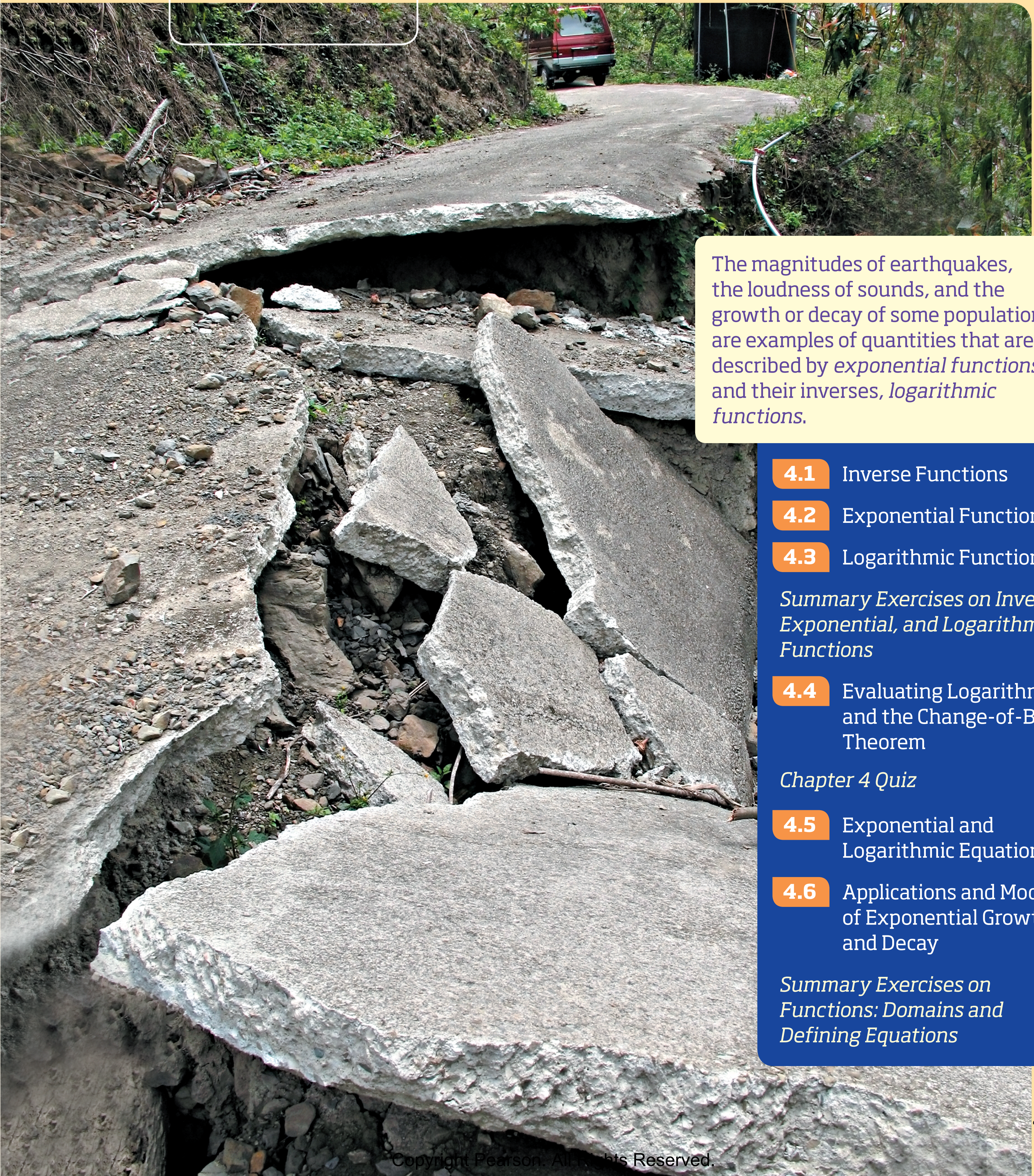
4 Inverse, Exponential, and Logarithmic Functions



The magnitudes of earthquakes, the loudness of sounds, and the growth or decay of some populations are examples of quantities that are described by *exponential functions* and their inverses, *logarithmic functions*.

4.1 Inverse Functions

- One-to-One Functions
- Inverse Functions
- Equations of Inverses
- An Application of Inverse Functions to Cryptography

One-to-One Functions

Suppose we define the following function F.

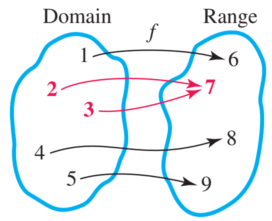
$$F = \{(-2, 2), (-1, 1), (0, 0), (1, 3), (2, 5)\}$$

(We have defined F so that each *second* component is used only once.) We can form another set of ordered pairs from F by interchanging the x- and y-values of each pair in F. We call this set G.

$$G = \{(2, -2), (1, -1), (0, 0), (3, 1), (5, 2)\}$$

G is the *inverse* of F. Function F was defined with each *second* component used only once, so set G will also be a function. (Each *first* component must be used only once.) In order for a function to have an inverse that is also a function, it must exhibit this one-to-one relationship.

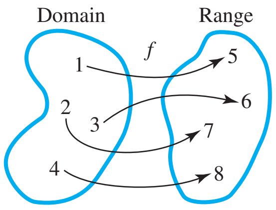
In a one-to-one function, each x-value corresponds to only one y-value, and each y-value corresponds to only one x-value.



Not One-to-One

Figure 1

The function f shown in **Figure 1** is not one-to-one because the y-value 7 corresponds to *two* x-values, 2 and 3. That is, the ordered pairs $(2, 7)$ and $(3, 7)$ both belong to the function. The function f in **Figure 2** is one-to-one.



One-to-One

Figure 2

One-to-One Function

A function f is a **one-to-one function** if, for elements a and b in the domain of f,

$$a \neq b \quad \text{implies} \quad f(a) \neq f(b).$$

That is, different values of the domain correspond to different values of the range.

Using the concept of the *contrapositive* from the study of logic, the boldface statement in the preceding box is equivalent to

$$f(a) = f(b) \quad \text{implies} \quad a = b.$$

This means that if two range values are equal, then their corresponding domain values are equal. We use this statement to show that a function f is one-to-one in **Example 1(a).**

EXAMPLE 1 Deciding Whether Functions Are One-to-One

Determine whether each function is one-to-one.

(a) $f(x) = -4x + 12$ **(b)** $f(x) = \sqrt{25 - x^2}$

SOLUTION

(a) We can determine that the function $f(x) = -4x + 12$ is one-to-one by showing that $f(a) = f(b)$ leads to the result $a = b$.

$$\begin{aligned} f(a) &= f(b) & & \\ -4a + 12 &= -4b + 12 & & f(x) = -4x + 12 \\ -4a &= -4b & & \text{Subtract 12.} \\ a &= b & & \text{Divide by } -4. \end{aligned}$$

By the definition, $f(x) = -4x + 12$ is one-to-one.

(b) We can determine that the function $f(x) = \sqrt{25 - x^2}$ is not one-to-one by showing that *different* values of the domain correspond to the *same* value of the range. If we choose $a = 3$ and $b = -3$, then $3 \neq -3$, but

$$f(3) = \sqrt{25 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4$$

and $$f(-3) = \sqrt{25 - (-3)^2} = \sqrt{25 - 9} = 4.$$

Here, even though $3 \neq -3$, $f(3) = f(-3) = 4$. By the definition, f is *not* a one-to-one function.

✔ Now Try Exercises 17 and 19.

As illustrated in **Example 1(b),** a way to show that a function is *not* one-to-one is to produce a pair of different domain elements that lead to the same function value. There is a useful graphical test for this, the **horizontal line test.**

Horizontal Line Test

A function is one-to-one if every horizontal line intersects the graph of the function at most once.

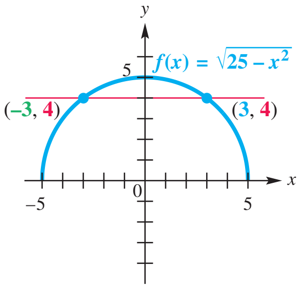


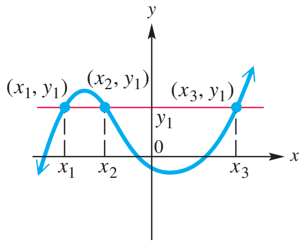
Figure 3

NOTE In **Example 1(b),** the graph of the function is a semicircle, as shown in **Figure 3.** Because there is at least one horizontal line that intersects the graph in more than one point, this function is not one-to-one.

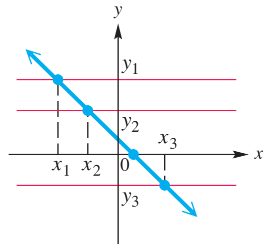
EXAMPLE 2 Using the Horizontal Line Test

Determine whether each graph is the graph of a one-to-one function.

(a)



(b)



SOLUTION

(a) Each point where the horizontal line intersects the graph has the same value of y but a different value of x. Because more than one different value of x (here three) lead to the same value of y, the function is not one-to-one.

(b) Every horizontal line will intersect the graph at exactly one point, so this function is one-to-one.

✔ Now Try Exercises 11 and 13.

The function graphed in **Example 2(b)** decreases on its entire domain.

In general, a function that is either increasing or decreasing on its entire domain, such as $f(x) = -x$, $g(x) = x^3$, and $h(x) = \frac{1}{x}$, must be one-to-one.

Tests to Determine Whether a Function Is One-to-One

1. Show that $f(a) = f(b)$ implies $a = b$. This means that f is one-to-one. **(See Example 1(a).)**
2. In a one-to-one function, every y-value corresponds to no more than one x-value. To show that a function is not one-to-one, find at least two x-values that produce the same y-value. **(See Example 1(b).)**
3. Sketch the graph and use the horizontal line test. **(See Example 2.)**
4. If the function either increases or decreases on its entire domain, then it is one-to-one. A sketch is helpful here, too. **(See Example 2(b).)**

Inverse Functions Certain pairs of one-to-one functions “undo” each other. For example, consider the functions

$$g(x) = 8x + 5 \quad \text{and} \quad f(x) = \frac{1}{8}x - \frac{5}{8}.$$

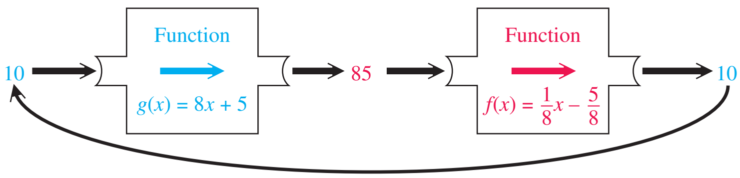
We choose an arbitrary element from the domain of g, say 10. Evaluate $g(10)$.

$$g(x) = 8x + 5 \qquad \text{Given function}$$
$$g(10) = 8 \cdot 10 + 5 \qquad \text{Let } x = 10.$$
$$g(10) = 85 \qquad \text{Multiply and then add.}$$

Now, we evaluate $f(85)$.

$$f(x) = \frac{1}{8}x - \frac{5}{8} \qquad \text{Given function}$$
$$f(85) = \frac{1}{8}(85) - \frac{5}{8} \qquad \text{Let } x = 85.$$
$$f(85) = \frac{85}{8} - \frac{5}{8} \qquad \text{Multiply.}$$
$$f(85) = 10 \qquad \text{Subtract and then divide.}$$

Starting with 10, we “applied” function g and then “applied” function f to the result, which returned the number 10. See **Figure 4.**



These functions contain inverse operations that “undo” each other.

Figure 4

As further examples, confirm the following.

$$g(3) = 29 \quad \text{and} \quad f(29) = 3$$
$$g(-5) = -35 \quad \text{and} \quad f(-35) = -5$$
$$g(2) = 21 \quad \text{and} \quad f(21) = 2$$
$$f(2) = -\frac{3}{8} \quad \text{and} \quad g\left(-\frac{3}{8}\right) = 2$$

In particular, for the pair of functions $g(x) = 8x + 5$ and $f(x) = \frac{1}{8}x - \frac{5}{8}$,

$$f(g(2)) = 2 \quad \text{and} \quad g(f(2)) = 2.$$

In fact, for *any* value of x,

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x.$$

Using the notation for composition of functions, these two equations can be written as follows.

$$(f \circ g)(x) = x \quad \text{and} \quad (g \circ f)(x) = x$$ The result is the identity function.

Because the compositions of f and g yield the *identity* function, they are *inverses* of each other.

Inverse Function

Let f be a one-to-one function. Then g is the **inverse function** of f if

$$(f \circ g)(x) = x \quad \text{for every } x \text{ in the domain of } g,$$

and

$$(g \circ f)(x) = x \quad \text{for every } x \text{ in the domain of } f.$$

The condition that f is one-to-one in the definition of inverse function is essential. Otherwise, g will not define a function.

EXAMPLE 3 Determining Whether Two Functions Are Inverses

Let functions f and g be defined respectively by

$$f(x) = x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{x+1}.$$

Is g the inverse function of f?

SOLUTION As shown in **Figure 5,** the horizontal line test applied to the graph indicates that f is one-to-one, so the function has an inverse. Because it is one-to-one, we now find $(f \circ g)(x)$ and $(g \circ f)(x)$.

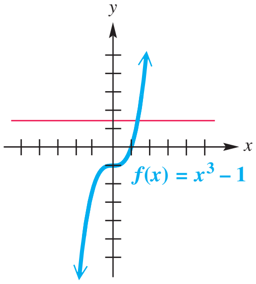


Figure 5

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= \left(\sqrt[3]{x+1}\right)^3 - 1 \\ &= x + 1 - 1 \\ &= x\end{aligned} \qquad \Bigg| \qquad \begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= \sqrt[3]{(x^3-1)+1} \\ &= \sqrt[3]{x^3} \\ &= x\end{aligned}$$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, function g is the inverse of function f.

✔ **Now Try Exercise 41.**

A special notation is used for inverse functions: If g is the inverse of a function f, then g is written as f^{-1} (read **"f-inverse"**).

$$f(x) = x^3 - 1 \quad \text{has inverse} \quad f^{-1}(x) = \sqrt[3]{x+1}.$$ See **Example 3.**

CAUTION ***Do not confuse the -1 in f^{-1} with a negative exponent.*** The symbol $f^{-1}(x)$ represents the inverse function of f, *not* $\frac{1}{f(x)}$.

By the definition of inverse function, the domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}. See **Figure 6.**

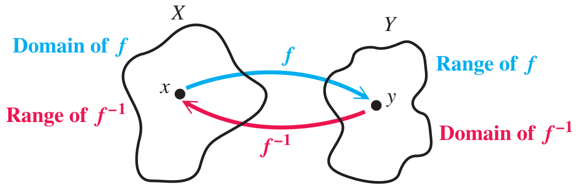


Figure 6

EXAMPLE 4 Finding Inverses of One-to-One Functions

Find the inverse of each function that is one-to-one.

(a) $F = \{(-2, 1), (-1, 0), (0, 1), (1, 2), (2, 2)\}$

(b) $G = \{(3, 1), (0, 2), (2, 3), (4, 0)\}$

(c) The table in the margin shows the number of hurricanes recorded in the North Atlantic during the years 2009–2013. Let f be the function defined in the table, with the years forming the domain and the numbers of hurricanes forming the range.

Year	Number of Hurricanes
2009	3
2010	12
2011	7
2012	10
2013	2

Source: www.wunderground.com

SOLUTION

(a) Each x-value in F corresponds to just one y-value. However, the y-value 2 corresponds to two x-values, 1 and 2. Also, the y-value 1 corresponds to both -2 and 0. Because at least one y-value corresponds to more than one x-value, F is not one-to-one and does not have an inverse.

(b) Every x-value in G corresponds to only one y-value, and every y-value corresponds to only one x-value, so G is a one-to-one function. The inverse function is found by interchanging the x- and y-values in each ordered pair.

$$G^{-1} = \{(1, 3), (2, 0), (3, 2), (0, 4)\}$$

Notice how the domain and range of G become the range and domain, respectively, of G^{-1}.

(c) Each x-value in f corresponds to only one y-value, and each y-value corresponds to only one x-value, so f is a one-to-one function. The inverse function is found by interchanging the x- and y-values in the table.

$$f^{-1}(x) = \{(3, 2009), (12, 2010), (7, 2011), (10, 2012), (2, 2013)\}$$

The domain and range of f become the range and domain of f^{-1}.

✔ **Now Try Exercises 37, 51, and 53.**

Equations of Inverses The inverse of a one-to-one function is found by interchanging the x- and y-values of each of its ordered pairs. The equation of the inverse of a function defined by $y = f(x)$ is found in the same way.

Finding the Equation of the Inverse of $y = f(x)$

For a one-to-one function f defined by an equation $y = f(x)$, find the defining equation of the inverse as follows. (If necessary, replace $f(x)$ with y first. Any restrictions on x and y should be considered.)

Step 1 Interchange x and y.

Step 2 Solve for y.

Step 3 Replace y with $f^{-1}(x)$.

EXAMPLE 5 Finding Equations of Inverses

Determine whether each equation defines a one-to-one function. If so, find the equation of the inverse.

(a) $f(x) = 2x + 5$ **(b)** $y = x^2 + 2$ **(c)** $f(x) = (x - 2)^3$

SOLUTION

(a) The graph of $y = 2x + 5$ is a nonhorizontal line, so by the horizontal line test, f is a one-to-one function. Find the equation of the inverse as follows.

$$f(x) = 2x + 5 \quad \text{Given function}$$

$$y = 2x + 5 \quad \text{Let } y = f(x).$$

Step 1 $$x = 2y + 5 \quad \text{Interchange } x \text{ and } y.$$

Step 2 $$x - 5 = 2y \quad \text{Subtract 5.}$$

$$y = \frac{x-5}{2} \quad \left.\begin{array}{l}\text{Divide by 2.}\\ \text{Rewrite.}\end{array}\right\} \text{Solve for } y.$$

Step 3 $$f^{-1}(x) = \frac{1}{2}x - \frac{5}{2} \quad \text{Replace } y \text{ with } f^{-1}(x). \quad \frac{a-b}{c} = \left(\frac{1}{c}\right)a - \frac{b}{c}$$

Thus, the equation $f^{-1}(x) = \frac{x-5}{2} = \frac{1}{2}x - \frac{5}{2}$ represents a linear function. In the function $y = 2x + 5$, the value of y is found by starting with a value of x, multiplying by 2, and adding 5.

The equation $f^{-1}(x) = \frac{x-5}{2}$ for the inverse *subtracts* 5 and then *divides* by 2. An inverse is used to "undo" what a function does to the variable x.

(b) The equation $y = x^2 + 2$ has a parabola opening up as its graph, so some horizontal lines will intersect the graph at two points. For example, both $x = 3$ and $x = -3$ correspond to $y = 11$. Because of the presence of the x^2-term, there are many pairs of x-values that correspond to the same y-value. This means that the function defined by $y = x^2 + 2$ is not one-to-one and does not have an inverse.

Proceeding with the steps for finding the equation of an inverse leads to

$$y = x^2 + 2$$

$$x = y^2 + 2 \quad \text{Interchange } x \text{ and } y.$$

$$x - 2 = y^2 \quad \text{Solve for } y.$$

Remember both roots.

$$\pm\sqrt{x-2} = y. \quad \text{Square root property}$$

The last equation shows that there are two y-values for each choice of x greater than 2, indicating that this is not a function.

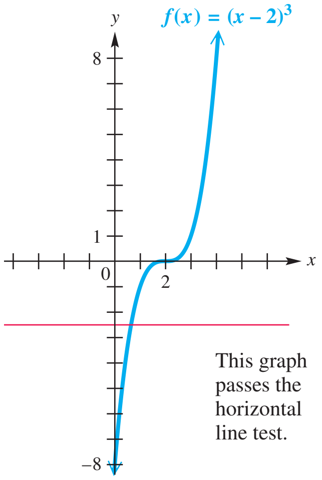


Figure 7

(c) **Figure 7** shows that the horizontal line test assures us that this horizontal translation of the graph of the cubing function is one-to-one.

$$
\begin{array}{lrll}
 & f(x) = (x-2)^3 & \text{Given function} & \\
 & y = (x-2)^3 & \text{Replace } f(x) \text{ with } y. & \\
\textit{Step 1} & x = (y-2)^3 & \text{Interchange } x \text{ and } y. & \\
\textit{Step 2} & \sqrt[3]{x} = \sqrt[3]{(y-2)^3} & \text{Take the cube root on each side.} & \\
 & \sqrt[3]{x} = y - 2 & \sqrt[3]{a^3} = a & \text{Solve for } y. \\
 & \sqrt[3]{x} + 2 = y & \text{Add 2.} & \\
\textit{Step 3} & f^{-1}(x) = \sqrt[3]{x} + 2 & \text{Replace } y \text{ with } f^{-1}(x). \text{ Rewrite.} &
\end{array}
$$

✔ **Now Try Exercises 59(a), 63(a), and 65(a).**

EXAMPLE 6 Finding the Equation of the Inverse of a Rational Function

The following rational function is one-to-one. Find its inverse.

$$f(x) = \frac{2x+3}{x-4}, \quad x \neq 4$$

SOLUTION

$$
\begin{array}{lrll}
 & f(x) = \dfrac{2x+3}{x-4}, \quad x \neq 4 & \text{Given function} & \\
 & y = \dfrac{2x+3}{x-4} & \text{Replace } f(x) \text{ with } y. & \\
\textit{Step 1} & x = \dfrac{2y+3}{y-4}, \quad y \neq 4 & \text{Interchange } x \text{ and } y. & \\
\textit{Step 2} & x(y-4) = 2y+3 & \text{Multiply by } y-4. & \\
 & xy - 4x = 2y + 3 & \text{Distributive property} & \\
 & xy - 2y = 4x + 3 & \text{Add } 4x \text{ and } -2y. & \text{Solve for } y. \\
 & y(x-2) = 4x + 3 & \text{Factor out } y. & \\
 & y = \dfrac{4x+3}{x-2}, \quad x \neq 2 & \text{Divide by } x-2. &
\end{array}
$$

Pay close attention here.

In the final line, we give the condition $x \neq 2$. (Note that 2 is not in the *range* of f, so it is not in the domain of f^{-1}.)

$$
\begin{array}{lrl}
\textit{Step 3} & f^{-1}(x) = \dfrac{4x+3}{x-2}, \quad x \neq 2 & \text{Replace } y \text{ with } f^{-1}(x).
\end{array}
$$

✔ **Now Try Exercise 71(a).**

One way to graph the inverse of a function f whose equation is known follows.

Step 1 Find some ordered pairs that are on the graph of f.

Step 2 Interchange x and y to find ordered pairs that are on the graph of f^{-1}.

Step 3 Plot those points, and sketch the graph of f^{-1} through them.

Another way is to select points on the graph of f and use symmetry to find corresponding points on the graph of f^{-1}.

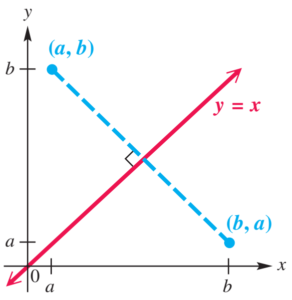


Figure 8

For example, suppose the point (a, b) shown in **Figure 8** is on the graph of a one-to-one function f. Then the point (b, a) is on the graph of f^{-1}. The line segment connecting (a, b) and (b, a) is perpendicular to, and cut in half by, the line $y = x$. The points (a, b) and (b, a) are "mirror images" of each other with respect to $y = x$.

Thus, we can find the graph of f^{-1} from the graph of f by locating the mirror image of each point in f with respect to the line $y = x$.

EXAMPLE 7 Graphing f^{-1} Given the Graph of f

In each set of axes in **Figure 9**, the graph of a one-to-one function f is shown in blue. Graph f^{-1} in red.

SOLUTION In **Figure 9**, the graphs of two functions f shown in blue are given with their inverses shown in red. In each case, the graph of f^{-1} is a reflection of the graph of f with respect to the line $y = x$.

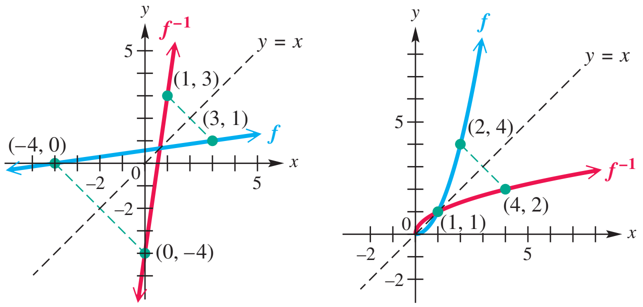


Figure 9

Now Try Exercises 77 and 81.

EXAMPLE 8 Finding the Inverse of a Function (Restricted Domain)

Let $f(x) = \sqrt{x + 5}, \quad x \geq -5$. Find $f^{-1}(x)$.

SOLUTION The domain of f is restricted to the interval $[-5, \infty)$. Function f is one-to-one because it is an increasing function and thus has an inverse function. Now we find the equation of the inverse.

$$f(x) = \sqrt{x+5}, \quad x \geq -5 \qquad \text{Given function}$$

$$y = \sqrt{x+5}, \quad x \geq -5 \qquad \text{Replace } f(x) \text{ with } y.$$

Step 1 $$x = \sqrt{y+5}, \quad y \geq -5 \qquad \text{Interchange } x \text{ and } y.$$

Step 2
$$\left.\begin{aligned} x^2 &= \left(\sqrt{y+5}\right)^2 && \text{Square each side.} \\ x^2 &= y + 5 && \left(\sqrt{a}\right)^2 = a \text{ for } a \geq 0 \\ y &= x^2 - 5 && \text{Subtract 5. Rewrite.} \end{aligned}\right\} \text{Solve for } y.$$

However, we cannot define $f^{-1}(x)$ as $x^2 - 5$. The domain of f is $[-5, \infty)$, and its range is $[0, \infty)$. The range of f is the domain of f^{-1}, so f^{-1} must be defined as follows.

Step 3 $$f^{-1}(x) = x^2 - 5, \quad x \geq 0$$

As a check, the range of f^{-1}, $[-5, \infty)$, is the domain of f.

Graphs of f and f^{-1} are shown in **Figures 10 and 11.** The line $y = x$ is included on the graphs to show that the graphs of f and f^{-1} are mirror images with respect to this line.

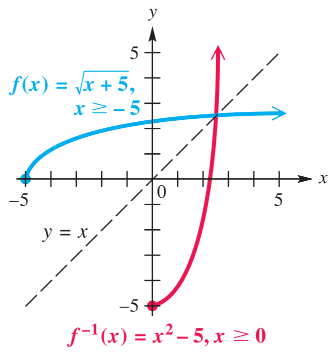


Figure 10

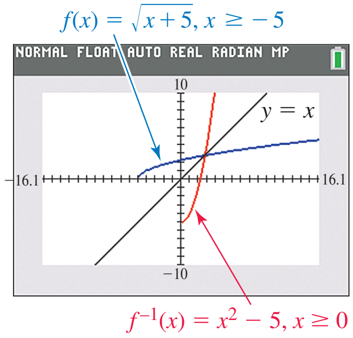
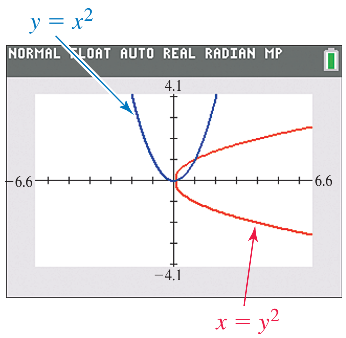


Figure 11

✔ **Now Try Exercise 75.**

Important Facts about Inverses

1. If f is one-to-one, then f^{-1} exists.
2. The domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.
3. If the point (a, b) lies on the graph of f, then (b, a) lies on the graph of f^{-1}. The graphs of f and f^{-1} are reflections of each other across the line $y = x$.
4. To find the equation for f^{-1}, replace $f(x)$ with y, interchange x and y, and solve for y. This gives $f^{-1}(x)$.



Despite the fact that $y = x^2$ is not one-to-one, the calculator will draw its "inverse," $x = y^2$.

Figure 12

Some graphing calculators have the capability of "drawing" the reflection of a graph across the line $y = x$. This feature does not require that the function be one-to-one, however, so the resulting figure may not be the graph of a function. See **Figure 12.** ***It is necessary to understand the mathematics to interpret results correctly.*** ■

An Application of Inverse Functions to Cryptography A one-to-one function and its inverse can be used to make information secure. The function is used to encode a message, and its inverse is used to decode the coded message. In practice, complicated functions are used.

EXAMPLE 9 Using Functions to Encode and Decode a Message

Use the one-to-one function $f(x) = 3x + 1$ and the following numerical values assigned to each letter of the alphabet to encode and decode the message BE MY FACEBOOK FRIEND.

A	1	**H**	8	**O**	15	**V**	22
B	2	**I**	9	**P**	16	**W**	23
C	3	**J**	10	**Q**	17	**X**	24
D	4	**K**	11	**R**	18	**Y**	25
E	5	**L**	12	**S**	19	**Z**	26
F	6	**M**	13	**T**	20		
G	7	**N**	14	**U**	21		

A	1	**N**	14
B	2	**O**	15
C	3	**P**	16
D	4	**Q**	17
E	5	**R**	18
F	6	**S**	19
G	7	**T**	20
H	8	**U**	21
I	9	**V**	22
J	10	**W**	23
K	11	**X**	24
L	12	**Y**	25
M	13	**Z**	26

SOLUTION The message BE MY FACEBOOK FRIEND would be encoded as

7 16 40 76 19 4 10 16 7
46 46 34 19 55 28 16 43 13

because

B corresponds to 2 and $f(2) = 3(2) + 1 = 7$,

E corresponds to 5 and $f(5) = 3(5) + 1 = 16$, and so on.

Using the inverse $f^{-1}(x) = \frac{1}{3}x - \frac{1}{3}$ to decode yields

$$f^{-1}(7) = \frac{1}{3}(7) - \frac{1}{3} = 2, \quad \text{which corresponds to B,}$$

$$f^{-1}(16) = \frac{1}{3}(16) - \frac{1}{3} = 5, \quad \text{which corresponds to E,} \quad \text{and so on.}$$

Now Try Exercise 97.

4.1 Exercises

CONCEPT PREVIEW *Determine whether the function represented in each table is one-to-one.*

1. The table shows the number of registered passenger cars in the United States for the years 2008–2012.

Year	Registered Passenger Cars (in thousands)
2008	137,080
2009	134,880
2010	139,892
2011	125,657
2012	111,290

Source: U.S. Federal Highway Administration.

2. The table gives the number of representatives currently in Congress from each of five New England states.

State	Number of Representatives
Connecticut	5
Maine	2
Massachusetts	9
New Hampshire	2
Vermont	1

Source: www.house.gov

CONCEPT PREVIEW *Fill in the blank(s) to correctly complete each sentence.*

3. For a function to have an inverse, it must be ________.

4. If two functions f and g are inverses, then $(f \circ g)(x) =$ ________ and ________ $= x$.

5. The domain of f is equal to the ________ of f^{-1}, and the range of f is equal to the ________ of f^{-1}.

6. If the point (a, b) lies on the graph of f, and f has an inverse, then the point ________ lies on the graph of f^{-1}.

