers to Quick Checks

1. Absolute maximum is 20 at $x = 3$; absolute minimum is 0 at $x = 1$. **2. (a)** Absolute maximum is 0 at $x = 0$; absolute minimum is -25 at $x = 5$. **(b)** Absolute maximum is 0 at $x = 10$; absolute minimum is -25 at $x = 5$.
3. The derivative is $f'(x) = nx^{n-1}$. If n is odd, $n - 1$ is even. Thus, nx^{n-1} is always positive or zero, never negative.
4. No absolute maximum; absolute minimum is 12 at $x = 3$.

2.5 Maximum–Minimum Problems; Business and Economics Applications

OBJECTIVE

- Solve maximum–minimum problems using calculus.

An important use of calculus is the solving of maximum–minimum problems, that is, finding the absolute maximum or minimum value of some varying quantity Q and the point at which that maximum or minimum occurs.

Exampl E 1 **Maximizing Area.** A hobby store has 20 ft of fencing to fence off a rectangular area for an electric train in one corner of its display room. The two sides up against the wall require no fence. What dimensions of the rectangle will maximize the area? What is the maximum area?

Solution At first glance, we might think that it does not matter what dimensions we use: They will all yield the same area. This is not the case. Let's first make a drawing and express the area in terms of one variable. If we let $x =$ the length, in feet, of one side and $y =$ the length, in feet, of the other, then, since the sum of the lengths must be 20 ft, we have

$$x + y = 20 \quad \text{and} \quad y = 20 - x.$$

Thus, the area is given by

$$\begin{aligned} A &= xy \\ &= x(20 - x) \\ &= 20x - x^2. \end{aligned}$$

TECHNOLOGY CONNECTION

EXERCISES

1. Complete this table, using a calculator as needed.

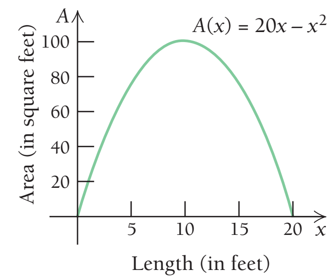
x	$y = 20 - x$	$A = x(20 - x)$
0		
4		
6.5		
8		
10		
12		
13.2		
20		

2. Graph $A(x) = x(20 - x)$ over the interval $[0, 20]$.

3. Estimate the maximum value, and state where it occurs.

We are trying to find the maximum value of

$$A(x) = 20x - x^2$$



over the interval $(0, 20)$. We consider the interval $(0, 20)$ because x is a length and cannot be negative or 0. Since there is only 20 ft of fencing, x cannot be greater than 20. Also, x cannot be 20 because then the length of y would be 0.

a) We first find $A'(x)$: $\quad A'(x) = 20 - 2x.$

b) This derivative exists for all values of x in $(0, 20)$. Thus, the only critical values are where

$$A'(x) = 20 - 2x = 0$$
$$-2x = -20$$
$$x = 10.$$

Since there is only one critical value, we can use the second derivative to determine whether we have a maximum. Note that

$$A''(x) = -2,$$

which is a constant. Thus, $A''(10)$ is negative, so $A(10)$ is a maximum. Now

$$A(10) = 10(20 - 10) = 10 \cdot 10 = 100.$$

Thus, the maximum area of 100 ft^2 is obtained using 10 ft for the length of one side and $20 - 10$, or 10 ft for the other. Note that $A(5) = 75$, $A(16) = 64$, and $A(12) = 96$; so length does affect area.

Quick Check 1

Repeat Example 1 starting with 50 ft of fencing, and again starting with 100 ft of fencing. Do you detect a pattern? If you had n feet of fencing, what would be the dimensions of the maximum area (in terms of n)?

Quick Check 1

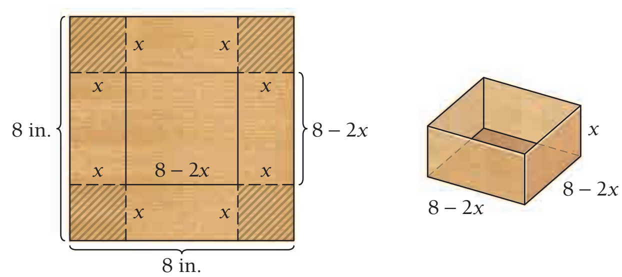
Here is a general strategy for solving maximum–minimum problems. Although it may not guarantee success, it should certainly improve your chances.

A Strategy for Solving Maximum–Minimum Problems

1. Read the problem carefully. If relevant, make a drawing.
2. Make a list of appropriate variables and constants, noting what varies, what stays fixed, and what units are used. Label the measurements on your drawing, if one exists.
3. Translate the problem to an equation involving a quantity Q to be maximized or minimized. Try to represent Q in terms of the variables of step 2.
4. Try to express Q as a function of one variable. Use the procedures developed in Sections 2.1–2.4 to determine the maximum or minimum values and the points at which they occur.

Exampl E 2 **Maximizing Volume.** From a thin piece of cardboard 8 in. by 8 in., square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

Solution We might again think at first that it does not matter what the dimensions are, but our experience with Example 1 suggests otherwise. We make a drawing in which x is the length, in inches, of each square to be cut. It is important to note that since the original square is 8 in. by 8 in., after the smaller squares are removed, the lengths of the sides of the box will be $(8 - 2x)$ in. by $(8 - 2x)$ in.



After the four small squares are removed and the sides are folded up, the volume V of the resulting box is

$$V = l \cdot w \cdot h = (8 - 2x) \cdot (8 - 2x) \cdot x,$$

or $$V(x) = (64 - 32x + 4x^2)x = 4x^3 - 32x^2 + 64x.$$

Since $8 - 2x > 0$, this means that $x < 4$. Thus, we need to maximize

$$V(x) = 4x^3 - 32x^2 + 64x \quad \text{over the interval } (0, 4).$$

To do so, we first find $V'(x)$:

$$V'(x) = 12x^2 - 64x + 64.$$

Since $V'(x)$ exists for all x in the interval $(0, 4)$, we can set it equal to 0 to find the critical values:

$$\begin{aligned} V'(x) = 12x^2 - 64x + 64 &= 0 \\ 4(3x^2 - 16x + 16) &= 0 \\ 4(3x - 4)(x - 4) &= 0 \\ 3x - 4 = 0 \quad \text{or} \quad x - 4 &= 0 \\ 3x = 4 \quad \text{or} \quad x &= 4 \\ x = \tfrac{4}{3} \quad \text{or} \quad x &= 4. \end{aligned}$$

The only critical value in $(0, 4)$ is $\frac{4}{3}$. Thus, we can use the second derivative,

$$V''(x) = 24x - 64,$$

to determine whether we have a maximum. Since

$$V''\left(\tfrac{4}{3}\right) = 24 \cdot \tfrac{4}{3} - 64 = 32 - 64 < 0,$$

we know that $V\left(\frac{4}{3}\right)$ is a maximum.

TECHNOLOGY CONNECTION

EXERCISES

1. Complete this table to help visualize Example 2.

x	$8 - 2x$	$4x^3 - 32x^2 + 64x$
0		
0.5		
1.0		
1.5		
2.0		
2.5		
3.0		
3.5		
4.0		

2. Graph $V(x) = 4x^3 - 32x^2 + 64x$ over the interval $(0, 4)$.

3. Estimate a maximum value, and state where it occurs.

Thus, to maximize the box's volume, small squares with edges measuring $\frac{4}{3}$ in., or $1\frac{1}{3}$ in., should be cut from each corner of the original 8 in. by 8 in. piece of cardboard. When the sides are folded up, the resulting box will have sides of length

$$8 - 2x = 8 - 2 \cdot \frac{4}{3} = \frac{24}{3} - \frac{8}{3} = \frac{16}{3} = 5\frac{1}{3} \text{ in.}$$

and a height of $1\frac{1}{3}$ in. The maximum volume is

$$V\left(\frac{4}{3}\right) = 4\left(\frac{4}{3}\right)^3 - 32\left(\frac{4}{3}\right)^2 + 64\left(\frac{4}{3}\right) = \frac{1024}{27} = 37\frac{25}{27} \text{ in}^3.$$

❯ Quick Check 2

Repeat Example 2 starting with a sheet of cardboard measuring 8.5 in. by 11 in. (the size of a typical sheet of paper). Will this box hold 1 liter (L) of liquid? (*Hint*: $1 \text{ L} = 1000 \text{ cm}^3$ and $1 \text{ in}^3 = 16.38 \text{ cm}^3$.)

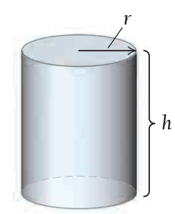
❮ Quick Check 2

In manufacturing, minimizing the amount of material used is always preferred, both from a cost standpoint and in terms of efficiency.

Exampl E 3

Minimizing Material: Surface Area. A manufacturer of food-storage containers makes a cylindrical can with a volume of 500 milliliters (mL; $1 \text{ mL} = 1 \text{ cm}^3$). What dimensions (height and radius) will minimize the material needed to produce each can, that is, minimize the surface area?

Solution We let h = height of the can and r = radius, both measured in centimeters. The formula for volume of a cylinder is

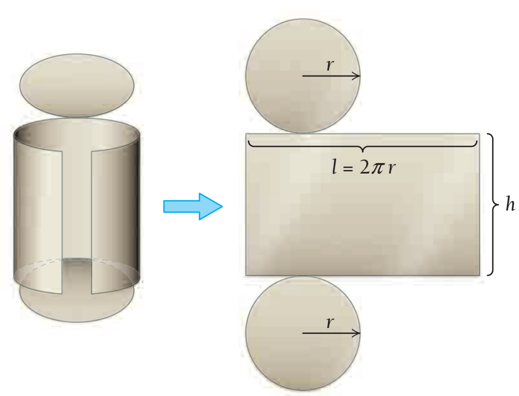


$$V = \pi r^2 h.$$

Since we know the volume is 500 cm^3, this formula allows us to relate h and r, expressing one in terms of the other. It is easier to solve for h in terms of r:

$$\pi r^2 h = 500$$
$$h = \frac{500}{\pi r^2}.$$

The can is composed of two circular ends, each with an area equal to πr^2, and a side wall that, when laid out flat, is a rectangle with a height h and a length the same as the circumference of the circular ends, or $2\pi r$. Thus, the area of this rectangle is $2\pi rh$.



The total surface area A is the sum of the areas of the two circular ends and the side wall:

$$A = 2(\pi r^2) + (2\pi rh)$$
$$= 2\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right). \qquad \text{Substituting for } h.$$

Simplifying, we have area A as a function of radius r:

$$A(r) = 2\pi r^2 + \frac{1000}{r}.$$

The nature of this problem situation requires that $r > 0$. We differentiate:

$$A'(r) = 4\pi r - \frac{1000}{r^2} \qquad \text{Note that } \frac{d}{dr}\left(\frac{1000}{r}\right) = \frac{d}{dr}(1000r^{-1}) = -1000r^{-2} = -\frac{1000}{r^2}.$$

We set the derivative equal to 0 and solve for r to determine the critical values:

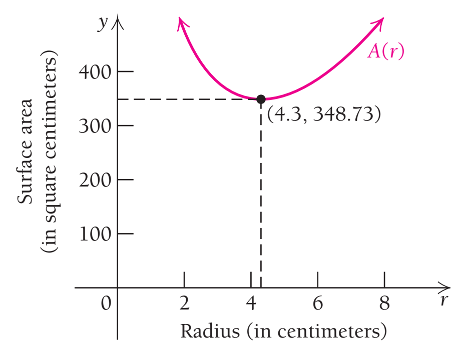
$$4\pi r - \frac{1000}{r^2} = 0$$
$$4\pi r = \frac{1000}{r^2}$$
$$4\pi r^3 = 1000$$
$$r^3 = \frac{1000}{4\pi} = \frac{250}{\pi}$$
$$r = \sqrt[3]{\frac{250}{\pi}} \approx 4.3 \text{ cm}.$$

The critical value $r \approx 4.3$ is the only critical value in the interval $(0, \infty)$. The second derivative is

$$A''(r) = 4\pi + \frac{2000}{r^3}.$$

Evaluating $A''(x)$ at the critical value, we get a positive value:

$$A''(4.3) = 4\pi + \frac{2000}{(4.3)^3} > 0.$$



The graph is concave up at the critical value, so the critical value indicates a minimum point. Thus, the radius should be $\sqrt[3]{\frac{250}{\pi}}$, or approximately 4.3 cm. The height is approximately $h = \frac{500}{\pi(4.3)^2} \approx 8.6$ cm, and the minimum total surface area is approximately 348.73 cm^2. Assuming that the material used for the side and the ends costs the same, minimizing the surface area will also minimize the cost to produce each can.

❮ Quick Check 3

❯ Quick Check 3

Repeat Example 3 for a cylindrical can with a volume of 1000 cm^3. What do you notice about the relationship between the cylinder's radius and height? Repeat the example again for any other volume. Does the relationship between radius and height still hold? State this relationship.

Exampl E 4 **Business: Maximizing Revenue.** A stereo manufacturer determines that in order to sell x units of a new stereo, the price per unit, in dollars, must be

$$p(x) = 1000 - x.$$

The manufacturer also determines that the total cost of producing x units is given by

$$C(x) = 3000 + 20x.$$

a) Find the total revenue $R(x)$.

b) Find the total profit $P(x)$.

c) How many units must the company produce and sell in order to maximize profit?

d) What is the maximum profit?

e) What price per unit must be charged in order to make this maximum profit?

Solution

a)
$$\begin{aligned} R(x) &= \text{Total revenue} \\ &= (\text{Number of units}) \cdot (\text{Price per unit}) \\ &= \quad x \quad \cdot \quad p \\ &= x(1000 - x) = 1000x - x^2 \end{aligned}$$

b)
$$\begin{aligned} P(x) &= \text{Total revenue} - \text{Total cost} \\ &= R(x) - C(x) \\ &= (1000x - x^2) - (3000 + 20x) \\ &= -x^2 + 980x - 3000 \end{aligned}$$

c) To find the maximum value of $P(x)$, we first find $P'(x)$:

$$P'(x) = -2x + 980.$$

This is defined for all real numbers, so the only critical values will come from solving $P'(x) = 0$:

$$\begin{aligned} P'(x) = -2x + 980 &= 0 \\ -2x &= -980 \\ x &= 490. \end{aligned}$$

There is only one critical value. We can therefore try to use the second derivative to determine whether we have an absolute maximum. Note that

$$P''(x) = -2, \text{ a constant.}$$

Thus, $P''(490)$ is negative, and so profit is maximized when 490 units are produced and sold.

d) The maximum profit is given by

$$\begin{aligned} P(490) &= -(490)^2 + 980 \cdot 490 - 3000 \\ &= \$237{,}100. \end{aligned}$$

Thus, the stereo manufacturer makes a maximum profit of $237,100 by producing and selling 490 stereos.

e) The price per unit needed to make the maximum profit is

$$p = 1000 - 490 = \$510.$$

❮ Quick Check 4

❯ Quick Check 4

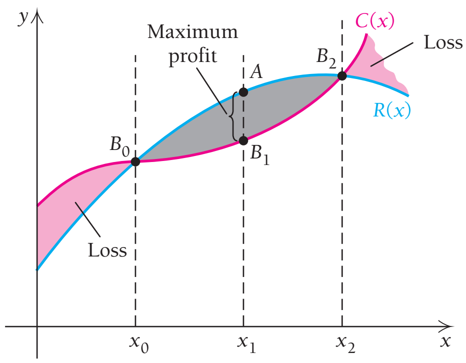
Repeat Example 4 with the price function

$$p(x) = 1750 - 2x$$

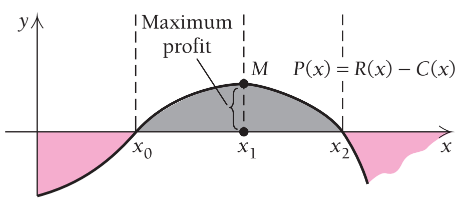
and the cost function

$$C(x) = 2250 + 15x.$$

Round your answers when necessary.



Figur E 1



Figur E 2

Let's take a general look at the total-profit function and its related functions.

Figure 1 shows an example of total-cost and total-revenue functions. We can estimate what the maximum profit might be by looking for the widest gap between $R(x)$ and $C(x)$, when $R(x) > C(x)$. Points B_0 and B_2 are break-even points.

Figure 2 shows the related total-profit function. Note that when production is too low ($< x_0$), there is a loss, perhaps due to high fixed or initial costs and low revenue. When production is too high ($> x_2$), there is also a loss, perhaps due to the increased cost of overtime pay or expansion.

The business operates at a profit everywhere between x_0 and x_2. Note that maximum profit occurs at a critical value x_1 of $P(x)$. If we assume that $P'(x)$ exists for all x in some interval, usually $[0, \infty)$, this critical value occurs at some number x such that

$$P'(x) = 0 \quad \text{and} \quad P''(x) < 0.$$

Since $P(x) = R(x) - C(x)$, it follows that

$$P'(x) = R'(x) - C'(x) \quad \text{and} \quad P''(x) = R''(x) - C''(x).$$

Thus, the maximum profit occurs at some number x such that

$$P'(x) = R'(x) - C'(x) = 0 \quad \text{and} \quad P''(x) = R''(x) - C''(x) < 0,$$

or

$$R'(x) = C'(x) \quad \text{and} \quad R''(x) < C''(x).$$

In summary, we have the following theorem.

THEOREM 10

Maximum profit occurs at those x-values for which

$$R'(x) = C'(x) \quad \text{and} \quad R''(x) < C''(x).*$$

You can check that the results in parts (c) and (d) of Example 4 can be easily found using Theorem 10.

Exampl E 5 **Business: Determining a Ticket Price.** Promoters of international fund-raising concerts must walk a fine line between profit and loss, especially when determining the price to charge for admission to closed-circuit TV showings in local theaters. By keeping records, a theater determines that at an admission price of \$26, it averages 1000 people in attendance. For every drop in price of \$1, it gains 50 customers. Each customer spends an average of \$4 on concessions. What admission price should the theater charge in order to maximize total revenue?

Solution Let x be the number of dollars by which the price of \$26 should be decreased. (If x is negative, the price is increased.) We first express the total revenue R as a function of x. Note that the increase in ticket sales is $50x$ when the price drops x dollars:

$$\begin{aligned} R(x) &= (\text{Revenue from tickets}) + (\text{Revenue from concessions}) \\ &= (\text{Number of people}) \cdot (\text{Ticket price}) + (\text{Number of people}) \cdot 4 \\ &= (1000 + 50x)(26 - x) + (1000 + 50x) \cdot 4 \\ &= 26{,}000 - 1000x + 1300x - 50x^2 + 4000 + 200x, \end{aligned}$$

or $$R(x) = -50x^2 + 500x + 30{,}000.$$

*In Section 2.6, the concepts of *marginal revenue* and *marginal cost* are introduced, allowing $R'(x) = C'(x)$ to be regarded as Marginal revenue = Marginal cost.