7. If $f(x) = x^3$, then $f^{-1}(x) =$ ________.

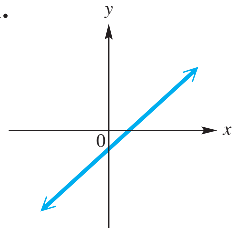
8. If a function f has an inverse, then the graph of f^{-1} may be obtained by reflecting the graph of f across the line with equation ________.

9. If a function f has an inverse and $f(-3) = 6$, then $f^{-1}(6) =$ ________.

10. If $f(-4) = 16$ and $f(4) = 16$, then f ____________ (does/does not) have an inverse because ________.

Determine whether each function graphed or defined is one-to-one. ***See Examples 1 and 2.***

11.



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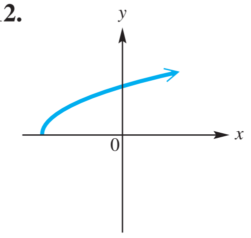

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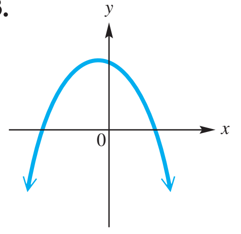

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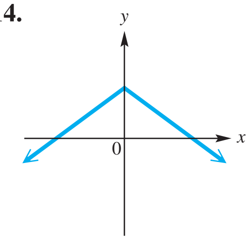

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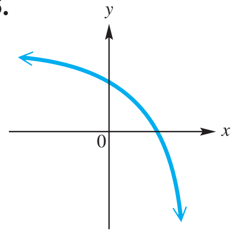
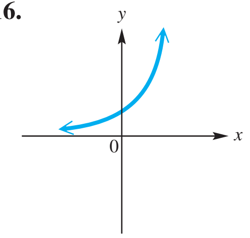


16.



17. $y = 2x - 8$

18. $y = 4x + 20$

19. $y = \sqrt{36 - x^2}$

20. $y = -\sqrt{100 - x^2}$

21. $y = 2x^3 - 1$

22. $y = 3x^3 - 6$

23. $y = \dfrac{-1}{x + 2}$

24. $y = \dfrac{4}{x - 8}$

25. $y = 2(x + 1)^2 - 6$

26. $y = -3(x - 6)^2 + 8$

27. $y = \sqrt[3]{x + 1} - 3$

28. $y = -\sqrt[3]{x + 2} - 8$

Concept Check *Answer each question.*

29. Can a constant function, such as $f(x) = 3$, defined over the set of real numbers, be one-to-one?

30. Can a polynomial function of even degree defined over the set of real numbers have an inverse?

Concept Check *An everyday activity is described. Keeping in mind that an inverse operation "undoes" what an operation does, describe each inverse activity.*

31. tying your shoelaces

32. starting a car

33. entering a room

34. climbing the stairs

35. screwing in a light bulb

36. filling a cup

Determine whether the given functions are inverses. **See Example 4.**

37.

x	$f(x)$
3	−4
2	−6
5	8
1	9
4	3

x	$g(x)$
−4	3
−6	2
8	5
9	1
3	4

38.

x	$f(x)$
−2	−8
−1	−1
0	0
1	1
2	8

x	$g(x)$
8	−2
1	−1
0	0
−1	1
−8	2

39. $f = \{(2, 5), (3, 5), (4, 5)\}; \quad g = \{(5, 2)\}$

40. $f = \{(1, 1), (3, 3), (5, 5)\}; \quad g = \{(1, 1), (3, 3), (5, 5)\}$

Use the definition of inverses to determine whether f and g are inverses. **See Example 3.**

41. $f(x) = 2x + 4, \quad g(x) = \frac{1}{2}x - 2$

42. $f(x) = 3x + 9, \quad g(x) = \frac{1}{3}x - 3$

43. $f(x) = -3x + 12, \quad g(x) = -\frac{1}{3}x - 12$

44. $f(x) = -4x + 2, \quad g(x) = -\frac{1}{4}x - 2$

45. $f(x) = \frac{x+1}{x-2}, \quad g(x) = \frac{2x+1}{x-1}$

46. $f(x) = \frac{x-3}{x+4}, \quad g(x) = \frac{4x+3}{1-x}$

47. $f(x) = \frac{2}{x+6}, \quad g(x) = \frac{6x+2}{x}$

48. $f(x) = \frac{-1}{x+1}, \quad g(x) = \frac{1-x}{x}$

49. $f(x) = x^2 + 3, \quad x \geq 0; \quad g(x) = \sqrt{x-3}, \quad x \geq 3$

50. $f(x) = \sqrt{x+8}, \quad x \geq -8; \quad g(x) = x^2 - 8, \quad x \geq 0$

Find the inverse of each function that is one-to-one. **See Example 4.**

51. $\{(-3, 6), (2, 1), (5, 8)\}$

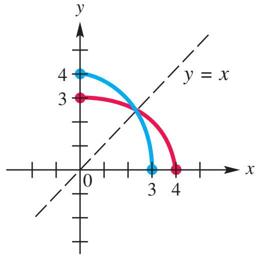
52. $\left\{(3, -1), (5, 0), (0, 5), \left(4, \frac{2}{3}\right)\right\}$

53. $\{(1, -3), (2, -7), (4, -3), (5, -5)\}$

54. $\{(6, -8), (3, -4), (0, -8), (5, -4)\}$

Determine whether each pair of functions graphed are inverses. **See Example 7.**

55.



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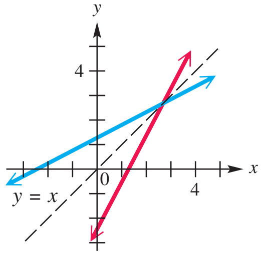
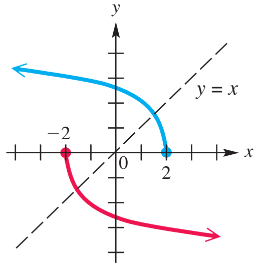
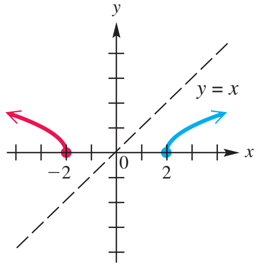


57.



58.



*For each function that is one-to-one, **(a)** write an equation for the inverse function, **(b)** graph f and f^{-1} on the same axes, and **(c)** give the domain and range of both f and f^{-1}. If the function is not one-to-one, say so.* ***See Examples 5–8.***

59. $f(x) = 3x - 4$

60. $f(x) = 4x - 5$

61. $f(x) = -4x + 3$

62. $f(x) = -6x - 8$

63. $f(x) = x^3 + 1$

64. $f(x) = -x^3 - 2$

65. $f(x) = x^2 + 8$

66. $f(x) = -x^2 + 2$

67. $f(x) = \dfrac{1}{x}, \quad x \neq 0$

68. $f(x) = \dfrac{4}{x}, \quad x \neq 0$

69. $f(x) = \dfrac{1}{x - 3}, \quad x \neq 3$

70. $f(x) = \dfrac{1}{x + 2}, \quad x \neq -2$

71. $f(x) = \dfrac{x + 1}{x - 3}, \quad x \neq 3$

72. $f(x) = \dfrac{x + 2}{x - 1}, \quad x \neq 1$

73. $f(x) = \dfrac{2x + 6}{x - 3}, \quad x \neq 3$

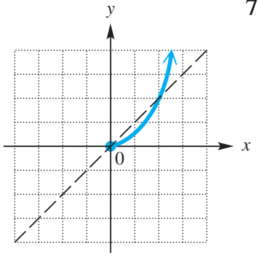
74. $f(x) = \dfrac{-3x + 12}{x - 6}, \quad x \neq 6$

75. $f(x) = \sqrt{x + 6}, \quad x \geq -6$

76. $f(x) = -\sqrt{x^2 - 16}, \quad x \geq 4$

Graph the inverse of each one-to-one function. ***See Example 7.***

77.



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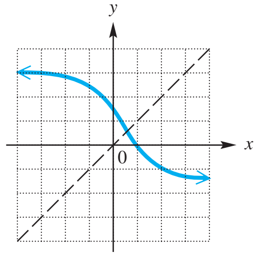

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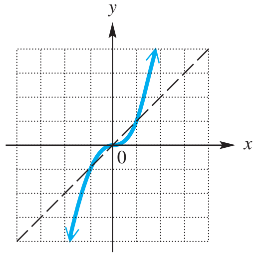

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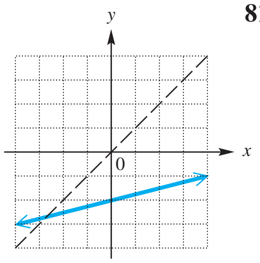
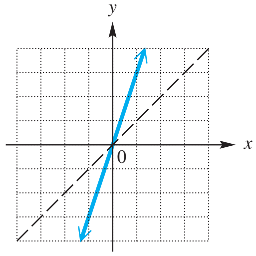
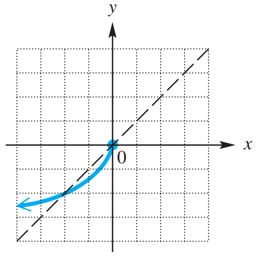


81.



82.



Concept Check *The graph of a function f is shown in the figure. Use the graph to find each value.*

83. $f^{-1}(4)$

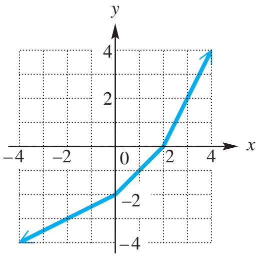
84. $f^{-1}(2)$

85. $f^{-1}(0)$

86. $f^{-1}(-2)$

87. $f^{-1}(-3)$

88. $f^{-1}(-4)$



Concept Check *Answer each of the following.*

89. Suppose $f(x)$ is the number of cars that can be built for x dollars. What does $f^{-1}(1000)$ represent?

90. Suppose $f(r)$ is the volume (in cubic inches) of a sphere of radius r inches. What does $f^{-1}(5)$ represent?

91. If a line has slope a, what is the slope of its reflection across the line $y = x$?

92. For a one-to-one function f, find $(f^{-1} \circ f)(2)$, where $f(2) = 3$.

Use a graphing calculator to graph each function defined as follows, using the given viewing window. Use the graph to decide which functions are one-to-one. If a function is one-to-one, give the equation of its inverse.

93. $f(x) = 6x^3 + 11x^2 - 6;$
$[-3, 2]$ by $[-10, 10]$

94. $f(x) = x^4 - 5x^2;$
$[-3, 3]$ by $[-8, 8]$

95. $f(x) = \dfrac{x-5}{x+3}, \quad x \neq -3;$
$[-8, 8]$ by $[-6, 8]$

96. $f(x) = \dfrac{-x}{x-4}, \quad x \neq 4;$
$[-1, 8]$ by $[-6, 6]$

Use the following alphabet coding assignment to work each problem. ***See Example 9.***

A	1	**H**	8	**O**	15	**V**	22
B	2	**I**	9	**P**	16	**W**	23
C	3	**J**	10	**Q**	17	**X**	24
D	4	**K**	11	**R**	18	**Y**	25
E	5	**L**	12	**S**	19	**Z**	26
F	6	**M**	13	**T**	20		
G	7	**N**	14	**U**	21		

97. The function $f(x) = 3x - 2$ was used to encode a message as

37 25 19 61 13 34 22 1 55 1 52 52 25 64 13 10.

Find the inverse function and determine the message.

98. The function $f(x) = 2x - 9$ was used to encode a message as

-5 9 5 5 9 27 15 29 -1 21 19 31 -3 27 41.

Find the inverse function and determine the message.

99. Encode the message SEND HELP, using the one-to-one function

$$f(x) = x^3 - 1.$$

Give the inverse function that the decoder will need when the message is received.

100. Encode the message SAILOR BEWARE, using the one-to-one function

$$f(x) = (x+1)^3.$$

Give the inverse function that the decoder will need when the message is received.

4.2 Exponential Functions

- Exponents and Properties
- Exponential Functions
- Exponential Equations
- Compound Interest
- The Number *e* and Continuous Compounding
- Exponential Models

Exponents and Properties

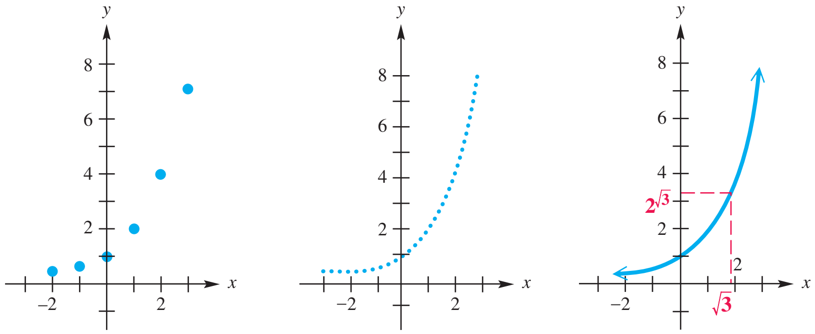
Recall the definition of $a^{m/n}$: If a is a real number, m is an integer, n is a positive integer, and $\sqrt[n]{a}$ is a real number, then

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m.$$

For example, $$16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8,$$

$$27^{-1/3} = \frac{1}{27^{1/3}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3}, \quad \text{and} \quad 64^{-1/2} = \frac{1}{64^{1/2}} = \frac{1}{\sqrt{64}} = \frac{1}{8}.$$

In this section, we extend the definition of a^r to include all *real* (not just rational) values of the exponent r. Consider the graphs of $y = 2^x$ for different domains in **Figure 13.**



$y = 2^x$; integers as domain

$y = 2^x$; selected rational numbers as domain

$y = 2^x$; real numbers as domain

Figure 13

The equations that use just integers or selected rational numbers as domain in **Figure 13** leave holes in the graphs. In order for the graph to be continuous, we must extend the domain to include irrational numbers such as $\sqrt{3}$. We might evaluate $2^{\sqrt{3}}$ by approximating the exponent with the rational numbers 1.7, 1.73, 1.732, and so on. Because these values approach the value of $\sqrt{3}$ more and more closely, it is reasonable that $2^{\sqrt{3}}$ should be approximated more and more closely by the numbers $2^{1.7}$, $2^{1.73}$, $2^{1.732}$, and so on. These expressions can be evaluated using rational exponents as follows.

$$2^{1.7} = 2^{17/10} = \left(\sqrt[10]{2}\right)^{17} \approx 3.249009585$$

Because any irrational number may be approximated more and more closely using rational numbers, we can extend the definition of a^r to include all real number exponents and apply all previous theorems for exponents. In addition to the rules for exponents presented earlier, we use several new properties in this chapter.

Additional Properties of Exponents

For any real number $a > 0$, $a \neq 1$, the following statements hold.

Property	Description
(a) a^x **is a unique real number for all real numbers** x**.**	$y = a^x$ can be considered a function $f(x) = a^x$ with domain $(-\infty, \infty)$.
(b) $a^b = a^c$ **if and only if** $b = c$**.**	The function $f(x) = a^x$ is one-to-one.
(c) If $a > 1$ **and** $m < n$**, then** $a^m < a^n$**.**	*Example:* $2^3 < 2^4$ $(a > 1)$ Increasing the exponent leads to a *greater* number. The function $f(x) = 2^x$ is an *increasing* function.
(d) If $0 < a < 1$ **and** $m < n$**, then** $a^m > a^n$**.**	*Example:* $\left(\frac{1}{2}\right)^2 > \left(\frac{1}{2}\right)^3$ $(0 < a < 1)$ Increasing the exponent leads to a *lesser* number. The function $f(x) = \left(\frac{1}{2}\right)^x$ is a *decreasing* function.

Exponential Functions We now define a function $f(x) = a^x$ whose domain is the set of all real numbers. Notice how the independent variable x appears in the exponent in this function. In earlier chapters, this was not the case.

Exponential Function

If $a > 0$ and $a \neq 1$, then the **exponential function with base a** is

$$f(x) = a^x.$$

NOTE The restrictions on a in the definition of an exponential function are important. Consider the outcome of breaking each restriction.

If $a < 0$, say $a = -2$, and we let $x = \frac{1}{2}$, then $f\left(\frac{1}{2}\right) = (-2)^{1/2} = \sqrt{-2}$, which is *not* a real number.

If $a = 1$, then the function becomes the constant function $f(x) = 1^x = 1$, which is *not* an exponential function.

EXAMPLE 1 Evaluating an Exponential Function

For $f(x) = 2^x$, find each of the following.

(a) $f(-1)$ **(b)** $f(3)$ **(c)** $f\left(\frac{5}{2}\right)$ **(d)** $f(4.92)$

SOLUTION

(a) $f(-1) = 2^{-1} = \frac{1}{2}$ Replace x with -1. **(b)** $f(3) = 2^3 = 8$

(c) $f\left(\frac{5}{2}\right) = 2^{5/2} = (2^5)^{1/2} = 32^{1/2} = \sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$

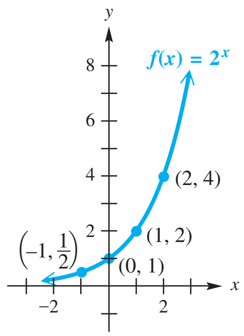
(d) $f(4.92) = 2^{4.92} \approx 30.2738447$ Use a calculator.

Now Try Exercises 13, 19, and 23.

We repeat the final graph of $y = 2^x$ (with real numbers as domain) from **Figure 13** and summarize important details of the function $f(x) = 2^x$ here.

- The y-intercept is $(0, 1)$.
- Because $2^x > 0$ for all x and $2^x \to 0$ as $x \to -\infty$, the x-axis is a horizontal asymptote.
- As the graph suggests, the domain of the function is $(-\infty, \infty)$ and the range is $(0, \infty)$.
- The function is increasing on its entire domain. Therefore, it is one-to-one.

These observations lead to the following generalizations about the graphs of exponential functions.



Graph of $f(x) = 2^x$ with domain $(-\infty, \infty)$

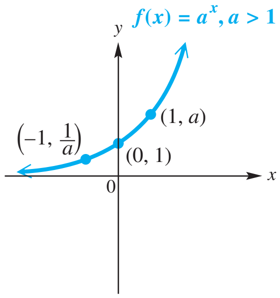
Figure 13 (repeated)

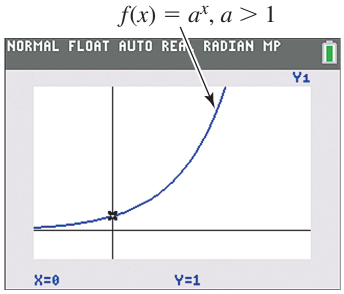
Exponential Function $f(x) = a^x$

Domain: $(-\infty, \infty)$ Range: $(0, \infty)$

For $f(x) = 2^x$:

x	$f(x)$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8





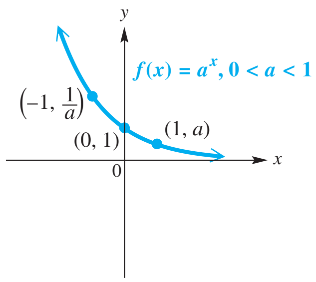
This is the general behavior seen on a calculator graph for **any base *a*, for *a* > 1.**

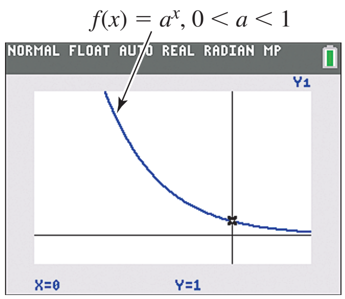
Figure 14

- **$f(x) = a^x$, for $a > 1$,** is increasing and continuous on its entire domain, $(-\infty, \infty)$.
- The x-axis is a horizontal asymptote as $x \to -\infty$.
- The graph passes through the points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.

For $f(x) = \left(\frac{1}{2}\right)^x$:

x	$f(x)$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$





This is the general behavior seen on a calculator graph for **any base *a*, for 0 < *a* < 1.**

Figure 15

- **$f(x) = a^x$, for $0 < a < 1$,** is decreasing and continuous on its entire domain, $(-\infty, \infty)$.
- The x-axis is a horizontal asymptote as $x \to \infty$.
- The graph passes through the points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.

Recall that the graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis. Thus, we have the following.

$$\text{If } f(x) = 2^x, \quad \text{then} \quad f(-x) = 2^{-x} = 2^{-1 \cdot x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x.$$

This is supported by the graphs in **Figures 14 and 15.**

The graph of $f(x) = 2^x$ is typical of graphs of $f(x) = a^x$ where $a > 1$. For larger values of a, the graphs rise more steeply, but the general shape is similar to the graph in **Figure 14.** When $0 < a < 1$, the graph decreases in a manner similar to the graph of $f(x) = \left(\frac{1}{2}\right)^x$ in **Figure 15.**

In **Figure 16,** the graphs of several typical exponential functions illustrate these facts.

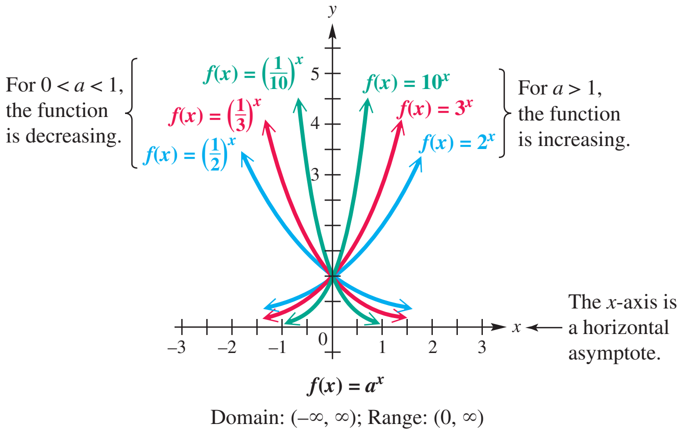


Figure 16

In summary, the graph of a function of the form $f(x) = a^x$ has the following features.

Characteristics of the Graph of $f(x) = a^x$

1. The points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$ are on the graph.
2. If $a > 1$, then f is an increasing function.

   If $0 < a < 1$, then f is a decreasing function.
3. The x-axis is a horizontal asymptote.
4. The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

EXAMPLE 2 Graphing an Exponential Function

Graph $f(x) = \left(\frac{1}{5}\right)^x$. Give the domain and range.

SOLUTION The y-intercept is $(0, 1)$, and the x-axis is a horizontal asymptote. Plot a few ordered pairs, and draw a smooth curve through them as shown in **Figure 17.**

x	$f(x)$
-2	25
-1	5
0	1
1	$\frac{1}{5}$
2	$\frac{1}{25}$

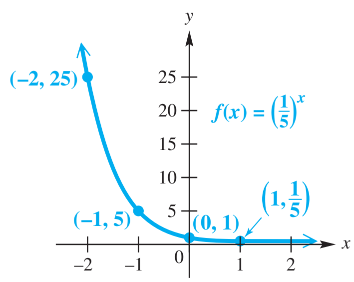


Figure 17

This function has domain $(-\infty, \infty)$, range $(0, \infty)$, and is one-to-one. It is decreasing on its entire domain.

✔ **Now Try Exercise 29.**

EXAMPLE 3 Graphing Reflections and Translations

Graph each function. Show the graph of $y = 2^x$ for comparison. Give the domain and range.

(a) $f(x) = -2^x$ **(b)** $f(x) = 2^{x+3}$ **(c)** $f(x) = 2^{x-2} - 1$

SOLUTION In each graph, we show in particular how the point $(0, 1)$ on the graph of $y = 2^x$ has been translated.

(a) The graph of $f(x) = -2^x$ is that of $f(x) = 2^x$ reflected across the x-axis. See **Figure 18.** The domain is $(-\infty, \infty)$, and the range is $(-\infty, 0)$.

(b) The graph of $f(x) = 2^{x+3}$ is the graph of $f(x) = 2^x$ translated 3 units to the left, as shown in **Figure 19.** The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

(c) The graph of $f(x) = 2^{x-2} - 1$ is that of $f(x) = 2^x$ translated 2 units to the right and 1 unit down. See **Figure 20.** The domain is $(-\infty, \infty)$, and the range is $(-1, \infty)$.

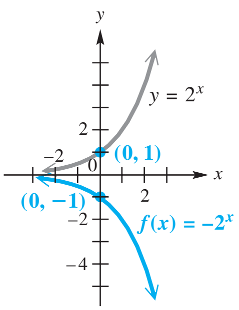


Figure 18

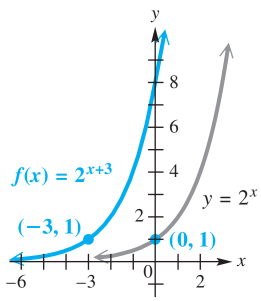


Figure 19

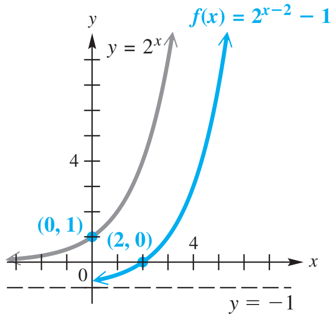


Figure 20

✔ **Now Try Exercises 39, 41, and 47.**

Exponential Equations Because the graph of $f(x) = a^x$ is that of a one-to-one function, to solve $a^{x_1} = a^{x_2}$, we need only show that $x_1 = x_2$. This property is used to solve an **exponential equation,** which is an equation with a variable as exponent.

EXAMPLE 4 Solving an Exponential Equation

Solve $\left(\frac{1}{3}\right)^x = 81$.

SOLUTION ***Write each side of the equation using a common base.***

$$\left(\frac{1}{3}\right)^x = 81$$

$$(3^{-1})^x = 81 \quad \text{Definition of negative exponent}$$

$$3^{-x} = 81 \quad (a^m)^n = a^{mn}$$

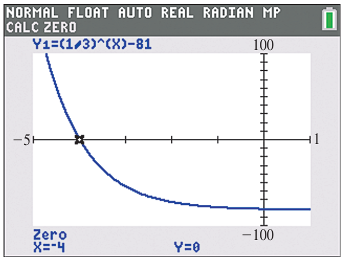
$$3^{-x} = 3^4 \quad \text{Write 81 as a power of 3.}$$

$$-x = 4 \quad \text{Set exponents equal (Property (b) given earlier).}$$

$$x = -4 \quad \text{Multiply by } -1.$$

Check by substituting -4 for x in the original equation. The solution set is $\{-4\}$.

✔ **Now Try Exercise 73.**



The x-intercept of the graph of $y = \left(\frac{1}{3}\right)^x - 81$ can be used to verify the solution in **Example 4.**

EXAMPLE 5 Solving an Exponential Equation

Solve $2^{x+4} = 8^{x-6}$.

SOLUTION Write each side of the equation using a common base.

$$2^{x+4} = 8^{x-6}$$

$$2^{x+4} = (2^3)^{x-6} \quad \text{Write 8 as a power of 2.}$$

$$2^{x+4} = 2^{3x-18} \quad (a^m)^n = a^{mn}$$

$$x + 4 = 3x - 18 \quad \text{Set exponents equal (Property (b)).}$$

$$-2x = -22 \quad \text{Subtract } 3x \text{ and } 4.$$

$$x = 11 \quad \text{Divide by } -2.$$

Check by substituting 11 for x in the original equation. The solution set is $\{11\}$.

✔ **Now Try Exercise 81.**

Later in this chapter, we describe a general method for solving exponential equations where the approach used in **Examples 4 and 5** is not possible. For instance, the above method could not be used to solve an equation like

$$7^x = 12$$

because it is not easy to express both sides as exponential expressions with the same base.

In **Example 6,** we solve an equation that has the variable as the base of an exponential expression.

EXAMPLE 6 Solving an Equation with a Fractional Exponent

Solve $x^{4/3} = 81$.

SOLUTION Notice that the variable is in the base rather than in the exponent.

$$x^{4/3} = 81$$

$$\left(\sqrt[3]{x}\right)^4 = 81 \quad \text{Radical notation for } a^{m/n}$$

$$\sqrt[3]{x} = \pm 3 \quad \text{Take fourth roots on each side. Remember to use } \pm.$$

$$x = \pm 27 \quad \text{Cube each side.}$$

Check *both* solutions in the original equation. Both check, so the solution set is $\{\pm 27\}$.

Alternative Method There may be more than one way to solve an exponential equation, as shown here.

$$x^{4/3} = 81$$

$$(x^{4/3})^3 = 81^3 \quad \text{Cube each side.}$$

$$x^4 = (3^4)^3 \quad \text{Write 81 as } 3^4.$$

$$x^4 = 3^{12} \quad (a^m)^n = a^{mn}$$

$$x = \pm\sqrt[4]{3^{12}} \quad \text{Take fourth roots on each side.}$$

$$x = \pm 3^3 \quad \text{Simplify the radical.}$$

$$x = \pm 27 \quad \text{Apply the exponent.}$$

The same solution set, $\{\pm 27\}$, results.

✔ **Now Try Exercise 83.**

Compound Interest Recall the formula for simple interest, $I = Prt$, where P is principal (amount deposited), r is annual rate of interest expressed as a decimal, and t is time in years that the principal earns interest. Suppose $t = 1$ yr. Then at the end of the year, the amount has grown to the following.

$$P + Pr = P(1 + r) \quad \text{Original principal plus interest}$$

If this balance earns interest at the same interest rate for another year, the balance at the end of *that* year will increase as follows.

$$\begin{aligned} [P(1 + r)] + [P(1 + r)]r &= [P(1 + r)](1 + r) && \text{Factor.} \\ &= P(1 + r)^2 && a \cdot a = a^2 \end{aligned}$$

After the third year, the balance will grow in a similar pattern.

$$\begin{aligned} [P(1 + r)^2] + [P(1 + r)^2]r &= [P(1 + r)^2](1 + r) && \text{Factor.} \\ &= P(1 + r)^3 && a^2 \cdot a = a^3 \end{aligned}$$

Continuing in this way produces a formula for interest compounded annually.

$$\boldsymbol{A = P(1 + r)^t}$$

The general formula for compound interest can be derived in the same way.

Compound Interest

If P dollars are deposited in an account paying an annual rate of interest r compounded (paid) n times per year, then after t years the account will contain A dollars, according to the following formula.

$$\boldsymbol{A = P\left(1 + \frac{r}{n}\right)^{tn}}$$

EXAMPLE 7 Using the Compound Interest Formula

Suppose \$1000 is deposited in an account paying 4% interest per year compounded quarterly (four times per year).

(a) Find the amount in the account after 10 yr with no withdrawals.

(b) How much interest is earned over the 10-yr period?

SOLUTION

(a)

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{tn} && \text{Compound interest formula} \\ A &= 1000\left(1 + \frac{0.04}{4}\right)^{10(4)} && \text{Let } P = 1000,\ r = 0.04,\ n = 4, \text{ and } t = 10. \\ A &= 1000(1 + 0.01)^{40} && \text{Simplify.} \\ A &= 1488.86 && \text{Round to the nearest cent.} \end{aligned}$$

Thus, \$1488.86 is in the account after 10 yr.

(b) The interest earned for that period is

$$\$1488.86 - \$1000 = \$488.86.$$

✔ **Now Try Exercise 97(a).**

In the formula for compound interest

$$A = P\left(1 + \frac{r}{n}\right)^{tn},$$

A is sometimes called the **future value** and P the **present value.** A is also called the **compound amount** and is the balance *after* interest has been earned.

EXAMPLE 8 Finding Present Value

Becky must pay a lump sum of \$6000 in 5 yr.

(a) What amount deposited today (present value) at 3.1% compounded annually will grow to \$6000 in 5 yr?

(b) If only \$5000 is available to deposit now, what annual interest rate is necessary for the money to increase to \$6000 in 5 yr?

SOLUTION

(a)

$$A = P\left(1 + \frac{r}{n}\right)^{tn} \quad \text{Compound interest formula}$$

$$6000 = P\left(1 + \frac{0.031}{1}\right)^{5(1)} \quad \text{Let } A = 6000,\ r = 0.031,\ n = 1, \text{ and } t = 5.$$

$$6000 = P(1.031)^5 \quad \text{Simplify.}$$

$$P = \frac{6000}{(1.031)^5} \quad \text{Divide by } (1.031)^5 \text{ to solve for } P.$$

$$P \approx 5150.60 \quad \text{Use a calculator.}$$

If Becky leaves \$5150.60 for 5 yr in an account paying 3.1% compounded annually, she will have \$6000 when she needs it. Thus, \$5150.60 is the present value of \$6000 if interest of 3.1% is compounded annually for 5 yr.

(b)

$$A = P\left(1 + \frac{r}{n}\right)^{tn} \quad \text{Compound interest formula}$$

$$6000 = 5000(1 + r)^5 \quad \text{Let } A = 6000,\ P = 5000,\ n = 1, \text{ and } t = 5.$$

$$\frac{6}{5} = (1 + r)^5 \quad \text{Divide by 5000.}$$

$$\left(\frac{6}{5}\right)^{1/5} = 1 + r \quad \text{Take the fifth root on each side.}$$

$$\left(\frac{6}{5}\right)^{1/5} - 1 = r \quad \text{Subtract 1.}$$

$$r \approx 0.0371 \quad \text{Use a calculator.}$$

An interest rate of 3.71% will produce enough interest to increase the \$5000 to \$6000 by the end of 5 yr.

✔ **Now Try Exercises 99 and 103.**

CAUTION When performing the computations in problems like those in **Examples 7 and 8,** do not round off during intermediate steps. Keep all calculator digits and round at the end of the process.

n	$\left(1+\frac{1}{n}\right)^n$ (rounded)
1	2
2	2.25
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
1,000,000	2.71828

The Number e and Continuous Compounding The more often interest is compounded within a given time period, the more interest will be earned. Surprisingly, however, there is a limit on the amount of interest, no matter how often it is compounded.

Suppose that \$1 is invested at 100% interest per year, compounded n times per year. Then the interest rate (in decimal form) is 1.00, and the interest rate per period is $\frac{1}{n}$. According to the formula (with $P = 1$), the compound amount at the end of 1 yr will be

$$A = \left(1 + \frac{1}{n}\right)^n.$$

A calculator gives the results in the margin for various values of n. The table suggests that as n increases, the value of $\left(1 + \frac{1}{n}\right)^n$ gets closer and closer to some fixed number. This is indeed the case. This fixed number is called e. ***(In mathematics, e is a real number and not a variable.)***

Value of e

$$e \approx 2.718281828459045$$

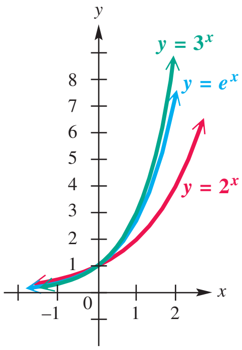


Figure 21

Figure 21 shows graphs of the functions

$$y = 2^x, \quad y = 3^x, \quad \text{and} \quad y = e^x.$$

Because $2 < e < 3$, the graph of $y = e^x$ lies "between" the other two graphs.

As mentioned above, the amount of interest earned increases with the frequency of compounding, but the value of the expression $\left(1 + \frac{1}{n}\right)^n$ approaches e as n gets larger. Consequently, the formula for compound interest approaches a limit as well, called the compound amount from **continuous compounding.**

Continuous Compounding

If P dollars are deposited at a rate of interest r compounded continuously for t years, then the compound amount A in dollars on deposit is given by the following formula.

$$A = Pe^{rt}$$

EXAMPLE 9 Solving a Continuous Compounding Problem

Suppose \$5000 is deposited in an account paying 3% interest compounded continuously for 5 yr. Find the total amount on deposit at the end of 5 yr.

SOLUTION

$A = Pe^{rt}$	Continuous compounding formula
$A = 5000e^{0.03(5)}$	Let $P = 5000$, $r = 0.03$, and $t = 5$.
$A = 5000e^{0.15}$	Multiply exponents.
$A \approx 5809.17$ or \$5809.17	Use a calculator.

Check that daily compounding would have produced a compound amount about \$0.03 less.

✔ **Now Try Exercise 97(b).**

EXAMPLE 10 Comparing Interest Earned as Compounding Is More Frequent

In **Example 7,** we found that \$1000 invested at 4% compounded quarterly for 10 yr grew to \$1488.86. Compare this same investment compounded annually, semiannually, monthly, daily, and continuously.

SOLUTION Substitute 0.04 for r, 10 for t, and the appropriate number of compounding periods for n into the formulas

$$A = P\left(1 + \frac{r}{n}\right)^{tn} \quad \text{Compound interest formula}$$

and

$$A = Pe^{rt}. \quad \text{Continuous compounding formula}$$

The results for amounts of \$1 and \$1000 are given in the table.

Compounded	\$1	\$1000
Annually	$(1 + 0.04)^{10} \approx 1.48024$	\$1480.24
Semiannually	$\left(1 + \frac{0.04}{2}\right)^{10(2)} \approx 1.48595$	\$1485.95
Quarterly	$\left(1 + \frac{0.04}{4}\right)^{10(4)} \approx 1.48886$	\$1488.86
Monthly	$\left(1 + \frac{0.04}{12}\right)^{10(12)} \approx 1.49083$	\$1490.83
Daily	$\left(1 + \frac{0.04}{365}\right)^{10(365)} \approx 1.49179$	\$1491.79
Continuously	$e^{10(0.04)} \approx 1.49182$	\$1491.82

Comparing the results for a \$1000 investment, we notice the following.

- Compounding semiannually rather than annually increases the value of the account after 10 yr by \$5.71.
- Quarterly compounding grows to \$2.91 more than semiannual compounding after 10 yr.
- Daily compounding yields only \$0.96 more than monthly compounding.
- Continuous compounding yields only \$0.03 more than daily compounding.

Each increase in compounding frequency earns less additional interest.

✔ **Now Try Exercise 105.**

LOOKING AHEAD TO CALCULUS
In calculus, the derivative allows us to determine the slope of a tangent line to the graph of a function. For the function

$$f(x) = e^x,$$

the derivative is the function f itself:

$$f'(x) = e^x.$$

Therefore, in calculus the exponential function with base e is much easier to work with than exponential functions having other bases.

Exponential Models

The number e is important as the base of an exponential function in many practical applications. In situations involving growth or decay of a quantity, the amount or number present at time t often can be closely modeled by a function of the form

$$y = y_0 e^{kt},$$

where y_0 is the amount or number present at time $t = 0$ and k is a constant.

Exponential functions are used to model the growth of microorganisms in a culture, the growth of certain populations, and the decay of radioactive material.

EXAMPLE 11 Using Data to Model Exponential Growth

Data from recent years indicate that future amounts of carbon dioxide in the atmosphere may grow according to the table. Amounts are given in parts per million.

Year	Carbon Dioxide (ppm)
1990	353
2000	375
2075	590
2175	1090
2275	2000

Source: International Panel on Climate Change (IPCC).

(a) Make a scatter diagram of the data. Do the carbon dioxide levels appear to grow exponentially?

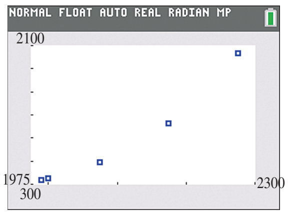
(b) One model for the data is the function

$$y = 0.001942e^{0.00609x},$$

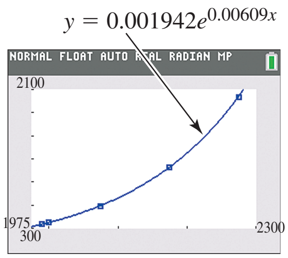
where x is the year and $1990 \le x \le 2275$. Use a graph of this model to estimate when future levels of carbon dioxide will double and triple over the preindustrial level of 280 ppm.

SOLUTION

(a) We show a calculator graph for the data in **Figure 22(a).** The data appear to resemble the graph of an increasing exponential function.



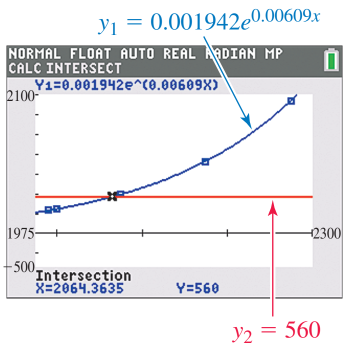
(a)



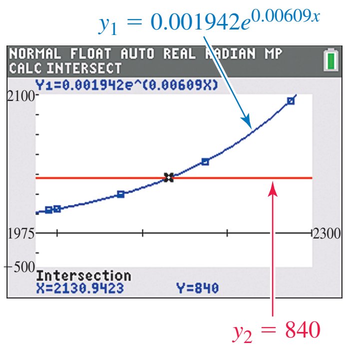
(b)

Figure 22

(b) A graph of $y = 0.001942e^{0.00609x}$ in **Figure 22(b)** shows that it is very close to the data points. We graph $y_2 = 2 \cdot 280 = 560$ in **Figure 23(a)** and $y_2 = 3 \cdot 280 = 840$ in **Figure 23(b)** on the same coordinate axes as the given function, and we use the calculator to find the intersection points.



(a)

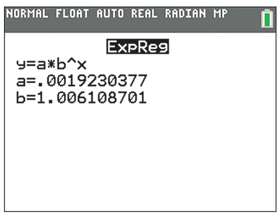


(b)

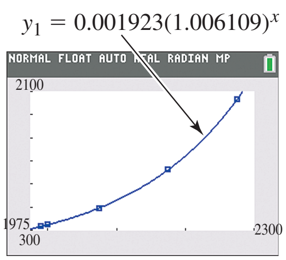
Figure 23

The graph of the function intersects the horizontal lines at x-values of approximately 2064.4 and 2130.9. According to this model, carbon dioxide levels will have doubled during 2064 and tripled by 2131.

Now Try Exercise 107.



(a)



(b)

Figure 24

Graphing calculators are capable of fitting exponential curves to scatter diagrams like the one found in **Example 11.** The TI-84 Plus displays another (different) equation in **Figure 24(a)** for the atmospheric carbon dioxide example, approximated as follows.

$$y = 0.001923(1.006109)^x$$

This calculator form differs from the model in **Example 11. Figure 24(b)** shows the data points and the graph of this exponential regression equation. ■

4.2 Exercises

CONCEPT PREVIEW *Fill in the blank(s) to correctly complete each sentence.*

1. If $f(x) = 4^x$, then $f(2) =$ _____ and $f(-2) =$ _____.
2. If $a > 1$, then the graph of $f(x) = a^x$ __________ (rises/falls) from left to right.
3. If $0 < a < 1$, then the graph of $f(x) = a^x$ __________ (rises/falls) from left to right.
4. The domain of $f(x) = 4^x$ is ________ and the range is ________.
5. The graph of $f(x) = 8^x$ passes through the points $(-1,$ ___$)$, $(0,$ ___$)$, and $(1,$ ___$)$.
6. The graph of $f(x) = -\left(\frac{1}{3}\right)^{x+4} - 5$ is that of $f(x) = \left(\frac{1}{3}\right)^x$ reflected across the _____-axis, translated _____ units to the left and _____ units down.

CONCEPT PREVIEW *Solve each equation. Round answers to the nearest hundredth as needed.*

7. $\left(\dfrac{1}{4}\right)^x = 64$
8. $x^{2/3} = 36$
9. $A = 2000\left(1 + \dfrac{0.03}{4}\right)^{8(4)}$
10. $10{,}000 = 5000(1 + r)^{25}$

For $f(x) = 3^x$ and $g(x) = \left(\frac{1}{4}\right)^x$, find each of the following. Round answers to the nearest thousandth as needed. ***See Example 1.***

11. $f(2)$
12. $f(3)$
13. $f(-2)$
14. $f(-3)$
15. $g(2)$
16. $g(3)$
17. $g(-2)$
18. $g(-3)$
19. $f\left(\dfrac{3}{2}\right)$
20. $f\left(-\dfrac{5}{2}\right)$
21. $g\left(\dfrac{3}{2}\right)$
22. $g\left(-\dfrac{5}{2}\right)$
23. $f(2.34)$
24. $f(-1.68)$
25. $g(-1.68)$
26. $g(2.34)$

Graph each function. ***See Example 2.***

27. $f(x) = 3^x$
28. $f(x) = 4^x$
29. $f(x) = \left(\dfrac{1}{3}\right)^x$
30. $f(x) = \left(\dfrac{1}{4}\right)^x$
31. $f(x) = \left(\dfrac{3}{2}\right)^x$
32. $f(x) = \left(\dfrac{5}{3}\right)^x$
33. $f(x) = \left(\dfrac{1}{10}\right)^{-x}$
34. $f(x) = \left(\dfrac{1}{6}\right)^{-x}$
35. $f(x) = 4^{-x}$
36. $f(x) = 10^{-x}$
37. $f(x) = 2^{|x|}$
38. $f(x) = 2^{-|x|}$

Graph each function. Give the domain and range. ***See Example 3.***

39. $f(x) = 2^x + 1$
40. $f(x) = 2^x - 4$
41. $f(x) = 2^{x+1}$
42. $f(x) = 2^{x-4}$
43. $f(x) = -2^{x+2}$
44. $f(x) = -2^{x-3}$
45. $f(x) = 2^{-x}$
46. $f(x) = -2^{-x}$
47. $f(x) = 2^{x-1} + 2$
48. $f(x) = 2^{x+3} + 1$
49. $f(x) = 2^{x+2} - 4$
50. $f(x) = 2^{x-3} - 1$

Graph each function. Give the domain and range. ***See Example 3.***

51. $f(x) = \left(\frac{1}{3}\right)^{x} - 2$ **52.** $f(x) = \left(\frac{1}{3}\right)^{x} + 4$ **53.** $f(x) = \left(\frac{1}{3}\right)^{x+2}$

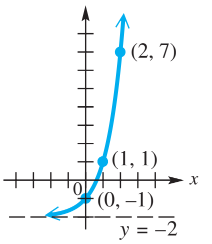
54. $f(x) = \left(\frac{1}{3}\right)^{x-4}$ **55.** $f(x) = \left(\frac{1}{3}\right)^{-x+1}$ **56.** $f(x) = \left(\frac{1}{3}\right)^{-x-2}$

57. $f(x) = \left(\frac{1}{3}\right)^{-x}$ **58.** $f(x) = -\left(\frac{1}{3}\right)^{-x}$ **59.** $f(x) = \left(\frac{1}{3}\right)^{x-2} + 2$

60. $f(x) = \left(\frac{1}{3}\right)^{x-1} + 3$ **61.** $f(x) = \left(\frac{1}{3}\right)^{x+2} - 1$ **62.** $f(x) = \left(\frac{1}{3}\right)^{x+3} - 2$

Connecting Graphs with Equations *Write an equation for the graph given. Each represents an exponential function f with base 2 or 3, translated and/or reflected.*

63.



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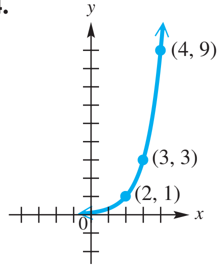

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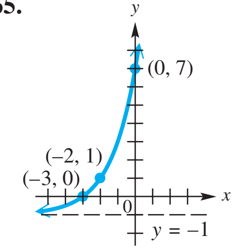

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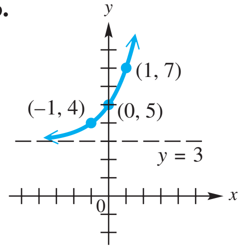
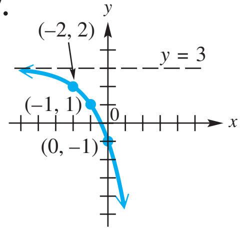


67.



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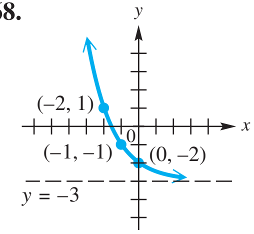

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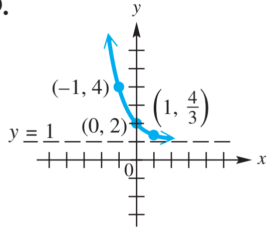

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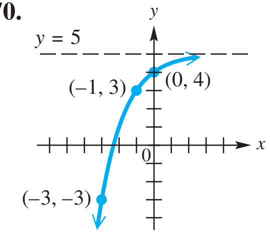


Solve each equation. ***See Examples 4–6.***

71. $4^x = 2$ **72.** $125^x = 5$ **73.** $\left(\frac{5}{2}\right)^x = \frac{4}{25}$ **74.** $\left(\frac{2}{3}\right)^x = \frac{9}{4}$

75. $2^{3-2x} = 8$ **76.** $5^{2+2x} = 25$ **77.** $e^{4x-1} = (e^2)^x$ **78.** $e^{3-x} = (e^3)^{-x}$

79. $27^{4x} = 9^{x+1}$ **80.** $32^{2x} = 16^{x-1}$ **81.** $4^{x-2} = 2^{3x+3}$ **82.** $2^{6-3x} = 8^{x+1}$

83. $x^{2/3} = 4$ **84.** $x^{2/5} = 16$ **85.** $x^{5/2} = 32$ **86.** $x^{3/2} = 27$

87. $x^{-6} = \frac{1}{64}$ **88.** $x^{-4} = \frac{1}{256}$ **89.** $x^{5/3} = -243$ **90.** $x^{7/5} = -128$

91. $\left(\frac{1}{e}\right)^{-x} = \left(\frac{1}{e^2}\right)^{x+1}$ **92.** $e^{x-1} = \left(\frac{1}{e^4}\right)^{x+1}$ **93.** $\left(\sqrt{2}\right)^{x+4} = 4^x$

94. $\left(\sqrt[3]{5}\right)^{-x} = \left(\frac{1}{5}\right)^{x+2}$ **95.** $\frac{1}{27} = x^{-3}$ **96.** $\frac{1}{32} = x^{-5}$

Solve each problem. ***See Examples 7–9.***

97. ***Future Value*** Find the future value and interest earned if \$8906.54 is invested for 9 yr at 3% compounded

(a) semiannually **(b)** continuously.

98. ***Future Value*** Find the future value and interest earned if \$56,780 is invested at 2.8% compounded

(a) quarterly for 23 quarters **(b)** continuously for 15 yr.

99. ***Present Value*** Find the present value that will grow to \$25,000 if interest is 3.2% compounded quarterly for 11 quarters.

100. ***Present Value*** Find the present value that will grow to \$45,000 if interest is 3.6% compounded monthly for 1 yr.

101. ***Present Value*** Find the present value that will grow to \$5000 if interest is 3.5% compounded quarterly for 10 yr.

102. ***Interest Rate*** Find the required annual interest rate to the nearest tenth of a percent for \$65,000 to grow to \$65,783.91 if interest is compounded monthly for 6 months.

103. ***Interest Rate*** Find the required annual interest rate to the nearest tenth of a percent for \$1200 to grow to \$1500 if interest is compounded quarterly for 9 yr.

104. ***Interest Rate*** Find the required annual interest rate to the nearest tenth of a percent for \$5000 to grow to \$6200 if interest is compounded quarterly for 8 yr.

Solve each problem. ***See Example 10.***

105. ***Comparing Loans*** Bank A is lending money at 6.4% interest compounded annually. The rate at Bank B is 6.3% compounded monthly, and the rate at Bank C is 6.35% compounded quarterly. At which bank will we pay the *least* interest?

106. ***Future Value*** Suppose \$10,000 is invested at an annual rate of 2.4% for 10 yr. Find the future value if interest is compounded as follows.

(a) annually **(b)** quarterly **(c)** monthly **(d)** daily (365 days)

(Modeling) *Solve each problem.* ***See Example 11.***

107. ***Atmospheric Pressure*** The atmospheric pressure (in millibars) at a given altitude (in meters) is shown in the table.

Altitude	Pressure	Altitude	Pressure
0	1013	6000	472
1000	899	7000	411
2000	795	8000	357
3000	701	9000	308
4000	617	10,000	265
5000	541		

Source: Miller, A. and J. Thompson, *Elements of Meteorology,* Fourth Edition, Charles E. Merrill Publishing Company, Columbus, Ohio.

(a) Use a graphing calculator to make a scatter diagram of the data for atmospheric pressure P at altitude x.

(b) Would a linear or an exponential function fit the data better?

(c) The following function approximates the data.

$$P(x) = 1013e^{-0.0001341x}$$

Use a graphing calculator to graph P and the data on the same coordinate axes.

(d) Use P to predict the pressures at 1500 m and 11,000 m, and compare them to the actual values of 846 millibars and 227 millibars, respectively.

108. ***World Population Growth*** World population in millions closely fits the exponential function

$$f(x) = 6084e^{0.0120x},$$

where x is the number of years since 2000. (*Source:* U.S. Census Bureau.)

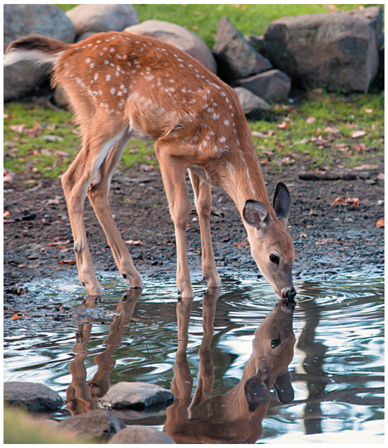
(a) The world population was about 6853 million in 2010. How closely does the function approximate this value?

(b) Use this model to predict world population in 2020 and 2030.

109. ***Deer Population*** The exponential growth of the deer population in Massachusetts can be approximated using the model

$$f(x) = 50{,}000(1 + 0.06)^x,$$

where 50,000 is the initial deer population and 0.06 is the rate of growth. $f(x)$ is the total population after x years have passed. Find each value to the nearest thousand.



(a) Predict the total population after 4 yr.

(b) If the initial population was 30,000 and the growth rate was 0.12, how many deer would be present after 3 yr?

(c) How many additional deer can we expect in 5 yr if the initial population is 45,000 and the current growth rate is 0.08?

110. ***Employee Training*** A person learning certain skills involving repetition tends to learn quickly at first. Then learning tapers off and skill acquisition approaches some upper limit. Suppose the number of symbols per minute that a person using a keyboard can type is given by

$$f(t) = 250 - 120(2.8)^{-0.5t},$$

where t is the number of months the operator has been in training. Find each value to the nearest whole number.

(a) $f(2)$ **(b)** $f(4)$ **(c)** $f(10)$

(d) What happens to the number of symbols per minute after several months of training?

Use a graphing calculator to find the solution set of each equation. Approximate the solution(s) to the nearest tenth.

111. $5e^{3x} = 75$ **112.** $6^{-x} = 1 - x$ **113.** $3x + 2 = 4^x$ **114.** $x = 2^x$

115. A function of the form $\boldsymbol{f(x) = x^r}$, where r is a constant, is a **power function.** Discuss the difference between an exponential function and a power function.

116. ***Concept Check*** If $f(x) = a^x$ and $f(3) = 27$, determine each function value.

(a) $f(1)$ **(b)** $f(-1)$ **(c)** $f(2)$ **(d)** $f(0)$

Concept Check *Give an equation of the form $f(x) = a^x$ to define the exponential function whose graph contains the given point.*

117. $(3, 8)$ **118.** $(3, 125)$ **119.** $(-3, 64)$ **120.** $(-2, 36)$

Concept Check *Use properties of exponents to write each function in the form $f(t) = ka^t$, where k is a constant. (Hint: Recall that $a^{x+y} = a^x \cdot a^y$.)*

121. $f(t) = 3^{2t+3}$ **122.** $f(t) = 2^{3t+2}$ **123.** $f(t) = \left(\frac{1}{3}\right)^{1-2t}$ **124.** $f(t) = \left(\frac{1}{2}\right)^{1-2t}$

In calculus, the following can be shown.

$$e^x = 1 + x + \frac{x^2}{2 \cdot 1} + \frac{x^3}{3 \cdot 2 \cdot 1} + \frac{x^4}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{x^5}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \cdots$$

Using more terms, one can obtain a more accurate approximation for e^x.

125. Use the terms shown, and replace x with 1 to approximate $e^1 = e$ to three decimal places. Check the result with a calculator.

126. Use the terms shown, and replace x with -0.05 to approximate $e^{-0.05}$ to four decimal places. Check the result with a calculator.

Relating Concepts

For individual or collaborative investigation *(Exercises 127–132)*

Consider $f(x) = a^x$, where $a > 1$. ***Work these exercises in order.***

127. Is f a one-to-one function? If so, what kind of related function exists for f?

128. If f has an inverse function f^{-1}, sketch f and f^{-1} on the same set of axes.

129. If f^{-1} exists, find an equation for $y = f^{-1}(x)$. (You need not solve for y.)

130. If $a = 10$, what is the equation for $y = f^{-1}(x)$? (You need not solve for y.)

131. If $a = e$, what is the equation for $y = f^{-1}(x)$? (You need not solve for y.)

132. If the point (p, q) is on the graph of f, then the point ________ is on the graph of f^{-1}.

4.3 Logarithmic Functions

- Logarithms
- Logarithmic Equations
- Logarithmic Functions
- Properties of Logarithms

Logarithms

The previous section dealt with exponential functions of the form $y = a^x$ for all positive values of a, where $a \neq 1$. The horizontal line test shows that exponential functions are one-to-one and thus have inverse functions. The equation defining the inverse of a function is found by interchanging x and y in the equation that defines the function. Starting with $y = a^x$ and interchanging x and y yields

$$x = a^y.$$

Here y is the exponent to which a must be raised in order to obtain x. We call this exponent a **logarithm,** symbolized by the abbreviation **"log."** The expression $\log_a x$ represents the logarithm in this discussion. The number a is the **base** of the logarithm, and x is the **argument** of the expression. It is read **"logarithm with base a of x,"** or **"logarithm of x with base a,"** or **"base a logarithm of x."**

Logarithm

For all real numbers y and all positive numbers a and x, where $a \neq 1$,

$$y = \log_a x \quad \text{is equivalent to} \quad x = a^y.$$

The expression $\log_a x$ represents the exponent to which the base a must be raised in order to obtain x.

EXAMPLE 1 Writing Equivalent Logarithmic and Exponential Forms

The table shows several pairs of equivalent statements, written in both logarithmic and exponential forms.

SOLUTION

Logarithmic Form	Exponential Form
$\log_2 8 = 3$	$2^3 = 8$
$\log_{1/2} 16 = -4$	$\left(\frac{1}{2}\right)^{-4} = 16$
$\log_{10} 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_3 \frac{1}{81} = -4$	$3^{-4} = \frac{1}{81}$
$\log_5 5 = 1$	$5^1 = 5$
$\log_{3/4} 1 = 0$	$\left(\frac{3}{4}\right)^0 = 1$

To remember the relationships among a, x, and y in the two equivalent forms $y = \log_a x$ and $x = a^y$, refer to these diagrams.

A logarithm is an exponent.

Exponent

Logarithmic form: $y = \log_a x$

Base

Exponent

Exponential form: $a^y = x$

Base

✔ **Now Try Exercises 11, 13, 15, and 17.**

Logarithmic Equations

The definition of logarithm can be used to solve a **logarithmic equation,** which is an equation with a logarithm in at least one term.

EXAMPLE 2 Solving Logarithmic Equations

Solve each equation.

(a) $\log_x \frac{8}{27} = 3$ **(b)** $\log_4 x = \frac{5}{2}$ **(c)** $\log_{49} \sqrt[3]{7} = x$

SOLUTION Many logarithmic equations can be solved by first writing the equation in exponential form.

(a)

$$\log_x \frac{8}{27} = 3$$

$$x^3 = \frac{8}{27} \quad \text{Write in exponential form.}$$

$$x^3 = \left(\frac{2}{3}\right)^3 \quad \frac{8}{27} = \left(\frac{2}{3}\right)^3$$

$$x = \frac{2}{3} \quad \text{Take cube roots.}$$

CHECK

$$\log_x \frac{8}{27} = 3 \quad \text{Original equation}$$

$$\log_{2/3} \frac{8}{27} \stackrel{?}{=} 3 \quad \text{Let } x = \tfrac{2}{3}.$$

$$\left(\frac{2}{3}\right)^3 \stackrel{?}{=} \frac{8}{27} \quad \text{Write in exponential form.}$$

$$\frac{8}{27} = \frac{8}{27} \ ✓ \quad \text{True}$$

The solution set is $\left\{\frac{2}{3}\right\}$.

(b)

	$\log_4 x = \frac{5}{2}$	
	$4^{5/2} = x$	Write in exponential form.
	$(4^{1/2})^5 = x$	$a^{mn} = (a^m)^n$
	$2^5 = x$	$4^{1/2} = (2^2)^{1/2} = 2$
	$32 = x$	Apply the exponent.
CHECK	$\log_4 32 \stackrel{?}{=} \frac{5}{2}$	Let $x = 32$.
	$4^{5/2} \stackrel{?}{=} 32$	
	$2^5 \stackrel{?}{=} 32$	$4^{5/2} = (\sqrt{4})^5 = 2^5$
	$32 = 32$ ✓	True

The solution set is $\{32\}$.

(c)

$\log_{49} \sqrt[3]{7} = x$	
$49^x = \sqrt[3]{7}$	Write in exponential form.
$(7^2)^x = 7^{1/3}$	Write with the same base.
$7^{2x} = 7^{1/3}$	Power rule for exponents
$2x = \frac{1}{3}$	Set exponents equal.
$x = \frac{1}{6}$	Divide by 2.

A check shows that the solution set is $\left\{\frac{1}{6}\right\}$.

✔ **Now Try Exercises 19, 29, and 35.**

Logarithmic Functions

We define the logarithmic function with base a.

Logarithmic Function

If $a > 0$, $a \neq 1$, and $x > 0$, then the **logarithmic function with base *a*** is

$$f(x) = \log_a x.$$

Exponential and logarithmic functions are inverses of each other. To show this, we use the three steps for finding the inverse of a function.