To find x such that $R(x)$ is a maximum, we first find $R'(x)$:

$$R'(x) = -100x + 500.$$

This derivative exists for all real numbers x. Thus, the only critical values are where $R'(x) = 0$; so we solve that equation:

$$-100x + 500 = 0$$
$$-100x = -500$$
$$x = 5 \qquad \text{This corresponds to lowering the price by \$5.}$$

Since this is the only critical value, we can use the second derivative,

$$R''(x) = -100,$$

to determine whether we have a maximum. Since $R''(5)$ is negative, $R(5)$ is a maximum. Therefore, in order to maximize revenue, the theater should charge

$26 − $5, or $21 per ticket.

❮ Quick Check 5

❯ Quick Check 5

A baseball team charges $30 per ticket and averages 20,000 people in attendance per game. Each person spends an average of $8 on concessions. For every drop of $1 in the ticket price, the attendance rises by 800 people. What ticket price should the team charge to maximize total revenue?

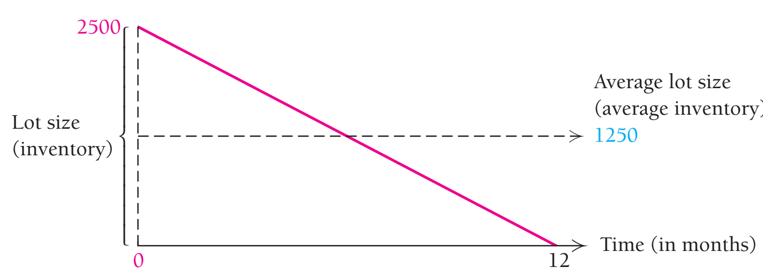
Minimizing Inventory Costs

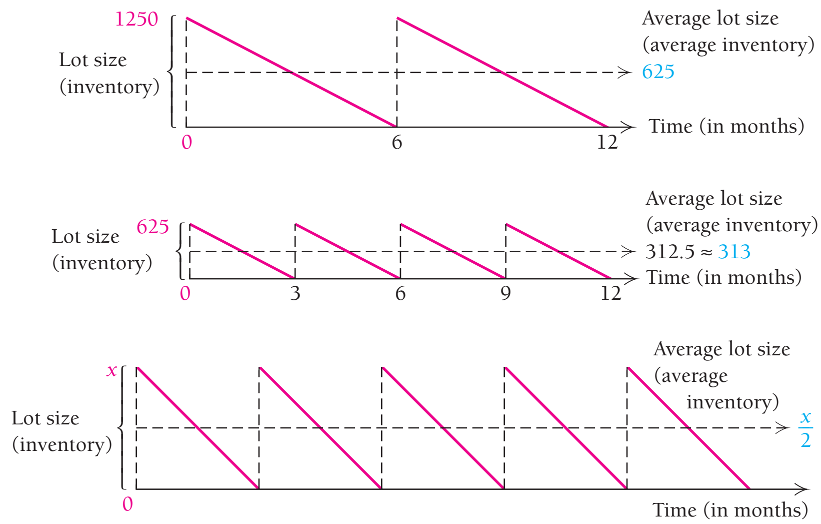
A retail business outlet needs to be concerned about inventory costs. Suppose, for example, that an appliance store sells 2500 television sets per year. It *could* operate by ordering all the sets at once. But then the owners would face the carrying costs (insurance, building space, and so on) of storing them all. Thus, they might make several, say 5, smaller orders, so that the largest number they would ever have to store is 500. However, each time they reorder, there are costs for paperwork, delivery charges, labor, and so on. It seems, therefore, that there must be some balance between carrying costs and reorder costs. Let's see how calculus can help determine what that balance might be. We are trying to minimize the following function:

Total inventory costs = (Yearly carrying costs) + (Yearly reorder costs).

The *lot size* x is the largest number ordered each reordering period. If x units are ordered each period, then during that time somewhere between 0 and x units are in stock. To have a representative expression for the amount in stock at any one time in the period, we can use the average, $x/2$. This represents the average amount held in stock over the course of each time period.

Refer to the graphs shown below and on the next page. If the lot size is 2500, then during the period between orders, there are somewhere between 0 and 2500 units in stock. On average, there are 2500/2, or 1250 units in stock. If the lot size is 1250, then during the period between orders, there are somewhere between 0 and 1250 units in stock. On average, there are 1250/2, or 625 units in stock. In general, if the lot size is x, the average inventory is $x/2$.





■ **Exampl E 6** **Business: Minimizing Inventory Costs.** A retail appliance store sells 2500 television sets per year. It costs \$10 to store one set for a year. To reorder, there is a fixed cost of \$20, plus a fee of \$9 per set. How many times per year should the store reorder, and in what lot size, to minimize inventory costs?

Solution Let $x =$ the lot size. Inventory costs are given by

$$C(x) = (\text{Yearly carrying costs}) + (\text{Yearly reorder costs}).$$

We consider each component of inventory costs separately.

a) *Yearly carrying costs.* The average amount held in stock is $x/2$, and it costs \$10 per set for storage. Thus,

$$\text{Yearly carrying costs} = \begin{pmatrix}\text{Yearly cost}\\ \text{per item}\end{pmatrix} \cdot \begin{pmatrix}\text{Average number}\\ \text{of items}\end{pmatrix}$$
$$= 10 \cdot \frac{x}{2}.$$

b) *Yearly reorder costs.* We know that x is the lot size, and we let N be the number of reorders each year. Then $Nx = 2500$, and $N = 2500/x$. Thus,

$$\text{Yearly reorder costs} = \begin{pmatrix}\text{Cost of each}\\ \text{order}\end{pmatrix} \cdot \begin{pmatrix}\text{Number of}\\ \text{reorders}\end{pmatrix}$$
$$= (20 + 9x)\frac{2500}{x}.$$

c) Thus, we have

$$C(x) = 10 \cdot \frac{x}{2} + (20 + 9x)\frac{2500}{x}$$
$$= 5x + \frac{50{,}000}{x} + 22{,}500 = 5x + 50{,}000x^{-1} + 22{,}500.$$

d) To find a minimum value of C over $[1, 2500]$, we first find $C'(x)$:

$$C'(x) = 5 - \frac{50{,}000}{x^2}.$$

e) $C'(x)$ exists for all x in $[1, 2500]$, so the only critical values are those x-values such that $C'(x) = 0$. We solve $C'(x) = 0$:

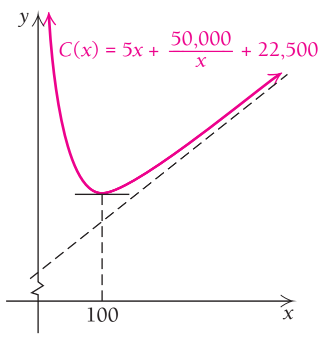
$$5 - \frac{50,000}{x^2} = 0$$

$$5 = \frac{50,000}{x^2}$$

$$5x^2 = 50,000$$

$$x^2 = 10,000$$

$$x = \pm 100.$$



Since there is only one critical value in $[1, 2500]$, that is, $x = 100$, we can use the second derivative to see whether it yields a maximum or a minimum:

$$C''(x) = \frac{100,000}{x^3}.$$

$C''(x)$ is positive for all x in $[1, 2500]$, so we have a minimum at $x = 100$. Thus, to minimize inventory costs, the store should order 2500/100, or 25, times per year. The lot size is 100 sets.

TECHNOLOGY CONNECTION

Exploratory

Many calculators can make tables and/or spreadsheets of function values. In reference to Example 6, without using calculus, one might make an estimate of the lot size that will minimize total inventory costs by using a table like the one below. Complete the table, and estimate the solution of Example 6.

EXERCISES

1. Graph $C(x)$ over the interval $[1, 2500]$.

2. Graphically estimate the minimum value, and note where it occurs. Does the table confirm the graph?

Lot Size, x	Number of Reorders, $\frac{2500}{x}$	Average Inventory, $\frac{x}{2}$	Carrying Costs, $10 \cdot \frac{x}{2}$	Cost of Each Order, $20 + 9x$	Reorder Costs, $(20 + 9x)\frac{2500}{x}$	Total Inventory Costs, $C(x) = 10 \cdot \frac{x}{2} + (20 + 9x)\frac{2500}{x}$
2500	1	1250	\$12,500	\$22,520	\$22,520	\$35,020
1250	2	625	6,250	11,270	22,540	
500	5	250	2,500	4,520		
250	10	125				
167	15	84				
125	20					
100	25					
90	28					
50	50					

What happens in problems like Example 6 if the answer is not a whole number? For those cases, we consider the two whole numbers closest to the answer and substitute them into $C(x)$. The value that yields the smaller $C(x)$ is the lot size.

Exampl E 7 **Business: Minimizing Inventory Costs.** Reconsider Example 6, but change the $10 storage cost to $20. How many times per year should the store reorder television sets, and in what lot size, in order to minimize inventory costs?

Solution Comparing this situation with that in Example 6, we find that the inventory cost function becomes

$$C(x) = 20 \cdot \frac{x}{2} + (20 + 9x)\frac{2500}{x}$$

$$= 10x + \frac{50{,}000}{x} + 22{,}500 = 10x + 50{,}000x^{-1} + 22{,}500.$$

Then we find $C'(x)$, set it equal to 0, and solve for x:

$$C'(x) = 10 - \frac{50{,}000}{x^2} = 0$$

$$10 = \frac{50{,}000}{x^2}$$

$$10x^2 = 50{,}000$$

$$x^2 = 5000$$

$$x = \sqrt{5000}$$

$$\approx 70.7.$$

It is impossible to reorder 70.7 sets each time, so we consider the two numbers closest to 70.7, which are 70 and 71. Since

$$C(70) \approx \$23{,}914.29 \quad \text{and} \quad C(71) \approx \$23{,}914.23,$$

it follows that the lot size that will minimize cost is 71, although the difference, $0.06, is not much. (*Note*: Such a procedure will not work for all functions but will work for the type we are considering here.) The number of times an order should be placed is 2500/71 with a remainder of 15, indicating that 35 orders should be placed. Of those, $35 - 15 = 20$ will be for 71 items and 15 will be for 72 items.

❮ Quick Check 6

❯ **Quick Check 6**

Repeat Example 7 with a storage cost of $30 per set and assuming that the store sells 3000 sets per year.

The lot size that minimizes total inventory costs is often referred to as the *economic ordering quantity.* Three assumptions are made in using the preceding method to determine the economic ordering quantity. First, the demand for the product is the same year round. For television sets, this may be reasonable, but for seasonal items such as clothing or skis, this assumption is unrealistic. Second, the time between the placing of an order and its receipt remains consistent throughout the year. Finally, the various costs involved, such as storage, shipping charges, and so on, do not vary. This assumption may not be reasonable in a time of inflation, although variation in these costs can be allowed for by anticipating what they might be and using average costs. Regardless, the model described above is useful, allowing us to analyze a seemingly difficult problem using calculus.

Section Summary

- In many real-life applications, we wish to determine the minimum or maximum value of some function modeling a situation.
- Identify a realistic interval for the domain of the input variable. If it is a closed interval, its endpoints should be considered as possible critical values.

EXERCISE SET 2.5

1. Of all numbers whose sum is 50, find the two that have the maximum product. That is, maximize $Q = xy$, where $x + y = 50$.

2. Of all numbers whose sum is 70, find the two that have the maximum product. That is, maximize $Q = xy$, where $x + y = 70$.

3. In Exercise 1, can there be a minimum product? Why or why not?

4. In Exercise 2, can there be a minimum product? Why or why not?

5. Of all numbers whose difference is 4, find the two that have the minimum product.

6. Of all numbers whose difference is 6, find the two that have the minimum product.

7. Maximize $Q = xy^2$, where x and y are positive numbers such that $x + y^2 = 1$.

8. Maximize $Q = xy^2$, where x and y are positive numbers such that $x + y^2 = 4$.

9. Minimize $Q = 2x^2 + 3y^2$, where $x + y = 5$.

10. Minimize $Q = x^2 + 2y^2$, where $x + y = 3$.

11. Maximize $Q = xy$, where x and y are positive numbers such that $\frac{4}{3}x^2 + y = 16$.

12. Maximize $Q = xy$, where x and y are positive numbers such that $x + \frac{4}{3}y^2 = 1$.

13. **Maximizing area.** A lifeguard needs to rope off a rectangular swimming area in front of Long Lake Beach, using 180 yd of rope and floats. What dimensions of the rectangle will maximize the area? What is the maximum area? (Note that the shoreline is one side of the rectangle.)

14. **Maximizing area.** A rancher wants to enclose two rectangular areas near a river, one for sheep and one for cattle. There are 240 yd of fencing available. What is the largest total area that can be enclosed?

15. **Maximizing area.** A carpenter is building a rectangular shed with a fixed perimeter of 54 ft. What are the dimensions of the largest shed that can be built? What is its area?

16. **Maximizing area.** Of all rectangles that have a perimeter of 42 ft, find the dimensions of the one with the largest area. What is its area?

17. **Maximizing volume.** From a 50-cm-by-50-cm sheet of aluminum, square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

18. **Maximizing volume.** From a thin piece of cardboard 20 in. by 20 in., square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

19. **Minimizing surface area.** Drum Tight Containers is designing an open-top, square-based, rectangular box that will have a volume of 62.5 in^3. What dimensions will minimize surface area? What is the minimum surface area?

20. **Minimizing surface area.** A soup company is constructing an open-top, square-based, rectangular metal tank that will have a volume of 32 ft^3. What dimensions will minimize surface area? What is the minimum surface area?

21. Minimizing surface area. Open Air Waste Management is designing a rectangular construction dumpster that will be twice as long as it is wide and must hold 12 yd^3 of debris. Find the dimensions of the dumpster that will minimize its surface area.

22. Minimizing surface area. Ever Green Gardening is designing a rectangular compost container that will be twice as tall as it is wide and must hold 18 ft^3 of composted food scraps. Find the dimensions of the compost container with minimal surface area (include the bottom and top).

APPLICATIONS

Business and Economics

Maximizing profit. *Find the maximum profit and the number of units that must be produced and sold in order to yield the maximum profit. Assume that revenue, $R(x)$, and cost, $C(x)$, are in dollars for Exercises 23–26.*

23. $R(x) = 50x - 0.5x^2$, $C(x) = 4x + 10$

24. $R(x) = 50x - 0.5x^2$, $C(x) = 10x + 3$

25. $R(x) = 2x$, $C(x) = 0.01x^2 + 0.6x + 30$

26. $R(x) = 5x$, $C(x) = 0.001x^2 + 1.2x + 60$

27. $R(x) = 9x - 2x^2$, $C(x) = x^3 - 3x^2 + 4x + 1$; assume that $R(x)$ and $C(x)$ are in thousands of dollars, and x is in thousands of units.

28. $R(x) = 100x - x^2$,
$C(x) = \frac{1}{3}x^3 - 6x^2 + 89x + 100$;
assume that $R(x)$ and $C(x)$ are in thousands of dollars, and x is in thousands of units.

29. Maximizing profit. Raggs, Ltd., a clothing firm, determines that in order to sell x suits, the price per suit must be

$$p = 150 - 0.5x.$$

It also determines that the total cost of producing x suits is given by

$$C(x) = 4000 + 0.25x^2.$$

a) Find the total revenue, $R(x)$.
b) Find the total profit, $P(x)$.
c) How many suits must the company produce and sell in order to maximize profit?
d) What is the maximum profit?
e) What price per suit must be charged in order to maximize profit?

30. Maximizing profit. Riverside Appliances is marketing a new refrigerator. It determines that in order to sell x refrigerators, the price per refrigerator must be

$$p = 280 - 0.4x.$$

It also determines that the total cost of producing x refrigerators is given by

$$C(x) = 5000 + 0.6x^2.$$

a) Find the total revenue, $R(x)$.
b) Find the total profit, $P(x)$.
c) How many refrigerators must the company produce and sell in order to maximize profit?
d) What is the maximum profit?
e) What price per refrigerator must be charged in order to maximize profit?

31. Maximizing revenue. A university is trying to determine what price to charge for tickets to football games. At a price of \$18 per ticket, attendance averages 40,000 people per game. Every decrease of \$3 adds 10,000 people to the average number. Every person at the game spends an average of \$4.50 on concessions. What price per ticket should be charged in order to maximize revenue? How many people will attend at that price?

32. Maximizing profit. Gritz-Charlston is a 300-unit luxury hotel. All rooms are occupied when the hotel charges \$80 per day for a room. For every increase of x dollars in the daily room rate, there are x rooms vacant. Each occupied room costs \$22 per day to service and maintain. What should the hotel charge per day in order to maximize profit?

33. Maximizing yield. An apple farm yields an average of 30 bushels of apples per tree when 20 trees are planted on an acre of ground. Each time 1 more tree is planted per acre, the yield decreases by 1 bushel (bu) per tree as a result of crowding. How many trees should be planted on an acre in order to get the highest yield?

34. Nitrogen prices. During 2001, nitrogen prices fell by 41%. Over the same period of time, nitrogen demand went up by 12%. (*Source*: *Chemical Week*.)

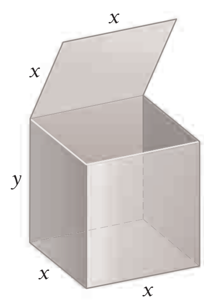
a) Assuming a linear change in demand, find the demand function, $q(x)$, by finding the equation of the line that passes through the points $(1, 1)$ and $(0.59, 1.12)$. Here x is the price as a fraction of the January 2001 price, and $q(x)$ is the demand as a fraction of the demand in January.
b) As a percentage of the January 2001 price, what should the price of nitrogen be to maximize revenue?

35. Vanity license plates. According to a pricing model, increasing the fee for vanity license plates by \$1 decreases the percentage of a state's population that will request them by 0.04%. (*Source*: E. D. Craft, "The demand for vanity (plates): Elasticities, net revenue maximization, and deadweight loss," *Contemporary Economic Policy*, Vol. 20, 133–144 (2002).)

a) Recently, the fee for vanity license plates in Maryland was \$25, and the percentage of the state's population that had vanity plates was 2.13%. Use this information to construct the demand function, $q(x)$, for the percentage of Maryland's population that will request vanity license plates for a fee of x dollars.
b) Find the fee, x, that will maximize revenue from vanity plates.