	$f(x) = 2^x$	Exponential function with base 2
	$y = 2^x$	Let $y = f(x)$.
Step 1	$x = 2^y$	Interchange x and y.
Step 2	$y = \log_2 x$	Solve for y by writing in equivalent logarithmic form.
Step 3	$f^{-1}(x) = \log_2 x$	Replace y with $f^{-1}(x)$.

The graph of $f(x) = 2^x$ has the x-axis as horizontal asymptote and is shown in red in **Figure 25.** Its inverse, $f^{-1}(x) = \log_2 x$, has the y-axis as vertical asymptote and is shown in blue. The graphs are reflections of each other across the line $y = x$. As a result, their domains and ranges are interchanged.

x	$f(x) = 2^x$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4

x	$f^{-1}(x) = \log_2 x$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2

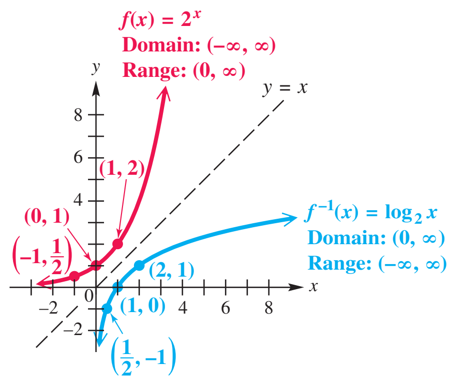


Figure 25

The domain of an exponential function is the set of all real numbers, so the range of a logarithmic function also will be the set of all real numbers. In the same way, both the range of an exponential function and the domain of a logarithmic function are the set of all positive real numbers.

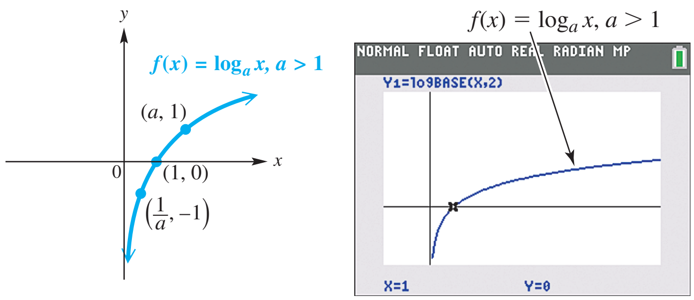
Thus, logarithms can be found for positive numbers only.

Logarithmic Function $f(x) = \log_a x$

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

For $f(x) = \log_2 x$:

x	$f(x)$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



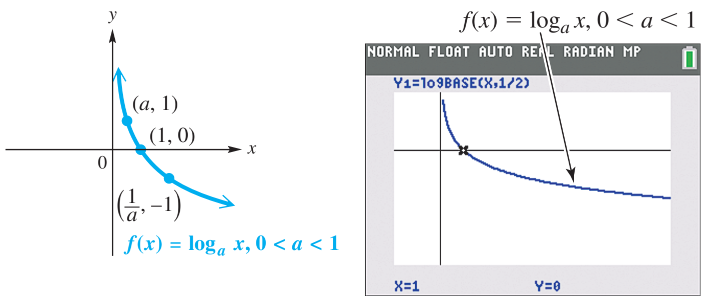
This is the general behavior seen on a calculator graph for **any base a, for $a > 1$.**

Figure 26

- $f(x) = \log_a x$**, for $a > 1$,** is increasing and continuous on its entire domain, $(0, \infty)$.
- The y-axis is a vertical asymptote as $x \to 0$ from the right.
- The graph passes through the points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$.

For $f(x) = \log_{1/2} x$:

x	$f(x)$
$\frac{1}{4}$	2
$\frac{1}{2}$	1
1	0
2	-1
4	-2
8	-3



This is the general behavior seen on a calculator graph for **any base a, for $0 < a < 1$.**

Figure 27

- $f(x) = \log_a x$**, for $0 < a < 1$,** is decreasing and continuous on its entire domain, $(0, \infty)$.
- The y-axis is a vertical asymptote as $x \to 0$ from the right.
- The graph passes through the points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$.

Calculator graphs of logarithmic functions sometimes do not give an accurate picture of the behavior of the graphs near the vertical asymptotes. While it may seem as if the graph has an endpoint, this is not the case. The resolution of the calculator screen is not precise enough to indicate that the graph approaches the vertical asymptote as the value of x gets closer to it. Do not draw incorrect conclusions just because the calculator does not show this behavior. ■

The graphs in **Figures 26 and 27** and the information with them suggest the following generalizations about the graphs of logarithmic functions of the form $f(x) = \log_a x$.

Characteristics of the Graph of $f(x) = \log_a x$

1. The points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$ are on the graph.
2. If $a > 1$, then f is an increasing function.
   If $0 < a < 1$, then f is a decreasing function.
3. The y-axis is a vertical asymptote.
4. The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$.

EXAMPLE 3 Graphing Logarithmic Functions

Graph each function.

(a) $f(x) = \log_{1/2} x$ **(b)** $f(x) = \log_3 x$

SOLUTION

(a) One approach is to first graph $y = \left(\frac{1}{2}\right)^x$, which defines the inverse function of f, by plotting points. Some ordered pairs are given in the table with the graph shown in red in **Figure 28.**

The graph of $f(x) = \log_{1/2} x$ is the reflection of the graph of $y = \left(\frac{1}{2}\right)^x$ across the line $y = x$. The ordered pairs for $y = \log_{1/2} x$ are found by interchanging the x- and y-values in the ordered pairs for $y = \left(\frac{1}{2}\right)^x$. See the graph in blue in **Figure 28.**

x	$y = \left(\frac{1}{2}\right)^x$
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
4	$\frac{1}{16}$

x	$f(x) = \log_{1/2} x$
4	-2
2	-1
1	0
$\frac{1}{2}$	1
$\frac{1}{4}$	2
$\frac{1}{16}$	4

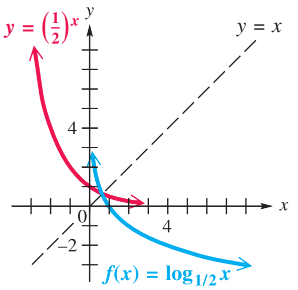


Figure 28

x	$f(x) = \log_3 x$
$\frac{1}{3}$	-1
1	0
3	1
9	2

Think: $x = 3^y$

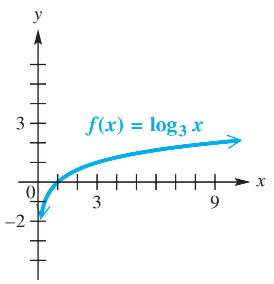


Figure 29

(b) Another way to graph a logarithmic function is to write $f(x) = y = \log_3 x$ in exponential form as $x = 3^y$, and then select y-values and calculate corresponding x-values. Several selected ordered pairs are shown in the table for the graph in **Figure 29.**

✔ **Now Try Exercise 55.**

CAUTION If we write a logarithmic function in exponential form in order to graph it, as in **Example 3(b),** we start *first* with y-values to calculate corresponding x-values. ***Be careful to write the values in the ordered pairs in the correct order.***

More general logarithmic functions can be obtained by forming the composition of $f(x) = \log_a x$ with a function $g(x)$. For example, if $f(x) = \log_2 x$ and $g(x) = x - 1$, then

$$(f \circ g)(x) = f(g(x)) = \log_2 (x - 1).$$

The next example shows how to graph such functions.

EXAMPLE 4 Graphing Translated Logarithmic Functions

Graph each function. Give the domain and range.

(a) $f(x) = \log_2 (x - 1)$ **(b)** $f(x) = (\log_3 x) - 1$

(c) $f(x) = \log_4 (x + 2) + 1$

SOLUTION

(a) The graph of $f(x) = \log_2 (x - 1)$ is the graph of $g(x) = \log_2 x$ translated 1 unit to the right. The vertical asymptote has equation $x = 1$. Because logarithms can be found only for positive numbers, we solve $x - 1 > 0$ to find the domain, $(1, \infty)$. To determine ordered pairs to plot, use the equivalent exponential form of the equation $y = \log_2 (x - 1)$.

$$
\begin{aligned}
y &= \log_2 (x - 1) \\
x - 1 &= 2^y && \text{Write in exponential form.} \\
x &= 2^y + 1 && \text{Add 1.}
\end{aligned}
$$

We first choose values for y and then calculate each of the corresponding x-values. The range is $(-\infty, \infty)$. See **Figure 30.**

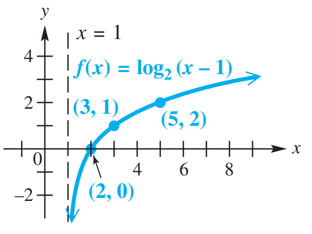


Figure 30

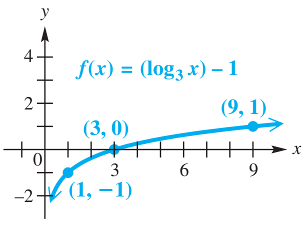


Figure 31

(b) The function $f(x) = (\log_3 x) - 1$ has the same graph as $g(x) = \log_3 x$ translated 1 unit down. We find ordered pairs to plot by writing the equation $y = (\log_3 x) - 1$ in exponential form.

$$
\begin{aligned}
y &= (\log_3 x) - 1 \\
y + 1 &= \log_3 x && \text{Add 1.} \\
x &= 3^{y+1} && \text{Write in exponential form.}
\end{aligned}
$$

Again, choose y-values and calculate the corresponding x-values. The graph is shown in **Figure 31.** The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$.

(c) The graph of $f(x) = \log_4(x + 2) + 1$ is obtained by shifting the graph of $y = \log_4 x$ to the left 2 units and up 1 unit. The domain is found by solving

$$x + 2 > 0,$$

which yields $(-2, \infty)$. The vertical asymptote has been shifted to the left 2 units as well, and it has equation $x = -2$. The range is unaffected by the vertical shift and remains $(-\infty, \infty)$. See **Figure 32.**

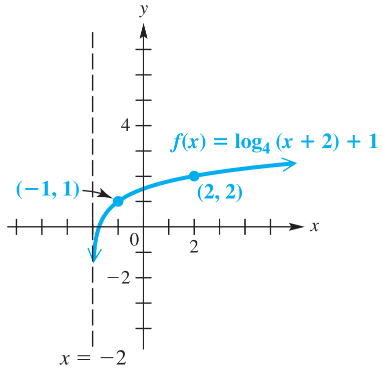


Figure 32

✔ **Now Try Exercises 43, 47, and 61.**

NOTE If we are given a graph such as the one in **Figure 31** and asked to find its equation, we could reason as follows: The point $(1, 0)$ on the basic logarithmic graph has been shifted *down* 1 unit, and the point $(3, 0)$ on the given graph is 1 unit lower than $(3, 1)$, which is on the graph of $y = \log_3 x$. Thus, the equation will be

$$y = (\log_3 x) - 1.$$

Properties of Logarithms

The properties of logarithms enable us to change the form of logarithmic statements so that products can be converted to sums, quotients can be converted to differences, and powers can be converted to products.

Properties of Logarithms

For $x > 0$, $y > 0$, $a > 0$, $a \neq 1$, and any real number r, the following properties hold.

Property	Description
Product Property $\log_a xy = \log_a x + \log_a y$	The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.
Quotient Property $\log_a \frac{x}{y} = \log_a x - \log_a y$	The logarithm of the quotient of two numbers is equal to the difference between the logarithms of the numbers.
Power Property $\log_a x^r = r \log_a x$	The logarithm of a number raised to a power is equal to the exponent multiplied by the logarithm of the number.
Logarithm of 1 $\log_a 1 = 0$	The base a logarithm of 1 is 0.
Base *a* Logarithm of *a* $\log_a a = 1$	The base a logarithm of a is 1.

Proof To prove the product property, let $m = \log_a x$ and $n = \log_a y$.

$\log_a x = m$ means $a^m = x$ — Write in exponential form.

$\log_a y = n$ means $a^n = y$

LOOKING AHEAD TO CALCULUS

A technique called **logarithmic differentiation,** which uses the properties of logarithms, can often be used to differentiate complicated functions.

Now consider the product xy.

$$xy = a^m \cdot a^n \qquad x = a^m \text{ and } y = a^n\text{; Substitute.}$$

$$xy = a^{m+n} \qquad \text{Product rule for exponents}$$

$$\log_a xy = m + n \qquad \text{Write in logarithmic form.}$$

$$\log_a xy = \log_a x + \log_a y \qquad \text{Substitute.}$$

The last statement is the result we wished to prove. The quotient and power properties are proved similarly and are left as exercises.

EXAMPLE 5 Using Properties of Logarithms

Use the properties of logarithms to rewrite each expression. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(a) $\log_6 (7 \cdot 9)$ **(b)** $\log_9 \frac{15}{7}$ **(c)** $\log_5 \sqrt{8}$

(d) $\log_a \sqrt[3]{m^2}$ **(e)** $\log_a \frac{mnq}{p^2t^4}$ **(f)** $\log_b \sqrt[n]{\frac{x^3y^5}{z^m}}$

SOLUTION

(a) $\log_6 (7 \cdot 9)$

$= \log_6 7 + \log_6 9$ Product property

(b) $\log_9 \frac{15}{7}$

$= \log_9 15 - \log_9 7$ Quotient property

(c) $\log_5 \sqrt{8}$

$= \log_5 (8^{1/2})$ $\sqrt{a} = a^{1/2}$

$= \frac{1}{2} \log_5 8$ Power property

(d) $\log_a \sqrt[3]{m^2}$

$= \log_a m^{2/3}$ $\sqrt[n]{a^m} = a^{m/n}$

$= \frac{2}{3} \log_a m$ Power property

(e) $\log_a \frac{mnq}{p^2t^4}$

Use parentheses to avoid errors.

$= \log_a m + \log_a n + \log_a q - (\log_a p^2 + \log_a t^4)$ Product and quotient properties

$= \log_a m + \log_a n + \log_a q - (2 \log_a p + 4 \log_a t)$ Power property

$= \log_a m + \log_a n + \log_a q - 2 \log_a p - 4 \log_a t$ Distributive property

Be careful with signs.

(f) $\log_b \sqrt[n]{\frac{x^3y^5}{z^m}}$

$= \log_b \left(\frac{x^3y^5}{z^m}\right)^{1/n}$ $\sqrt[n]{a} = a^{1/n}$

$= \frac{1}{n} \log_b \frac{x^3y^5}{z^m}$ Power property

$= \frac{1}{n}(\log_b x^3 + \log_b y^5 - \log_b z^m)$ Product and quotient properties

$= \frac{1}{n}(3 \log_b x + 5 \log_b y - m \log_b z)$ Power property

$= \frac{3}{n} \log_b x + \frac{5}{n} \log_b y - \frac{m}{n} \log_b z$ Distributive property

✔ **Now Try Exercises 71, 73, and 77.**

EXAMPLE 6 Using Properties of Logarithms

Write each expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(a) $\log_3 (x+2) + \log_3 x - \log_3 2$ **(b)** $2 \log_a m - 3 \log_a n$

(c) $\frac{1}{2} \log_b m + \frac{3}{2} \log_b 2n - \log_b m^2 n$

SOLUTION

(a) $\log_3 (x+2) + \log_3 x - \log_3 2$

$= \log_3 \frac{(x+2)x}{2}$ Product and quotient properties

(b) $2 \log_a m - 3 \log_a n$

$= \log_a m^2 - \log_a n^3$ Power property

$= \log_a \frac{m^2}{n^3}$ Quotient property

(c) $\frac{1}{2} \log_b m + \frac{3}{2} \log_b 2n - \log_b m^2 n$

$= \log_b m^{1/2} + \log_b (2n)^{3/2} - \log_b m^2 n$ Power property

$= \log_b \frac{m^{1/2}(2n)^{3/2}}{m^2 n}$ Product and quotient properties

Use parentheses around $2n$.

$= \log_b \frac{2^{3/2} n^{1/2}}{m^{3/2}}$ Rules for exponents

$= \log_b \left(\frac{2^3 n}{m^3}\right)^{1/2}$ Rules for exponents

$= \log_b \sqrt{\frac{8n}{m^3}}$ Definition of $a^{1/n}$

✔ **Now Try Exercises 83, 87, and 89.**

CAUTION ***There is no property of logarithms to rewrite a logarithm of a sum or difference.*** That is why, in **Example 6(a),**

$$\log_3 (x+2) \quad \text{cannot be written as} \quad \log_3 x + \log_3 2.$$

The distributive property does not apply here because $\log_3 (x+y)$ is one term. The abbreviation "log" is a function name, *not* a factor.

Napier's Rods

The search for ways to make calculations easier has been a long, ongoing process. Machines built by Charles Babbage and Blaise Pascal, a system of "rods" used by John Napier, and slide rules were the forerunners of today's calculators and computers. The invention of logarithms by John Napier in the 16th century was a great breakthrough in the search for easier calculation methods.

Source: IBM Corporate Archives.

EXAMPLE 7 Using Properties of Logarithms with Numerical Values

Given that $\log_{10} 2 \approx 0.3010$, find each logarithm without using a calculator.

(a) $\log_{10} 4$ **(b)** $\log_{10} 5$

SOLUTION

(a) $\log_{10} 4$

$= \log_{10} 2^2$

$= 2 \log_{10} 2$

$\approx 2(0.3010)$

≈ 0.6020

(b) $\log_{10} 5$

$= \log_{10} \frac{10}{2}$

$= \log_{10} 10 - \log_{10} 2$

$\approx 1 - 0.3010$

≈ 0.6990

✔ **Now Try Exercises 93 and 95.**

NOTE The values in **Example 7** are approximations of logarithms, so the final digit may differ from the actual 4-decimal-place approximation after properties of logarithms are applied.

Recall that for inverse functions f and g, $(f \circ g)(x) = (g \circ f)(x) = x$. We can use this property with exponential and logarithmic functions to state two more properties. If $f(x) = a^x$ and $g(x) = \log_a x$, then

$$(f \circ g)(x) = a^{\log_a x} = x \quad \text{and} \quad (g \circ f)(x) = \log_a (a^x) = x.$$

Theorem on Inverses

For $a > 0$, $a \neq 1$, the following properties hold.

$$\boldsymbol{a^{\log_a x} = x \text{ (for } x > 0)} \quad \text{and} \quad \boldsymbol{\log_a a^x = x}$$

Examples: $7^{\log_7 10} = 10$, $\log_5 5^3 = 3$, and $\log_r r^{k+1} = k + 1$

The second statement in the theorem will be useful when we solve logarithmic and exponential equations.

4.3 Exercises

CONCEPT PREVIEW *Match the logarithm in Column I with its value in Column II. Remember that* $\log_a x$ *is the exponent to which a must be raised in order to obtain x.*

	I	II
1.	**(a)** $\log_2 16$	**A.** 0
	(b) $\log_3 1$	**B.** $\frac{1}{2}$
	(c) $\log_{10} 0.1$	**C.** 4
	(d) $\log_2 \sqrt{2}$	**D.** -3
	(e) $\log_e \frac{1}{e^2}$	**E.** -1
	(f) $\log_{1/2} 8$	**F.** -2

	I	II
2.	**(a)** $\log_3 81$	**A.** -2
	(b) $\log_3 \frac{1}{3}$	**B.** -1
	(c) $\log_{10} 0.01$	**C.** 0
	(d) $\log_6 \sqrt{6}$	**D.** $\frac{1}{2}$
	(e) $\log_e 1$	**E.** $\frac{9}{2}$
	(f) $\log_3 27^{3/2}$	**F.** 4

CONCEPT PREVIEW *Write each equivalent form.*

3. Write $\log_2 8 = 3$ in exponential form.

4. Write $10^3 = 1000$ in logarithmic form.

CONCEPT PREVIEW *Solve each logarithmic equation.*

5. $\log_x \frac{16}{81} = 2$

6. $\log_{36} \sqrt[3]{6} = x$

CONCEPT PREVIEW *Sketch the graph of each function. Give the domain and range.*

7. $f(x) = \log_5 x$

8. $g(x) = \log_{1/5} x$

CONCEPT PREVIEW *Use the properties of logarithms to rewrite each expression. Assume all variables represent positive real numbers.*

9. $\log_{10} \frac{2x}{7}$

10. $3 \log_4 x - 5 \log_4 y$

If the statement is in exponential form, write it in an equivalent logarithmic form. If the statement is in logarithmic form, write it in exponential form. ***See Example 1.***

11. $3^4 = 81$

12. $2^5 = 32$

13. $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$

14. $10^{-4} = 0.0001$

15. $\log_6 36 = 2$

16. $\log_5 5 = 1$

17. $\log_{\sqrt{3}} 81 = 8$

18. $\log_4 \frac{1}{64} = -3$

Solve each equation. ***See Example 2.***

19. $x = \log_5 \frac{1}{625}$

20. $x = \log_3 \frac{1}{81}$

21. $\log_x \frac{1}{32} = 5$

22. $\log_x \frac{27}{64} = 3$

23. $x = \log_8 \sqrt[4]{8}$

24. $x = \log_7 \sqrt[5]{7}$

25. $x = 3^{\log_3 8}$

26. $x = 12^{\log_{12} 5}$

27. $x = 2^{\log_2 9}$

28. $x = 8^{\log_8 11}$

29. $\log_x 25 = -2$

30. $\log_x 16 = -2$

31. $\log_4 x = 3$

32. $\log_2 x = 3$

33. $x = \log_4 \sqrt[3]{16}$

34. $x = \log_5 \sqrt[4]{25}$

35. $\log_9 x = \frac{5}{2}$

36. $\log_4 x = \frac{7}{2}$

37. $\log_{1/2}(x + 3) = -4$

38. $\log_{1/3}(x + 6) = -2$

39. $\log_{(x+3)} 6 = 1$

40. $\log_{(x-4)} 19 = 1$

41. $3x - 15 = \log_x 1 \quad (x > 0, x \neq 1)$

42. $4x - 24 = \log_x 1 \quad (x > 0, x \neq 1)$

Graph each function. Give the domain and range. ***See Example 4.***

43. $f(x) = (\log_2 x) + 3$

44. $f(x) = \log_2 (x + 3)$

45. $f(x) = |\log_2 (x + 3)|$

Graph each function. Give the domain and range. ***See Example 4.***

46. $f(x) = (\log_{1/2} x) - 2$

47. $f(x) = \log_{1/2} (x - 2)$

48. $f(x) = |\log_{1/2} (x - 2)|$

Concept Check *In Exercises 49–54, match the function with its graph from choices A–F.*

49. $f(x) = \log_2 x$

50. $f(x) = \log_2 2x$

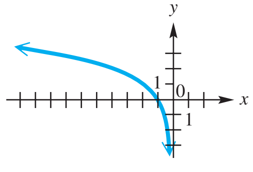
51. $f(x) = \log_2 \frac{1}{x}$

52. $f(x) = \log_2 \left(\frac{1}{2}x\right)$

53. $f(x) = \log_2 (x - 1)$

54. $f(x) = \log_2 (-x)$

A.



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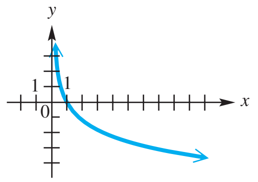

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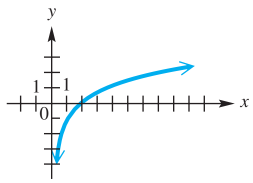

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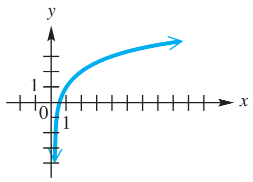
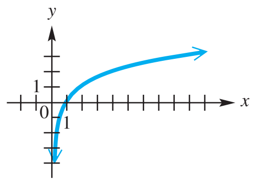
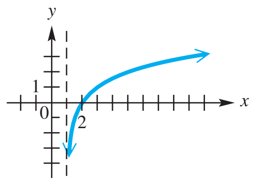


E.



F.



Graph each function. **See Examples 3 and 4.**

55. $f(x) = \log_5 x$

56. $f(x) = \log_{10} x$

57. $f(x) = \log_5 (x + 1)$

58. $f(x) = \log_6 (x - 2)$

59. $f(x) = \log_{1/2} (1 - x)$

60. $f(x) = \log_{1/3} (3 - x)$

61. $f(x) = \log_3 (x - 1) + 2$

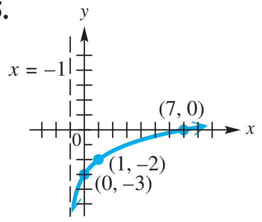
62. $f(x) = \log_2 (x + 2) - 3$

63. $f(x) = \log_{1/2} (x + 3) - 2$

64. ***Concept Check*** To graph the function $f(x) = -\log_5 (x - 7) - 4$, reflect the graph of $y = \log_5 x$ across the ______-axis, then shift the graph ______ units to the right and ______ units down.

Connecting Graphs with Equations *Write an equation for the graph given. Each represents a logarithmic function f with base* 2 *or* 3*, translated and/or reflected.* ***See the Note following Example 4.***

65.



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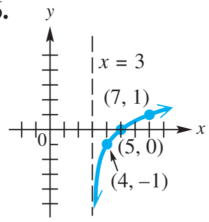

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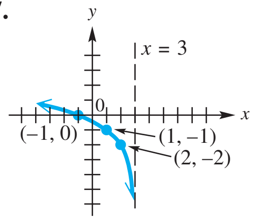
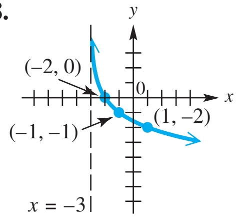
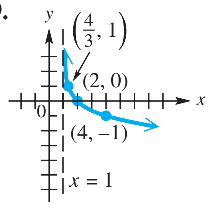


68.



69.



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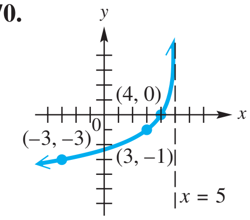


Use the properties of logarithms to rewrite each expression. Simplify the result if possible. Assume all variables represent positive real numbers. **See Example 5.**

71. $\log_2 \dfrac{6x}{y}$

72. $\log_3 \dfrac{4p}{q}$

73. $\log_5 \dfrac{5\sqrt{7}}{3}$

74. $\log_2 \dfrac{2\sqrt{3}}{5}$

75. $\log_4 (2x + 5y)$

76. $\log_6 (7m + 3q)$

77. $\log_2 \sqrt{\dfrac{5r^3}{z^5}}$

78. $\log_3 \sqrt[3]{\dfrac{m^5n^4}{t^2}}$

79. $\log_2 \dfrac{ab}{cd}$

80. $\log_2 \dfrac{xy}{tqr}$

81. $\log_3 \dfrac{\sqrt{x} \cdot \sqrt[3]{y}}{w^2\sqrt{z}}$

82. $\log_4 \dfrac{\sqrt[3]{a} \cdot \sqrt[4]{b}}{\sqrt{c} \cdot \sqrt[3]{d^2}}$

Write each expression as a single logarithm with coefficient 1*. Assume all variables represent positive real numbers, with* $a \neq 1$ and $b \neq 1$. **See Example 6.**

83. $\log_a x + \log_a y - \log_a m$

84. $\log_b k + \log_b m - \log_b a$

85. $\log_a m - \log_a n - \log_a t$

86. $\log_b p - \log_b q - \log_b r$

87. $\frac{1}{3} \log_b x^4y^5 - \frac{3}{4} \log_b x^2y$

88. $\frac{1}{2} \log_a p^3q^4 - \frac{2}{3} \log_a p^4q^3$

89. $2 \log_a (z + 1) + \log_a (3z + 2)$

90. $5 \log_a (z + 7) + \log_a (2z + 9)$

91. $-\frac{2}{3} \log_5 5m^2 + \frac{1}{2} \log_5 25m^2$

92. $-\frac{3}{4} \log_3 16p^4 - \frac{2}{3} \log_3 8p^3$

Given that $\log_{10} 2 \approx 0.3010$ *and* $\log_{10} 3 \approx 0.4771$, *find each logarithm without using a calculator.* ***See Example 7.***

93. $\log_{10} 6$ **94.** $\log_{10} 12$ **95.** $\log_{10} \frac{3}{2}$ **96.** $\log_{10} \frac{2}{9}$

97. $\log_{10} \frac{9}{4}$ **98.** $\log_{10} \frac{20}{27}$ **99.** $\log_{10} \sqrt{30}$ **100.** $\log_{10} 36^{1/3}$

Solve each problem.

101. ***(Modeling) Interest Rates of Treasury Securities*** The table gives interest rates for various U.S. Treasury Securities on January 2, 2015.

(a) Make a scatter diagram of the data.

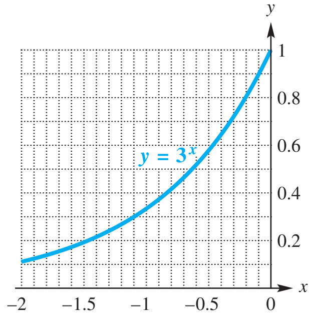
(b) Which type of function will model this data best: linear, exponential, or logarithmic?

Time	Yield
3-month	0.02%
6-month	0.10%
2-year	0.66%
5-year	1.61%
10-year	2.11%
30-year	2.60%

Source: www.federal reserve.gov

102. ***Concept Check*** Use the graph to estimate each logarithm.

(a) $\log_3 0.3$ **(b)** $\log_3 0.8$



103. ***Concept Check*** Suppose $f(x) = \log_a x$ and $f(3) = 2$. Determine each function value.

(a) $f\left(\frac{1}{9}\right)$ **(b)** $f(27)$ **(c)** $f(9)$ **(d)** $f\left(\frac{\sqrt{3}}{3}\right)$

104. Use properties of logarithms to evaluate each expression.

(a) $100^{\log_{10} 3}$ **(b)** $\log_{10}(0.01)^3$ **(c)** $\log_{10}(0.0001)^5$ **(d)** $1000^{\log_{10} 5}$

105. Using the compound interest formula $A = P\left(1 + \frac{r}{n}\right)^{tn}$, show that the amount of time required for a deposit to double is

$$\frac{1}{\log_2\left(1 + \frac{r}{n}\right)^n}.$$

106. ***Concept Check*** If $(5, 4)$ is on the graph of the logarithmic function with base a, which of the following statements is true:

$$5 = \log_a 4 \quad \text{or} \quad 4 = \log_a 5?$$

Use a graphing calculator to find the solution set of each equation. Give solutions to the nearest hundredth.

107. $\log_{10} x = x - 2$ **108.** $2^{-x} = \log_{10} x$

109. Prove the quotient property of logarithms: $\log_a \frac{x}{y} = \log_a x - \log_a y$.

110. Prove the power property of logarithms: $\log_a x^r = r \log_a x$.

Summary Exercises on Inverse, Exponential, and Logarithmic Functions

The following exercises are designed to help solidify your understanding of inverse, exponential, and logarithmic functions from **Sections 4.1–4.3.**

Determine whether the functions in each pair are inverses of each other.

1. $f(x) = 3x - 4, \quad g(x) = \frac{1}{3}x + \frac{4}{3}$

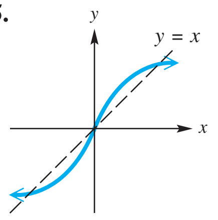
2. $f(x) = 8 - 5x, \quad g(x) = 8 + \frac{1}{5}x$

3. $f(x) = 1 + \log_2 x, \quad g(x) = 2^{x-1}$

4. $f(x) = 3^{x/5} - 2, \quad g(x) = 5\log_3(x + 2)$

Determine whether each function is one-to-one. If it is, then sketch the graph of its inverse function.