36. Maximizing revenue. When a theater owner charges $5 for admission, there is an average attendance of 180 people. For every $0.10 increase in admission, there is a loss of 1 customer from the average number. What admission should be charged in order to maximize revenue?

37. Minimizing costs. A rectangular box with a volume of 320 ft^3 is to be constructed with a square base and top. The cost per square foot for the bottom is 15¢, for the top is 10¢, and for the sides is 2.5¢. What dimensions will minimize the cost?



38. Maximizing area. A publisher decides that each page in a new book must have an area of 73.125 in^2, a 0.75-in. margin at the top and at the bottom of each page, and a 0.5-in. margin on each of the sides. What should the outside dimensions of each page be so that the printed area is a maximum?

39. Minimizing inventory costs. A sporting goods store sells 100 pool tables per year. It costs $20 to store one pool table for a year. To reorder, there is a fixed cost of $40 per shipment plus $16 for each pool table. How many times per year should the store order pool tables, and in what lot size, in order to minimize inventory costs?

40. Minimizing inventory costs. A pro shop in a bowling center sells 200 bowling balls per year. It costs $4 to store one bowling ball for a year. To reorder, there is a fixed cost of $1, plus $0.50 for each bowling ball. How many times per year should the shop order bowling balls, and in what lot size, in order to minimize inventory costs?

41. Minimizing inventory costs. A retail outlet for Boxowitz Calculators sells 720 calculators per year. It costs $2 to store one calculator for a year. To reorder, there is a fixed cost of $5, plus $2.50 for each calculator. How many times per year should the store order calculators, and in what lot size, in order to minimize inventory costs?

42. Minimizing inventory costs. Bon Temps Surf and Scuba Shop sells 360 surfboards per year. It costs $8 to store one surfboard for a year. Each reorder costs $10, plus an additional $5 for each surfboard ordered. How many times per year should the store order surfboards, and in what lot size, in order to minimize inventory costs?

43. Minimizing inventory costs. Repeat Exercise 41 using the same data, but assume yearly sales of 256 calculators with the fixed cost of each reorder set at $4.

44. Minimizing inventory costs. Repeat Exercise 42 using the same data, but change the reorder costs from an additional $5 per surfboard to $6 per surfboard.

45. Minimizing surface area. A closed-top cylindrical container is to have a volume of 250 in^2. What dimensions (radius and height) will minimize the surface area?

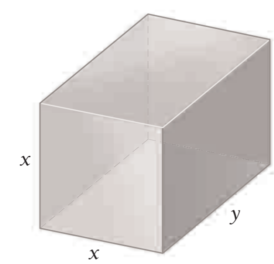
46. Minimizing surface area. An open-top cylindrical container is to have a volume of 400 cm^2. What dimensions (radius and height) will minimize the surface area?

47. Minimizing cost. Assume that the costs of the materials for making the cylindrical container described in Exercise 45 are $\$0.005/in^2$ for the circular base and top and $\$0.003/in^2$ for the wall. What dimensions will minimize the cost of materials?

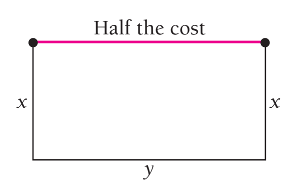
48. Minimizing cost. Assume that the costs of the materials for making the cylindrical container described in Exercise 46 are $\$0.0015/cm^2$ for the base and $\$0.008/cm^2$ for the wall. What dimensions will minimize the cost of materials?

General Interest

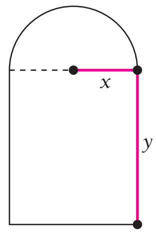
49. Maximizing volume. The postal service places a limit of 84 in. on the combined length and girth of (distance around) a package to be sent parcel post. What dimensions of a rectangular box with square cross-section will contain the largest volume that can be mailed? (*Hint*: There are two different girths.)



50. Minimizing cost. A rectangular play area is to be fenced off in a person's yard and is to contain 48 yd^2. The next-door neighbor agrees to pay half the cost of the fence on the side of the play area that lies along the property line. What dimensions will minimize the cost of the fence?



51. Maximizing light. A Norman window is a rectangle with a semicircle on top. Suppose that the perimeter of a particular Norman window is to be 24 ft. What should its dimensions be in order to allow the maximum amount of light to enter through the window?



52. Maximizing light. Repeat Exercise 51, but assume that the semicircle is to be stained glass, which transmits only half as much light as clear glass does.

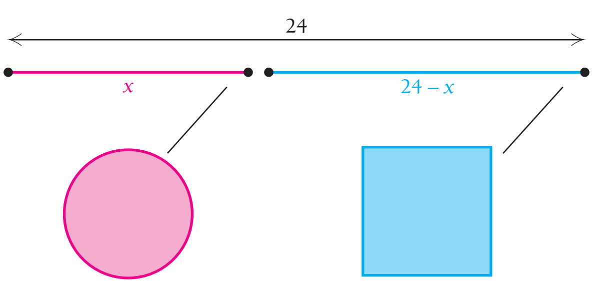
SYNTHESIS

53. For what positive number is the sum of its reciprocal and five times its square a minimum?

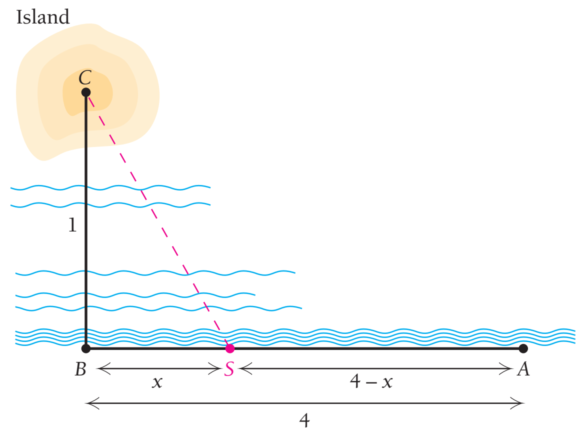
54. For what positive number is the sum of its reciprocal and four times its square a minimum?

55. Business: maximizing profit. The amount of money that customers deposit in a bank in savings accounts is directly proportional to the interest rate that the bank pays on that money. Suppose that a bank was able to turn around and loan out all the money deposited in its savings accounts at an interest rate of 18%. What interest rate should it pay on its savings accounts in order to maximize profit?

56. A 24-in. piece of wire is cut in two pieces. One piece is used to form a circle and the other to form a square. How should the wire be cut so that the sum of the areas is a minimum? A maximum?

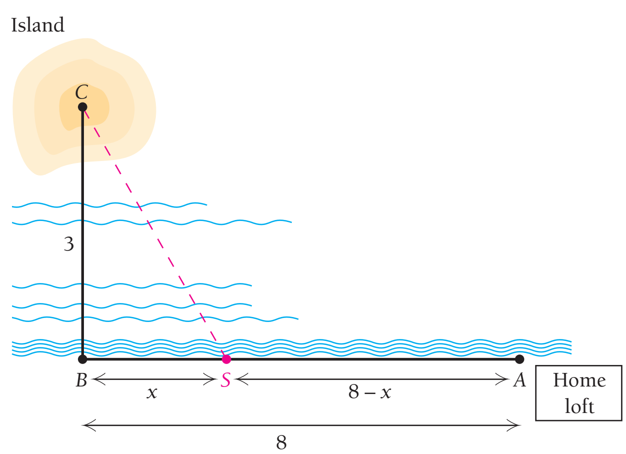


57. Business: minimizing costs. A power line is to be constructed from a power station at point A to an island at point C, which is 1 mi directly out in the water from a point B on the shore. Point B is 4 mi downshore from the power station at A. It costs \$5000 per mile to lay the power line under water and \$3000 per mile to lay the line under ground. At what point S downshore from A should the line come to the shore in order to minimize cost? Note that S could very well be B or A. (*Hint*: The length of CS is $\sqrt{1 + x^2}$.)

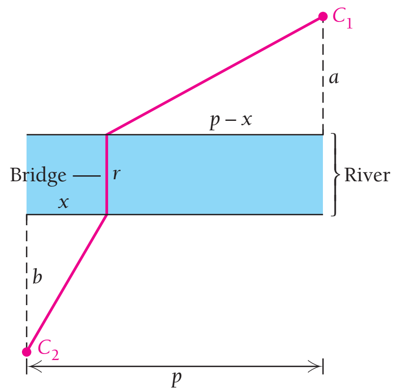


58. Life science: flights of homing pigeons. It is known that homing pigeons tend to avoid flying over water in the daytime, perhaps because the downdrafts of air over water make flying difficult. Suppose that a homing pigeon is released on an island at point C, which is 3 mi directly out in the water from a point B on shore. Point B is 8 mi downshore from the pigeon's home loft at point A. Assume that a pigeon flying over water uses energy at a rate 1.28 times the rate over land. Toward what point S downshore from A should the pigeon fly in order to minimize the total energy required to get to the home loft at A? Assume that

Total energy =
(Energy rate over water) · (Distance over water)
+ (Energy rate over land) · (Distance over land).



59. Business: minimizing distance. A road is to be built between two cities C_1 and C_2, which are on opposite sides of a river of uniform width r. C_1 is a units from the river, and C_2 is b units from the river, with $a \leq b$. A bridge will carry the traffic across the river. Where should the bridge be located in order to minimize the total distance between the cities? Give a general solution using the constants a, b, p, and r as shown in the figure.



60. Business: minimizing cost. The total cost, in dollars, of producing x units of a certain product is given by

$$C(x) = 8x + 20 + \frac{x^3}{100}.$$

a) Find the average cost, $A(x) = C(x)/x$.
b) Find $C'(x)$ and $A'(x)$.
c) Find the minimum of $A(x)$ and the value x_0 at which it occurs. Find $C'(x_0)$.
d) Compare $A(x_0)$ and $C'(x_0)$.

61. Business: minimizing cost. Consider

$$A(x) = C(x)/x.$$

a) Find $A'(x)$ in terms of $C'(x)$ and $C(x)$.
b) Show that if $A(x)$ has a minimum, then it will occur at that value of x_0 for which

$$C'(x_0) = A(x_0) = \frac{C(x_0)}{x_0}.$$

This result shows that if average cost can be minimized, such a minimum will occur when marginal cost equals average cost.

62. Minimize $Q = x^3 + 2y^3$, where x and y are positive numbers, such that $x + y = 1$.

63. Minimize $Q = 3x + y^3$, where $x^2 + y^2 = 2$.

64. Business: minimizing inventory costs—a general solution. A store sells Q units of a product per year. It costs a dollars to store one unit for a year. To reorder, there is a fixed cost of b dollars, plus c dollars for each unit. How many times per year should the store reorder, and in what lot size, in order to minimize inventory costs?

65. Business: minimizing inventory costs. Use the general solution found in Exercise 64 to find how many times per year a store should reorder, and in what lot size, when $Q = 2500$, $a = \$10$, $b = \$20$, and $c = \$9$.

Answers to Quick Checks

1. With 50 ft of fencing, the dimensions are 25 ft by 25 ft (625 ft^2 area); with 100 ft of fencing, they are 50 ft by 50 ft (2500 ft^2 area); in general, n feet of fencing gives $n/2$ ft by $n/2$ ft ($n^2/4$ ft^2 area).
2. The dimensions are approximately 1.585 in. by 5.33 in. by 7.83 in.; the volume is 66.15 in^3, or 1083.5 cm^3, slightly more than 1 L.
3. $r \approx 5.42$ cm, $h \approx 10.84$ cm, surface area $\approx 553.58\ cm^2$; the relationship is $h = 2r$ (height equals diameter).
4. **(a)** $R(x) = 1750x - 2x^2$
(b) $P(x) = -2x^2 + 1735x - 2250$ **(c)** $x = 434$ units
(d) Maximum profit $= \$374{,}028$
(e) Price per unit $= \$882.00$
5. \$23.50 **6.** $x \approx 63$; the store should place 8 orders for 63 sets and 39 orders for 64 sets.

2.6 Marginals and Differentials

OBJECTIVES

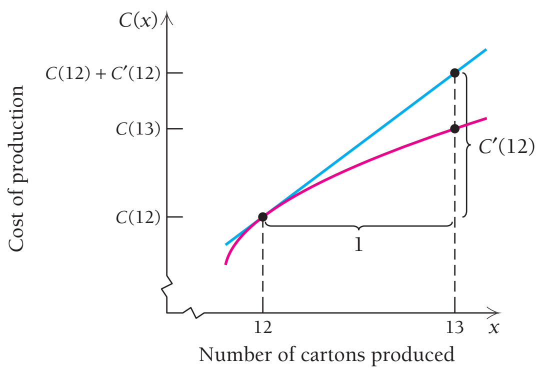
- Find marginal cost, revenue, and profit.
- Find Δy and dy.
- Use differentials for approximations.

In this section, we consider ways of using calculus to make linear approximations. Suppose, for example, that a company is considering an increase in production. Usually the company wants at least an approximation of what the resulting changes in *cost*, *revenue*, and *profit* will be.

Marginal Cost, Revenue, and Profit

Suppose that a band is producing its own CD and considering an increase in monthly production from 12 cartons to 13. To estimate the resulting increase in cost, it would be reasonable to find the rate at which cost is increasing when 12 cartons are produced and add that to the cost of producing 12 cartons. That is,

$$C(13) \approx C(12) + C'(12).$$



The number $C'(12)$ is called the *marginal cost at 12*. Remember that $C'(12)$ is the slope of the tangent line at the point $(12, C(12))$. If, for example, this slope is $\frac{3}{4}$, we can regard it as a vertical change of 3 with a horizontal change of 4, or a vertical change of $\frac{3}{4}$ with a horizontal change of 1. It is this latter interpretation that we use for estimating. Graphically, this interpretation can be viewed as shown at the left. Note in the figure that $C'(12)$ is slightly more than the difference between $C(13)$ and $C(12)$, or $C(13) - C(12)$. For other curves, $C'(12)$ may be slightly less than $C(13) - C(12)$. Almost always, however, it is simpler to compute $C'(12)$ than it is to compute $C(13) - C(12)$.

Generalizing, we have the following.

DEFINITIONS

Let $C(x)$, $R(x)$, and $P(x)$ represent, respectively, the total cost, revenue, and profit from the production and sale of x items.

The **marginal cost*** at x, given by $C'(x)$, is the approximate cost of the $(x + 1)$st item:

$$C'(x) \approx C(x + 1) - C(x), \text{ or } C(x + 1) \approx C(x) + C'(x).$$

The **marginal revenue** at x, given by $R'(x)$, is the approximate revenue from the $(x + 1)$st item:

$$R'(x) \approx R(x + 1) - R(x), \text{ or } R(x + 1) \approx R(x) + R'(x).$$

The **marginal profit** at x, given by $P'(x)$, is the approximate profit from the $(x + 1)$st item:

$$P'(x) \approx P(x + 1) - P(x), \text{ or } P(x + 1) \approx P(x) + P'(x).$$

You can confirm that $P'(x) = R'(x) - C'(x)$.

Exampl E 1 **Business: Marginal Cost, Revenue, and Profit.** Given

$$C(x) = 62x^2 + 27{,}500 \quad \text{and}$$
$$R(x) = x^3 - 12x^2 + 40x + 10,$$

find each of the following.

a) Total profit, $P(x)$

b) Total cost, revenue, and profit from the production and sale of 50 units of the product

c) The marginal cost, revenue, and profit when 50 units are produced and sold

Solution

a) Total profit $= P(x) = R(x) - C(x)$

$$= x^3 - 12x^2 + 40x + 10 - (62x^2 + 27{,}500)$$
$$= x^3 - 74x^2 + 40x - 27{,}490$$

*The term “marginal” comes from the Marginalist School of Economic Thought, which originated in Austria for the purpose of applying mathematics and statistics to the study of economics.

TECHNOLOGY CONNECTION

Business: Marginal Revenue, Cost, and Profit

EXERCISE

1. Using the viewing window $[0, 100, 0, 2000]$, graph these total-revenue and total-cost functions:

$$R(x) = 50x - 0.5x^2$$

and

$$C(x) = 10x + 3.$$

Then find $P(x)$ and graph it using the same viewing window. Find $R'(x)$, $C'(x)$, and $P'(x)$, and graph them using $[0, 60, 0, 60]$. Then find $R(40)$, $C(40)$, $P(40)$, $R'(40)$, $C'(40)$, and $P'(40)$. Which marginal function is constant?

b) $C(50) = 62 \cdot 50^2 + 27{,}500 = \$182{,}500$ (the total cost of producing the first 50 units);

$R(50) = 50^3 - 12 \cdot 50^2 + 40 \cdot 50 + 10 = \$97{,}010$ (the total revenue from the sale of the first 50 units);

$$\begin{aligned} P(50) &= R(50) - C(50) \\ &= \$97{,}010 - \$182{,}500 \\ &= -\$85{,}490 \end{aligned}$$

We could also use $P(x)$ from part (a).

There is a *loss* of \$85,490 when 50 units are produced and sold.

c) $C'(x) = 124x$, so $C'(50) = 124 \cdot 50 = \$6200$. Once 50 units have been made, the approximate cost of the 51st unit (marginal cost) is \$6200.

$$R'(x) = 3x^2 - 24x + 40, \text{ so } R'(50) = 3 \cdot 50^2 - 24 \cdot 50 + 40 = \$6340.$$

Once 50 units have been sold, the approximate revenue from the 51st unit (marginal revenue) is \$6340.

$$P'(x) = 3x^2 - 148x + 40, \text{ so } P'(50) = 3 \cdot 50^2 - 148 \cdot 50 + 40 = \$140.$$

Once 50 units have been produced and sold, the approximate profit from the sale of the 51st item (marginal profit) is \$140.

Often, in business, formulas for $C(x)$, $R(x)$, and $P(x)$ are not known, but information may exist about the cost, revenue, and profit trends at a particular value $x = a$. For example, $C(a)$ and $C'(a)$ may be known, allowing a reasonable prediction to be made about $C(a + 1)$. In a similar manner, predictions can be made for $R(a + 1)$ and $P(a + 1)$. In Example 1, formulas *do* exist, so it is possible to see how accurate our predictions were. We check $C(51) - C(50)$ and leave the checks of $R(51) - R(50)$ and $P(51) - P(50)$ for you (see the Technology Connection below, at left):

$$\begin{aligned} C(51) - C(50) &= 62 \cdot 51^2 + 27{,}500 - (62 \cdot 50^2 + 27{,}500) \\ &= 6262, \end{aligned}$$

whereas $\quad C'(50) = 6200.$

In this case, $C'(50)$ provides an approximation of $C(51) - C(50)$ that is within 1% of the actual value.

Note that marginal cost is different from *average* cost:

$$\text{Average cost per unit for 50 units} = \frac{C(50)}{50} = \frac{182{,}500}{50} = \$3650,$$

$C(50)$ ← Total cost of 50 units

50 ← The number of units, 50

whereas

$$\begin{aligned} \text{Marginal cost when 50 units are produced} &= \$6200 \\ &\approx \text{cost of the 51st unit.} \end{aligned}$$

TECHNOLOGY CONNECTION

To check the accuracy of $R'(50)$ as an estimate of $R(51) - R(50)$, let $y_1 = x^3 - 12x^2 + 40x + 10$, $y_2 = y_1(x + 1) - y_1(x)$, and $y_3 = \text{nDeriv}(y_1, x, x)$. By using TABLE with Indpnt: Ask, we can display a table in which y_2 (the difference between $y_1(x + 1)$ and $y_1(x)$) can be compared with $y_1'(x)$.

X	Y2	Y3
40	3989	3880
48	5933	5800
50	6479	6340

X =

EXERCISE

1. Create a table to check the effectiveness of using $P'(50)$ to approximate $P(51) - P(50)$.

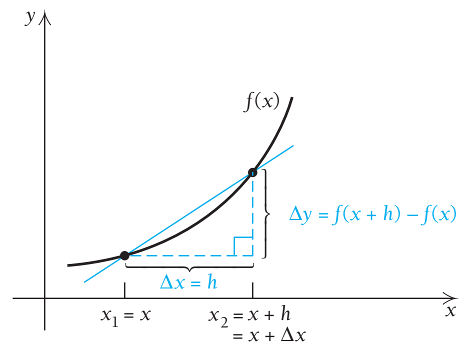
Differentials and Delta Notation

Just as the marginal cost $C'(x_0)$ can be used to estimate $C(x_0 + 1)$, the value of the derivative of any continuous function, $f'(x_0)$, can be used to estimate values of $f(x)$ for x-values near x_0. Before we do so, however, we need to develop some notation.

Recall the difference quotient

$$\frac{f(x + h) - f(x)}{h},$$

illustrated in the graph at the right. The difference quotient is used to define the derivative of a function at x. The number h is considered to be a *change* in x. Another notation for such a change is Δx, read "delta x" and called **delta notation.** The expression Δx is *not* the product of Δ and x; it is a new type of variable that represents the *change* in the value of x from a *first* value to a *second*. Thus,



$$\Delta x = (x + h) - x = h.$$

If subscripts are used for the first and second values of x, we have

$$\Delta x = x_2 - x_1, \quad \text{or} \quad x_2 = x_1 + \Delta x.$$

Note that the value of Δx can be positive or negative. For example,

if $x_1 = 4$ and $\Delta x = 0.7$, then $x_2 = 4.7$,

and if $x_1 = 4$ and $\Delta x = -0.7$, then $x_2 = 3.3$.

We generally omit the subscripts and use x and $x + \Delta x$. Now suppose that we have a function given by $y = f(x)$. A change in x from x to $x + \Delta x$ yields a change in y from $f(x)$ to $f(x + \Delta x)$. The change in y is given by

$$\Delta y = f(x + \Delta x) - f(x).$$

Exampl E 2 For $y = x^2$, $x = 4$, and $\Delta x = 0.1$, find Δy.

Solution We have

$$\Delta y = (4 + 0.1)^2 - 4^2$$
$$= (4.1)^2 - 4^2 = 16.81 - 16 = 0.81.$$

Exampl E 3 For $y = x^3$, $x = 2$, and $\Delta x = -0.1$, find Δy.

Solution We have

$$\Delta y = [2 + (-0.1)]^3 - 2^3$$
$$= (1.9)^3 - 2^3 = 6.859 - 8 = -1.141.$$

❮ Quick Check 1

❯ Quick Check 1

For $y = 2x^4 + x$, $x = 2$, and $\Delta x = -0.05$, find Δy.

Let's now use calculus to predict function values. If delta notation is used, the difference quotient

$$\frac{f(x + h) - f(x)}{h}$$

becomes

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}.$$

We can then express the derivative as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Note that the delta notation resembles Leibniz notation (see Section 1.5).

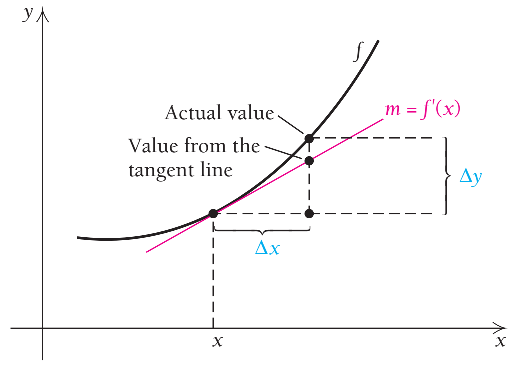
For values of Δx close to 0, we have the approximation

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}, \quad \text{or} \quad f'(x) \approx \frac{\Delta y}{\Delta x}.$$

Multiplying both sides of the second expression by Δx gives us

$$\Delta y \approx f'(x)\,\Delta x.$$

We can see this in the graph at the right.



From this graph, it seems reasonable to assume that, for small values of Δx, the y-values on the tangent line can be used to estimate function values on the curve.

For f, a continuous, differentiable function, and small Δx,

$$f'(x) \approx \frac{\Delta y}{\Delta x} \quad \text{and} \quad \Delta y \approx f'(x) \cdot \Delta x.$$

Let's illustrate this idea by considering the square-root function, $f(x) = \sqrt{x}$. We know how to approximate $\sqrt{27}$ using a calculator. But suppose we didn't. We could begin with $\sqrt{25}$ and use as a change in input $\Delta x = 2$. We would use the corresponding change in y, that is, $\Delta y \approx f'(x)\,\Delta x$, to estimate $\sqrt{27}$.

Exampl E 4 Approximate $\sqrt{27}$ using $\Delta y \approx f'(x)\,\Delta x$.

Solution We first think of the number closest to 27 that is a perfect square. This is 25. What we will do is approximate how $y = \sqrt{x}$ changes when x changes from 25 to 27.

From the box above, we have

$$\begin{aligned} \Delta y &\approx f'(x) \cdot \Delta x \\ &= \tfrac{1}{2}x^{-1/2} \cdot \Delta x \end{aligned} \quad \Big\} \text{ Using } y = \sqrt{x} = x^{1/2} \text{ as } f(x)$$

We are interested in Δy as x changes from 25 to 27, so

$$\begin{aligned} \Delta y &\approx \frac{1}{2\sqrt{25}} \cdot 2 \qquad \text{Replacing } x \text{ with 25 and } \Delta x \text{ with 2} \\ &= \frac{1}{5} = 0.2. \end{aligned}$$

We can now approximate $\sqrt{27}$:

$$\begin{aligned} \sqrt{27} &= \sqrt{25} + \Delta y \\ &= 5 + \Delta y \\ &\approx 5 + 0.2 \\ &\approx 5.2. \end{aligned}$$

To five decimal places, $\sqrt{27} = 5.19615$. Thus, our approximation is fairly accurate.

❮ Quick Check 2

❯ Quick Check 2

Approximate $\sqrt{98}$ using $\Delta y \approx f'(x)\,\Delta x$. To five decimal places, $\sqrt{98} = 9.89949$. How close is your approximation?

Up to now, we have not defined the symbols dy and dx as separate entities, but have treated dy/dx as one symbol. We now define dy and dx. These symbols are called **differentials**.

DEFINITION

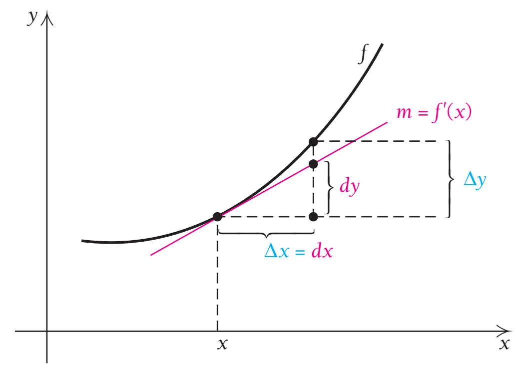
For $y = f(x)$, we define

dx, called the **differential of x**, by $dx = \Delta x$

and

dy, called the **differential of y**, by $dy = f'(x)\,dx$.

We can illustrate dx and dy as shown at the right. Note that $dx = \Delta x$, but $dy \neq \Delta y$, though $dy \approx \Delta y$, for small values of dx.



Exampl E 5

For $y = x(4 - x)^3$:

a) Find dy.

b) Find dy when $x = 5$ and $dx = 0.01$.

c) Compare dy to Δy.

Solution

a) First, we find dy/dx:

$$\frac{dy}{dx} = x[3(4 - x)^2(-1)] + (4 - x)^3 \quad \text{Using the Product and Chain Rules}$$
$$= -3x(4 - x)^2 + (4 - x)^3$$
$$= (4 - x)^2[-3x + (4 - x)] \quad \text{Factoring out } (4 - x)^2$$
$$= (4 - x)^2[-4x + 4]$$
$$= -4(4 - x)^2(x - 1). \quad \text{Factoring out } -4$$

Then we solve for dy:

$$dy = -4(4 - x)^2(x - 1)\,dx.$$

b) When $x = 5$ and $dx = 0.01$,

$$dy = -4(4 - 5)^2(5 - 1)(0.01) = -4(-1)^2(4)(0.01) = -0.16.$$

c) The value $dy = -0.16$ is the approximate change in y between $x_1 = 5$ and $x_2 = 5.01$ (that is, $x_2 = x_1 + dx = 5 + 0.01$). The actual change in y is determined by evaluating the function for x_2 and x_1 and subtracting:

$$\Delta y = [5.01(4 - 5.01)^3] - [5(4 - 5)^3]$$
$$= [5.01(-1.01)^3] - [5(-1)^3]$$
$$= [5.01(-1.030301)] - [5(-1)]$$
$$= -0.16180801.$$

We see that the approximation dy and the actual change Δy are reasonably close. It is easier to calculate the approximation since that involves fewer steps, but the trade-off is some loss in accuracy. As long as dx is small, this loss in accuracy is acceptable for many applications.

Differentials are often used in applications involving measurements and tolerance. When we measure an object, we accept that our measurements are not exact, and we allow for a small tolerance in our measurements. If x represents a measurement (a length, a weight, a volume, etc.), then dx represents the tolerance. Even a small tolerance for the input can have a significant effect on the output, as the following example shows.

Exampl E 6 **Business: Cost and Tolerance.** In preparation for laying new tile, Michelle measures the floor of a large conference room and finds it to be square, measuring 100 ft by 100 ft. Suppose her measurements are accurate to ± 6 in. (the tolerance).

a) Use a differential to estimate the difference in area (dA) due to the tolerance.

b) Compare the result from part (a) with the actual difference in area (ΔA).

c) If each tile covers 1 ft^2 and a box of 12 tiles costs \$24, how much extra cost should Michelle allot for the potential overage in floor area?

Solution

a) The floor is a square, with a presumed measurement of 100 ft per side and a tolerance of ± 6 in. $= \pm 0.5$ ft. The area A in square feet (ft^2) for a square of side length x ft is

$$A(x) = x^2.$$

The derivative is $dA/dx = 2x$, and solving for dA gives the differential of A:

$$dA = 2x\,dx.$$

To find dA, we substitute $x = 100$ and $dx = \pm 0.5$:

$$dA = 2(100)(\pm 0.5) = \pm 100.$$

The value of dA is interpreted as the approximate difference in area due to the inexactness in measuring. Therefore, if Michelle's measurements are off by half a foot, the total area can differ by approximately ± 100 ft^2. A small "error" in measurement can lead to quite a large difference in the resulting area.

b) The actual difference in area (ΔA) is calculated directly. We set $x_1 = 100$ ft, the presumed length measurement and let x_2 represent the length plus or minus the tolerance.

If the true length is at the low end, we have $x_2 = 99.5$ ft, that is, 100 ft minus the tolerance of 0.5 ft. The floor's area is then $99.5^2 = 9900.25$ ft^2. The actual difference in area is

$$\begin{aligned}\Delta A &= A(x_2) - A(x_1)\\ &= A(99.5) - A(100)\\ &= 99.5^2 - 100^2\\ &= 9900.25 - 10{,}000\\ &= -99.75\text{ ft}^2.\end{aligned}$$

Thus, the actual difference in area is $\Delta A = -99.75$ ft^2, which compares well with the approximate value of $dA = -100$ ft^2.

If the true length is at the high end, we have $x_2 = 100.5$ ft. The floor's area is then $100.5^2 = 10{,}100.25$ ft^2. The actual difference in area is

$$\begin{aligned}\Delta A &= A(x_2) - A(x_1)\\ &= A(100.5) - A(100)\\ &= 100.5^2 - 100^2\\ &= 10{,}100.25 - 10{,}000\\ &= 100.25 \text{ ft}^2.\end{aligned}$$

In this case, the actual difference in area is $\Delta A = 100.25$ ft^2, which again compares well with the approximate value of $dA = +100$ ft^2.

c) The tiles (each measuring 1 ft^2) come 12 to a box. Thus, if the room were exactly 100 ft by 100 ft (an area of 10,000 ft^2), Michelle would need $10{,}000/12 = 833.33\ldots$, or 834 boxes to cover the floor. To take into account the possibility that the room is larger by 100 ft^2, she needs a total of $10{,}100/12 = 841.67\ldots$, or 842 boxes of tiles. Therefore, she should buy 8 extra boxes of tiles, meaning an extra cost of $(8)(24) = \$192$.

❯ Quick Check 3

The four walls of a room measure 10 ft by 10 ft each, with a tolerance of ± 0.25 ft.

a) Calculate the approximate difference in area, dA, for the four walls.

b) Workers will be texturing the four walls using "knockdown" spray. Each bottle of knockdown spray costs \$9 and covers 12 ft^2. How much extra cost for knockdown spray should the workers allot for the potential overage in wall area?

❮ Quick Check 3

We see that there is an advantage to using a differential to calculate an approximate difference in an output variable. There is less actual calculating, and the result is often quite accurate. Compare the arithmetic steps needed in parts (a) and (b) of Example 6. Even though dA is an approximation, it is accurate enough for Michelle's needs: it is sufficient for her to know that the area can be off by as much as "about" 100 ft^2.

Historically, differentials were quite valuable when used to make approximations. However, with the advent of computers and graphing calculators, such use has diminished considerably. The use of marginals remains important in the study of business and economics.

Section Summary

- If $C(x)$ represents the cost for producing x items, then *marginal cost* $C'(x)$ is its derivative, and $C'(x) \approx C(x+1) - C(x)$. Thus, the cost to produce the $(x+1)$st item can be approximated by $C(x+1) \approx C(x) + C'(x)$.
- If $R(x)$ represents the revenue from selling x items, then *marginal revenue* $R'(x)$ is its derivative, and $R'(x) \approx R(x+1) - R(x)$. Thus, the revenue from the $(x+1)$st item can be approximated by $R(x+1) \approx R(x) + R'(x)$.
- If $P(x)$ represents profit from selling x items, then *marginal profit* $P'(x)$ is its derivative, and $P'(x) \approx P(x+1) - P(x)$. Thus, the profit from the $(x+1)$st item can be approximated by $P(x+1) \approx P(x) + P'(x)$.
- In general, profit = revenue − cost, or $P(x) = R(x) - C(x)$.
- In *delta notation*, $\Delta x = (x+h) - x = h$, and $\Delta y = f(x+h) - f(x)$. For small values of Δx, we have $\dfrac{\Delta y}{\Delta x} \approx f'(x)$, which is equivalent to $\Delta y \approx f'(x)\,\Delta x$.
- The *differential* of x is $dx = \Delta x$. Since $\dfrac{dy}{dx} = f'(x)$, we have $dy = f'(x)\,dx$. In general, $dy \approx \Delta y$, and the approximation can be very close for sufficiently small dx.

EXERCISE SET 2.6

APPLICATIONS

Business and Economics

1. Marginal revenue, cost, and profit. Let $R(x)$, $C(x)$, and $P(x)$ be, respectively, the revenue, cost, and profit, in dollars, from the production and sale of x items. If

$$R(x) = 5x \quad \text{and} \quad C(x) = 0.001x^2 + 1.2x + 60,$$

find each of the following.

a) $P(x)$
b) $R(100)$, $C(100)$, and $P(100)$
c) $R'(x)$, $C'(x)$, and $P'(x)$
d) $R'(100)$, $C'(100)$, and $P'(100)$
e) Describe in words the meaning of each quantity in parts (b) and (d).

2. Marginal revenue, cost, and profit. Let $R(x)$, $C(x)$, and $P(x)$ be, respectively, the revenue, cost, and profit, in dollars, from the production and sale of x items. If

$$R(x) = 50x - 0.5x^2 \quad \text{and} \quad C(x) = 4x + 10,$$

find each of the following.

a) $P(x)$
b) $R(20)$, $C(20)$, and $P(20)$
c) $R'(x)$, $C'(x)$, and $P'(x)$
d) $R'(20)$, $C'(20)$, and $P'(20)$

3. Marginal cost. Suppose that the monthly cost, in dollars, of producing x chairs is

$$C(x) = 0.001x^3 + 0.07x^2 + 19x + 700,$$

and currently 25 chairs are produced monthly.

a) What is the current monthly cost?
b) What would be the additional cost of increasing production to 26 chairs monthly?
c) What is the marginal cost when $x = 25$?
d) Use marginal cost to estimate the difference in cost between producing 25 and 27 chairs per month.
e) Use the answer from part (d) to predict $C(27)$.

4. Marginal cost. Suppose that the daily cost, in dollars, of producing x radios is

$$C(x) = 0.002x^3 + 0.1x^2 + 42x + 300,$$

and currently 40 radios are produced daily.

a) What is the current daily cost?
b) What would be the additional daily cost of increasing production to 41 radios daily?
c) What is the marginal cost when $x = 40$?
d) Use marginal cost to estimate the daily cost of increasing production to 42 radios daily.

5. Marginal revenue. Pierce Manufacturing determines that the daily revenue, in dollars, from the sale of x lawn chairs is

$$R(x) = 0.005x^3 + 0.01x^2 + 0.5x.$$

Currently, Pierce sells 70 lawn chairs daily.

a) What is the current daily revenue?
b) How much would revenue increase if 73 lawn chairs were sold each day?
c) What is the marginal revenue when 70 lawn chairs are sold daily?
d) Use the answer from part (c) to estimate $R(71)$, $R(72)$, and $R(73)$.

6. Marginal profit. For Sunshine Motors, the weekly profit, in dollars, of selling x cars is

$$P(x) = -0.006x^3 - 0.2x^2 + 900x - 1200,$$

and currently 60 cars are sold weekly.

a) What is the current weekly profit?
b) How much profit would be lost if the dealership were able to sell only 59 cars weekly?
c) What is the marginal profit when $x = 60$?
d) Use marginal profit to estimate the weekly profit if sales increase to 61 cars weekly.