5.



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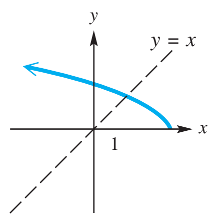

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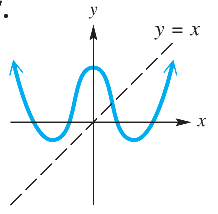

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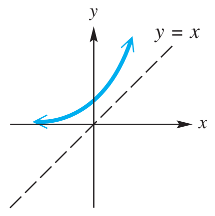


In Exercises 9–12, match each function with its graph from choices A–D.

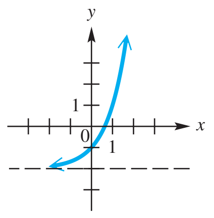
9. $y = \log_3(x + 2)$

10. $y = 5 - 2^x$

11. $y = \log_2(5 - x)$

12. $y = 3^x - 2$

A.



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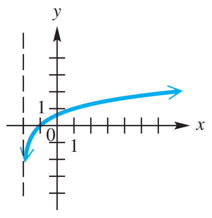

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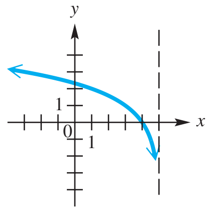
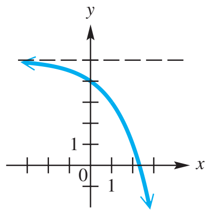


D.



13. The functions in **Exercises 9–12** form two pairs of inverse functions. Determine which functions are inverses of each other.

14. Determine the inverse of the function $f(x) = \log_5 x$. (*Hint:* Replace $f(x)$ with y, and write in exponential form.)

For each function that is one-to-one, write an equation for the inverse function. Give the domain and range of both f and f^{-1}. If the function is not one-to-one, say so.

15. $f(x) = 3x - 6$

16. $f(x) = 2(x+1)^3$

17. $f(x) = 3x^2$

18. $f(x) = \dfrac{2x-1}{5-3x}$

19. $f(x) = \sqrt[3]{5 - x^4}$

20. $f(x) = \sqrt{x^2 - 9}, \quad x \geq 3$

Write an equivalent statement in logarithmic form.

21. $\left(\dfrac{1}{10}\right)^{-3} = 1000$

22. $a^b = c$

23. $\left(\sqrt{3}\right)^4 = 9$

24. $4^{-3/2} = \dfrac{1}{8}$

25. $2^x = 32$

26. $27^{4/3} = 81$

Solve each equation.

27. $3x = 7^{\log_7 6}$

28. $x = \log_{10} 0.001$

29. $x = \log_6 \dfrac{1}{216}$

30. $\log_x 5 = \dfrac{1}{2}$

31. $\log_{10} 0.01 = x$

32. $\log_x 3 = -1$

33. $\log_x 1 = 0$

34. $x = \log_2 \sqrt{8}$

35. $\log_x \sqrt[3]{5} = \dfrac{1}{3}$

36. $\log_{1/3} x = -5$

37. $\log_{10}(\log_2 2^{10}) = x$

38. $x = \log_{4/5} \dfrac{25}{16}$

39. $2x - 1 = \log_6 6^x$

40. $x = \sqrt{\log_{1/2} \dfrac{1}{16}}$

41. $2^x = \log_2 16$

42. $\log_3 x = -2$

43. $\left(\dfrac{1}{3}\right)^{x+1} = 9^x$

44. $5^{2x-6} = 25^{x-3}$

4.4 Evaluating Logarithms and the Change-of-Base Theorem

- Common Logarithms
- Applications and Models with Common Logarithms
- Natural Logarithms
- Applications and Models with Natural Logarithms
- Logarithms with Other Bases

Common Logarithms

Two of the most important bases for logarithms are 10 and e. Base 10 logarithms are **common logarithms.** The common logarithm of x is written **$\log x$,** where the base is understood to be 10.

Common Logarithm

For all positive numbers x,

$$\log x = \log_{10} x.$$

A calculator with a log key can be used to find the base 10 logarithm of any positive number.

EXAMPLE 1 Evaluating Common Logarithms with a Calculator

Use a calculator to find the values of

$$\log 1000, \quad \log 142, \quad \text{and} \quad \log 0.005832.$$

SOLUTION **Figure 33** shows that the exact value of log 1000 is 3 (because $10^3 = 1000$), and that

$$\log 142 \approx 2.152288344$$

and $$\log 0.005832 \approx -2.234182485.$$

Most common logarithms that appear in calculations are approximations, as seen in the second and third displays.

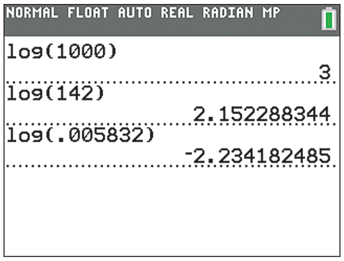


Figure 33

✔ **Now Try Exercises 11, 15, and 17.**

For $a > 1$, base a logarithms of numbers between 0 and 1 are always negative, and base a logarithms of numbers greater than 1 are always positive.

Applications and Models with Common Logarithms In chemistry, the **pH** of a solution is defined as

$$\mathbf{pH = -\log[H_3O^+]},$$

where $[H_3O^+]$ is the hydronium ion concentration in moles* per liter. The pH value is a measure of the acidity or alkalinity of a solution. Pure water has pH 7.0, substances with pH values greater than 7.0 are alkaline, and substances with pH values less than 7.0 are acidic. See **Figure 34.** It is customary to round pH values to the nearest tenth.

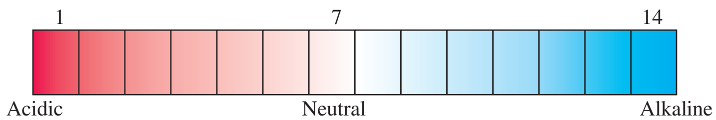


Figure 34

EXAMPLE 2 Finding pH

(a) Find the pH of a solution with $[H_3O^+] = 2.5 \times 10^{-4}$.

(b) Find the hydronium ion concentration of a solution with pH = 7.1.

SOLUTION

(a)
$$\begin{aligned}
\text{pH} &= -\log[H_3O^+] & \\
\text{pH} &= -\log(2.5 \times 10^{-4}) & &\text{Substitute } [H_3O^+] = 2.5 \times 10^{-4}. \\
\text{pH} &= -(\log 2.5 + \log 10^{-4}) & &\text{Product property} \\
\text{pH} &= -(0.3979 - 4) & &\log 10^{-4} = -4 \\
\text{pH} &= -0.3979 + 4 & &\text{Distributive property} \\
\text{pH} &\approx 3.6 & &\text{Add.}
\end{aligned}$$

*A *mole* is the amount of a substance that contains the same number of molecules as the number of atoms in exactly 12 grams of carbon-12.

(b)

$$\begin{aligned} \text{pH} &= -\log[\text{H}_3\text{O}^+] \\ 7.1 &= -\log[\text{H}_3\text{O}^+] && \text{Substitute pH} = 7.1. \\ -7.1 &= \log[\text{H}_3\text{O}^+] && \text{Multiply by } -1. \\ [\text{H}_3\text{O}^+] &= 10^{-7.1} && \text{Write in exponential form.} \\ [\text{H}_3\text{O}^+] &\approx 7.9 \times 10^{-8} && \text{Evaluate } 10^{-7.1} \text{ with a calculator.} \end{aligned}$$

✔ **Now Try Exercises 29 and 33.**

NOTE In the fourth line of the solution in **Example 2(a),** we use the equality symbol, =, rather than the approximate equality symbol, ≈, when replacing log 2.5 with 0.3979. This is often done for convenience, despite the fact that most logarithms used in applications are indeed approximations.

EXAMPLE 3 Using pH in an Application

Wetlands are classified as *bogs, fens, marshes,* and *swamps* based on pH values. A pH value between 6.0 and 7.5 indicates that the wetland is a "rich fen." When the pH is between 3.0 and 6.0, it is a "poor fen," and if the pH falls to 3.0 or less, the wetland is a "bog." (*Source:* R. Mohlenbrock, "Summerby Swamp, Michigan," *Natural History*.)

Suppose that the hydronium ion concentration of a sample of water from a wetland is 6.3×10^{-5}. How would this wetland be classified?

SOLUTION

$$\begin{aligned} \text{pH} &= -\log[\text{H}_3\text{O}^+] && \text{Definition of pH} \\ \text{pH} &= -\log(6.3 \times 10^{-5}) && \text{Substitute for } [\text{H}_3\text{O}^+]. \\ \text{pH} &= -(\log 6.3 + \log 10^{-5}) && \text{Product property} \\ \text{pH} &= -\log 6.3 - (-5) && \text{Distributive property; } \log 10^n = n \\ \text{pH} &= -\log 6.3 + 5 && \text{Definition of subtraction} \\ \text{pH} &\approx 4.2 && \text{Use a calculator.} \end{aligned}$$

The pH is between 3.0 and 6.0, so the wetland is a poor fen.

✔ **Now Try Exercise 37.**

EXAMPLE 4 Measuring the Loudness of Sound

The loudness of sounds is measured in **decibels.** We first assign an intensity of I_0 to a very faint **threshold sound.** If a particular sound has intensity I, then the decibel rating d of this louder sound is given by the following formula.

$$\boldsymbol{d = 10 \log \frac{I}{I_0}}$$

Find the decibel rating d of a sound with intensity $10{,}000I_0$.

SOLUTION

$$\begin{aligned} d &= 10 \log \frac{10{,}000 I_0}{I_0} && \text{Let } I = 10{,}000 I_0. \\ d &= 10 \log 10{,}000 && \tfrac{I_0}{I_0} = 1 \\ d &= 10(4) && \log 10{,}000 = \log 10^4 = 4 \\ d &= 40 && \text{Multiply.} \end{aligned}$$

The sound has a decibel rating of 40.

✔ **Now Try Exercise 63.**

Natural Logarithms

In most practical applications of logarithms, the irrational number e is used as the base. Logarithms with base e are **natural logarithms** because they occur in the life sciences and economics in natural situations that involve growth and decay. The base e logarithm of x is written **ln *x*** (read **"el-en *x*"**). ***The expression* ln *x* represents the exponent to which e must be raised in order to obtain x.**

Natural Logarithm

For all positive numbers x,

$$\ln x = \log_e x.$$

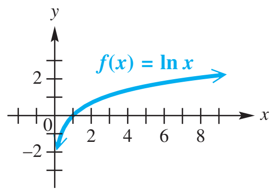


Figure 35

A graph of the natural logarithmic function $f(x) = \ln x$ is given in **Figure 35.**

EXAMPLE 5 Evaluating Natural Logarithms with a Calculator

Use a calculator to find the values of

$$\ln e^3, \quad \ln 142, \quad \text{and} \quad \ln 0.005832.$$

SOLUTION **Figure 36** shows that the exact value of $\ln e^3$ is 3, and that

$$\ln 142 \approx 4.955827058$$

and

$$\ln 0.005832 \approx -5.144395284.$$

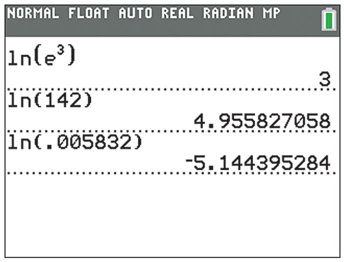


Figure 36

✔ **Now Try Exercises 45, 51, and 53.**

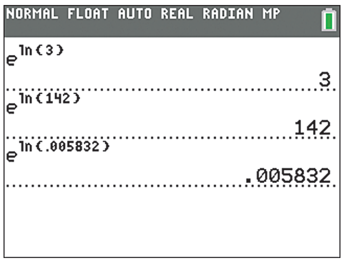


Figure 37

Figure 37 illustrates that **ln *x* is the exponent to which e must be raised in order to obtain *x*.**

Applications and Models with Natural Logarithms

EXAMPLE 6 Measuring the Age of Rocks

Geologists sometimes measure the age of rocks by using "atomic clocks." By measuring the amounts of argon-40 and potassium-40 in a rock, it is possible to find the age t of the specimen in years with the formula

$$t = (1.26 \times 10^9)\frac{\ln\left(1 + 8.33\left(\frac{A}{K}\right)\right)}{\ln 2},$$

where A and K are the numbers of atoms of argon-40 and potassium-40, respectively, in the specimen.

(a) How old is a rock in which $A = 0$ and $K > 0$?

(b) The ratio $\frac{A}{K}$ for a sample of granite from New Hampshire is 0.212. How old is the sample?

LOOKING AHEAD TO CALCULUS

The natural logarithmic function $f(x) = \ln x$ and the reciprocal function $g(x) = \frac{1}{x}$ have an important relationship in calculus. The derivative of the natural logarithmic function is the reciprocal function. Using **Leibniz notation** (named after one of the co-inventors of calculus), we write this fact as $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

SOLUTION

(a) If $A = 0$, then $\frac{A}{K} = 0$ and the equation is as follows.

$$t = (1.26 \times 10^9)\frac{\ln\left(1 + 8.33\left(\frac{A}{K}\right)\right)}{\ln 2} \qquad \text{Given formula}$$

$$t = (1.26 \times 10^9)\frac{\ln 1}{\ln 2} \qquad \frac{A}{K} = 0, \text{ so } \ln(1+0) = \ln 1$$

$$t = (1.26 \times 10^9)(0) \qquad \ln 1 = 0$$

$$t = 0$$

The rock is new (0 yr old).

(b) Because $\frac{A}{K} = 0.212$, we have the following.

$$t = (1.26 \times 10^9)\frac{\ln(1 + 8.33(0.212))}{\ln 2} \qquad \text{Substitute.}$$

$$t \approx 1.85 \times 10^9 \qquad \text{Use a calculator.}$$

The granite is about 1.85 billion yr old. ✔ **Now Try Exercise 77.**

EXAMPLE 7 Modeling Global Temperature Increase

Carbon dioxide in the atmosphere traps heat from the sun. The additional solar radiation trapped by carbon dioxide is **radiative forcing.** It is measured in watts per square meter (w/m^2). In 1896 the Swedish scientist Svante Arrhenius modeled radiative forcing R caused by additional atmospheric carbon dioxide, using the logarithmic equation

$$R = k \ln \frac{C}{C_0},$$

where C_0 is the preindustrial amount of carbon dioxide, C is the current carbon dioxide level, and k is a constant. Arrhenius determined that $10 \le k \le 16$ when $C = 2C_0$. (*Source:* Clime, W., *The Economics of Global Warming*, Institute for International Economics, Washington, D.C.)

(a) Let $C = 2C_0$. Is the relationship between R and k linear or logarithmic?

(b) The average global temperature increase T (in °F) is given by $T(R) = 1.03R$. Write T as a function of k.

SOLUTION

(a) If $C = 2C_0$, then $\frac{C}{C_0} = 2$, so $R = k \ln 2$ is a linear relation, because $\ln 2$ is a constant.

(b)

$$T(R) = 1.03R$$

$$T(k) = 1.03k \ln \frac{C}{C_0} \qquad \text{Use the given expression for } R.$$

✔ **Now Try Exercise 75.**

Logarithms with Other Bases We can use a calculator to find the values of either natural logarithms (base e) or common logarithms (base 10). However, sometimes we must use logarithms with other bases. The change-of-base theorem can be used to convert logarithms from one base to another.

LOOKING AHEAD TO CALCULUS

In calculus, natural logarithms are more convenient to work with than logarithms with other bases. The change-of-base theorem enables us to convert any logarithmic function to a *natural* logarithmic function.

Change-of-Base Theorem

For any positive real numbers x, a, and b, where $a \neq 1$ and $b \neq 1$, the following holds.

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Proof Let $y = \log_a x.$

Then $a^y = x$ Write in exponential form.

$\log_b a^y = \log_b x$ Take the base b logarithm on each side.

$y \log_b a = \log_b x$ Power property

$y = \dfrac{\log_b x}{\log_b a}$ Divide each side by $\log_b a$.

$\log_a x = \dfrac{\log_b x}{\log_b a}.$ Substitute $\log_a x$ for y.

Any positive number other than 1 can be used for base b in the change-of-base theorem, but usually the only practical bases are e and 10 since most calculators give logarithms for these two bases only.

Using the change-of-base theorem, we can graph an equation such as $y = \log_2 x$ by directing the calculator to graph $y = \frac{\log x}{\log 2}$, or, equivalently, $y = \frac{\ln x}{\ln 2}$. ■

EXAMPLE 8 Using the Change-of-Base Theorem

Use the change-of-base theorem to find an approximation to four decimal places for each logarithm.

(a) $\log_5 17$ **(b)** $\log_2 0.1$

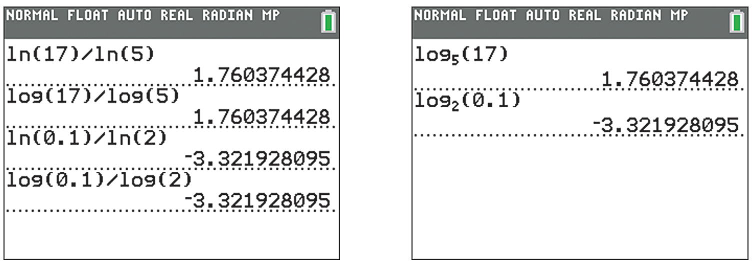
SOLUTION

(a) We use natural logarithms to approximate this logarithm. Because $\log_5 5 = 1$ and $\log_5 25 = 2$, we can estimate $\log_5 17$ to be a number between 1 and 2.

$$\log_5 17 = \frac{\ln 17}{\ln 5} \approx 1.7604$$

Check: $5^{1.7604} \approx 17$

The first two entries in **Figure 38(a)** show that the results are the same whether natural or common logarithms are used.



(a) **(b)**

Figure 38

(b) We use common logarithms for this approximation.

$$\log_2 0.1 = \frac{\log 0.1}{\log 2} \approx -3.3219 \quad \text{Check: } 2^{-3.3219} \approx 0.1$$

The last two entries in **Figure 38(a)** show that the results are the same whether natural or common logarithms are used.

Some calculators, such as the TI-84 Plus, evaluate these logarithms directly without using the change-of-base theorem. See **Figure 38(b).**

✔ Now Try Exercises 79 and 81.

EXAMPLE 9 Modeling Diversity of Species

One measure of the diversity of the species in an ecological community is modeled by the formula

$$H = -[P_1 \log_2 P_1 + P_2 \log_2 P_2 + \cdots + P_n \log_2 P_n],$$

where $P_1, P_2, \ldots, P_n$ are the proportions of a sample that belong to each of n species found in the sample. (*Source:* Ludwig, J., and J. Reynolds, *Statistical Ecology: A Primer on Methods and Computing*, © 1988, John Wiley & Sons, NY.)

Find the measure of diversity in a community with two species where there are 90 of one species and 10 of the other.

SOLUTION There are 100 members in the community, so $P_1 = \frac{90}{100} = 0.9$ and $P_2 = \frac{10}{100} = 0.1$.

$$H = -[0.9 \log_2 0.9 + 0.1 \log_2 0.1] \quad \text{Substitute for } P_1 \text{ and } P_2.$$

In **Example 8(b),** we found that $\log_2 0.1 \approx -3.32$. Now we find $\log_2 0.9$.

$$\log_2 0.9 = \frac{\log 0.9}{\log 2} \approx -0.152 \quad \text{Change-of-base theorem}$$

Now evaluate H.

$$\begin{aligned} H &= -[0.9 \log_2 0.9 + 0.1 \log_2 0.1] \\ H &\approx -[0.9(-0.152) + 0.1(-3.32)] && \text{Substitute approximate values.} \\ H &\approx 0.469 && \text{Simplify.} \end{aligned}$$

Verify that $H \approx 0.971$ if there are 60 of one species and 40 of the other. As the proportions of n species get closer to $\frac{1}{n}$ each, the measure of diversity increases to a maximum of $\log_2 n$.

✔ Now Try Exercise 73.

We saw previously that graphing calculators are capable of fitting exponential curves to data that suggest such behavior. The same is true for logarithmic curves. For example, during the early 2000s on one particular day, interest rates for various U.S. Treasury Securities were as shown in the table.

Time	3-mo	6-mo	2-yr	5-yr	10-yr	30-yr
Yield	0.83%	0.91%	1.35%	2.46%	3.54%	4.58%

Source: U.S. Treasury.

Figure 39 shows how a calculator gives the best-fitting natural logarithmic curve for the data, as well as the data points and the graph of this curve. ■

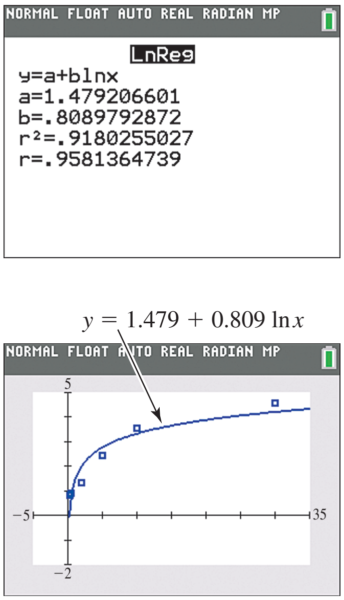


Figure 39

4.4 Exercises

CONCEPT PREVIEW *Answer each of the following.*

1. For the exponential function $f(x) = a^x$, where $a > 1$, is the function increasing or decreasing over its entire domain?

2. For the logarithmic function $g(x) = \log_a x$, where $a > 1$, is the function increasing or decreasing over its entire domain?

3. If $f(x) = 5^x$, what is the rule for $f^{-1}(x)$?

4. What is the name given to the exponent to which 4 must be raised to obtain 11?

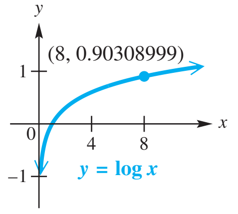
5. A base e logarithm is called a(n) ________ logarithm, and a base 10 logarithm is called a(n) ________ logarithm.

6. How is $\log_3 12$ written in terms of natural logarithms using the change-of-base theorem?

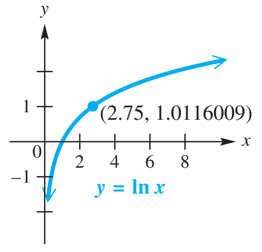
7. Why is $\log_2 0$ undefined?

8. Between what two consecutive integers must $\log_2 12$ lie?

9. The graph of $y = \log x$ shows a point on the graph. Write the logarithmic equation associated with that point.



10. The graph of $y = \ln x$ shows a point on the graph. Write the logarithmic equation associated with that point.



Find each value. If applicable, give an approximation to four decimal places. ***See Example 1.***

11. $\log 10^{12}$
12. $\log 10^7$
13. $\log 0.1$
14. $\log 0.01$
15. $\log 63$
16. $\log 94$
17. $\log 0.0022$
18. $\log 0.0055$
19. $\log(387 \times 23)$
20. $\log(296 \times 12)$
21. $\log \frac{518}{342}$
22. $\log \frac{643}{287}$
23. $\log 387 + \log 23$
24. $\log 296 + \log 12$
25. $\log 518 - \log 342$
26. $\log 643 - \log 287$

Answer each question.

27. Why is the result in **Exercise 23** the same as that in **Exercise 19?**

28. Why is the result in **Exercise 25** the same as that in **Exercise 21?**

For each substance, find the pH from the given hydronium ion concentration. **See Example 2(a).**

29. grapefruit, 6.3×10^{-4}

30. limes, 1.6×10^{-2}

31. crackers, 3.9×10^{-9}

32. sodium hydroxide (lye), 3.2×10^{-14}

Find the $[H_3O^+]$ *for each substance with the given pH.* **See Example 2(b).**

33. soda pop, 2.7

34. wine, 3.4

35. beer, 4.8

36. drinking water, 6.5

Suppose that water from a wetland area is sampled and found to have the given hydronium ion concentration. Determine whether the wetland is a rich fen, a poor fen, or a bog. **See Example 3.**

37. 2.49×10^{-5}

38. 6.22×10^{-5}

39. 2.49×10^{-2}

40. 3.14×10^{-2}

41. 2.49×10^{-7}

42. 5.86×10^{-7}

Solve each problem.

43. Use a calculator to find an approximation for each logarithm.

(a) log 398.4 **(b)** log 39.84 **(c)** log 3.984

(d) From the answers to parts (a)–(c), make a conjecture concerning the decimal values in the approximations of common logarithms of numbers greater than 1 that have the same digits.

44. Given that $\log 25 \approx 1.3979$, $\log 250 \approx 2.3979$, and $\log 2500 \approx 3.3979$, make a conjecture for an approximation of log 25,000. Why does this pattern continue?

Find each value. If applicable, give an approximation to four decimal places. **See Example 5.**

45. $\ln e^{1.6}$

46. $\ln e^{5.8}$

47. $\ln \frac{1}{e^2}$

48. $\ln \frac{1}{e^4}$

49. $\ln \sqrt{e}$

50. $\ln \sqrt[3]{e}$

51. ln 28

52. ln 39

53. ln 0.00013

54. ln 0.0077

55. $\ln(27 \times 943)$

56. $\ln(33 \times 568)$

57. $\ln \frac{98}{13}$

58. $\ln \frac{84}{17}$

59. $\ln 27 + \ln 943$

60. $\ln 33 + \ln 568$

61. $\ln 98 - \ln 13$

62. $\ln 84 - \ln 17$

Solve each problem. **See Examples 4, 6, 7, and 9.**

63. ***Decibel Levels*** Find the decibel ratings of sounds having the following intensities.

(a) $100I_0$ **(b)** $1000I_0$ **(c)** $100{,}000I_0$ **(d)** $1{,}000{,}000I_0$

(e) If the intensity of a sound is doubled, by how much is the decibel rating increased? Round to the nearest whole number.

64. ***Decibel Levels*** Find the decibel ratings of the following sounds, having intensities as given. Round each answer to the nearest whole number.

(a) whisper, $115I_0$

(b) busy street, $9{,}500{,}000I_0$

(c) heavy truck, 20 m away, $1{,}200{,}000{,}000I_0$

(d) rock music, $895{,}000{,}000{,}000I_0$

(e) jetliner at takeoff, $109{,}000{,}000{,}000{,}000I_0$

65. ***Earthquake Intensity*** The magnitude of an earthquake, measured on the Richter scale, is $\log_{10} \frac{I}{I_0}$, where I is the amplitude registered on a seismograph 100 km from the epicenter of the earthquake, and I_0 is the amplitude of an earthquake of a certain (small) size. Find the Richter scale ratings for earthquakes having the following amplitudes.

(a) $1000I_0$ **(b)** $1{,}000{,}000I_0$ **(c)** $100{,}000{,}000I_0$

66. ***Earthquake Intensity*** On December 26, 2004, an earthquake struck in the Indian Ocean with a magnitude of 9.1 on the Richter scale. The resulting tsunami killed an estimated 229,900 people in several countries. Express this reading in terms of I_0 to the nearest hundred thousand.

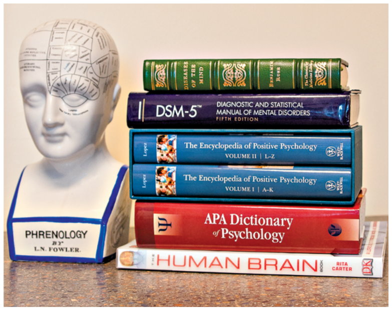
67. ***Earthquake Intensity*** On February 27, 2010, a massive earthquake struck Chile with a magnitude of 8.8 on the Richter scale. Express this reading in terms of I_0 to the nearest hundred thousand.

68. ***Earthquake Intensity Comparison*** Compare the answers to **Exercises 66 and 67.** How many times greater was the force of the 2004 earthquake than that of the 2010 earthquake?

69. ***(Modeling) Bachelor's Degrees in Psychology*** The table gives the number of bachelor's degrees in psychology (in thousands) earned at U.S. colleges and universities for selected years from 1980 through 2012. Suppose x represents the number of years since 1950. Thus, 1980 is represented by 30, 1990 is represented by 40, and so on.

Year	Degrees Earned (in thousands)
1980	42.1
1990	54.0
2000	74.2
2010	97.2
2011	100.9
2012	109.0

Source: National Center for Education Statistics.



The following function is a logarithmic model for the data.

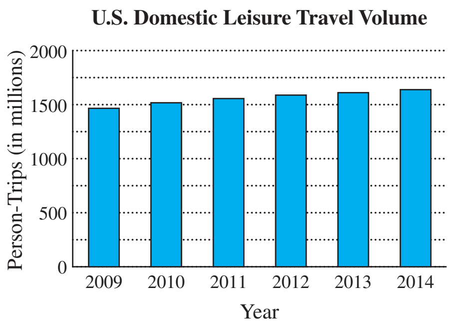
$$f(x) = -273 + 90.6 \ln x$$

Use this function to estimate the number of bachelor's degrees in psychology earned in the year 2016 to the nearest tenth thousand. What assumption must we make to estimate the number of degrees in years beyond 2012?

70. ***(Modeling) Domestic Leisure Travel*** The bar graph shows numbers of leisure trips within the United States (in millions of person-trips of 50 or more miles one-way) over the years 2009–2014. The function

$$f(t) = 1458 + 95.42 \ln t, \quad t \geq 1,$$

where t represents the number of years since 2008 and $f(t)$ is the number of person-trips, in millions, approximates the curve reasonably well.



Source: Statista 2014.

Use the function to approximate the number of person-trips in 2012 to the nearest million. How does this approximation compare to the actual number of 1588 million?

71. ***(Modeling) Diversity of Species*** The number of species $S(n)$ in a sample is given by

$$S(n) = a \ln\left(1 + \frac{n}{a}\right),$$

where n is the number of individuals in the sample, and a is a constant that indicates the diversity of species in the community. If $a = 0.36$, find $S(n)$ for each value of n. (*Hint:* $S(n)$ must be a whole number.)

(a) 100 **(b)** 200 **(c)** 150 **(d)** 10

72. ***(Modeling) Diversity of Species*** In **Exercise 71,** find $S(n)$ if a changes to 0.88. Use the following values of n.

(a) 50 **(b)** 100 **(c)** 250

73. ***(Modeling) Diversity of Species*** Suppose a sample of a small community shows two species with 50 individuals each. Find the measure of diversity H.

74. ***(Modeling) Diversity of Species*** A virgin forest in northwestern Pennsylvania has 4 species of large trees with the following proportions of each:

hemlock, 0.521; beech, 0.324; birch, 0.081; maple, 0.074.

Find the measure of diversity H to the nearest thousandth.