7. Marginal profit. Crawford Computing finds that its weekly profit, in dollars, from the production and sale of x laptop computers is

$$P(x) = -0.004x^3 - 0.3x^2 + 600x - 800.$$

Currently Crawford builds and sells 9 laptops weekly.

a) What is the current weekly profit?
b) How much profit would be lost if production and sales dropped to 8 laptops weekly?
c) What is the marginal profit when $x = 9$?
d) Use the answers from parts (a)–(c) to estimate the profit resulting from the production and sale of 10 laptops weekly.

8. Marginal revenue. Solano Carriers finds that its monthly revenue, in dollars, from the sale of x carry-on suitcases is

$$R(x) = 0.007x^3 - 0.5x^2 + 150x.$$

Currently Solano is selling 26 carry-on suitcases monthly.

a) What is the current monthly revenue?
b) How much would revenue increase if sales increased from 26 to 28 suitcases?
c) What is the marginal revenue when 26 suitcases are sold?
d) Use the answers from parts (a)–(c) to estimate the revenue resulting from selling 27 suitcases per month.

9. Sales. Let $N(x)$ be the number of computers sold annually when the price is x dollars per computer. Explain in words what occurs if $N(1000) = 500{,}000$ and $N'(1000) = -100$.

10. Sales. Estimate the number of computers sold in Exercise 9 if the price is raised to \$1025.

For Exercises 11–16, assume that $C(x)$ and $R(x)$ are in dollars and x is the number of units produced and sold.

11. For the total-cost function

$$C(x) = 0.01x^2 + 0.6x + 30,$$

find ΔC and $C'(x)$ when $x = 70$ and $\Delta x = 1$.

12. For the total-cost function

$$C(x) = 0.01x^2 + 1.6x + 100,$$

find ΔC and $C'(x)$ when $x = 80$ and $\Delta x = 1$.

13. For the total-revenue function

$$R(x) = 2x,$$

find ΔR and $R'(x)$ when $x = 70$ and $\Delta x = 1$.

14. For the total-revenue function

$$R(x) = 3x,$$

find ΔR and $R'(x)$ when $x = 80$ and $\Delta x = 1$.

15. a) Using $C(x)$ from Exercise 11 and $R(x)$ from Exercise 13, find the total profit, $P(x)$.

b) Find ΔP and $P'(x)$ when $x = 70$ and $\Delta x = 1$.

16. a) Using $C(x)$ from Exercise 12 and $R(x)$ from Exercise 14, find the total profit, $P(x)$.

b) Find ΔP and $P'(x)$ when $x = 80$ and $\Delta x = 1$.

17. Marginal supply. The supply, S, of a new rollerball pen is given by

$$S = 0.007p^3 - 0.5p^2 + 150p,$$

where p is the price in dollars.

a) Find the rate of change of quantity with respect to price, dS/dp.

b) How many units will producers want to supply when the price is \$25 per unit?

c) Find the rate of change at $p = 25$, and interpret this result.

d) Would you expect dS/dp to be positive or negative? Why?

18. Marginal productivity. An employee's monthly productivity, M, in number of units produced, is found to be a function of the number of years of service, t. For a certain product, the productivity function is given by

$$M(t) = -2t^2 + 100t + 180.$$

a) Find the productivity of an employee after 5 yr, 10 yr, 25 yr, and 45 yr of service.

b) Find the marginal productivity.

c) Find the marginal productivity at $t = 5$; $t = 10$; $t = 25$; $t = 45$; and interpret the results.

d) Explain how the employee's marginal productivity might be related to experience and to age.

19. Average cost. The average cost for a company to produce x units of a product is given by the function

$$A(x) = \frac{13x + 100}{x}.$$

Use $A'(x)$ to estimate the change in average cost as production goes from 100 units to 101 units.

20. Supply. A supply function for a certain product is given by

$$S(p) = 0.08p^3 + 2p^2 + 10p + 11,$$

where $S(p)$ is the number of items produced when the price is p dollars. Use $S'(p)$ to estimate how many more units a producer will supply when the price changes from \$18.00 per unit to \$18.20 per unit.

21. Gross domestic product. The U.S. gross domestic product, in billions of current dollars, may be modeled by the function

$$P(x) = 567 + x(36x^{0.6} - 104),$$

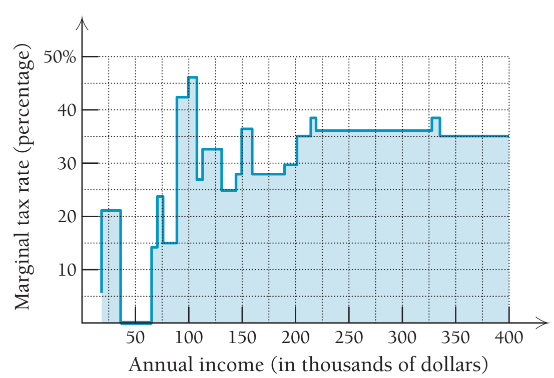
where x is the number of years since 1960. (*Source*: U.S. Bureau for Economic Analysis.) Use $P'(x)$ to estimate how much the gross domestic product increased from 2009 to 2010.

22. Advertising. Norris Inc. finds that it sells N units of a product after spending x thousands of dollars on advertising, where

$$N(x) = -x^2 + 300x + 6.$$

Use $N'(x)$ to estimate how many more units Norris will sell by increasing its advertising expenditure from \$100,000 to \$101,000.

Marginal tax rate. *Businesses and individuals are frequently concerned about their marginal tax rate, or the rate at which the next dollar earned is taxed. In progressive taxation, the 80,001st dollar earned is taxed at a higher rate than the 25,001st dollar earned and at a lower rate than the 140,001st dollar earned. Use the graph below, showing the marginal tax rate for 2005, to answer Exercises 23–26.*



(*Source*: "Towards Fundamental Tax Reform" by Alan Auerbach and Kevin Hassett, *New York Times*, 5/5/05, p. C2.)

23. Was the taxation in 2005 progressive? Why or why not?

24. Marcy and Tyrone work for the same marketing agency. Because she is not yet a partner, Marcy's year-end income is approximately \$95,000; Tyrone's year-end income is approximately \$150,000. Suppose that one of them is to receive another \$5000 in income for the year. Which one would keep more of that \$5000 after taxes? Why?

25. Alan earns \$25,000 per year and is considering a second job that would earn him another \$2000 annually. At what rate will his tax liability (the amount he must pay in taxes) change if he takes the extra job? Express your answer in tax dollars paid per dollar earned.

26. Iris earns \$50,000 per year and is considering extra work that would earn her an extra \$3000 annually. At what rate will her tax liability grow if she takes the extra work (see Exercise 25)?

Find Δy and $f'(x)\Delta x$. Round to four and two decimal places, respectively.

27. For $y = f(x) = x^2$, $x = 2$, and $\Delta x = 0.01$

28. For $y = f(x) = x^3$, $x = 2$, and $\Delta x = 0.01$

29. For $y = f(x) = x + x^2$, $x = 3$, and $\Delta x = 0.04$

30. For $y = f(x) = x - x^2$, $x = 3$, and $\Delta x = 0.02$

31. For $y = f(x) = 1/x^2$, $x = 1$, and $\Delta x = 0.5$

32. For $y = f(x) = 1/x$, $x = 1$, and $\Delta x = 0.2$

33. For $y = f(x) = 3x - 1$, $x = 4$, and $\Delta x = 2$

34. For $y = f(x) = 2x - 3$, $x = 8$, and $\Delta x = 0.5$

Use $\Delta y \approx f'(x)\Delta x$ to find a decimal approximation of each radical expression. Round to three decimal places.

35. $\sqrt{26}$

36. $\sqrt{10}$

37. $\sqrt{102}$

38. $\sqrt{103}$

39. $\sqrt[3]{1005}$

40. $\sqrt[3]{28}$

Find dy.

41. $y = \sqrt{x + 1}$

42. $y = \sqrt{3x - 2}$

43. $y = (2x^3 + 1)^{3/2}$

44. $y = x^3(2x + 5)^2$

45. $y = \sqrt[5]{x + 27}$

46. $y = \dfrac{x^3 + x + 2}{x^2 + 3}$

47. $y = x^4 - 2x^3 + 5x^2 + 3x - 4$

48. $y = (7 - x)^8$

49. In Exercise 47, find dy when $x = 2$ and $dx = 0.1$.

50. In Exercise 48, find dy when $x = 1$ and $dx = 0.01$.

51. For $y = (3x - 10)^5$, find dy when $x = 4$ and $dx = 0.03$.

52. For $y = x^5 - 2x^3 - 7x$, find dy when $x = 3$ and $dx = 0.02$.

53. For $f(x) = x^4 - x^2 + 8$, use a differential to approximate $f(5.1)$.

54. For $f(x) = x^3 - 5x + 9$, use a differential to approximate $f(3.2)$.

SYNTHESIS

Life and Physical Sciences

55. **Body surface area.** Certain chemotherapy dosages depend on a patient's surface area. According to the Gehan and George model,

$$S = 0.02235h^{0.42246}w^{0.51456},$$

where h is the patient's height in centimeters, w is his or her weight in kilograms, and S is the approximation to his or her surface area in square meters. (*Source*: www.halls.md.) Joanne is 160 cm tall and weighs 60 kg. Use a differential to estimate how much her surface area changes after her weight decreases by 1 kg.

56. **Healing wound.** The circular area of a healing wound is given by $A = \pi r^2$, where r is the radius, in centimeters. By approximately how much does the area decrease when the radius is decreased from 2 cm to 1.9 cm? Use 3.14 for π.

57. **Medical dosage.** The function

$$N(t) = \frac{0.8t + 1000}{5t + 4}$$

gives the bodily concentration $N(t)$, in parts per million, of a dosage of medication after time t, in hours. Use differentials to determine whether the concentration changes more from 1.0 hr to 1.1 hr or from 2.8 hr to 2.9 hr.

General Interest

58. **Major League ticket prices.** The average ticket price of a major league baseball game can be modeled by the function

$$p(x) = 0.09x^2 - 0.19x + 9.41,$$

where x is the number of years after 1990. (*Source*: Major League Baseball.) Use differentials to predict whether ticket price will increase more between 2010 and 2012 or between 2030 and 2031.

59. Suppose that a rope surrounds the earth at the equator. The rope is lengthened by 10 ft. By about how much is the rope raised above the earth?

Business and Economics

60. **Marginal average cost.** In Section 1.6, we defined the average cost of producing x units of a product in terms of the total cost $C(x)$ by $A(x) = C(x)/x$. Find a general expression for *marginal average cost*, $A'(x)$.

61. **Cost and tolerance.** A painting firm contracts to paint the exterior of a large water tank in the shape of a half-dome (a hemisphere). The radius of the tank is measured to be 100 ft with a tolerance of ± 6 in. (± 0.5 ft). (The formula for the surface area of a hemisphere is $A = 2\pi r^2$; use 3.14 as an approximation for π.) Each can of paint costs \$30 and covers 300 ft^2.

a) Calculate dA, the approximate difference in the surface area due to the tolerance.

b) Assuming the painters cannot bring partial cans of paint to the job, how many extra cans should they bring to cover the extra area they may encounter?

c) How much extra should the painters plan to spend on paint to account for the possible extra area?

62. **Strategic oil supply.** The U.S. Strategic Petroleum Reserve (SPR) stores petroleum in large spherical caverns built into salt deposits along the Gulf of Mexico. (*Source*: U.S. Department of Energy.) These caverns can be enlarged by filling the void with water, which dissolves the surrounding salt, and then pumping brine out. Suppose a cavern has a radius of 400 ft, which engineers want to enlarge by 5 ft. Use a differential to estimate how much volume will be added to form the enlarged cavern. (The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$; use 3.14 as an approximation for π.)

Marginal revenue. *In each of Exercises 63–67, a demand function, $p = D(x)$, expresses price, in dollars, as a function of the number of items produced and sold. Find the marginal revenue.*

63. $p = 100 - \sqrt{x}$

64. $p = 400 - x$

65. $p = 500 - x$

66. $p = \dfrac{4000}{x} + 3$

67. $p = \dfrac{3000}{x} + 5$

68. Look up "differential" in a book or Web site devoted to math history. In a short paragraph, describe your findings.

69. Explain the uses of the differential.

Answers to Quick Checks

1. $\Delta y = -3.1319875$ **2.** $\sqrt{98} \approx \frac{99}{10}$, or 9.9. This is within 0.001 of the actual value of $\sqrt{98}$.
3. (a) $\pm 20\ \text{ft}^2$ **(b)** \$18 (for 2 extra bottles)

2.7 Implicit Differentiation and Related Rates*

OBJECTIVES

- Differentiate implicitly.
- Solve related-rate problems.

We often write a function with the output variable (usually y) isolated on one side of the equation. For example, if we write $y = x^3$, we have expressed y as an *explicit* function of x. Sometimes, with an equation like $y^3 + x^2y^5 - x^4 = 27$, it may be cumbersome or nearly impossible to isolate the output variable; in such a case, we have an implicit relationship between the variables x and y. Then, we can find the derivative of y with respect to x using a process called *implicit differentiation*.

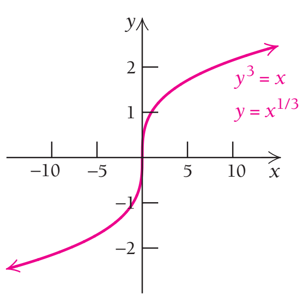
Implicit Differentiation

Consider the equation

$$y^3 = x.$$

This equation *implies* that y is a function of x, for if we solve for y, we get

$$y = \sqrt[3]{x} = x^{1/3}.$$



We know from our earlier work that

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3}. \qquad (1)$$

A method known as **implicit differentiation** allows us to find dy/dx *without* solving for y. To do so, we use the Chain Rule, treating y as a function of x, and differentiate both sides of

$$y^3 = x$$

with respect to x:

$$\frac{d}{dx}y^3 = \frac{d}{dx}x.$$

The derivative on the left side is found using the Extended Power Rule:

$$3y^2\frac{dy}{dx} = 1. \qquad \text{Remembering that the derivative of } y \text{ with respect to } x \text{ is written } dy/dx$$

Finally, we solve for dy/dx by dividing both sides by $3y^2$:

$$\frac{dy}{dx} = \frac{1}{3y^2}, \quad \text{or} \quad \frac{1}{3}y^{-2}.$$

We can show that this indeed gives us the same answer as equation (1) by replacing y with $x^{1/3}$:

$$\frac{dy}{dx} = \frac{1}{3}y^{-2} = \frac{1}{3}(x^{1/3})^{-2} = \frac{1}{3}x^{-2/3}.$$

*This section can be omitted without loss of continuity.

Often, it is difficult or impossible to solve for y and to express dy/dx solely in terms of x. For example, the equation

$$y^3 + x^2y^5 - x^4 = 27$$

determines y as a function of x, but it would be difficult to solve for y. We can nevertheless find a formula for the derivative of y *without* solving for y. To do so usually involves computing $\frac{d}{dx}y^n$ for various integers n, and hence involves the Extended Power Rule in the form

$$\frac{d}{dx}y^n = ny^{n-1} \cdot \frac{dy}{dx}.$$

TECHNOLOGY CONNECTION

Exploratory

Graphicus can be used to graph equations that relate x and y implicity. Press [+] and then [g(x,y)=0], and enter $y^3 + x^2y^3 - x^4 = 27$ as

y^3+x^2(y^3)-x^4=27

EXERCISES

Graph each equation.

1. $y^3 + x^2y^3 - x^4 = 27$
2. $y^2x + 2x^3y^3 = y + 1$

Exampl E 1

For $y^3 + x^2y^5 - x^4 = 27$:

a) Find dy/dx using implicit differentiation.

b) Find the slope of the tangent line to the curve at the point $(0, 3)$.

Solution

a) We differentiate the term x^2y^5 using the Product Rule. Because y is a function of x, it is critical that dy/dx is included as a factor in the result any time a term involving y is differentiated. When an expression involving just x is differentiated, there is no factor dy/dx.

$$\frac{d}{dx}(y^3 + x^2y^5 - x^4) = \frac{d}{dx}(27) \qquad \text{Differentiating both sides with respect to } x$$

$$\frac{d}{dx}y^3 + \frac{d}{dx}x^2y^5 - \frac{d}{dx}x^4 = 0$$

$$3y^2 \cdot \frac{dy}{dx} + x^2 \cdot 5y^4 \cdot \frac{dy}{dx} + y^5 \cdot 2x - 4x^3 = 0. \qquad \text{Using the Extended Power Rule and the Product Rule}$$

We next isolate those terms with dy/dx as a factor on one side:

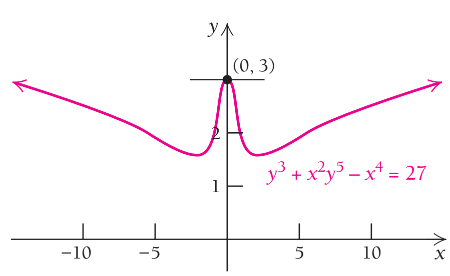
$$3y^2 \cdot \frac{dy}{dx} + 5x^2y^4 \cdot \frac{dy}{dx} = 4x^3 - 2xy^5 \qquad \text{Adding } 4x^3 - 2xy^5 \text{ to both sides}$$

$$(3y^2 + 5x^2y^4)\frac{dy}{dx} = 4x^3 - 2xy^5 \qquad \text{Factoring out } dy/dx$$

$$\frac{dy}{dx} = \frac{4x^3 - 2xy^5}{3y^2 + 5x^2y^4}. \qquad \text{Solving for } dy/dx \text{ and leaving the answer in terms of } x \text{ and } y$$

b) To find the slope of the tangent line to the curve at $(0, 3)$, we replace x with 0 and y with 3:

$$\frac{dy}{dx} = \frac{4 \cdot 0^3 - 2 \cdot 0 \cdot 3^5}{3 \cdot 3^2 + 5 \cdot 0^2 \cdot 3^4} = 0.$$



Quick Check 1

For $y^2x + 2x^3y^3 = y + 1$, find dy/dx using implicit differentiation.

Quick Check 1

It is not uncommon for the expression for dy/dx to contain *both* variables x and y. When using the derivative to calculate a slope, we must evaluate it at both the x-value and the y-value of the point of tangency.

The steps in Example 1 are typical of those used when differentiating implicitly.

To differentiate implicitly:

a) Differentiate both sides of the equation with respect to x (or whatever variable you are differentiating with respect to).
b) Apply the rules for differentiation (the Power, Product, Quotient, and Chain Rules) as necessary. Any time an expression involving y is differentiated, dy/dx will be a factor in the result.
c) Isolate all terms with dy/dx as a factor on one side of the equation.
d) If necessary, factor out dy/dx.
e) If necessary, divide both sides of the equation to isolate dy/dx.

The demand function for a product (see Section R.5) is often given implicitly.

Exampl E 2 For the demand equation $x = \sqrt{200 - p^3}$, differentiate implicitly to find dp/dx.

Solution

$$\frac{d}{dx}x = \frac{d}{dx}\sqrt{200 - p^3}$$

$$1 = \frac{1}{2}(200 - p^3)^{-1/2} \cdot (-3p^2) \cdot \frac{dp}{dx} \qquad \text{Using the Extended Power Rule twice}$$

$$1 = \frac{-3p^2}{2\sqrt{200 - p^3}} \cdot \frac{dp}{dx}$$

$$\frac{2\sqrt{200 - p^3}}{-3p^2} = \frac{dp}{dx}$$

Related Rates

Suppose that y is a function of x, say

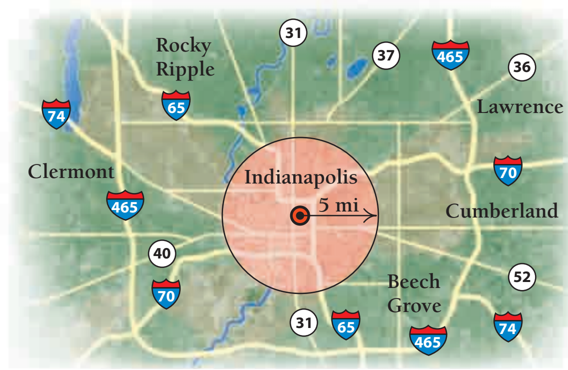
$$y = f(x),$$

and x is a function of time, t. Since y depends on x and x depends on t, it follows that y depends on t. The Chain Rule gives the following:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Thus, the rate of change of y is *related* to the rate of change of x. Let's see how this comes up in problems. It helps to keep in mind that any variable can be thought of as a function of time t, even though a specific expression in terms of t may not be given or its rate of change with respect to t may be 0.

Exampl E 3 **Business: Service Area.** A restaurant supplier services the restaurants in a circular area in such a way that the radius r is increasing at the rate of 2 mi per year at the moment when $r = 5$ mi. At that moment, how fast is the area increasing?



Solution The area A and the radius r are always related by the equation for the area of a circle:

$$A = \pi r^2.$$

We take the derivative of both sides with respect to t:

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}.$$ The factor dr/dt results from the Chain Rule and the fact that r is assumed to be a function of t.

At the moment in question, $dr/dt = 2$ mi/yr and $r = 5$ mi, so

$$\frac{dA}{dt} = 2\pi(5\text{mi})\left(2\,\frac{\text{mi}}{\text{yr}}\right)$$
$$= 20\pi\frac{\text{mi}^2}{\text{yr}} \approx 63 \text{ square miles per year.}$$

❯ Quick Check 2

A spherical balloon is deflating, losing 20 cm^3 of air per minute. At the moment when the radius of the balloon is 8 cm, how fast is the radius decreasing? (*Hint*: $V = \frac{4}{3}\pi r^3$.)

❮ Quick Check 2

Exampl E 4 **Business: Rates of Change of Revenue, Cost, and Profit.**

For Luce Landscaping, the total revenue from the yard maintenance of x homes is given by

$$R(x) = 1000x - x^2,$$

and the total cost is given by

$$C(x) = 3000 + 20x.$$

Suppose that Luce is adding 10 homes per day at the moment when the 400th customer is signed. At that moment, what is the rate of change of (a) total revenue, (b) total cost, and (c) total profit?

Solution

a) $$\frac{dR}{dt} = 1000 \cdot \frac{dx}{dt} - 2x \cdot \frac{dx}{dt}$$ Differentiating both sides with respect to time

$$= 1000 \cdot 10 - 2(400)10$$ Substituting 10 for dx/dt and 400 for x

$$= \$2000 \text{ per day}$$

b) $$\frac{dC}{dt} = 20 \cdot \frac{dx}{dt}$$ Differentiating both sides with respect to time

$$= 20(10)$$
$$= \$200 \text{ per day}$$

c) Since $P = R - C$,

$$\frac{dP}{dt} = \frac{dR}{dt} - \frac{dC}{dt}$$
$$= \$2000 \text{ per day} - \$200 \text{ per day}$$
$$= \$1800 \text{ per day.}$$

Section Summary

- If variables x and y are related to one another by an equation but neither variable is isolated on one side of the equation, we say that x and y have an implicit relationship. To find dy/dx without solving such an equation for y, we use *implicit differentiation*.
- Whenever we implicitly differentiate y with respect to x, the factor dy/dx will appear as a result of the Chain Rule.
- To determine the slope of a tangent line at a point on the graph of an implicit relationship, we may need to evaluate the derivative by inserting both the x-value and the y-value of the point of tangency.

EXERCISE SET 2.7

Differentiate implicitly to find dy/dx. Then find the slope of the curve at the given point.

1. $x^3 + 2y^3 = 6$; $(2, -1)$

2. $3x^3 - y^2 = 8$; $(2, 4)$

3. $2x^2 - 3y^3 = 5$; $(-2, 1)$

4. $2x^3 + 4y^2 = -12$; $(-2, -1)$

5. $x^2 - y^2 = 1$; $(\sqrt{3}, \sqrt{2})$

6. $x^2 + y^2 = 1$; $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

7. $3x^2y^4 = 12$; $(2, -1)$

8. $2x^3y^2 = -18$; $(-1, 3)$

9. $x^3 - x^2y^2 = -9$; $(3, -2)$

10. $x^4 - x^2y^3 = 12$; $(-2, 1)$

11. $xy - x + 2y = 3$; $\left(-5, \frac{2}{3}\right)$

12. $xy + y^2 - 2x = 0$; $(1, -2)$

13. $x^2y - 2x^3 - y^3 + 1 = 0$; $(2, -3)$

14. $4x^3 - y^4 - 3y + 5x + 1 = 0$; $(1, -2)$

Differentiate implicitly to find dy/dx.

15. $2xy + 3 = 0$

16. $x^2 + 2xy = 3y^2$

17. $x^2 - y^2 = 16$

18. $x^2 + y^2 = 25$

19. $y^5 = x^3$

20. $y^3 = x^5$

21. $x^2y^3 + x^3y^4 = 11$

22. $x^3y^2 + x^5y^3 = -19$

For each demand equation in Exercises 23–30, differentiate implicitly to find dp/dx.

23. $p^3 + p - 3x = 50$

24. $p^2 + p + 2x = 40$

25. $xp^3 = 24$

26. $x^3p^2 = 108$

27. $\frac{xp}{x + p} = 2$ (*Hint*: Clear the fraction first.)

28. $\frac{x^2p + xp + 1}{2x + p} = 1$ (*Hint*: Clear the fraction first.)

29. $(p + 4)(x + 3) = 48$

30. $1000 - 300p + 25p^2 = x$

31. Two variable quantities A and B are found to be related by the equation

$$A^3 + B^3 = 9.$$

What is the rate of change dA/dt at the moment when $A = 2$ and $dB/dt = 3$?

32. Two nonnegative variable quantities G and H are found to be related by the equation

$$G^2 + H^2 = 25.$$

What is the rate of change dH/dt when $dG/dt = 3$ and $G = 0$? $G = 1$? $G = 3$?

APPLICATIONS

Business and Economics

Rates of change of total revenue, cost, and profit. *In Exercises 33–36, find the rates of change of total revenue, cost, and profit with respect to time. Assume that R(x) and C(x) are in dollars.*

33. $R(x) = 50x - 0.5x^2$,
$C(x) = 4x + 10$,
when $x = 30$ and $dx/dt = 20$ units per day

34. $R(x) = 50x - 0.5x^2$,
$C(x) = 10x + 3$,
when $x = 10$ and $dx/dt = 5$ units per day

35. $R(x) = 2x$,
$C(x) = 0.01x^2 + 0.6x + 30$,
when $x = 20$ and $dx/dt = 8$ units per day

36. $R(x) = 280x - 0.4x^2$,
$C(x) = 5000 + 0.6x^2$,
when $x = 200$ and $dx/dt = 300$ units per day

37. Change of sales. Suppose that the price p, in dollars, and number of sales, x, of a certain item follow the equation

$$5p + 4x + 2px = 60.$$

Suppose also that p and x are both functions of time, measured in days. Find the rate at which x is changing when

$$x = 3, p = 5, \text{ and } \frac{dp}{dt} = 1.5.$$

38. Change of revenue. For x and p as described in Exercise 37, find the rate at which the total revenue $R = xp$ is changing when

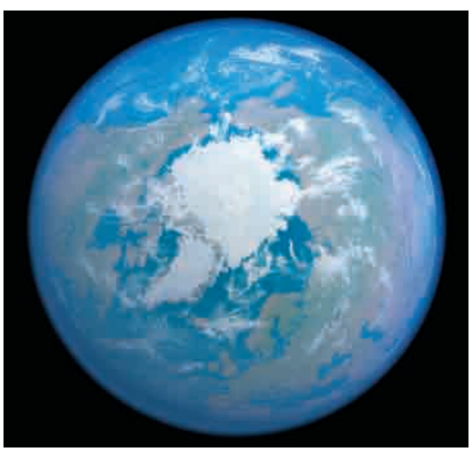
$$x = 3, p = 5, \text{ and } \frac{dp}{dt} = 1.5.$$

Life and Natural Sciences

39. Rate of change of the Arctic ice cap. In a trend that scientists attribute, at least in part, to global warming, the floating cap of sea ice on the Arctic Ocean has been shrinking since 1950. The ice cap always shrinks in summer and grows in winter. Average minimum size of the ice cap, in square miles, can be approximated by

$$A = \pi r^2.$$

In 2012, the radius of the ice cap was approximately 648 mi and was shrinking at a rate of approximately 4.4 mi/yr. (*Source*: National Snow and Ice Data Center, Sept. 19, 2012.) How fast was the area changing at that time?



40. Rate of change of a healing wound. The area of a healing wound is given by

$$A = \pi r^2.$$

The radius is decreasing at the rate of 1 millimeter per day (-1 mm/day) at the moment when $r = 25$ mm. How fast is the area decreasing at that moment?

41. Body surface area. Certain chemotherapy dosages depend on a patient's surface area. According to the Mosteller model,

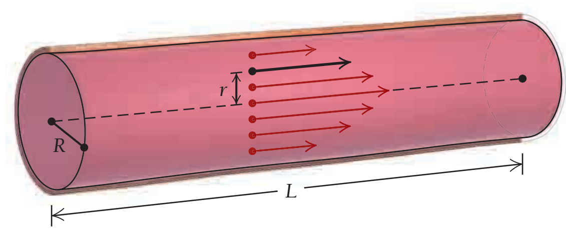
$$S = \frac{\sqrt{hw}}{60},$$

where h is the patient's height in centimeters, w is the patient's weight in kilograms, and S is the approximation to the patient's surface area in square meters. (*Source*: www.halls.md.) Assume that Tom's height is a constant 165 cm, but he is on a diet. If he loses 2 kg per month, how fast is his surface area decreasing at the instant he weighs 70 kg?

Poiseuille's Law. *The flow of blood in a blood vessel is faster toward the center of the vessel and slower toward the outside. The speed of the blood V, in millimeters per second (mm/sec), is given by*

$$V = \frac{p}{4Lv}(R^2 - r^2),$$

where R is the radius of the blood vessel, r is the distance of the blood from the center of the vessel, and p, L, and v are physical constants related to pressure, length, and viscosity of the blood vessels, respectively. Assume that dV/dt is measured in millimeters per second squared (mm/sec^2). Use this formula for Exercises 42 and 43.



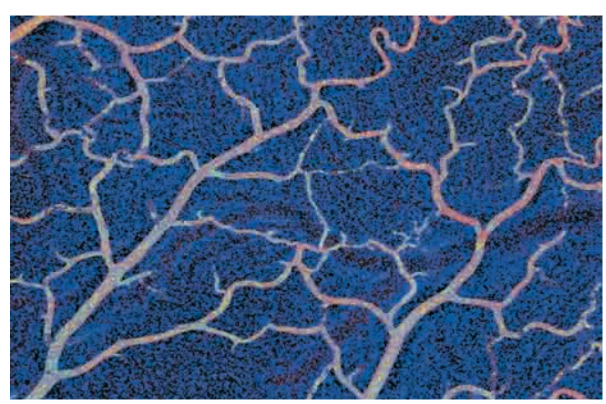
42. Assume that r is a constant as well as p, L, and v.

a) Find the rate of change dV/dt in terms of R and dR/dt when $L = 80$ mm, $p = 500$, and $v = 0.003$.

b) A person goes out into the cold to shovel snow. Cold air has the effect of contracting blood vessels far from the heart. Suppose that a blood vessel contracts at a rate of

$$\frac{dR}{dt} = -0.0002 \text{ mm/sec}$$

at a place in the blood vessel where the radius $R = 0.075$ mm. Find the rate of change, dV/dt, at that location.



The flow of blood in a blood vessel can be modeled by Poiseuille's Law.

43. Assume that r is a constant as well as p, L, and v.

a) Find the rate of change dV/dt in terms of R and dR/dt when $L = 70$ mm, $p = 400$, and $v = 0.003$.

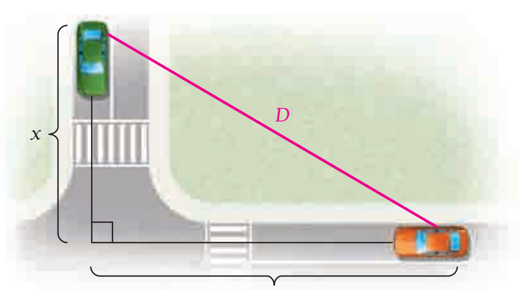
b) When shoveling snow in cold air, a person with a history of heart trouble can develop angina (chest pains) due to contracting blood vessels. To counteract this, he or she may take a nitroglycerin tablet, which dilates the blood vessels. Suppose that after a nitroglycerin tablet is taken, a blood vessel dilates at a rate of

$$\frac{dR}{dt} = 0.00015 \text{ mm/sec}$$

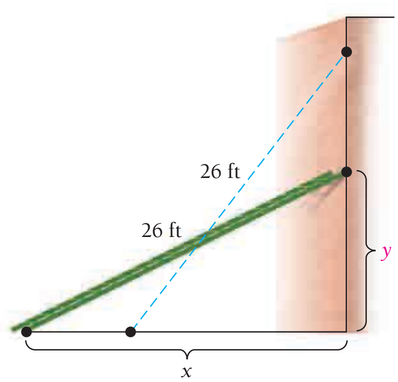
at a place in the blood vessel where the radius $R = 0.1$ mm. Find the rate of change, dV/dt.

General Interest

44. Two cars start from the same point at the same time. One travels north at 25 mph, and the other travels east at 60 mph. How fast is the distance between them increasing at the end of 1 hr? (*Hint*: $D^2 = x^2 + y^2$. To find D after 1 hr, solve $D^2 = 25^2 + 60^2$.)



45. A ladder 26 ft long leans against a vertical wall. If the lower end is being moved away from the wall at the rate of 5 ft/s, how fast is the height of the top changing (this will be a negative rate) when the lower end is 10 ft from the wall?



46. An inner city revitalization zone is a rectangle that is twice as long as it is wide. A diagonal through the region is growing at a rate of 90 m per year at a time when the region is 440 m wide. How fast is the area changing at that point in time?

47. The volume of a cantaloupe is given by

$$V = \tfrac{4}{3}\pi r^3.$$

The radius is growing at the rate of 0.7 cm/week, at a time when the radius is 7.5 cm. How fast is the volume changing at that moment?

SYNTHESIS

Differentiate implicitly to find dy/dx.

48. $\sqrt{x} + \sqrt{y} = 1$

49. $\frac{1}{x^2} + \frac{1}{y^2} = 5$

50. $y^3 = \frac{x - 1}{x + 1}$

51. $y^2 = \frac{x^2 - 1}{x^2 + 1}$

52. $x^{3/2} + y^{2/3} = 1$

53. $(x - y)^3 + (x + y)^3 = x^5 + y^5$

Differentiate implicitly to find d^2y/dx^2.

54. $xy + x - 2y = 4$

55. $y^2 - xy + x^2 = 5$

56. $x^2 - y^2 = 5$

57. $x^3 - y^3 = 8$

58. Explain the usefulness of implicit differentiation.

59. Look up the word "implicit" in a dictionary. Explain how that definition can be related to the concept of a function that is defined "implicitly."

TECHNOLOGY CONNECTION

Graph each of the following equations. Equations must be solved for y before they can be entered into most calculators. Graphicus does not require that equations be solved for y.

60. $x^2 + y^2 = 4$
Note: You will probably need to sketch the graph in two parts: $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$. Then graph the tangent line to the graph at the point $(-1, \sqrt{3})$.

61. $x^4 = y^2 + x^6$
Then graph the tangent line to the graph at the point $(-0.8, 0.384)$.

62. $y^4 = y^2 - x^2$

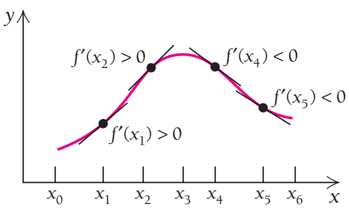
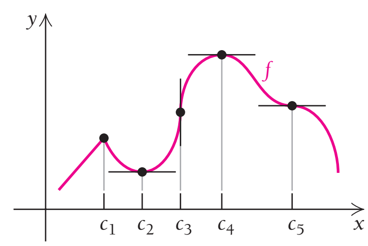
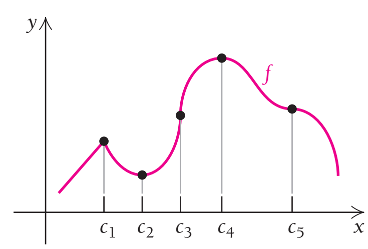
63. $x^3 = y^2(2 - x)$

64. $y^2 = x^3$

Answers to Quick Checks

1. $\frac{dy}{dx} = \frac{y^2 + 6x^2y^3}{1 - 2xy - 6x^3y^2}$ **2.** Approximately -0.025 cm/min

CHAPTER 2 SUMMARY

KEY TERMS AND CONCEPTS	EXAMPLES
SECTION 2.1	
A function is **increasing** over an open interval I if, for all x in I, $f'(x) > 0$. A function is **decreasing** over an open interval I if, for all x in I, $f'(x) < 0$.	 f is increasing over the interval (x_0, x_3) and decreasing over (x_3, x_6).
If f is a continuous function, then a **critical value** is any number c for which $f'(c) = 0$ or $f'(c)$ does not exist. If $f'(c)$ does not exist, then the graph of f may have a corner or a vertical tangent at $(c, f(c))$. The ordered pair $(c, f(c))$ is called a *critical point*.	 The values c_1, c_2, c_3, c_4, and c_5 are critical values of f. • $f'(c_1)$ does not exist (corner). • $f'(c_2) = 0$. • $f'(c_3)$ does not exist (vertical tangent). • $f'(c_4) = 0$. • $f'(c_5) = 0$.
If f is a continuous function, then a **relative extremum (maximum** or **minimum)** always occurs at a critical value. The converse is not true: a critical value may not correspond to an extremum.	 • The critical point $(c_1, f(c_1))$ is a relative maximum. • The critical point $(c_2, f(c_2))$ is a relative minimum. • The critical point $(c_3, f(c_3))$ is neither a relative maximum nor a relative minimum. • The critical point $(c_4, f(c_4))$ is a relative maximum. • The critical point $(c_5, f(c_5))$ is neither a relative maximum nor a relative minimum.