75. ***(Modeling) Global Temperature Increase*** In **Example 7,** we expressed the average global temperature increase T (in °F) as

$$T(k) = 1.03k \ln \frac{C}{C_0},$$

where C_0 is the preindustrial amount of carbon dioxide, C is the current carbon dioxide level, and k is a constant. Arrhenius determined that $10 \le k \le 16$ when C was double the value C_0. Use $T(k)$ to find the range of the rise in global temperature T (rounded to the nearest degree) that Arrhenius predicted. (*Source:* Clime, W., *The Economics of Global Warming,* Institute for International Economics, Washington, D.C.)

76. ***(Modeling) Global Temperature Increase*** (Refer to **Exercise 75.**) According to one study by the IPCC, future increases in average global temperatures (in °F) can be modeled by

$$T(C) = 6.489 \ln \frac{C}{280},$$

where C is the concentration of atmospheric carbon dioxide (in ppm). C can be modeled by the function

$$C(x) = 353(1.006)^{x-1990},$$

where x is the year. (*Source:* International Panel on Climate Change (IPCC).)

(a) Write T as a function of x.

(b) Using a graphing calculator, graph $C(x)$ and $T(x)$ on the interval [1990, 2275] using different coordinate axes. Describe the graph of each function. How are C and T related?

(c) Approximate the slope of the graph of T. What does this slope represent?

(d) Use graphing to estimate x and $C(x)$ when $T(x) = 10°F$.

77. ***Age of Rocks*** Use the formula of **Example 6** to estimate the age of a rock sample having $\frac{A}{K} = 0.103$. Give the answer in billions of years, rounded to the nearest hundredth.

78. ***(Modeling) Planets' Distances from the Sun and Periods of Revolution*** The table contains the planets' average distances D from the sun and their periods P of revolution around the sun in years. The distances have been normalized so that Earth is one unit away from the sun. For example, since Jupiter's distance is 5.2, its distance from the sun is 5.2 times farther than Earth's.

Planet	D	P
Mercury	0.39	0.24
Venus	0.72	0.62
Earth	1	1
Mars	1.52	1.89
Jupiter	5.2	11.9
Saturn	9.54	29.5
Uranus	19.2	84.0
Neptune	30.1	164.8

Source: Ronan, C., *The Natural History of the Universe*, MacMillan Publishing Co., New York.

(a) Using a graphing calculator, make a scatter diagram by plotting the point $(\ln D, \ln P)$ for each planet on the xy-coordinate axes. Do the data points appear to be linear?

(b) Determine a linear equation that models the data points. Graph the line and the data on the same coordinate axes.

(c) Use this linear model to predict the period of Pluto if its distance is 39.5. Compare the answer to the actual value of 248.5 yr.

Use the change-of-base theorem to find an approximation to four decimal places for each logarithm. ***See Example 8.***

79. $\log_2 5$ **80.** $\log_2 9$ **81.** $\log_8 0.59$ **82.** $\log_8 0.71$

83. $\log_{1/2} 3$ **84.** $\log_{1/3} 2$ **85.** $\log_\pi e$ **86.** $\log_\pi \sqrt{2}$

87. $\log_{\sqrt{13}} 12$ **88.** $\log_{\sqrt{19}} 5$ **89.** $\log_{0.32} 5$ **90.** $\log_{0.91} 8$

Let $u = \ln a$ and $v = \ln b$. Write each expression in terms of u and v without using the ln *function.*

91. $\ln\left(b^4\sqrt{a}\right)$ **92.** $\ln \dfrac{a^3}{b^2}$ **93.** $\ln\sqrt{\dfrac{a^3}{b^5}}$ **94.** $\ln\left(\sqrt[3]{a}\cdot b^4\right)$

Concept Check *Use the various properties of exponential and logarithmic functions to evaluate the expressions in parts (a)–(c).*

95. Given $g(x) = e^x$, find **(a)** $g(\ln 4)$ **(b)** $g(\ln 5^2)$ **(c)** $g\left(\ln \frac{1}{e}\right)$.

96. Given $f(x) = 3^x$, find **(a)** $f(\log_3 2)$ **(b)** $f(\log_3(\ln 3))$ **(c)** $f(\log_3(2 \ln 3))$.

97. Given $f(x) = \ln x$, find **(a)** $f(e^6)$ **(b)** $f(e^{\ln 3})$ **(c)** $f(e^{2 \ln 3})$.

98. Given $f(x) = \log_2 x$, find **(a)** $f(2^7)$ **(b)** $f(2^{\log_2 2})$ **(c)** $f(2^{2 \log_2 2})$.

Work each problem.

99. ***Concept Check*** Which of the following is equivalent to $2 \ln(3x)$ for $x > 0$?

A. $\ln 9 + \ln x$ **B.** $\ln 6x$ **C.** $\ln 6 + \ln x$ **D.** $\ln 9x^2$

100. ***Concept Check*** Which of the following is equivalent to $\ln(4x) - \ln(2x)$ for $x > 0$?

A. $2 \ln x$ **B.** $\ln 2x$ **C.** $\dfrac{\ln 4x}{\ln 2x}$ **D.** $\ln 2$

101. The function $f(x) = \ln|x|$ plays a prominent role in calculus. Find its domain, its range, and the symmetries of its graph.

102. Consider the function $f(x) = \log_3|x|$.

(a) What is the domain of this function?

(b) Use a graphing calculator to graph $f(x) = \log_3|x|$ in the window $[-4, 4]$ by $[-4, 4]$.

(c) How might one easily misinterpret the domain of the function by merely observing the calculator graph?

Use properties of logarithms to rewrite each function, and describe how the graph of the given function compares to the graph of $g(x) = \ln x$.

103. $f(x) = \ln(e^2 x)$

104. $f(x) = \ln \frac{x}{e}$

105. $f(x) = \ln \frac{x}{e^2}$

Chapter 4 Quiz (Sections 4.1–4.4)

1. For the one-to-one function $f(x) = \sqrt[3]{3x - 6}$, find $f^{-1}(x)$.

2. Solve $4^{2x+1} = 8^{3x-6}$.

3. Graph $f(x) = -3^x$. Give the domain and range.

4. Graph $f(x) = \log_4(x + 2)$. Give the domain and range.

5. ***Future Value*** Suppose that \$15,000 is deposited in a bank certificate of deposit at an annual rate of 2.7% for 8 yr. Find the future value if interest is compounded as follows.

(a) annually **(b)** quarterly **(c)** monthly **(d)** daily (365 days)

6. Use a calculator to evaluate each logarithm to four decimal places.

(a) $\log 34.56$ **(b)** $\ln 34.56$

7. What is the meaning of the expression $\log_6 25$?

8. Solve each equation.

(a) $x = 3^{\log_3 4}$ **(b)** $\log_x 25 = 2$ **(c)** $\log_4 x = -2$

9. Assuming all variables represent positive real numbers, use properties of logarithms to rewrite

$$\log_3 \frac{\sqrt{x} \cdot y}{pq^4}.$$

10. Given $\log_b 9 = 3.1699$ and $\log_b 5 = 2.3219$, find the value of $\log_b 225$.

11. Find the value of $\log_3 40$ to four decimal places.

12. If $f(x) = 4^x$, what is the value of $f(\log_4 12)$?

4.5 Exponential and Logarithmic Equations

- Exponential Equations
- Logarithmic Equations
- Applications and Models

Exponential Equations

We solved exponential equations in earlier sections. General methods for solving these equations depend on the property below, which follows from the fact that logarithmic functions are one-to-one.

Property of Logarithms

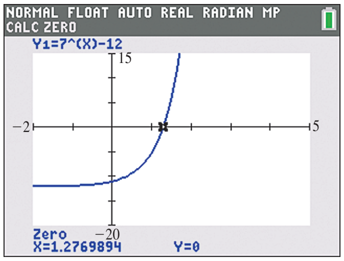
If $x > 0$, $y > 0$, $a > 0$, and $a \neq 1$, then the following holds.

$$x = y \quad \text{is equivalent to} \quad \log_a x = \log_a y.$$

EXAMPLE 1 Solving an Exponential Equation

Solve $7^x = 12$. Give the solution to the nearest thousandth.

SOLUTION The properties of exponents cannot be used to solve this equation, so we apply the preceding property of logarithms. While any appropriate base b can be used, the best practical base is base 10 or base e. We choose base e (natural) logarithms here.



As seen in the display at the bottom of the screen, when rounded to three decimal places, the solution of $7^x - 12 = 0$ agrees with that found in **Example 1.**

$$7^x = 12$$

$$\ln 7^x = \ln 12 \qquad \text{Property of logarithms}$$

$$x \ln 7 = \ln 12 \qquad \text{Power property}$$

This is *exact*.

$$x = \frac{\ln 12}{\ln 7} \qquad \text{Divide by } \ln 7.$$

$$x \approx 1.277 \qquad \text{Use a calculator.}$$

This is *approximate*.

The solution set is $\{1.277\}$.

Now Try Exercise 11.

CAUTION Do not confuse a quotient like $\frac{\ln 12}{\ln 7}$ in **Example 1** with $\ln \frac{12}{7}$, which can be written as $\ln 12 - \ln 7$. ***We cannot change the quotient of two logarithms to a difference of logarithms.***

$$\frac{\ln 12}{\ln 7} \neq \ln \frac{12}{7}$$

EXAMPLE 2 Solving an Exponential Equation

Solve $3^{2x-1} = 0.4^{x+2}$. Give the solution to the nearest thousandth.

SOLUTION

$$3^{2x-1} = 0.4^{x+2}$$

$$\ln 3^{2x-1} = \ln 0.4^{x+2} \qquad \text{Take the natural logarithm on each side.}$$

$$(2x-1)\ln 3 = (x+2)\ln 0.4 \qquad \text{Power property}$$

$$2x \ln 3 - \ln 3 = x \ln 0.4 + 2 \ln 0.4 \qquad \text{Distributive property}$$

$$2x \ln 3 - x \ln 0.4 = 2 \ln 0.4 + \ln 3 \qquad \text{Write so that the terms with } x \text{ are on one side.}$$

$$x(2 \ln 3 - \ln 0.4) = 2 \ln 0.4 + \ln 3 \qquad \text{Factor out } x.$$

$$x = \frac{2 \ln 0.4 + \ln 3}{2 \ln 3 - \ln 0.4} \qquad \text{Divide by } 2 \ln 3 - \ln 0.4.$$

$$x = \frac{\ln 0.4^2 + \ln 3}{\ln 3^2 - \ln 0.4} \qquad \text{Power property}$$

$$x = \frac{\ln 0.16 + \ln 3}{\ln 9 - \ln 0.4} \qquad \text{Apply the exponents.}$$

This is *exact*.

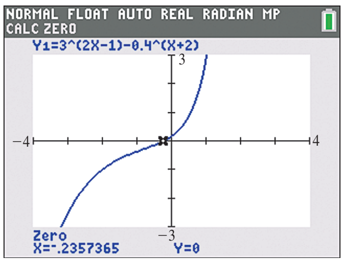
$$x = \frac{\ln 0.48}{\ln 22.5} \qquad \text{Product and quotient properties}$$

$$x \approx -0.236 \qquad \text{Use a calculator.}$$

This is *approximate*.

The solution set is $\{-0.236\}$.

Now Try Exercise 19.



This screen supports the solution found in **Example 2.**

EXAMPLE 3 Solving Base *e* Exponential Equations

Solve each equation. Give solutions to the nearest thousandth.

(a) $e^{x^2} = 200$ **(b)** $e^{2x+1} \cdot e^{-4x} = 3e$

SOLUTION

(a)

$$
\begin{aligned}
e^{x^2} &= 200 \\
\ln e^{x^2} &= \ln 200 && \text{Take the natural logarithm on each side.} \\
x^2 &= \ln 200 && \ln e^{x^2} = x^2
\end{aligned}
$$

Remember both roots.

$$
\begin{aligned}
x &= \pm\sqrt{\ln 200} && \text{Square root property} \\
x &\approx \pm 2.302 && \text{Use a calculator.}
\end{aligned}
$$

The solution set is $\{\pm 2.302\}$.

(b)

$$
\begin{aligned}
e^{2x+1} \cdot e^{-4x} &= 3e \\
e^{-2x+1} &= 3e && a^m \cdot a^n = a^{m+n} \\
e^{-2x} &= 3 && \text{Divide by } e;\ \tfrac{e^{-2x+1}}{e^1} = e^{-2x+1-1} = e^{-2x}. \\
\ln e^{-2x} &= \ln 3 && \text{Take the natural logarithm on each side.} \\
-2x \ln e &= \ln 3 && \text{Power property} \\
-2x &= \ln 3 && \ln e = 1 \\
x &= -\frac{1}{2}\ln 3 && \text{Multiply by } -\tfrac{1}{2}. \\
x &\approx -0.549 && \text{Use a calculator.}
\end{aligned}
$$

The solution set is $\{-0.549\}$.

✔ **Now Try Exercises 21 and 23.**

EXAMPLE 4 Solving an Exponential Equation (Quadratic in Form)

Solve $e^{2x} - 4e^x + 3 = 0$. Give exact value(s) for x.

SOLUTION If we substitute $u = e^x$, we notice that the equation is quadratic in form.

$$
\begin{aligned}
e^{2x} - 4e^x + 3 &= 0 \\
(e^x)^2 - 4e^x + 3 &= 0 && a^{mn} = (a^n)^m \\
u^2 - 4u + 3 &= 0 && \text{Let } u = e^x. \\
(u-1)(u-3) &= 0 && \text{Factor.} \\
u - 1 = 0 \quad \text{or} \quad u - 3 &= 0 && \text{Zero-factor property} \\
u = 1 \quad \text{or} \quad u &= 3 && \text{Solve for } u. \\
e^x = 1 \quad \text{or} \quad e^x &= 3 && \text{Substitute } e^x \text{ for } u. \\
\ln e^x = \ln 1 \quad \text{or} \quad \ln e^x &= \ln 3 && \text{Take the natural logarithm on each side.} \\
x = 0 \quad \text{or} \quad x &= \ln 3 && \ln e^x = x;\ \ln 1 = 0
\end{aligned}
$$

Both values check, so the solution set is $\{0, \ln 3\}$.

✔ **Now Try Exercise 35.**

Logarithmic Equations The following equations involve logarithms of variable expressions.

EXAMPLE 5 Solving Logarithmic Equations

Solve each equation. Give exact values.

(a) $7 \ln x = 28$ **(b)** $\log_2(x^3 - 19) = 3$

SOLUTION

(a)

$$7 \ln x = 28$$

$$\log_e x = 4 \quad \ln x = \log_e x; \text{ Divide by 7.}$$

$$x = e^4 \quad \text{Write in exponential form.}$$

The solution set is $\{e^4\}$.

(b)

$$\log_2(x^3 - 19) = 3$$

$$x^3 - 19 = 2^3 \quad \text{Write in exponential form.}$$

$$x^3 - 19 = 8 \quad \text{Apply the exponent.}$$

$$x^3 = 27 \quad \text{Add 19.}$$

$$x = \sqrt[3]{27} \quad \text{Take cube roots.}$$

$$x = 3 \quad \sqrt[3]{27} = 3$$

The solution set is $\{3\}$.

✔ **Now Try Exercises 41 and 49.**

EXAMPLE 6 Solving a Logarithmic Equation

Solve $\log(x + 6) - \log(x + 2) = \log x$. Give exact value(s).

SOLUTION Recall that logarithms are defined only for nonnegative numbers.

$$\log(x + 6) - \log(x + 2) = \log x$$

$$\log \frac{x + 6}{x + 2} = \log x \quad \text{Quotient property}$$

$$\frac{x + 6}{x + 2} = x \quad \text{Property of logarithms}$$

$$x + 6 = x(x + 2) \quad \text{Multiply by } x + 2.$$

$$x + 6 = x^2 + 2x \quad \text{Distributive property}$$

$$x^2 + x - 6 = 0 \quad \text{Standard form}$$

$$(x + 3)(x - 2) = 0 \quad \text{Factor.}$$

$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{Zero-factor property}$$

$$x = -3 \quad \text{or} \quad x = 2 \quad \text{Solve for } x.$$

The proposed negative solution (-3) is not in the domain of $\log x$ in the original equation, so the only valid solution is the positive number 2. The solution set is $\{2\}$.

✔ **Now Try Exercise 69.**

CAUTION Recall that the domain of $y = \log_a x$ is $(0, \infty)$. ***For this reason, it is always necessary to check that proposed solutions of a logarithmic equation result in logarithms of positive numbers in the original equation.***

EXAMPLE 7 Solving a Logarithmic Equation

Solve $\log_2 [(3x-7)(x-4)] = 3$. Give exact value(s).

SOLUTION

$$\log_2 [(3x-7)(x-4)] = 3$$

$$(3x-7)(x-4) = 2^3 \qquad \text{Write in exponential form.}$$

$$3x^2 - 19x + 28 = 8 \qquad \text{Multiply. Apply the exponent.}$$

$$3x^2 - 19x + 20 = 0 \qquad \text{Standard form}$$

$$(3x-4)(x-5) = 0 \qquad \text{Factor.}$$

$$3x - 4 = 0 \quad \text{or} \quad x - 5 = 0 \qquad \text{Zero-factor property}$$

$$x = \frac{4}{3} \quad \text{or} \quad x = 5 \qquad \text{Solve for } x.$$

A check is necessary to be sure that the argument of the logarithm in the given equation is positive. In both cases, the product $(3x-7)(x-4)$ leads to 8, and $\log_2 8 = 3$ is true. The solution set is $\left\{\frac{4}{3}, 5\right\}$.

✔ **Now Try Exercise 53.**

EXAMPLE 8 Solving a Logarithmic Equation

Solve $\log(3x+2) + \log(x-1) = 1$. Give exact value(s).

SOLUTION

$$\log(3x+2) + \log(x-1) = 1$$

$$\log_{10}[(3x+2)(x-1)] = 1 \qquad \log x = \log_{10} x\text{; product property}$$

$$(3x+2)(x-1) = 10^1 \qquad \text{Write in exponential form.}$$

$$3x^2 - x - 2 = 10 \qquad \text{Multiply; } 10^1 = 10.$$

$$3x^2 - x - 12 = 0 \qquad \text{Subtract 10.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic formula}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-12)}}{2(3)} \qquad \text{Substitute } a = 3,\ b = -1,\ c = -12.$$

The two proposed solutions are

$$\frac{1 - \sqrt{145}}{6} \quad \text{and} \quad \frac{1 + \sqrt{145}}{6}.$$

The first proposed solution, $\frac{1 - \sqrt{145}}{6}$, is negative. Substituting for x in $\log(x-1)$ results in a negative argument, which is not allowed. Therefore, this solution must be rejected.

The second proposed solution, $\frac{1 + \sqrt{145}}{6}$, is positive. Substituting it for x in $\log(3x+2)$ results in a positive argument. Substituting it for x in $\log(x+1)$ also results in a positive argument. Both are necessary conditions. Therefore, the solution set is $\left\{\frac{1 + \sqrt{145}}{6}\right\}$.

✔ **Now Try Exercise 77.**

NOTE We could have replaced 1 with $\log_{10} 10$ in **Example 8** by first writing

$$\log(3x + 2) + \log(x - 1) = 1 \qquad \text{Equation from Example 8}$$
$$\log_{10}[(3x + 2)(x - 1)] = \log_{10} 10 \qquad \text{Substitute.}$$
$$(3x + 2)(x - 1) = 10, \qquad \text{Property of logarithms}$$

and then continuing as shown on the preceding page.

EXAMPLE 9 Solving a Base *e* Logarithmic Equation

Solve $\ln e^{\ln x} - \ln(x - 3) = \ln 2$. Give exact value(s).

SOLUTION This logarithmic equation differs from those in **Examples 7 and 8** because the expression on the right side involves a logarithm.

$$\ln e^{\ln x} - \ln(x - 3) = \ln 2$$
$$\ln x - \ln(x - 3) = \ln 2 \qquad e^{\ln x} = x$$
$$\ln \frac{x}{x - 3} = \ln 2 \qquad \text{Quotient property}$$
$$\frac{x}{x - 3} = 2 \qquad \text{Property of logarithms}$$
$$x = 2(x - 3) \qquad \text{Multiply by } x - 3.$$
$$x = 2x - 6 \qquad \text{Distributive property}$$
$$x = 6 \qquad \text{Solve for } x.$$

Check that the solution set is $\{6\}$.

Now Try Exercise 79.

Solving an Exponential or Logarithmic Equation

To solve an exponential or logarithmic equation, change the given equation into one of the following forms, where a and b are real numbers, $a > 0$ and $a \neq 1$, and follow the guidelines.

1. $\boldsymbol{a^{f(x)} = b}$

   Solve by taking logarithms on each side.

2. $\boldsymbol{\log_a f(x) = b}$

   Solve by changing to exponential form $a^b = f(x)$.

3. $\boldsymbol{\log_a f(x) = \log_a g(x)}$

   The given equation is equivalent to the equation $f(x) = g(x)$. Solve algebraically.

4. In a more complicated equation, such as

   $$e^{2x+1} \cdot e^{-4x} = 3e, \qquad \text{See Example 3(b).}$$

   it may be necessary to first solve for $a^{f(x)}$ or $\log_a f(x)$ and then solve the resulting equation using one of the methods given above.

5. Check that each proposed solution is in the domain.

Applications and Models

EXAMPLE 10 Applying an Exponential Equation to the Strength of a Habit

The strength of a habit is a function of the number of times the habit is repeated. If N is the number of repetitions and H is the strength of the habit, then, according to psychologist C.L. Hull,

$$H = 1000(1 - e^{-kN}),$$

where k is a constant. Solve this equation for k.

SOLUTION

First solve for e^{-kN}.

$$H = 1000(1 - e^{-kN})$$

$$\frac{H}{1000} = 1 - e^{-kN} \quad \text{Divide by 1000.}$$

$$\frac{H}{1000} - 1 = -e^{-kN} \quad \text{Subtract 1.}$$

$$e^{-kN} = 1 - \frac{H}{1000} \quad \text{Multiply by } -1 \text{ and rewrite.}$$

Now solve for k.

$$\ln e^{-kN} = \ln\left(1 - \frac{H}{1000}\right) \quad \text{Take the natural logarithm on each side.}$$

$$-kN = \ln\left(1 - \frac{H}{1000}\right) \quad \ln e^x = x$$

$$k = -\frac{1}{N}\ln\left(1 - \frac{H}{1000}\right) \quad \text{Multiply by } -\tfrac{1}{N}.$$

With the final equation, if one pair of values for H and N is known, k can be found, and the equation can then be used to find either H or N for given values of the other variable.

✔ **Now Try Exercise 91.**

EXAMPLE 11 Modeling PC Tablet Sales in the U.S.

Year	Sales (in millions)
2010	10.3
2011	24.1
2012	35.1
2013	39.8
2014	42.1

Source: Forrester Research.

The table gives U.S. tablet sales (in millions) for several years. The data can be modeled by the function

$$f(t) = 20.57 \ln t + 10.58, \quad t \geq 1,$$

where t is the number of years after 2009.

(a) Use the function to estimate the number of tablets sold in the United States in 2015.

(b) If this trend continues, approximately when will annual sales reach 60 million?

SOLUTION

(a) The year 2015 is represented by $t = 2015 - 2009 = 6$.

$$f(t) = 20.57 \ln t + 10.58 \quad \text{Given function}$$

$$f(6) = 20.57 \ln 6 + 10.58 \quad \text{Let } t = 6.$$

$$f(6) \approx 47.4 \quad \text{Use a calculator.}$$

Based on this model, 47.4 million tablets were sold in 2015.

(b) Replace $f(t)$ with 60 and solve for t.

$$f(t) = 20.57 \ln t + 10.58 \quad \text{Given function}$$
$$60 = 20.57 \ln t + 10.58 \quad \text{Let } f(t) = 60.$$
$$49.42 = 20.57 \ln t \quad \text{Subtract 10.58.}$$
$$\ln t = \frac{49.42}{20.57} \quad \text{Divide by 20.57 and rewrite.}$$
$$t = e^{49.42/20.57} \quad \text{Write in exponential form.}$$
$$t \approx 11.05 \quad \text{Use a calculator.}$$

Adding 11 to 2009 gives the year 2020. Based on this model, annual sales will reach 60 million in 2020.

✔ **Now Try Exercise 111.**

4.5 Exercises

CONCEPT PREVIEW *Match each equation in Column I with the best first step for solving it in Column II.*

I	II
1. $10^x = 150$	**A.** Use the product rule for exponents.
2. $e^{2x-1} = 24$	**B.** Take the common logarithm on each side.
3. $\log_4(x^2 - 10) = 2$	**C.** Write the sum of logarithms as the logarithm of a product.
4. $e^{2x} \cdot e^x = 2e$	**D.** Let $u = e^x$ and write the equation in quadratic form.
5. $2e^{2x} - 5e^x - 3 = 0$	**E.** Change to exponential form.
6. $\log(2x - 1) + \log(x + 4) = 1$	**F.** Take the natural logarithm on each side.

CONCEPT PREVIEW *An exponential equation such as*

$$5^x = 9$$

can be solved for its exact solution using the meaning of logarithm and the change-of-base theorem. Because x is the exponent to which 5 must be raised in order to obtain 9, the exact solution is

$$\log_5 9, \quad or \quad \frac{\log 9}{\log 5}, \quad or \quad \frac{\ln 9}{\ln 5}.$$

For each equation, give the exact solution in three forms similar to the forms above.

7. $7^x = 19$ **8.** $3^x = 10$ **9.** $\left(\frac{1}{2}\right)^x = 12$ **10.** $\left(\frac{1}{3}\right)^x = 4$

Solve each equation. In Exercises 11–34, give irrational solutions as decimals correct to the nearest thousandth. In Exercises 35–40, give solutions in exact form. ***See Examples 1–4.***

11. $3^x = 7$ **12.** $5^x = 13$ **13.** $\left(\frac{1}{2}\right)^x = 5$

14. $\left(\frac{1}{3}\right)^x = 6$ **15.** $0.8^x = 4$ **16.** $0.6^x = 3$

17. $4^{x-1} = 3^{2x}$ **18.** $2^{x+3} = 5^{2x}$ **19.** $6^{x+1} = 4^{2x-1}$

20. $3^{x-4} = 7^{2x+5}$
21. $e^{x^2} = 100$
22. $e^{x^4} = 1000$
23. $e^{3x-7} \cdot e^{-2x} = 4e$
24. $e^{1-3x} \cdot e^{5x} = 2e$
25. $\left(\frac{1}{3}\right)^x = -3$
26. $\left(\frac{1}{9}\right)^x = -9$
27. $0.05(1.15)^x = 5$
28. $1.2(0.9)^x = 0.6$
29. $3(2)^{x-2} + 1 = 100$
30. $5(1.2)^{3x-2} + 1 = 7$
31. $2(1.05)^x + 3 = 10$
32. $3(1.4)^x - 4 = 60$
33. $5(1.015)^{x-1980} = 8$
34. $6(1.024)^{x-1900} = 9$
35. $e^{2x} - 6e^x + 8 = 0$
36. $e^{2x} - 8e^x + 15 = 0$
37. $2e^{2x} + e^x = 6$
38. $3e^{2x} + 2e^x = 1$
39. $5^{2x} + 3(5^x) = 28$
40. $3^{2x} - 12(3^x) = -35$

Solve each equation. Give solutions in exact form. ***See Examples 5–9.***

41. $5 \ln x = 10$
42. $3 \ln x = 9$
43. $\ln 4x = 1.5$
44. $\ln 2x = 5$
45. $\log(2 - x) = 0.5$
46. $\log(3 - x) = 0.75$
47. $\log_6(2x + 4) = 2$
48. $\log_5(8 - 3x) = 3$
49. $\log_4(x^3 + 37) = 3$
50. $\log_7(x^3 + 65) = 0$
51. $\ln x + \ln x^2 = 3$
52. $\log x + \log x^2 = 3$
53. $\log_3[(x + 5)(x - 3)] = 2$
54. $\log_4[(3x + 8)(x - 6)] = 3$
55. $\log_2[(2x + 8)(x + 4)] = 5$
56. $\log_5[(3x + 5)(x + 1)] = 1$
57. $\log x + \log(x + 15) = 2$
58. $\log x + \log(2x + 1) = 1$
59. $\log(x + 25) = \log(x + 10) + \log 4$
60. $\log(3x + 5) - \log(2x + 4) = 0$
61. $\log(x - 10) - \log(x - 6) = \log 2$
62. $\log(x^2 - 9) - \log(x - 3) = \log 5$
63. $\ln(7 - x) + \ln(1 - x) = \ln(25 - x)$
64. $\ln(3 - x) + \ln(5 - x) = \ln(50 - 6x)$
65. $\log_8(x + 2) + \log_8(x + 4) = \log_8 8$
66. $\log_2(5x - 6) - \log_2(x + 1) = \log_2 3$
67. $\log_2(x^2 - 100) - \log_2(x + 10) = 1$
68. $\log_2(x - 2) + \log_2(x - 1) = 1$
69. $\log x + \log(x - 21) = \log 100$
70. $\log x + \log(3x - 13) = \log 10$
71. $\log(9x + 5) = 3 + \log(x + 2)$
72. $\log(11x + 9) = 3 + \log(x + 3)$
73. $\ln(4x - 2) - \ln 4 = -\ln(x - 2)$
74. $\ln(5 + 4x) - \ln(3 + x) = \ln 3$
75. $\log_5(x + 2) + \log_5(x - 2) = 1$
76. $\log_2(x - 7) + \log_2 x = 3$
77. $\log_2(2x - 3) + \log_2(x + 1) = 1$
78. $\log_5(3x + 2) + \log_5(x - 1) = 1$
79. $\ln e^x - 2 \ln e = \ln e^4$
80. $\ln e^x - \ln e^3 = \ln e^3$
81. $\log_2(\log_2 x) = 1$
82. $\log x = \sqrt{\log x}$
83. $\log x^2 = (\log x)^2$
84. $\log_2 \sqrt{2x^2} = \frac{3}{2}$

85. ***Concept Check*** Consider the following statement: "We must reject any negative proposed solution when we solve an equation involving logarithms." Is this correct? Why or why not?

86. ***Concept Check*** What values of x could not possibly be solutions of the following equation?

$$\log_a(4x - 7) + \log_a(x^2 + 4) = 0$$

Solve each equation for the indicated variable. Use logarithms with the appropriate bases. ***See Example 10.***

87. $p = a + \dfrac{k}{\ln x}$, for x

88. $r = p - k \ln t$, for t

89. $T = T_0 + (T_1 - T_0)10^{-kt}$, for t

90. $A = \dfrac{Pr}{1 - (1 + r)^{-n}}$, for n

91. $I = \dfrac{E}{R}(1 - e^{-Rt/2})$, for t

92. $y = \dfrac{K}{1 + ae^{-bx}}$, for b

93. $y = A + B(1 - e^{-Cx})$, for x

94. $m = 6 - 2.5 \log \dfrac{M}{M_0}$, for M

95. $\log A = \log B - C \log x$, for A

96. $d = 10 \log \dfrac{I}{I_0}$, for I

97. $A = P\left(1 + \dfrac{r}{n}\right)^{tn}$, for t

98. $D = 160 + 10 \log x$, for x

To solve each problem, refer to the formulas for compound interest.

$$A = P\left(1 + \frac{r}{n}\right)^{tn} \quad \text{and} \quad A = Pe^{rt}$$

99. ***Compound Amount*** If \$10,000 is invested in an account at 3% annual interest compounded quarterly, how much will be in the account in 5 yr if no money is withdrawn?