(*continued*)

KEY TERMS AND CONCEPTS	EXAMPLES
SECTION 2.1 (*continued*)	
The **First-Derivative Test** allows us to classify a critical value as a relative maximum, a relative minimum, or neither.	Find the relative extrema of the function given by $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 20x + 7$. Critical values occur where $f'(x) = 0$ or $f'(x)$ does not exist. The derivative $f'(x) = x^2 - x - 20$ exists for all real numbers, so the only critical values occur when $f'(x) = 0$. Setting $x^2 - x - 20 = 0$ and solving, we have $x = -4$ and $x = 5$ as the critical values. To apply the First-Derivative Test, we check the sign of $f'(x)$ to the left and the right of each critical value, using test values:

Test Value	$x = -5$	$x = 0$	$x = 6$
Sign of $f'(x)$	$f'(-5) > 0$	$f'(0) < 0$	$f'(6) > 0$
Result	f increasing	f decreasing	f increasing

Therefore, there is a relative maximum at $x = -4$: $f(-4) = 57\frac{2}{3}$.There is a relative minimum at $x = 5$: $f(5) = -63\frac{5}{6}$.

SECTION 2.2

If the graph of f is smooth and continuous, then the **second derivative**, $f''(x)$,determines the **concavity** of the graph.

If $f''(x) > 0$ for all x in an open interval I, then the graph of f is **concave up** over I.

If $f''(x) < 0$ for all x in an open interval I, then the graph of f is **concave down** over I.

A **point of inflection** occurs at $(x_0, f(x_0))$ if $f''(x_0) = 0$ and there is a change in concavity on either side of x_0.

The function given by

$$f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 20x + 7$$

has the second derivative $f''(x) = 2x - 1$. Setting the second derivative equal to 0, we have $x_0 = \frac{1}{2}$. Using test values, we can check the concavity on either side of $x_0 = \frac{1}{2}$:

Test Value	$x = 0$	$x = 1$
Sign of $f''(x)$	$f''(0) < 0$	$f''(1) > 0$
Result	f is concave down	f is concave up

Therefore, the function is concave down over the interval $\left(-\infty, \frac{1}{2}\right)$ and concave up over the interval $\left(\frac{1}{2}, \infty\right)$. Since there is a change in concavity on either side of $x_0 = \frac{1}{2}$, we also conclude that the point $\left(\frac{1}{2}, -3\frac{1}{12}\right)$ is a point of inflection, where $f\left(\frac{1}{2}\right) = -3\frac{1}{12}$.

The **Second-Derivative Test** can also be used to classify relative extrema:

If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a relative minimum.

If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a relative maximum.

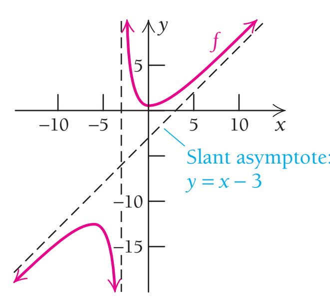
If $f'(c) = 0$ and $f''(c) = 0$, then the First-Derivative Test must be used to classify $f(c)$.

For the function

$$f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 20x + 7,$$

evaluating the second derivative, $f''(x) = 2x - 1$, at the critical values yields the following conclusions:

- At $x = -4$, we have $f''(-4) < 0$. Since $f'(-4) = 0$ and the graph is concave down, we conclude that there is a relative maximum at $x = -4$.
- At $x = 5$, we have $f''(5) > 0$. Since $f'(5) = 0$ and the graph is concave up, we conclude that there is a relative minimum at $x = 5$.

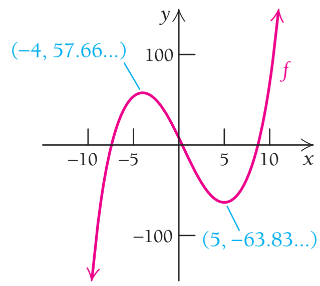
KEY TERMS AND CONCEPTS	EXAMPLES
SECTION 2.3	
A line $x = a$ is a **vertical asymptote** if $\lim_{x\to a^-} f(x) = \infty$, $\lim_{x\to a^-} f(x) = -\infty$, $\lim_{x\to a^+} f(x) = \infty$, or $\lim_{x\to a^+} f(x) = -\infty$. The graph of a rational function never crosses a vertical asymptote	Consider the function given by $f(x) = \frac{x^2 - 1}{x^2 + x - 6}$. Factoring, we have $f(x) = \frac{(x+1)(x-1)}{(x+3)(x-2)}$. This expression is simplified. Therefore, $x = -3$ and $x = 2$ are vertical asymptotes since $\lim_{x\to -3^-} f(x) = \infty$ and $\lim_{x\to -3^+} f(x) = -\infty$ and $\lim_{x\to 2^-} f(x) = -\infty$ and $\lim_{x\to 2^+} f(x) = \infty$.
A line $y = b$ is a **horizontal asymptote** if $\lim_{x\to -\infty} f(x) = b$ or $\lim_{x\to \infty} f(x) = b$. The graph of a function can cross a horizontal asymptote. An asymptote is usually sketched as a dashed line; it is not part of the graph itself.	Also, $y = 1$ is a horizontal asymptote since $\lim_{x\to \infty} f(x) = 1$ and $\lim_{x\to -\infty} f(x) = 1$.
For a rational function of the form $f(x) = p(x)/q(x)$, a **slant asymptote** occurs if the degree of the numerator is 1 greater than the degree of the denominator.	Let $f(x) = \frac{x^2 + 1}{x + 3}$. Long division yields $f(x) = x - 3 + \frac{10}{x + 3}$. As $x \to \infty$ or $x \to -\infty$, the remainder $\frac{10}{x+3} \to 0$. Therefore, the slant asymptote is $y = x - 3$. 
Asymptotes, extrema, x- and y-intercepts, points of inflection, concavity, and intervals of increasing or decreasing are all used in the strategy for accurate graph sketching.	Consider the function given by $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 20x + 7$. • f has a relative maximum point at $\left(-4, 57\frac{2}{3}\right)$ and a relative minimum point at $\left(5, -63\frac{5}{6}\right)$. • f has a point of inflection at $\left(\frac{1}{2}, -3\frac{1}{12}\right)$.

(*continued*)

KEY TERMS AND CONCEPTS	EXAMPLES

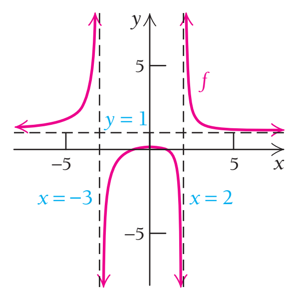
SECTION 2.3 (*continued*)

- f is increasing over the interval $(-\infty, -4)$ and over the interval $(5, \infty)$, decreasing over the interval $(-4, 5)$, concave down over the interval $\left(-\infty, \frac{1}{2}\right)$, and concave up over the interval $\left(\frac{1}{2}, \infty\right)$.
- f has a y-intercept at $(0, 7)$.



Consider the function given by $f(x) = \dfrac{x^2 - 1}{x^2 + x - 6}$.

- f has vertical asymptotes $x = -3$ and $x = 2$.
- f has a horizontal asymptote given by $y = 1$.
- f has a y-intercept at $\left(0, \frac{1}{6}\right)$.
- f has x-intercepts at $(-1, 0)$ and $(1, 0)$.



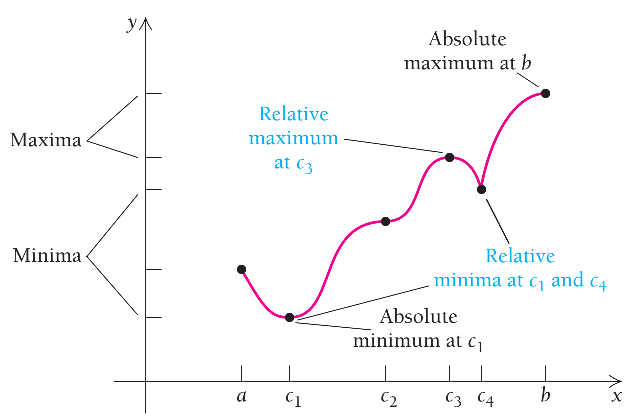
SECTION 2.4

If f is continuous over a closed interval $[a, b]$, then the **Extreme-Value Theorem** tells us that f will have both an absolute maximum value and an absolute minimum value over this interval. One or both points may occur at an endpoint of this interval.

Maximum–Minimum Principle 1 can be used to determine these absolute extrema: We find all critical values $c_1, c_2, c_3, \ldots, c_n$, in $[a, b]$, then evaluate

$$f(a), f(c_1), f(c_2), f(c_3), \ldots, f(c_n), f(b).$$

The largest of these is the **absolute maximum**, and the smallest is the **absolute minimum**.



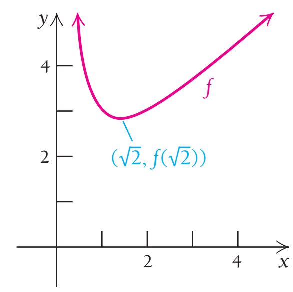
KEY TERMS AND CONCEPTS | **EXAMPLES**

If f is differentiable for all x in an interval I, and there is exactly one critical value c in I such that $f'(c) = 0$, then, according to **Maximum–Minimum Principle 2**, $f(c)$ is an absolute minimum if $f''(c) > 0$ or an absolute maximum if $f''(c) < 0$.

Let $f(x) = x + \frac{2}{x}$, for $x > 0$. The derivative is $f'(x) = 1 - \frac{2}{x^2}$. We solve for the critical value:

$$1 - \frac{2}{x^2} = 0$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}.$$

The only critical value over the interval where $x > 0$ is $x = \sqrt{2}$. The second derivative is $f''(x) = \frac{4}{x^3}$. We see that $f''(\sqrt{2}) = \frac{4}{(\sqrt{2})^3} > 0$. Therefore, $(\sqrt{2}, f(\sqrt{2}))$ is an absolute minimum.



SECTION 2.5

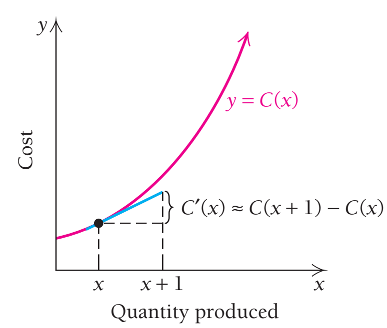
Many real-world applications involve maximum–minimum problems.

See Examples 1–7 in Section 2.5 and the problem-solving strategy on p. 263.

SECTION 2.6

Marginal cost, **marginal revenue**, and **marginal profit** are estimates of the cost, revenue, and profit for the $(x + 1)$st item produced:

- $C'(x) \approx C(x + 1) - C(x)$, so $C(x + 1) \approx C(x) + C'(x)$.
- $R'(x) \approx R(x + 1) - R(x)$, so $R(x + 1) \approx R(x) + R'(x)$.
- $P'(x) \approx P(x + 1) - P(x)$, so $P(x + 1) \approx P(x) + P'(x)$.



Delta notation represents the change in a variable:

$$\Delta x = x_2 - x_1$$

and

$$\Delta y = f(x_2) - f(x_1).$$

If $x_2 = x_1 + h$, then $\Delta x = h$.

If Δx is small, then the derivative can be used to approximate Δy:

$$\Delta y \approx f'(x) \cdot \Delta x.$$

For $f(x) = x^2$, let $x_1 = 2$ and $x_2 = 2.1$. Then, $\Delta x = 2.1 - 2 = 0.1$. Since $f(x_1) = f(2) = 4$ and $f(x_2) = f(2.1) = 4.41$, we have

$$\Delta y = f(x_2) - f(x_1) = 4.41 - 4 = 0.41.$$

Since $\Delta x = 0.1$ is small, we can approximate Δy by

$$\Delta y \approx f'(x)\,\Delta x.$$

The derivative is $f'(x) = 2x$. Therefore,

$$\Delta y \approx f'(2) \cdot (0.1) = 2(2) \cdot (0.1) = 0.4.$$

This approximation, 0.4, is very close to the actual difference, 0.41.

(continued)

KEY TERMS AND CONCEPTS	EXAMPLES
SECTION 2.6 (*continued*)	
Differentials allow us to approximate changes in the output variable y given a change in the input variable x: $dx = \Delta x$ and $dy = f'(x)\,dx$. If Δx is small, then $dy \approx \Delta y$. In practice, it is often simpler to calculate dy, and it will be very close to the true value of Δy.	For $y = \sqrt[3]{x}$, find dy when $x = 27$ and $dx = 2$. Note that $\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}}$. Thus, $dy = \frac{1}{3\sqrt[3]{x^2}}\,dx$. Evaluating, we have $dy = \frac{1}{3\sqrt[3]{(27)^2}}(2) = \frac{2}{27} \approx 0.074.$ This result can be used to approximate the value of $\sqrt[3]{29}$, using the fact that $\sqrt[3]{29} \approx \sqrt[3]{27} + dy$: $\sqrt[3]{29} = \sqrt[3]{27} + dy \approx 3 + \frac{2}{27} \approx 3.074.$ Thus, the approximation $\sqrt[3]{29} \approx 3.074$ is very close to the actual value, $\sqrt[3]{29} = 3.07231\ldots.$
SECTION 2.7	
If an equation has variables x and y and y is not isolated on one side of the equation, the derivative dy/dx can be found without solving for y by the method of **implicit differentiation.**	Find $\frac{dy}{dx}$ if $y^5 = x^3 + 7$. We differentiate both sides with respect to x, then solve for $\frac{dy}{dx}$. $\frac{d}{dx}y^5 = \frac{d}{dx}x^3 + \frac{d}{dx}7$; $5y^4\frac{dy}{dx} = 3x^2$; $\frac{dy}{dx} = \frac{3x^2}{5y^4}.$
A **related rate** occurs when the rate of change of one variable (with respect to time) can be calculated in terms of the rate of change (with respect to time) of another variable of which it is a function.	A cube of ice is melting, losing 30 cm^3 of its volume (V) per minute. When the side length (x) of the cube is 20 cm, how fast is the side length decreasing? Since $V = x^3$ and both V and x are changing with time, we differentiate each variable with respect to time: $\frac{dV}{dt} = 3x^2\frac{dx}{dt}.$ We have $x = 20$ and $\frac{dV}{dt} = -30$. Evaluating gives $-30 = 3(20)^2\frac{dx}{dt}$ or $\frac{dx}{dt} = -\frac{30}{3(20)^2} = -0.025$ cm/min.

CHAPTER 2 REVIEW EXERCISES

These review exercises are for test preparation. They can also be used as a practice test. Answers are at the back of the book. The blue bracketed section references tell you what part(s) of the chapter to restudy if your answer is incorrect.

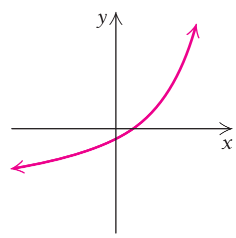
CONCEPT REINFORCEMENT

Match each description in column A with the most appropriate graph in column B. [2.1–2.4]

Column A / **Column B**

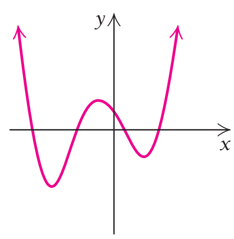
1. A function with a relative maximum but no absolute extrema

a)



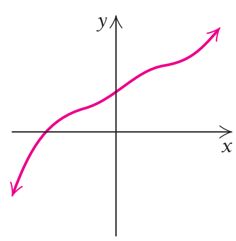
2. A function with both a vertical asymptote and a horizontal asymptote

b)



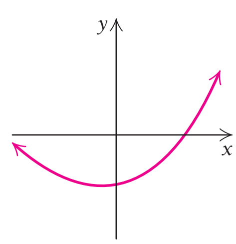
3. A function that is concave up and decreasing

c)



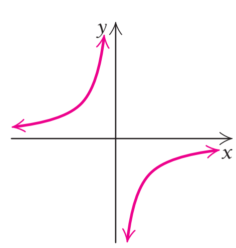
4. A function that is concave up and increasing

d)



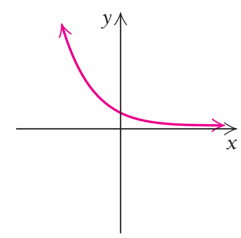
5. A function with three critical values

e)



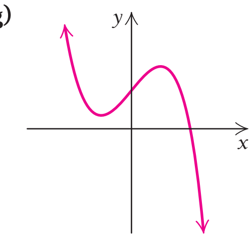
6. A function with one critical value and a second derivative that is always positive

f)



7. A function with a first derivative that is always positive

g)



In Exercises 8–13, classify each statement as either true or false.