100. ***Compound Amount*** If \$5000 is invested in an account at 4% annual interest compounded continuously, how much will be in the account in 8 yr if no money is withdrawn?

101. ***Investment Time*** Kurt wants to buy a \$30,000 truck. He has saved \$27,000. Find the number of years (to the nearest tenth) it will take for his \$27,000 to grow to \$30,000 at 4% interest compounded quarterly.

102. ***Investment Time*** Find t to the nearest hundredth of a year if \$1786 becomes \$2063 at 2.6%, with interest compounded monthly.

103. ***Interest Rate*** Find the interest rate to the nearest hundredth of a percent that will produce \$2500, if \$2000 is left at interest compounded semiannually for 8.5 yr.

104. ***Interest Rate*** At what interest rate, to the nearest hundredth of a percent, will \$16,000 grow to \$20,000 if invested for 7.25 yr and interest is compounded quarterly?

(Modeling) Solve each application. ***See Example 11.***

105. In the central Sierra Nevada (a mountain range in California), the percent of moisture that falls as snow rather than rain is approximated reasonably well by

$$f(x) = 86.3 \ln x - 680,$$

where x is the altitude in feet and $f(x)$ is the percent of moisture that falls as snow. Find the percent of moisture, to the nearest tenth, that falls as snow at each altitude.

(a) 3000 ft **(b)** 4000 ft **(c)** 7000 ft

106. Northwest Creations finds that its total sales in dollars, $T(x)$, from the distribution of x thousand catalogues is approximated by

$$T(x) = 5000 \log(x + 1).$$

Find the total sales, to the nearest dollar, resulting from the distribution of each number of catalogues.

(a) 5000 **(b)** 24,000 **(c)** 49,000

107. ***Average Annual Public University Costs*** The table shows the cost of a year's tuition, room and board, and fees at 4-year public colleges for the years 2006–2014. Letting y represent the cost in dollars and x the number of years since 2006, the function

$$f(x) = 13{,}017(1.05)^x$$

models the data quite well. According to this function, in what year will the 2006 cost be doubled?

Year	Average Annual Cost
2006	\$12,837
2007	\$13,558
2008	\$14,372
2009	\$15,235
2010	\$16,178
2011	\$17,156
2012	\$17,817
2013	\$18,383
2014	\$18,943

Source: The College Board, *Annual Survey of Colleges*.

108. ***Race Speed*** At the World Championship races held at Rome's Olympic Stadium in 1987, American sprinter Carl Lewis ran the 100-m race in 9.86 sec. His speed in meters per second after t seconds is closely modeled by the function

$$f(t) = 11.65(1 - e^{-t/1.27}).$$

(*Source:* Banks, Robert B., *Towing Icebergs, Falling Dominoes, and Other Adventures in Applied Mathematics,* Princeton University Press.)

(a) How fast, to the nearest hundredth, was he running as he crossed the finish line?

(b) After how many seconds, to the nearest hundredth, was he running at the rate of 10 m per sec?

109. ***Women in Labor Force*** The percent of women in the U.S. civilian labor force can be modeled fairly well by the function

$$f(x) = \frac{67.21}{1 + 1.081e^{-x/24.71}},$$

where x represents the number of years since 1950. (*Source: Monthly Labor Review,* U.S. Bureau of Labor Statistics.)

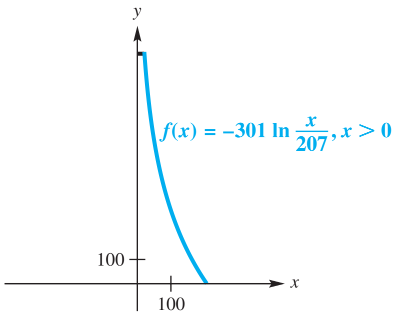
(a) What percent, to the nearest whole number, of U.S. women were in the civilian labor force in 2014?

(b) In what year were 55% of U.S. women in the civilian labor force?

110. ***Height of the Eiffel Tower*** One side of the Eiffel Tower in Paris has a shape that can be approximated by the graph of the function

$$f(x) = -301 \ln \frac{x}{207}, \quad x > 0,$$

where x and $f(x)$ are both measured in feet. (*Source:* Banks, Robert B., *Towing Icebergs, Falling Dominoes, and Other Adventures in Applied Mathematics,* Princeton University Press.)



(a) Why does the shape of the left side of the graph of the Eiffel Tower have the formula given by $f(-x)$?

(b) The short horizontal segment at the top of the figure has length 7.8744 ft. How tall, to the nearest foot, is the Eiffel Tower?

(c) How far from the center of the tower is the point on the right side that is 500 ft above the ground? Round to the nearest foot.

111. ***CO_2 Emissions Tax*** One action that government could take to reduce carbon emissions into the atmosphere is to levy a tax on fossil fuel. This tax would be based on the amount of carbon dioxide emitted into the air when the fuel is burned. The **cost-benefit equation**

$$\ln(1 - P) = -0.0034 - 0.0053x$$

models the approximate relationship between a tax of x dollars per ton of carbon and the corresponding percent reduction P (in decimal form) of emissions of carbon dioxide. (*Source:* Nordhause, W., "To Slow or Not to Slow: The Economics of the Greenhouse Effect," Yale University, New Haven, Connecticut.)

(a) Write P as a function of x.

(b) Graph P for $0 \le x \le 1000$. Discuss the benefit of continuing to raise taxes on carbon.

(c) Determine P, to the nearest tenth, when $x = \$60$. Interpret this result.

(d) What value of x will give a 50% reduction in carbon emissions?

112. ***Radiative Forcing*** Radiative forcing, R, measures the influence of carbon dioxide in altering the additional solar radiation trapped in Earth's atmosphere. The International Panel on Climate Change (IPCC) in 1990 estimated k to be 6.3 in the radiative forcing equation

$$R = k \ln \frac{C}{C_0},$$

where C_0 is the preindustrial amount of carbon dioxide and C is the current level. (*Source:* Clime, W., *The Economics of Global Warming,* Institute for International Economics, Washington, D.C.)

(a) Use the equation $R = 6.3 \ln \frac{C}{C_0}$ to determine the radiative forcing R (in watts per square meter to the nearest tenth) expected by the IPCC if the carbon dioxide level in the atmosphere doubles from its preindustrial level.

(b) Determine the global temperature increase T, to the nearest tenth, that the IPCC predicted would occur if atmospheric carbon dioxide levels were to double, given $T(R) = 1.03R$.

Find $f^{-1}(x)$, and give the domain and range.

113. $f(x) = e^{x-5}$

114. $f(x) = e^x + 10$

115. $f(x) = e^{x+1} - 4$

116. $f(x) = \ln(x + 2)$

117. $f(x) = 2 \ln 3x$

118. $f(x) = \ln(x - 1) + 6$

Use a graphing calculator to solve each equation. Give irrational solutions correct to the nearest hundredth.

119. $e^x + \ln x = 5$

120. $e^x - \ln(x + 1) = 3$

121. $2e^x + 1 = 3e^{-x}$

122. $e^x + 6e^{-x} = 5$

123. $\log x = x^2 - 8x + 14$

124. $\ln x = -\sqrt[3]{x + 3}$

125. Find the **error** in the following **"proof"** that $2 < 1$.

$$\frac{1}{9} < \frac{1}{3} \qquad \text{True statement}$$

$$\left(\frac{1}{3}\right)^2 < \frac{1}{3} \qquad \text{Rewrite the left side.}$$

$$\log\left(\frac{1}{3}\right)^2 < \log \frac{1}{3} \qquad \text{Take the logarithm on each side.}$$

$$2 \log \frac{1}{3} < 1 \log \frac{1}{3} \qquad \text{Property of logarithms; identity property}$$

$$2 < 1 \qquad \text{Divide each side by } \log \tfrac{1}{3}.$$

4.6 Applications and Models of Exponential Growth and Decay

- The Exponential Growth or Decay Function
- Growth Function Models
- Decay Function Models

LOOKING AHEAD TO CALCULUS
The exponential growth and decay function formulas are studied in calculus in conjunction with the topic known as **differential equations.**

The Exponential Growth or Decay Function

In many situations in ecology, biology, economics, and the social sciences, a quantity changes at a rate proportional to the amount present. The amount present at time t is a special function of t called an **exponential growth or decay function.**

Exponential Growth or Decay Function

Let y_0 be the amount or number present at time $t = 0$. Then, under certain conditions, the amount y present at any time t is modeled by

$$y = y_0 e^{kt}, \quad \text{where } k \text{ is a constant.}$$

The constant k determines the type of function.

- When $k > 0$, the function describes growth. Examples of exponential growth include compound interest and atmospheric carbon dioxide.
- When $k < 0$, the function describes decay. One example of exponential decay is radioactive decay.

Growth Function Models

The amount of time it takes for a quantity that grows exponentially to become twice its initial amount is its **doubling time.**

EXAMPLE 1 Determining a Function to Model Exponential Growth

Earlier in this chapter, we discussed the growth of atmospheric carbon dioxide over time using a function based on the data from the table. Now we determine such a function from the data.

Year	Carbon Dioxide (ppm)
1990	353
2000	375
2075	590
2175	1090
2275	2000

Source: International Panel on Climate Change (IPCC).

(a) Find an exponential function that gives the amount of carbon dioxide y in year x.

(b) Estimate the year when future levels of carbon dioxide will be double the preindustrial level of 280 ppm.

SOLUTION

(a) The data points exhibit exponential growth, so the equation will take the form

$$y = y_0 e^{kx}.$$

We must find the values of y_0 and k. The data begin with the year 1990, so to simplify our work we let 1990 correspond to $x = 0$, 1991 correspond to $x = 1$, and so on. Here y_0 is the initial amount and $y_0 = 353$ in 1990 when $x = 0$. Thus the equation is

$$y = 353e^{kx}. \quad \text{Let } y_0 = 353.$$

From the last pair of values in the table, we know that in 2275 the carbon dioxide level is expected to be 2000 ppm. The year 2275 corresponds to $2275 - 1990 = 285$. Substitute 2000 for y and 285 for x, and solve for k.

$$y = 353e^{kx} \quad \text{Solve for } k.$$
$$2000 = 353e^{k(285)} \quad \text{Substitute 2000 for } y \text{ and 285 for } x.$$
$$\frac{2000}{353} = e^{285k} \quad \text{Divide by 353.}$$
$$\ln \frac{2000}{353} = \ln e^{285k} \quad \text{Take the natural logarithm on each side.}$$
$$\ln \frac{2000}{353} = 285k \quad \ln e^x = x, \text{ for all } x.$$
$$k = \frac{1}{285} \cdot \ln \frac{2000}{353} \quad \text{Multiply by } \tfrac{1}{285} \text{ and rewrite.}$$
$$k \approx 0.00609 \quad \text{Use a calculator.}$$

A function that models the data is

$$y = 353e^{0.00609x}.$$

(b)

$$y = 353e^{0.00609x} \quad \text{Solve the model from part (a) for the year } x.$$
$$560 = 353e^{0.00609x} \quad \text{To double the level 280, let } y = 2(280) = 560.$$
$$\frac{560}{353} = e^{0.00609x} \quad \text{Divide by 353.}$$
$$\ln \frac{560}{353} = \ln e^{0.00609x} \quad \text{Take the natural logarithm on each side.}$$
$$\ln \frac{560}{353} = 0.00609x \quad \ln e^x = x, \text{ for all } x.$$
$$x = \frac{1}{0.00609} \cdot \ln \frac{560}{353} \quad \text{Multiply by } \tfrac{1}{0.00609} \text{ and rewrite.}$$
$$x \approx 75.8 \quad \text{Use a calculator.}$$

Since $x = 0$ corresponds to 1990, the preindustrial carbon dioxide level will double in the 75th year after 1990, or during 2065, according to this model.

✔ **Now Try Exercise 43.**

EXAMPLE 2 Finding Doubling Time for Money

How long will it take for money in an account that accrues interest at a rate of 3%, compounded continuously, to double?

SOLUTION

$$A = Pe^{rt} \quad \text{Continuous compounding formula}$$
$$2P = Pe^{0.03t} \quad \text{Let } A = 2P \text{ and } r = 0.03.$$
$$2 = e^{0.03t} \quad \text{Divide by } P.$$
$$\ln 2 = \ln e^{0.03t} \quad \text{Take the natural logarithm on each side.}$$
$$\ln 2 = 0.03t \quad \ln e^x = x$$
$$\frac{\ln 2}{0.03} = t \quad \text{Divide by 0.03.}$$
$$23.10 \approx t \quad \text{Use a calculator.}$$

It will take about 23 yr for the amount to double. ✔ **Now Try Exercise 31.**

EXAMPLE 3 Using an Exponential Function to Model Population Growth

According to the U.S. Census Bureau, the world population reached 6 billion people during 1999 and was growing exponentially. By the end of 2010, the population had grown to 6.947 billion. The projected world population (in billions of people) t years after 2010 is given by the function

$$f(t) = 6.947e^{0.00745t}.$$

(a) Based on this model, what will the world population be in 2025?

(b) If this trend continues, approximately when will the world population reach 9 billion?

SOLUTION

(a) Since $t = 0$ represents the year 2010, in 2025, t would be $2025 - 2010 = 15$ yr. We must find $f(t)$ when t is 15.

$$f(t) = 6.947e^{0.00745t} \quad \text{Given function}$$
$$f(15) = 6.947e^{0.00745(15)} \quad \text{Let } t = 15.$$
$$f(15) \approx 7.768 \quad \text{Use a calculator.}$$

The population will be 7.768 billion at the end of 2025.

(b)
$$f(t) = 6.947e^{0.00745t} \quad \text{Given function}$$
$$9 = 6.947e^{0.00745t} \quad \text{Let } f(t) = 9.$$
$$\frac{9}{6.947} = e^{0.00745t} \quad \text{Divide by 6.947.}$$
$$\ln \frac{9}{6.947} = \ln e^{0.00745t} \quad \text{Take the natural logarithm on each side.}$$
$$\ln \frac{9}{6.947} = 0.00745t \quad \ln e^x = x, \text{ for all } x.$$
$$t = \frac{\ln \frac{9}{6.947}}{0.00745} \quad \text{Divide by 0.00745 and rewrite.}$$
$$t \approx 34.8 \quad \text{Use a calculator.}$$

Thus, 34.8 yr after 2010, during the year 2044, world population will reach 9 billion.

✔ **Now Try Exercise 39.**

Decay Function Models **Half-life** is the amount of time it takes for a quantity that decays exponentially to become half its initial amount.

NOTE In **Example 4** on the next page, the initial amount of substance is given as 600 g. Because half-life is constant over the lifetime of a decaying quantity, starting with any initial amount, y_0, and substituting $\frac{1}{2}y_0$ for y in $y = y_0e^{kt}$ would allow the common factor y_0 to be divided out. The rest of the work would be the same.

EXAMPLE 4 **Determining an Exponential Function to Model Radioactive Decay**

Suppose 600 g of a radioactive substance are present initially and 3 yr later only 300 g remain.

(a) Determine an exponential function that models this decay.

(b) How much of the substance will be present after 6 yr?

SOLUTION

(a) We use the given values to find k in the exponential equation

$$y = y_0 e^{kt}.$$

Because the initial amount is 600 g, $y_0 = 600$, which gives $y = 600e^{kt}$. The initial amount (600 g) decays to half that amount (300 g) in 3 yr, so its half-life is 3 yr. Now we solve this exponential equation for k.

$$
\begin{aligned}
y &= 600e^{kt} && \text{Let } y_0 = 600.\\
300 &= 600e^{3k} && \text{Let } y = 300 \text{ and } t = 3.\\
0.5 &= e^{3k} && \text{Divide by 600.}\\
\ln 0.5 &= \ln e^{3k} && \text{Take the natural logarithm on each side.}\\
\ln 0.5 &= 3k && \ln e^x = x, \text{ for all } x.\\
\frac{\ln 0.5}{3} &= k && \text{Divide by 3.}\\
k &\approx -0.231 && \text{Use a calculator.}
\end{aligned}
$$

A function that models the situation is

$$y = 600e^{-0.231t}.$$

(b) To find the amount present after 6 yr, let $t = 6$.

$$
\begin{aligned}
y &= 600e^{-0.231t} && \text{Model from part (a)}\\
y &= 600e^{-0.231(6)} && \text{Let } t = 6.\\
y &= 600e^{-1.386} && \text{Multiply.}\\
y &\approx 150 && \text{Use a calculator.}
\end{aligned}
$$

After 6 yr, 150 g of the substance will remain. ✔ **Now Try Exercise 19.**

EXAMPLE 5 **Solving a Carbon Dating Problem**

Carbon-14, also known as radiocarbon, is a radioactive form of carbon that is found in all living plants and animals. After a plant or animal dies, the radiocarbon disintegrates. Scientists can determine the age of the remains by comparing the amount of radiocarbon with the amount present in living plants and animals. This technique is called **carbon dating.** The amount of radiocarbon present after t years is given by

$$y = y_0 e^{-0.0001216t},$$

where y_0 is the amount present in living plants and animals.

(a) Find the half-life of carbon-14.

(b) Charcoal from an ancient fire pit on Java contained $\frac{1}{4}$ the carbon-14 of a living sample of the same size. Estimate the age of the charcoal.

SOLUTION

(a) If y_0 is the amount of radiocarbon present in a living thing, then $\frac{1}{2}y_0$ is half this initial amount. We substitute and solve the given equation for t.

$$y = y_0 e^{-0.0001216t} \quad \text{Given equation}$$

$$\frac{1}{2}y_0 = y_0 e^{-0.0001216t} \quad \text{Let } y = \tfrac{1}{2}y_0.$$

$$\frac{1}{2} = e^{-0.0001216t} \quad \text{Divide by } y_0.$$

$$\ln\frac{1}{2} = \ln e^{-0.0001216t} \quad \text{Take the natural logarithm on each side.}$$

$$\ln\frac{1}{2} = -0.0001216t \quad \ln e^x = x, \text{ for all } x.$$

$$\frac{\ln\frac{1}{2}}{-0.0001216} = t \quad \text{Divide by } -0.0001216.$$

$$5700 \approx t \quad \text{Use a calculator.}$$

The half-life is 5700 yr.

(b) Solve again for t, this time letting the amount $y = \frac{1}{4}y_0$.

$$y = y_0 e^{-0.0001216t} \quad \text{Given equation}$$

$$\frac{1}{4}y_0 = y_0 e^{-0.0001216t} \quad \text{Let } y = \tfrac{1}{4}y_0.$$

$$\frac{1}{4} = e^{-0.0001216t} \quad \text{Divide by } y_0.$$

$$\ln\frac{1}{4} = \ln e^{-0.0001216t} \quad \text{Take the natural logarithm on each side.}$$

$$\frac{\ln\frac{1}{4}}{-0.0001216} = t \quad \ln e^x = x; \text{ Divide by } -0.0001216.$$

$$t \approx 11{,}400 \quad \text{Use a calculator.}$$

The charcoal is 11,400 yr old. ✔ **Now Try Exercise 23.**

EXAMPLE 6 Modeling Newton's Law of Cooling

Newton's law of cooling says that the rate at which a body cools is proportional to the difference in temperature between the body and the environment around it. The temperature $f(t)$ of the body at time t in appropriate units after being introduced into an environment having constant temperature T_0 is

$$f(t) = T_0 + Ce^{-kt}, \quad \text{where } C \text{ and } k \text{ are constants.}$$

A pot of coffee with a temperature of 100°C is set down in a room with a temperature of 20°C. The coffee cools to 60°C after 1 hr.

(a) Write an equation to model the data.

(b) Find the temperature after half an hour.

(c) How long will it take for the coffee to cool to 50°C?

SOLUTION

(a) We must find values for C and k in the given formula. As given, when $t = 0$, $T_0 = 20$, and the temperature of the coffee is $f(0) = 100$.

$$f(t) = T_0 + Ce^{-kt} \quad \text{Given function}$$
$$100 = 20 + Ce^{-0k} \quad \text{Let } t = 0,\ f(0) = 100, \text{ and } T_0 = 20.$$
$$100 = 20 + C \quad e^0 = 1$$
$$80 = C \quad \text{Subtract 20.}$$

The following function models the data.

$$f(t) = 20 + 80e^{-kt} \quad \text{Let } T_0 = 20 \text{ and } C = 80.$$

The coffee cools to 60°C after 1 hr, so when $t = 1$, $f(1) = 60$.

$$f(t) = 20 + 80e^{-kt} \quad \text{Above function with } T_0 = 20 \text{ and } C = 80$$
$$60 = 20 + 80e^{-1k} \quad \text{Let } t = 1 \text{ and } f(1) = 60.$$
$$40 = 80e^{-k} \quad \text{Subtract 20.}$$
$$\frac{1}{2} = e^{-k} \quad \text{Divide by 80.}$$
$$\ln \frac{1}{2} = \ln e^{-k} \quad \text{Take the natural logarithm on each side.}$$
$$\ln \frac{1}{2} = -k \quad \ln e^x = x, \text{ for all } x.$$
$$k \approx 0.693 \quad \text{Multiply by } -1\text{, rewrite, and use a calculator.}$$

Thus, the model is $f(t) = 20 + 80e^{-0.693t}$.

(b) To find the temperature after $\frac{1}{2}$ hr, let $t = \frac{1}{2}$ in the model from part (a).

$$f(t) = 20 + 80e^{-0.693t} \quad \text{Model from part (a)}$$
$$f\left(\frac{1}{2}\right) = 20 + 80e^{(-0.693)(1/2)} \quad \text{Let } t = \tfrac{1}{2}.$$
$$f\left(\frac{1}{2}\right) \approx 76.6°\text{C} \quad \text{Use a calculator.}$$

(c) To find how long it will take for the coffee to cool to 50°C, let $f(t) = 50$.

$$f(t) = 20 + 80e^{-0.693t} \quad \text{Model from part (a)}$$
$$50 = 20 + 80e^{-0.693t} \quad \text{Let } f(t) = 50.$$
$$30 = 80e^{-0.693t} \quad \text{Subtract 20.}$$
$$\frac{3}{8} = e^{-0.693t} \quad \text{Divide by 80.}$$
$$\ln \frac{3}{8} = \ln e^{-0.693t} \quad \text{Take the natural logarithm on each side.}$$
$$\ln \frac{3}{8} = -0.693t \quad \ln e^x = x, \text{ for all } x.$$
$$t = \frac{\ln \frac{3}{8}}{-0.693} \quad \text{Divide by } -0.693 \text{ and rewrite.}$$
$$t \approx 1.415 \text{ hr}, \quad \text{or} \quad \text{about 1 hr, 25 min}$$

✔ **Now Try Exercise 27.**

4.6 Exercises

CONCEPT PREVIEW ***Population Growth*** *A population is increasing according to the exponential function*

$$y = 2e^{0.02x},$$

where y is in millions and x is the number of years. Match each question in Column I with the correct procedure in Column II to answer the question.

I	II
1. How long will it take for the population to triple?	**A.** Evaluate $y = 2e^{0.02(1/3)}$.
2. When will the population reach 3 million?	**B.** Solve $2e^{0.02x} = 6$.
3. How large will the population be in 3 yr?	**C.** Evaluate $y = 2e^{0.02(3)}$.
4. How large will the population be in 4 months?	**D.** Solve $2e^{0.02x} = 3$.

CONCEPT PREVIEW ***Radioactive Decay*** *Strontium-90 decays according to the exponential function*

$$y = y_0e^{-0.0241t},$$

where t is time in years. Match each question in Column I with the correct procedure in Column II to answer the question.

I	II
5. If the initial amount of Strontium-90 is 200 g, how much will remain after 10 yr?	**A.** Solve $0.75y_0 = y_0e^{-0.0241t}$.
6. If the initial amount of Strontium-90 is 200 g, how much will remain after 20 yr?	**B.** Evaluate $y = 200e^{-0.0241(10)}$.
7. What is the half-life of Strontium-90?	**C.** Solve $\frac{1}{2}y_0 = y_0e^{-0.0241t}$.
8. How long will it take for any amount of Strontium-90 to decay to 75% of its initial amount?	**D.** Evaluate $y = 200e^{-0.0241(20)}$.

(Modeling) *The exercises in this set are grouped according to discipline. They involve exponential or logarithmic models.* ***See Examples 1–6.***

Physical Sciences *(Exercises 9–28)*

An initial amount of a radioactive substance y_0 is given, along with information about the amount remaining after a given time t in appropriate units. For an equation of the form $y = y_0e^{kt}$ that models the situation, give the exact value of k in terms of natural logarithms.

9. $y_0 = 60$ g; After 3 hr, 20 g remain.

10. $y_0 = 30$ g; After 6 hr, 10 g remain.

11. $y_0 = 10$ mg; The half-life is 100 days.

12. $y_0 = 20$ mg; The half-life is 200 days.

13. $y_0 = 2.4$ lb; After 2 yr, 0.6 lb remains.

14. $y_0 = 8.1$ kg; After 4 yr, 0.9 kg remains.

Solve each problem.

15. ***Decay of Lead*** A sample of 500 g of radioactive lead-210 decays to polonium-210 according to the function

$$A(t) = 500e^{-0.032t},$$

where t is time in years. Find the amount of radioactive lead remaining after

(a) 4 yr, **(b)** 8 yr, **(c)** 20 yr. **(d)** Find the half-life.

16. ***Decay of Plutonium*** Repeat **Exercise 15** for 500 g of plutonium-241, which decays according to the function $A(t) = A_0e^{-0.053t}$, where t is time in years.

17. ***Decay of Radium*** Find the half-life of radium-226, which decays according to the function $A(t) = A_0e^{-0.00043t}$, where t is time in years.

18. ***Decay of Tritium*** Find the half-life of tritium, a radioactive isotope of hydrogen, which decays according to the function $A(t) = A_0e^{-0.056t}$, where t is time in years.

19. ***Radioactive Decay*** If 12 g of a radioactive substance are present initially and 4 yr later only 6.0 g remain, how much of the substance will be present after 7 yr?

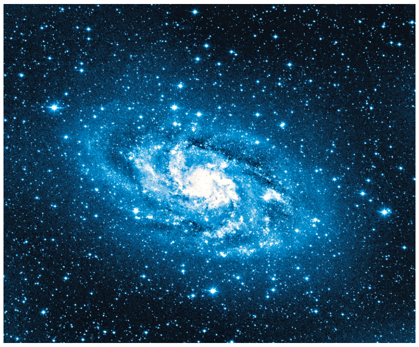
20. ***Radioactive Decay*** If 1 g of strontium-90 is present initially, and 2 yr later 0.95 g remains, how much strontium-90 will be present after 5 yr?

21. ***Decay of Iodine*** How long will it take any quantity of iodine-131 to decay to 25% of its initial amount, knowing that it decays according to the exponential function $A(t) = A_0e^{-0.087t}$, where t is time in days?

22. ***Magnitude of a Star*** The magnitude M of a star is modeled by

$$M = 6 - \frac{5}{2}\log\frac{I}{I_0},$$

where I_0 is the intensity of a just-visible star and I is the actual intensity of the star being measured. The dimmest stars are of magnitude 6, and the brightest are of magnitude 1. Determine the ratio of light intensities between a star of magnitude 1 and a star of magnitude 3.



23. ***Carbon-14 Dating*** Suppose an Egyptian mummy is discovered in which the amount of carbon-14 present is only about one-third the amount found in living human beings. How long ago did the Egyptian die?

24. ***Carbon-14 Dating*** A sample from a refuse deposit near the Strait of Magellan had 60% of the carbon-14 of a contemporary sample. How old was the sample?

25. ***Carbon-14 Dating*** Paint from the Lascaux caves of France contains 15% of the normal amount of carbon-14. Estimate the age of the paintings.

26. ***Dissolving a Chemical*** The amount of a chemical that will dissolve in a solution increases exponentially as the (Celsius) temperature t is increased according to the model

$$A(t) = 10e^{0.0095t}.$$

At what temperature will 15 g dissolve?

27. ***Newton's Law of Cooling*** Boiling water, at 100°C, is placed in a freezer at 0°C. The temperature of the water is 50°C after 24 min. Find the temperature of the water to the nearest hundredth after 96 min. (*Hint:* Change minutes to hours.)

28. ***Newton's Law of Cooling*** A piece of metal is heated to 300°C and then placed in a cooling liquid at 50°C. After 4 min, the metal has cooled to 175°C. Find its temperature to the nearest hundredth after 12 min. (*Hint:* Change minutes to hours.)

Finance *(Exercises 29–34)*

29. ***Comparing Investments*** Russ, who is self-employed, wants to invest \$60,000 in a pension plan. One investment offers 3% compounded quarterly. Another offers 2.75% compounded continuously.

(a) Which investment will earn more interest in 5 yr?

(b) How much more will the better plan earn?

30. ***Growth of an Account*** If Russ (see **Exercise 29**) chooses the plan with continuous compounding, how long will it take for his \$60,000 to grow to \$70,000?

31. ***Doubling Time*** Find the doubling time of an investment earning 2.5% interest if interest is compounded continuously.

32. ***Doubling Time*** If interest is compounded continuously and the interest rate is tripled, what effect will this have on the time required for an investment to double?

33. ***Growth of an Account*** How long will it take an investment to triple if interest is compounded continuously at 3%?

34. ***Growth of an Account*** Use the Table feature of a graphing calculator to find how long it will take \$1500 invested at 2.75% compounded daily to triple in value. Zoom in on the solution by systematically decreasing the increment for x. Find the answer to the nearest day. (Find the answer to the nearest day by eventually letting the increment of x equal $\frac{1}{365}$. The decimal part of the solution can be multiplied by 365 to determine the number of days greater than the nearest year. For example, if the solution is determined to be 16.2027 yr, then multiply 0.2027 by 365 to get 73.9855. The solution is then, to the nearest day, 16 yr, 74 days.) Confirm the answer algebraically.

Social Sciences *(Exercises 35–44)*

35. ***Legislative Turnover*** The turnover of legislators is a problem of interest to political scientists. It was found that one model of legislative turnover in a particular body was

$$M(t) = 434e^{-0.08t},$$

where $M(t)$ represents the number of continuously serving members at time t. Here, $t = 0$ represents 1965, $t = 1$ represents 1966, and so on. Use this model to approximate the number of continuously serving members in each year.

(a) 1969 **(b)** 1973 **(c)** 1979

36. ***Legislative Turnover*** Use the model in **Exercise 35** to determine the year in which the number of continuously serving members was 338.

37. ***Population Growth*** In 2000 India's population reached 1 billion, and it is projected to be 1.4 billion in 2025. (*Source:* U.S. Census Bureau.)

(a) Find values for P_0 and a so that $P(x) = P_0 a^{x-2000}$ models the population of India in year x. Round a to five decimal places.

(b) Predict India's population in 2020 to the nearest tenth of a billion.

(c) In what year is India's population expected to reach 1.5 billion?

38. ***Population Decline*** A midwestern city finds its residents moving to the suburbs. Its population is declining according to the function

$$P(t) = P_0 e^{-0.04t},$$

where t is time measured in years and P_0 is the population at time $t = 0$. Assume that $P_0 = 1{,}000{,}000$.