8. Every continuous function has at least one critical value. [2.1]

9. If a continuous function $y = f(x)$ has extrema, they will occur where $f'(x) = 0$. [2.1]

10. If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a relative minimum. [2.2]

11. If $f'(c) = 0$ and $f''(c) = 0$, then $f(c)$ cannot be a relative minimum. [2.2]

12. If the graph of $f(x) = P(x)/Q(x)$ has a horizontal asymptote, then the degree of the polynomial $P(x)$ must be the same as that of the polynomial $Q(x)$. [2.3]

13. Absolute extrema of a continuous function f always occur at the endpoints of a closed interval. [2.4]

REVIEW EXERCISES

For each function given, find any extrema, along with the x-value at which they occur. Then sketch a graph of the function. [2.1]

14. $f(x) = 4 - 3x - x^2$

15. $f(x) = x^4 - 2x^2 + 3$

16. $f(x) = \dfrac{-8x}{x^2 + 1}$

17. $f(x) = 4 + (x - 1)^3$

18. $f(x) = x^3 + x^2 - x + 3$

19. $f(x) = 3x^{2/3}$

20. $f(x) = 2x^3 - 3x^2 - 12x + 10$

21. $f(x) = x^3 - 3x + 2$

Sketch the graph of each function. List any minimum or maximum values and where they occur, as well as any points of inflection. State where the function is increasing or decreasing, as well as where it is concave up or concave down. [2.2]

22. $f(x) = \frac{1}{3}x^3 + 3x^2 + 9x + 2$

23. $f(x) = x^2 - 10x + 8$

24. $f(x) = 4x^3 - 6x^2 - 24x + 5$

25. $f(x) = x^4 - 2x^2$

26. $f(x) = 3x^4 + 2x^3 - 3x^2 + 1$ (Round to three decimal places where appropriate.)

27. $f(x) = \frac{1}{5}x^5 + \frac{3}{4}x^4 - \frac{4}{3}x^3 + 8$ (Round to three decimal places where appropriate.)

Sketch the graph of each function. Indicate where each function is increasing or decreasing, the coordinates at which relative extrema occur, where any asymptotes occur, where the graph is concave up or concave down, and where any intercepts occur. [2.3]

28. $f(x) = \frac{2x + 5}{x + 1}$

29. $f(x) = \frac{x}{x - 2}$

30. $f(x) = \frac{5}{x^2 - 16}$

31. $f(x) = -\frac{x + 1}{x^2 - x - 2}$

32. $f(x) = \frac{x^2 - 2x + 2}{x - 1}$

33. $f(x) = \frac{x^2 + 3}{x}$

Find the absolute maximum and minimum values of each function, if they exist, over the indicated interval. Indicate the x-value at which each extremum occurs. Where no interval is specified, use the real line. [2.4]

34. $f(x) = x^4 - 2x^2 + 3$; $[0, 3]$

35. $f(x) = 8x^2 - x^3$; $[-1, 8]$

36. $f(x) = x + \frac{50}{x}$; $(0, \infty)$

37. $f(x) = x^4 - 2x^2 + 1$

38. Of all numbers whose sum is 60, find the two that have the maximum product. [2.5]

39. Find the minimum value of $Q = x^2 - 2y^2$, where $x - 2y = 1$. [2.5]

40. Business: maximizing profit. If

$$R(x) = 52x - 0.5x^2 \quad \text{and} \quad C(x) = 22x - 1,$$

find the maximum profit and the number of units that must be produced and sold in order to yield this maximum profit. Assume that $R(x)$ and $C(x)$ are in dollars. [2.5]

41. Business: minimizing cost. A rectangular box with a square base and a cover is to have a volume of 2500 ft^3. If the cost per square foot for the bottom is \$2, for the top is \$3, and for the sides is \$1, what should the dimensions be in order to minimize the cost? [2.5]

42. Business: minimizing inventory cost. A store in California sells 360 hybrid bicycles per year. It costs \$8 to store one bicycle for a year. To reorder, there is a fixed cost of \$10, plus \$2 for each bicycle. How many times per year should the store order bicycles, and in what lot size, in order to minimize inventory costs? [2.5]

43. Business: marginal revenue. Crane Foods determines that its daily revenue, $R(x)$, in dollars, from the sale of x frozen dinners is

$$R(x) = 4x^{3/4}.$$

a) What is Crane's daily revenue when 81 frozen dinners are sold?

b) What is Crane's marginal revenue when 81 frozen dinners are sold?

c) Use the answers from parts (a) and (b) to estimate R(82). [2.6]

For Exercises 44 and 45, $y = f(x) = 2x^3 + x$. [2.6]

44. Find Δy and dy, given that $x = 1$ and $\Delta x = -0.05$.

45. a) Find dy.

b) Find dy when $x = -2$ and $dx = 0.01$.

46. Approximate $\sqrt{83}$ using $\Delta y \approx f'(x)\,\Delta x$. [2.6]

47. Physical science: waste storage. The Waste Isolation Pilot Plant (WIPP) in New Mexico consists of large rooms carved into a salt deposit and is used for long-term storage of radioactive waste. (*Source*: www.wipp.energy.gov.) A new storage room in the shape of a cube with an edge length of 200 ft is to be carved into the salt. Use a differential to estimate the potential difference in the volume of this room if the edge measurements have a tolerance of ± 2 ft. [2.6]

48. Differentiate the following implicitly to find dy/dx. Then find the slope of the curve at the given point. [2.7]

$$2x^3 + 2y^3 = -9xy; \quad (-1, -2)$$

49. A ladder 25 ft long leans against a vertical wall. If the lower end is being moved away from the wall at the rate of 6 ft/sec, how fast is the height of the top decreasing when the lower end is 7 ft from the wall? [2.7]

50. Business: total revenue, cost, and profit. Find the rates of change, with respect to time, of total revenue, cost, and profit for

$$R(x) = 120x - 0.5x^2 \quad \text{and} \quad C(x) = 15x + 6,$$

when $x = 100$ and $dx/dt = 30$ units per day. Assume that $R(x)$ and $C(x)$ are in dollars. [2.7]

SYNTHESIS

51. Find the absolute maximum and minimum values, if they exist, over the indicated interval. [2.4]

$$f(x) = (x - 3)^{2/5}; \quad (-\infty, \infty)$$

52. Find the absolute maximum and minimum values of the piecewise-defined function given by

$$f(x) = \begin{cases} 2 - x^2, & \text{for } -2 \le x \le 1, \\ 3x - 2, & \text{for } \ 1 < x < 2, \\ (x-4)^2, & \text{for } \ 2 \le x \le 6. \end{cases}$$ [2.4]

53. Differentiate implicitly to find dy/dx:

$(x - y)^4 + (x + y)^4 = x^6 + y^6$. [2.7]

54. Find the relative maxima and minima of

$y = x^4 - 8x^3 - 270x^2$. [2.1 and 2.2]

55. Determine a rational function f whose graph has a vertical asymptote at $x = -2$ and a horizontal asymptote at $y = 3$ and includes the point $(1, 2)$. [2.4]

TECHNOLOGY CONNECTION

Use a calculator to estimate the relative extrema of each function. [2.1 and 2.2]

56. $f(x) = 3.8x^5 - 18.6x^3$

57. $f(x) = \sqrt[3]{|9 - x^2|} - 1$

58. Life and physical sciences: incidence of breast cancer. The following table provides data relating the incidence of breast cancer per 100,000 women of various ages.

a) Use REGRESSION to fit linear, quadratic, cubic, and quartic functions to the data.

Age	Incidence per 100,000
0	0
27	10
32	25
37	60
42	125
47	187
52	224
57	270
62	340
67	408
72	437
77	475
82	460
87	420

(Source: *National Cancer Institute.*)

b) Which function best fits the data?

c) Determine the domain of the function on the basis of the function and the problem situation, and explain.

d) Determine the maximum value of the function on the domain. At what age is the incidence of breast cancer the greatest? [2.1 and 2.2]

Note: The function used in Exercise 28 of Section R.1 was found in this manner.

CHAPTER 2 TEST

Find all relative minimum or maximum values as well as the x-values at which they occur. State where each function is increasing or decreasing. Then sketch a graph of the function.

1. $f(x) = x^2 - 4x - 5$

2. $f(x) = 4 + 3x - x^3$

3. $f(x) = (x - 2)^{2/3} - 4$

4. $f(x) = \dfrac{16}{x^2 + 4}$

Sketch a graph of each function. List any extrema, and indicate any asymptotes or points of inflection.

5. $f(x) = x^3 + x^2 - x + 1$

6. $f(x) = 2x^4 - 4x^2 + 1$

7. $f(x) = (x - 2)^3 + 3$

8. $f(x) = x\sqrt{9 - x^2}$

9. $f(x) = \dfrac{2}{x - 1}$

10. $f(x) = \dfrac{-8}{x^2 - 4}$

11. $f(x) = \dfrac{x^2 - 1}{x}$

12. $f(x) = \dfrac{x - 3}{x + 2}$

Find the absolute maximum and minimum values, if they exist, of each function over the indicated interval. Where no interval is specified, use the real line.

13. $f(x) = x(6 - x)$

14. $f(x) = x^3 + x^2 - x + 1;\quad [-2, \frac{1}{2}]$

15. $f(x) = -x^2 + 8.6x + 10$

16. $f(x) = -2x + 5;\quad [-1, 1]$

17. $f(x) = -2x + 5$

18. $f(x) = 3x^2 - x - 1$

19. $f(x) = x^2 + \dfrac{128}{x};\quad (0, \infty)$

20. Of all numbers whose difference is 8, find the two that have the minimum product.

21. Minimize $Q = x^2 + y^2$, where $x - y = 10$.

22. Business: maximum profit. Find the maximum profit and the number of units, x, that must be produced and sold in order to yield the maximum profit. Assume that $R(x)$ and $C(x)$ are the revenue and cost, in dollars, when x units are produced:

$$R(x) = x^2 + 110x + 60,$$
$$C(x) = 1.1x^2 + 10x + 80.$$

23. Business: minimizing cost. From a thin piece of cardboard 60 in. by 60 in., square corners are cut out so that the sides can be folded up to make an open box. What dimensions will yield a box of maximum volume? What is the maximum volume?

24. Business: minimizing inventory costs. Ironside Sports sells 1225 tennis rackets per year. It costs \$2 to store one tennis racket for a year. To reorder, there is a fixed cost of \$1, plus \$0.50 for each tennis racket. How many times per year should Ironside order tennis rackets, and in what lot size, in order to minimize inventory costs?

25. For $y = f(x) = x^2 - 3$, $x = 5$, and $\Delta x = 0.1$, find Δy and $f'(x)\,\Delta x$.

26. Approximate $\sqrt{50}$ using $\Delta y \approx f'(x)\,\Delta x$.

27. For $y = \sqrt{x^2 + 3}$:

a) Find dy.

b) Find dy when $x = 2$ and $dx = 0.01$.

28. Differentiate the following implicitly to find dy/dx. Then find the slope of the curve at $(1, 2)$:

$$x^3 + y^3 = 9.$$

29. A spherical balloon has a radius of 15 cm. Use a differential to find the approximate change in the volume of the balloon if the radius is increased or decreased by 0.5 cm. (The volume of a sphere is $V = \frac{4}{3}\pi r^3$. Use 3.14 for π.)

30. A pole 13 ft long leans against a vertical wall. If the lower end is moving away from the wall at the rate of 0.4 ft/sec, how fast is the upper end coming down when the lower end is 12 ft from the wall?

SYNTHESIS

31. Find the absolute maximum and minimum values of the following function, if they exist, over $[0, \infty)$:

$$f(x) = \frac{x^2}{1 + x^3}.$$

32. Business: minimizing average cost. The total cost in dollars of producing x units of a product is given by

$$C(x) = 100x + 100\sqrt{x} + \frac{\sqrt{x^3}}{100}.$$

How many units should be produced to minimize the average cost?

TECHNOLOGY CONNECTION

33. Use a calculator to estimate any extrema of this function:

$$f(x) = 5x^3 - 30x^2 + 45x + 5\sqrt{x}.$$

34. Use a calculator to estimate any extrema of this function:

$$g(x) = x^5 - x^3.$$

35. Business: advertising. The business of manufacturing and selling bowling balls is one of frequent changes. Companies introduce new models to the market about every 3 to 4 months. Typically, a new model is created because of advances in technology such as new surface stock or a new way to place weight blocks in a ball. To decide how to best use advertising dollars, companies track the sales in relation to the amount spent on advertising. Suppose that a company has the following data from past sales.

Amount Spent on Advertising (in thousands)	Number of Bowling Balls Sold, N
\$ 0	8
50	13,115
100	19,780
150	22,612
200	20,083
250	12,430
300	4

a) Use REGRESSION to fit linear, quadratic, cubic, and quartic functions to the data.

b) Determine the domain of the function in part (a) that best fits the data and the problem situation. Justify your answer.

c) Determine the maximum value of the function on the domain. How much should the company spend on advertising its next new model in order to maximize the number of bowling balls sold?

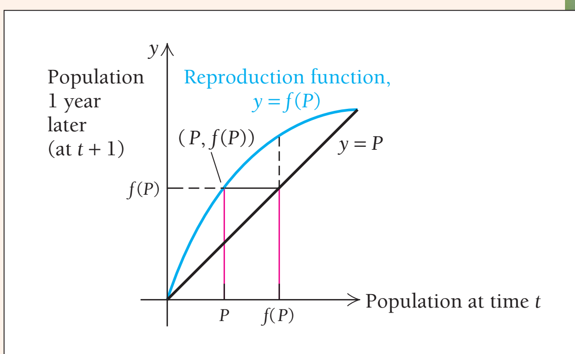
Extended Technology Application

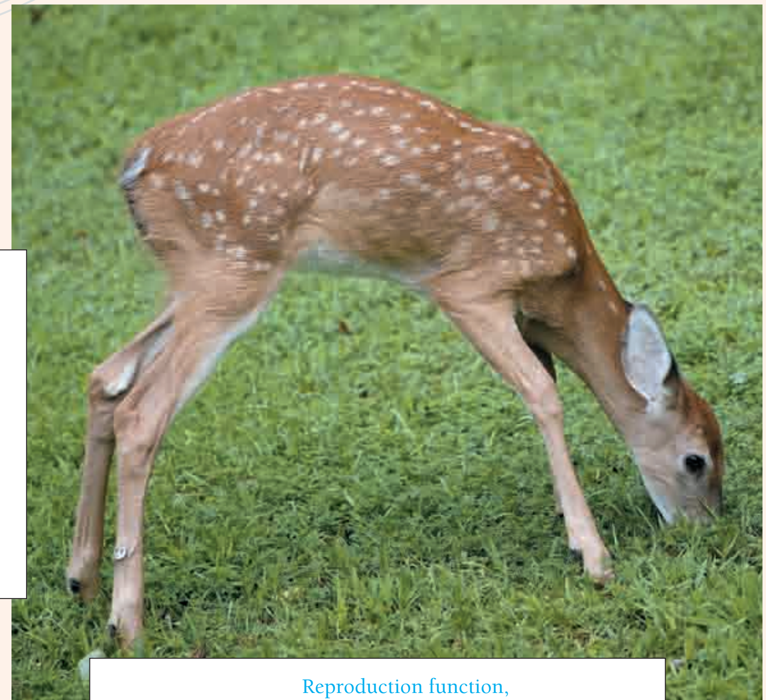
Maximum Sustainable Harvest

In certain situations, biologists are able to determine what is called a **reproduction curve**. This is a function

$$y = f(P)$$

such that if P is the population after P years, then $f(P)$ is the population a year later, at time $t + 1$. Such a curve is shown below.





The line $y = P$ is significant because if it ever coincides with the curve $y = f(P)$, then we know that the population stays the same from year to year. Here the graph of f lies mostly above the line, indicating that the population is increasing.

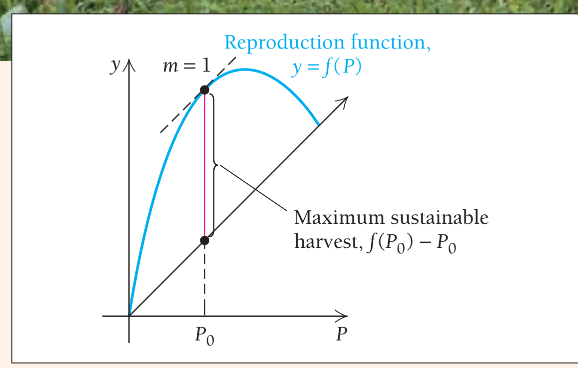
Too many deer in a forest can deplete the food supply and eventually cause the population to decrease for lack of food. In such cases, often with some controversy, hunters are allowed to "harvest" some of the deer. Then with a greater food supply, the remaining deer population may prosper and increase.

We know that a population P will grow to a population $f(P)$ in a year. If this were a population of fur-bearing animals and the population were increasing, then hunters could "harvest" the amount

$$f(P) - P$$

each year without shrinking the initial population P. If the population were remaining the same or decreasing, then such a harvest would deplete the population.

Suppose that we want to know the value of P_0 that would allow the harvest to be the largest. If we could determine that P_0, we could let the population grow until it reached that level and then begin harvesting year after year the amount $f(P_0) - P_0$.



Let the harvest function H be given by

$$H(P) = f(P) - P.$$

Then $H'(P) = f'(P) - 1$.

Now, if we assume that $H'(P)$ exists for all values of P and that there is only one critical value, it follows that the *maximum sustainable harvest* occurs at that value P_0 such that

$$H'(P_0) = f'(P_0) - 1 = 0$$

and $\quad H''(P_0) = f''(P_0) < 0.$

Or, equivalently, we have the following.

THEOREM

The **maximum sustainable harvest** occurs at P_0 such that

$$f'(P_0) = 1 \quad \text{and} \quad f''(P_0) < 0,$$

and is given by

$$H(P_0) = f(P_0) - P_0.$$

EXERCISES

For Exercises 1–3, do the following.

a) *Graph the reproduction curve, the line $y = P$, and the harvest function using the same viewing window.*
b) *Find the population at which the maximum sustainable harvest occurs. Use both a graphical solution and a calculus solution.*
c) *Find the maximum sustainable harvest.*

1. $f(P) = P(10 - P)$, where P is measured in thousands.

2. $f(P) = -0.025P^2 + 4P$, where P is measured in thousands. This is the reproduction curve in the Hudson Bay area for the snowshoe hare, a fur-bearing animal.

3. $f(P) = -0.01P^2 + 2P$, where P is measured in thousands. This is the reproduction curve in the Hudson Bay area for the lynx, a fur-bearing animal.

For Exercises 4 and 5, do the following.

a) *Graph the reproduction curve, the line $y = P$, and the harvest function using the same viewing window.*
b) *Graphically determine the population at which the maximum sustainable harvest occurs.*
c) *Find the maximum sustainable harvest.*

4. $f(P) = 40\sqrt{P}$, where P is measured in thousands. Assume that this is the reproduction curve for the brown trout population in a large lake.

5. $f(P) = 0.237P\sqrt{2000 - P^2}$, where P is measured in thousands.

6. The table below lists data regarding the reproduction of a certain animal.

a) Use REGRESSION to fit a cubic polynomial to these data.
b) Graph the reproduction curve, the line $y = P$, and the harvest function using the same viewing window.
c) Graphically determine the population at which the maximum sustainable harvest occurs.

POPULATION, P (in thousands)	POPULATION, $f(P)$, 1 YEAR LATER
10	9.7
20	23.1
30	37.4
40	46.2
50	42.6