(a) Find the population at time $t = 1$ to the nearest thousand.

(b) How long, to the nearest tenth of a year, will it take for the population to decline to 750,000?

(c) How long, to the nearest tenth of a year, will it take for the population to decline to half the initial number?

39. ***Health Care Spending*** Out-of-pocket spending in the United States for health care increased between 2008 and 2012. The function

$$f(x) = 7446e^{0.0305x}$$

models average annual expenditures per household, in dollars. In this model, x represents the year, where $x = 0$ corresponds to 2008. (*Source*: U.S. Bureau of Labor Statistics.)

(a) Estimate out-of-pocket household spending on health care in 2012 to the nearest dollar.

(b) In what year did spending reach \$7915 per household?

40. ***Recreational Expenditures*** Personal consumption expenditures for recreation in billions of dollars in the United States during the years 2000–2013 can be approximated by the function

$$A(t) = 632.37e^{0.0351t},$$

where $t = 0$ corresponds to the year 2000. Based on this model, how much were personal consumption expenditures in 2013 to the nearest billion? (*Source:* U.S. Bureau of Economic Analysis.)

41. ***Housing Costs*** Average annual per-household spending on housing over the years 2000–2012 is approximated by

$$H = 12{,}744e^{0.0264t},$$

where t is the number of years since 2000. Find H to the nearest dollar for each year. (*Source:* U.S. Bureau of Labor Statistics.)

(a) 2005 **(b)** 2009 **(c)** 2012

42. ***Evolution of Language*** The number of years, n, since two independently evolving languages split off from a common ancestral language is approximated by

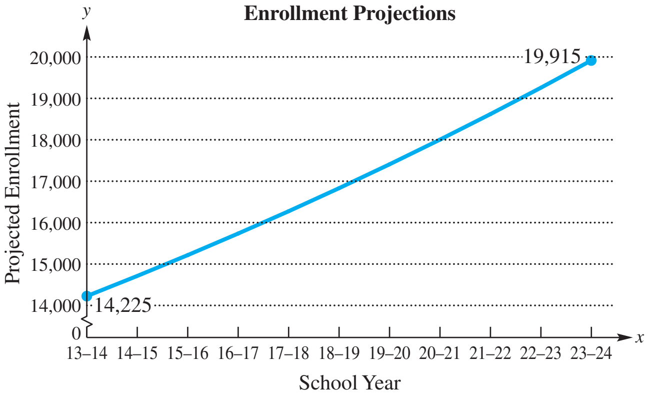
$$n \approx -7600 \log r,$$

where r is the proportion of words from the ancestral language common to both languages. Find each of the following to the nearest year.

(a) Find n if $r = 0.9$. **(b)** Find n if $r = 0.3$.

(c) How many years have elapsed since the split if half of the words of the ancestral language are common to both languages?

43. ***School District Growth*** Student enrollment in the Wentzville School District, one of the fastest-growing school districts in the state of Missouri, has projected growth as shown in the graph.



Source: Wentzville School District.

(a) Use the model $y = y_0 e^{kx}$ to find an exponential function that gives the projected enrollment y in school year x. Let the school year 2013–14 correspond to $x = 0$, 2014–15 correspond to $x = 1$, and so on, and use the two points indicated on the graph.

(b) Estimate the school year for which projected enrollment will be 21,500 students.

44. ***YouTube Views*** The number of views of a YouTube video increases after the number of hours posted as shown in the table.

Hour	Number of Views
20	100
25	517
30	2015
35	10,248

(a) Use the model $y = y_0 e^{kx}$ to find an exponential function that gives projected number of views y after number of hours x. Let hour 20 correspond to $x = 0$, hour 25 correspond to $x = 5$, and so on, and use the first and last data values given in the table.

(b) Estimate the number of views after 50 hr.

Life Sciences *(Exercises 45–50)*

45. ***Spread of Disease*** During an epidemic, the number of people who have never had the disease and who are not immune (they are *susceptible*) decreases exponentially according to the function

$$f(t) = 15{,}000e^{-0.05t},$$

where t is time in days. Find the number of susceptible people at each time.

(a) at the beginning of the epidemic **(b)** after 10 days **(c)** after 3 weeks

46. ***Spread of Disease*** Refer to **Exercise 45** and determine how long it will take, to the nearest day, for the initial number of people susceptible to decrease to half its amount.

47. ***Growth of Bacteria*** The growth of bacteria makes it necessary to time-date some food products so that they will be sold and consumed before the bacteria count is too high. Suppose for a certain product the number of bacteria present is given by

$$f(t) = 500e^{0.1t},$$

where t is time in days and the value of $f(t)$ is in millions. Find the number of bacteria, in millions, present at each time.

(a) 2 days **(b)** 4 days **(c)** 1 week

48. ***Growth of Bacteria*** How long will it take the bacteria population in **Exercise 47** to double? Round the answer to the nearest tenth.

49. ***Medication Effectiveness*** Drug effectiveness decreases over time. If, each hour, a drug is only 90% as effective as the previous hour, at some point the patient will not be receiving enough medication and must receive another dose. If the initial dose was 200 mg and the drug was administered 3 hr ago, the expression $200(0.90)^3$, which equals 145.8, represents the amount of effective medication still in the system. (The exponent is equal to the number of hours since the drug was administered.)

The amount of medication still available in the system is given by the function

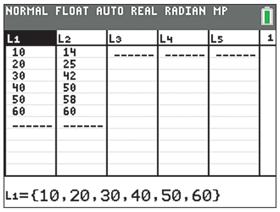
$$f(t) = 200(0.90)^t.$$

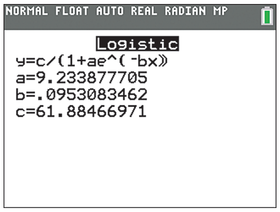
In this model, t is in hours and $f(t)$ is in milligrams. How long will it take for this initial dose to reach the dangerously low level of 50 mg? Round the answer to the nearest tenth.

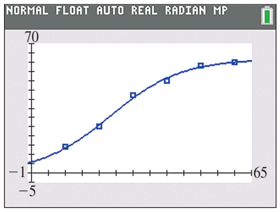
50. ***Population Size*** Many environmental situations place effective limits on the growth of the number of an organism in an area. Many such limited-growth situations are described by the **logistic function**

$$G(x) = \frac{MG_0}{G_0 + (M - G_0)e^{-kMx}},$$

where G_0 is the initial number present, M is the maximum possible size of the population, and k is a positive constant. The screens illustrate a typical logistic function calculation and graph.







Assume that $G_0 = 100$, $M = 2500$, $k = 0.0004$, and $x =$ time in decades (10-yr periods).

(a) Use a calculator to graph the function, using $0 \le x \le 8$ and $0 \le y \le 2500$.

(b) Estimate the value of $G(2)$ from the graph. Then evaluate $G(2)$ algebraically to find the population after 20 yr.

(c) Find the x-coordinate of the intersection of the curve with the horizontal line $y = 1000$ to estimate the number of decades required for the population to reach 1000. Then solve $G(x) = 1000$ algebraically to obtain the exact value of x.

Economics *(Exercises 51–56)*

51. ***Consumer Price Index*** The U.S. Consumer Price Index for the years 1990–2013 is approximated by

$$A(t) = 100e^{0.0264t},$$

where t represents the number of years after 1990. (Since $A(16)$ is about 153, the amount of goods that could be purchased for \$100 in 1990 cost about \$153 in 2006.) Use the function to determine the year in which costs will be 125% higher than in 1990. (*Source:* U.S. Bureau of Labor Statistics.)

52. ***Product Sales*** Sales of a product, under relatively stable market conditions but in the absence of promotional activities such as advertising, tend to decline at a constant yearly rate. This rate of sales decline varies considerably from product to product, but it seems to remain the same for any particular product. The sales decline can be expressed by the function

$$S(t) = S_0 e^{-at},$$

where $S(t)$ is the rate of sales at time t measured in years, S_0 is the rate of sales at time $t = 0$, and a is the sales decay constant.

(a) Suppose the sales decay constant for a particular product is $a = 0.10$. Let $S_0 = 50{,}000$ and find $S(1)$ and $S(3)$ to the nearest thousand.

(b) Find $S(2)$ and $S(10)$ to the nearest thousand if $S_0 = 80{,}000$ and $a = 0.05$.

53. ***Product Sales*** Use the sales decline function given in **Exercise 52.** If $a = 0.10$, $S_0 = 50{,}000$, and t is time measured in years, find the number of years it will take for sales to fall to half the initial sales. Round the answer to the nearest tenth.

54. ***Cost of Bread*** Assume the cost of a loaf of bread is \$4. With continuous compounding, find the number of years, to the nearest tenth, it would take for the cost to triple at an annual inflation rate of 4%.

55. ***Electricity Consumption*** Suppose that in a certain area the consumption of electricity has increased at a continuous rate of 6% per year. If it continued to increase at this rate, find the number of years, to the nearest tenth, before twice as much electricity would be needed.

56. ***Electricity Consumption*** Suppose a conservation campaign, together with higher rates, caused demand for electricity to increase at only 2% per year. (See **Exercise 55.**) Find the number of years, to the nearest tenth, before twice as much electricity would be needed.

(Modeling) *Solve each problem that uses a logistic function.*

57. ***Heart Disease*** As age increases, so does the likelihood of coronary heart disease (CHD). The fraction of people x years old with some CHD is modeled by

$$f(x)=\frac{0.9}{1+271e^{-0.122x}}.$$

(*Source:* Hosmer, D., and S. Lemeshow, *Applied Logistic Regression,* John Wiley and Sons.)

(a) Evaluate $f(25)$ and $f(65)$ to the nearest hundredth. Interpret the results.

(b) At what age, to the nearest year, does this likelihood equal 50%?

58. ***Tree Growth*** The height of a certain tree in feet after x years is modeled by

$$f(x)=\frac{50}{1+47.5e^{-0.22x}}.$$

(a) Make a table for f starting at $x=10$, and incrementing by 10. What appears to be the maximum height of the tree?

(b) Graph f and identify the horizontal asymptote. Explain its significance.

(c) After how many years was the tree 30 ft tall? Round to the nearest tenth.

Summary Exercises on Functions: Domains and Defining Equations

Finding the Domain of a Function: A Summary To find the domain of a function, given the equation that defines the function, remember that the value of x input into the equation must yield a real number for y when the function is evaluated. For the functions studied so far in this book, there are three cases to consider when determining domains.

Guidelines for Domain Restrictions

1. No input value can lead to 0 in a denominator, because division by 0 is undefined.
2. No input value can lead to an even root of a negative number, because this situation does not yield a real number.
3. No input value can lead to the logarithm of a negative number or 0, because this situation does not yield a real number.

Unless otherwise specified, we determine domains as follows.

- The domain of a **polynomial function** is the set of all real numbers.
- The domain of an **absolute value function** is the set of all real numbers for which the expression inside the absolute value bars (the argument) is defined.
- If a **function is defined by a rational expression,** the domain is the set of all real numbers for which the denominator is not zero.
- The domain of a **function defined by a radical with *even* root index** is the set of all real numbers that make the radicand greater than or equal to zero.

  If the root index is *odd*, the domain is the set of all real numbers for which the radicand is itself a real number.
- For an **exponential function** with constant base, the domain is the set of all real numbers for which the exponent is a real number.
- For a **logarithmic function,** the domain is the set of all real numbers that make the argument of the logarithm greater than zero.

Determining Whether an Equation Defines y as a Function of x

For y to be a function of x, it is necessary that every input value of x in the domain leads to one and only one value of y.

To determine whether an equation such as

$$x - y^3 = 0 \quad \text{or} \quad x - y^2 = 0$$

represents a function, solve the equation for y. In the first equation above, doing so leads to

$$y = \sqrt[3]{x}.$$

Notice that every value of x in the domain (that is, all real numbers) leads to one and only one value of y. So in the first equation, we can write y as a function of x. However, in the second equation above, solving for y leads to

$$y = \pm\sqrt{x}.$$

If we let $x = 4$, for example, we get two values of y: -2 and 2. Thus, in the second equation, we cannot write y as a function of x.

EXERCISES

Find the domain of each function. Write answers using interval notation.

1. $f(x) = 3x - 6$ **2.** $f(x) = \sqrt{2x - 7}$ **3.** $f(x) = |x + 4|$

4. $f(x) = \dfrac{x + 2}{x - 6}$ **5.** $f(x) = \dfrac{-2}{x^2 + 7}$ **6.** $f(x) = \sqrt{x^2 - 9}$

7. $f(x) = \dfrac{x^2 + 7}{x^2 - 9}$ **8.** $f(x) = \sqrt[3]{x^3 + 7x - 4}$ **9.** $f(x) = \log_5(16 - x^2)$

10. $f(x) = \log \dfrac{x + 7}{x - 3}$ **11.** $f(x) = \sqrt{x^2 - 7x - 8}$ **12.** $f(x) = 2^{1/x}$

13. $f(x) = \dfrac{1}{2x^2 - x + 7}$ **14.** $f(x) = \dfrac{x^2 - 25}{x + 5}$ **15.** $f(x) = \sqrt{x^3 - 1}$

16. $f(x) = \ln |x^2 - 5|$ **17.** $f(x) = e^{x^2+x+4}$ **18.** $f(x) = \dfrac{x^3 - 1}{x^2 - 1}$

19. $f(x) = \sqrt{\dfrac{-1}{x^3 - 1}}$ **20.** $f(x) = \sqrt[3]{\dfrac{1}{x^3 - 8}}$

21. $f(x) = \ln(x^2 + 1)$ **22.** $f(x) = \sqrt{(x-3)(x+2)(x-4)}$

23. $f(x) = \log\left(\dfrac{x+2}{x-3}\right)^2$ **24.** $f(x) = \sqrt[12]{(4-x)^2(x+3)}$

25. $f(x) = e^{|1/x|}$ **26.** $f(x) = \dfrac{1}{|x^2 - 7|}$

27. $f(x) = x^{100} - x^{50} + x^2 + 5$ **28.** $f(x) = \sqrt{-x^2 - 9}$

29. $f(x) = \sqrt[4]{16 - x^4}$ **30.** $f(x) = \sqrt[3]{16 - x^4}$

31. $f(x) = \sqrt{\dfrac{x^2 - 2x - 63}{x^2 + x - 12}}$ **32.** $f(x) = \sqrt[5]{5 - x}$

33. $f(x) = \left|\sqrt{5 - x}\right|$ **34.** $f(x) = \sqrt{\dfrac{-1}{x - 3}}$

35. $f(x) = \log\left|\dfrac{1}{4 - x}\right|$ **36.** $f(x) = 6^{x^2 - 9}$

37. $f(x) = 6^{\sqrt{x^2 - 25}}$ **38.** $f(x) = 6^{\sqrt[3]{x^2 - 25}}$

39. $f(x) = \ln\left(\dfrac{-3}{(x+2)(x-6)}\right)$ **40.** $f(x) = \dfrac{-2}{\log x}$

Determine which one of the choices (A, B, C, or D) is an equation in which y can be written as a function of x.

41. **A.** $3x + 2y = 6$ **B.** $x = \sqrt{|y|}$ **C.** $x = |y + 3|$ **D.** $x^2 + y^2 = 9$

42. **A.** $3x^2 + 2y^2 = 36$ **B.** $x^2 + y - 2 = 0$ **C.** $x - |y| = 0$ **D.** $x = y^2 - 4$

43. **A.** $x = \sqrt{y^2}$ **B.** $x = \log y^2$ **C.** $x^3 + y^3 = 5$ **D.** $x = \dfrac{1}{y^2 + 3}$

44. **A.** $\dfrac{x^2}{4} + \dfrac{y^2}{4} = 1$ **B.** $x = 5y^2 - 3$ **C.** $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ **D.** $x = 10^y$

45. **A.** $x = \dfrac{2 - y}{y + 3}$ **B.** $x = \ln(y + 1)^2$ **C.** $\sqrt{x} = |y + 1|$ **D.** $\sqrt[4]{x} = y^2$

46. **A.** $e^{y^2} = x$ **B.** $e^{y+2} = x$ **C.** $e^{|y|} = x$ **D.** $10^{|y+2|} = x$

47. **A.** $x^2 = \dfrac{1}{y^2}$ **B.** $x + 2 = \dfrac{1}{y^2}$ **C.** $3x = \dfrac{1}{y^4}$ **D.** $2x = \dfrac{1}{y^3}$

48. **A.** $|x| = |y|$ **B.** $x = |y^2|$ **C.** $x = \dfrac{1}{y}$ **D.** $x^4 + y^4 = 81$

49. **A.** $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ **B.** $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$ **C.** $\dfrac{x}{4} - \dfrac{y}{9} = 0$ **D.** $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 0$

50. **A.** $y^2 - \sqrt{(x + 2)^2} = 0$ **B.** $y - \sqrt{(x + 2)^2} = 0$

C. $y^6 - \sqrt{(x + 1)^2} = 0$ **D.** $y^4 - \sqrt{x^2} = 0$

Chapter 4 Test Prep

Key Terms

4.1 one-to-one function
inverse function

4.2 exponential function
exponential equation
compound interest
future value
present value
compound amount
continuous compounding

4.3 logarithm
base
argument
logarithmic equation
logarithmic function

4.4 common logarithm
pH
natural logarithm

4.6 doubling time
half-life

New Symbols

$f^{-1}(x)$	inverse of $f(x)$
e	a constant, approximately 2.718281828459045
$\log_a x$	logarithm of x with the base a
$\log x$	common (base 10) logarithm of x
$\ln x$	natural (base e) logarithm of x

Quick Review

Concepts	Examples

4.1 Inverse Functions

One-to-One Function

In a one-to-one function, each x-value corresponds to only one y-value, and each y-value corresponds to only one x-value.

A function f is one-to-one if, for elements a and b in the domain of f,

$$a \neq b \quad \text{implies} \quad f(a) \neq f(b).$$

Example: The function $y = f(x) = x^2$ is not one-to-one, because $y = 16$, for example, corresponds to both $x = 4$ and $x = -4$.

Horizontal Line Test

A function is one-to-one if every horizontal line intersects the graph of the function at most once.

Example: The graph of $f(x) = 2x - 1$ is a straight line with slope 2. f is a one-to-one function by the horizontal line test.

Inverse Functions

Let f be a one-to-one function. Then g is the inverse function of f if

$$(f \circ g)(x) = x \quad \text{for every } x \text{ in the domain of } g$$

and

$$(g \circ f)(x) = x \quad \text{for every } x \text{ in the domain of } f.$$

To find $g(x)$, interchange x and y in $y = f(x)$, solve for y, and replace y with $g(x)$, which is $f^{-1}(x)$.

Example: Find the inverse of f.

$f(x) = 2x - 1$	Given function
$y = 2x - 1$	Let $y = f(x)$.
$x = 2y - 1$	Interchange x and y.
$y = \frac{x+1}{2}$	Solve for y.
$f^{-1}(x) = \frac{x+1}{2}$	Replace y with $f^{-1}(x)$.
$f^{-1}(x) = \frac{1}{2}x + \frac{1}{2}$	$\frac{x+1}{2} = \frac{x}{2} + \frac{1}{2} = \frac{1}{2}x + \frac{1}{2}$

Concepts	Examples

4.2 Exponential Functions

Additional Properties of Exponents
For any real number $a > 0$, $a \neq 1$, the following hold true.

(a) a^x is a unique real number for all real numbers x.

(b) $a^b = a^c$ if and only if $b = c$.

(c) If $a > 1$ and $m < n$, then $a^m < a^n$.

(d) If $0 < a < 1$ and $m < n$, then $a^m > a^n$.

Examples:

(a) 2^x is a unique real number for all real numbers x.

(b) $2^x = 2^3$ if and only if $x = 3$.

(c) $2^5 < 2^{10}$, because $2 > 1$ and $5 < 10$.

(d) $\left(\frac{1}{2}\right)^5 > \left(\frac{1}{2}\right)^{10}$ because $0 < \frac{1}{2} < 1$ and $5 < 10$.

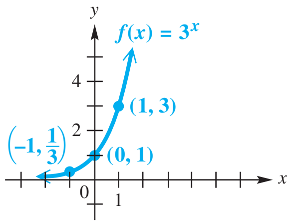
Exponential Function
If $a > 0$ and $a \neq 1$, then the exponential function with base a is $\boldsymbol{f(x) = a^x}$.

Graph of $f(x) = a^x$

1. The points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$ are on the graph.
2. If $a > 1$, then f is an increasing function.
   If $0 < a < 1$, then f is a decreasing function.
3. The x-axis is a horizontal asymptote.
4. The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

$f(x) = 3^x$ is the exponential function with base 3.

x	y
-1	$\frac{1}{3}$
0	1
1	3



4.3 Logarithmic Functions

Logarithm
For all real numbers y and all positive numbers a and x, where $a \neq 1$, $\boldsymbol{y = \log_a x}$ is equivalent to $\boldsymbol{x = a^y}$.

$\log_3 81 = 4$ is equivalent to $3^4 = 81$.

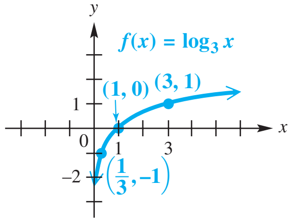
Logarithmic Function
If $a > 0$, $a \neq 1$, and $x > 0$, then the logarithmic function with base a is $\boldsymbol{f(x) = \log_a x}$.

Graph of $f(x) = \log_a x$

1. The points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$ are on the graph.
2. If $a > 1$, then f is an increasing function.
   If $0 < a < 1$, then f is a decreasing function.
3. The y-axis is a vertical asymptote.
4. The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$.

$f(x) = \log_3 x$ is the logarithmic function with base 3.

x	y
$\frac{1}{3}$	-1
1	0
3	1



Properties of Logarithms
For $x > 0$, $y > 0$, $a > 0$, $a \neq 1$, and any real number r, the following properties hold.

Property	Name	Example
$\log_a xy = \log_a x + \log_a y$	Product property	$\log_2 (3 \cdot 5) = \log_2 3 + \log_2 5$
$\log_a \frac{x}{y} = \log_a x - \log_a y$	Quotient property	$\log_2 \frac{3}{5} = \log_2 3 - \log_2 5$
$\log_a x^r = r \log_a x$	Power property	$\log_6 3^5 = 5 \log_6 3$
$\log_a 1 = 0$	Logarithm of 1	$\log_{10} 1 = 0$
$\log_a a = 1$	Base a logarithm of a	$\log_{10} 10 = 1$

Theorem on Inverses
For $a > 0$ and $a \neq 1$, the following properties hold.

$$a^{\log_a x} = x \quad (x > 0) \quad \text{and} \quad \log_a a^x = x$$

$$2^{\log_2 5} = 5 \quad \text{and} \quad \log_2 2^5 = 5$$

Concepts	Examples

4.4 Evaluating Logarithms and the Change-of-Base Theorem

Common and Natural Logarithms

For all positive numbers x, base 10 logarithms and base e logarithms are written as follows.

$$\log x = \log_{10} x \quad \text{Common logarithm}$$
$$\ln x = \log_e x \quad \text{Natural logarithm}$$

Approximate log 0.045 and ln 247.1.

$$\log 0.045 \approx -1.3468$$
$$\ln 247.1 \approx 5.5098$$
Use a calculator.

Change-of-Base Theorem

For any positive real numbers x, a, and b, where $a \neq 1$ and $b \neq 1$, the following holds.

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Approximate $\log_8 7$.

$$\log_8 7 = \frac{\log 7}{\log 8} = \frac{\ln 7}{\ln 8} \approx 0.9358 \quad \text{Use a calculator.}$$

4.5 Exponential and Logarithmic Equations

Property of Logarithms

If $x > 0$, $y > 0$, $a > 0$, and $a \neq 1$, then the following holds.

$$x = y \quad \text{is equivalent to} \quad \log_a x = \log_a y.$$

Solve.

$$e^{5x} = 10$$
$$\ln e^{5x} = \ln 10 \quad \text{Take natural logarithms.}$$
$$5x = \ln 10 \quad \ln e^x = x, \text{ for all } x.$$
$$x = \frac{\ln 10}{5} \quad \text{Divide by 5.}$$
$$x \approx 0.461 \quad \text{Use a calculator.}$$

The solution set can be written with the exact value, $\left\{\frac{\ln 10}{5}\right\}$, or with the approximate value, $\{0.461\}$.

$$\log_2(x^2 - 3) = \log_2 6$$
$$x^2 - 3 = 6 \quad \text{Property of logarithms}$$
$$x^2 = 9 \quad \text{Add 3.}$$
$$x = \pm 3 \quad \text{Take square roots.}$$

Both values check, so the solution set is $\{\pm 3\}$.

4.6 Applications and Models of Exponential Growth and Decay

Exponential Growth or Decay Function

Let y_0 be the amount or number present at time $t = 0$. Then, under certain conditions, the amount present at any time t is modeled by

$$y = y_0 e^{kt}, \quad \text{where } k \text{ is a constant.}$$

The formula for continuous compounding,

$$A = Pe^{rt},$$

is an example of exponential growth. Here, A is the compound amount if P dollars are invested at an annual interest rate r for t years.

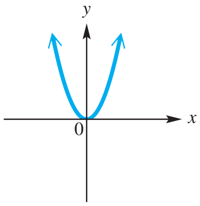
If $P = \$200$, $r = 3\%$, and $t = 5$ yr, find A.

$$A = Pe^{rt}$$
$$A = 200e^{0.03(5)} \quad \text{Substitute.}$$
$$A \approx \$232.37 \quad \text{Use a calculator.}$$

Chapter 4 Review Exercises

Determine whether each function as graphed or defined is one-to-one.

1.



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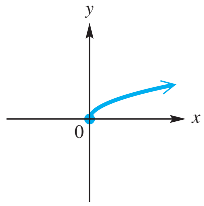

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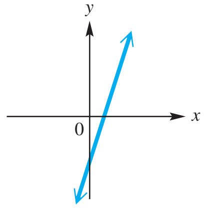


4. $y = x^3 + 1$

5. $y = (x + 3)^2$

6. $y = \sqrt{3x^2 + 2}$

Find the inverse of each function that is one-to-one.

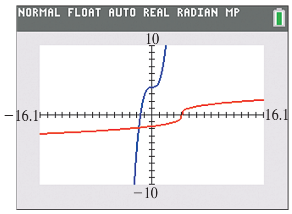
7. $f(x) = x^3 - 3$

8. $f(x) = \sqrt{25 - x^2}$

Concept Check *Work each problem.*

9. Suppose $f(t)$ is the amount an investment will grow to t years after 2004. What does $f^{-1}(\$50{,}000)$ represent?

10. The graphs of two functions are shown. Based on their graphs, are these functions inverses?



11. To have an inverse, a function must be a(n) ________ function.

12. *True* or *false*? The x-coordinate of the x-intercept of the graph of $y = f(x)$ is the y-coordinate of the y-intercept of the graph of $y = f^{-1}(x)$.

Match each equation with the figure that most closely resembles its graph.

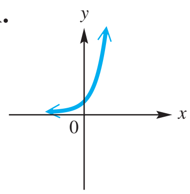
13. $y = \log_{0.3} x$

14. $y = e^x$

15. $y = \ln x$

16. $y = 0.3^x$

A.



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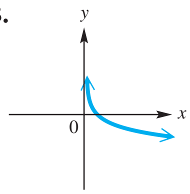

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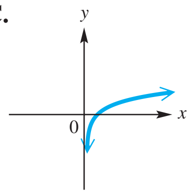

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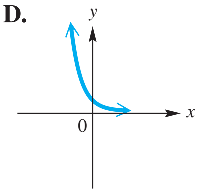


Write each equation in logarithmic form.

17. $2^5 = 32$

18. $100^{1/2} = 10$

19. $\left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$

20. Graph $f(x) = \left(\frac{1}{5}\right)^{x+2} - 1$. Give the domain and range.

Write each equation in exponential form.

21. $\log 1000 = 3$

22. $\log_9 27 = \frac{3}{2}$

23. $\ln \sqrt{e} = \frac{1}{2}$

24. *Concept Check* What is the base of the logarithmic function whose graph contains the point $(81, 4)$?

25. *Concept Check* What is the base of the exponential function whose graph contains the point $\left(-4, \frac{1}{16}\right)$?

Use properties of logarithms to rewrite each expression. Simplify the result if possible. Assume all variables represent positive real numbers.

26. $\log_5\left(x^2y^4\sqrt[5]{m^3p}\right)$ **27.** $\log_3 \dfrac{mn}{5r}$ **28.** $\log_7(7k + 5r^2)$

Use a calculator to find an approximation to four decimal places for each logarithm.

29. $\log 0.0411$ **30.** $\log 45.6$ **31.** $\ln 144{,}000$

32. $\ln 470$ **33.** $\log_{2/3} \dfrac{5}{8}$ **34.** $\log_3 769$

Solve each equation. Unless otherwise specified, give irrational solutions as decimals correct to the nearest thousandth.

35. $16^{x+4} = 8^{3x-2}$ **36.** $4^x = 12$ **37.** $3^{2x-5} = 13$

38. $2^{x+3} = 5^x$ **39.** $6^{x+3} = 4^x$ **40.** $e^{x-1} = 4$

41. $e^{2-x} = 12$ **42.** $2e^{5x+2} = 8$ **43.** $10e^{3x-7} = 5$

44. $5^{x+2} = 2^{2x-1}$ **45.** $6^{x-3} = 3^{4x+1}$ **46.** $e^{8x} \cdot e^{2x} = e^{20}$

47. $e^{6x} \cdot e^x = e^{21}$ **48.** $100(1.02)^{x/4} = 200$ **49.** $2e^{2x} - 5e^x - 3 = 0$ (Give exact form.)

50. $\left(\dfrac{1}{2}\right)^x + 2 = 0$ **51.** $4(1.06)^x + 2 = 8$

52. *Concept Check* Which one or more of the following choices is the solution set of $5^x = 9$?

A. $\{\log_5 9\}$ **B.** $\{\log_9 5\}$ **C.** $\left\{\dfrac{\log 9}{\log 5}\right\}$ **D.** $\left\{\dfrac{\ln 9}{\ln 5}\right\}$

Solve each equation. Give solutions in exact form.

53. $3 \ln x = 13$ **54.** $\ln 5x = 16$

55. $\log(2x + 7) = 0.25$ **56.** $\ln x + \ln x^3 = 12$

57. $\log_2(x^3 + 5) = 5$ **58.** $\log_3(x^2 - 9) = 3$

59. $\log_4[(3x + 1)(x - 4)] = 2$ **60.** $\ln e^{\ln x} - \ln(x - 4) = \ln 3$

61. $\log x + \log(13 - 3x) = 1$ **62.** $\log_7(3x + 2) - \log_7(x - 2) = 1$

63. $\ln(6x) - \ln(x + 1) = \ln 4$ **64.** $\log_{16}\sqrt{x + 1} = \dfrac{1}{4}$

65. $\ln[\ln e^{-x}] = \ln 3$ **66.** $S = a \ln\left(1 + \dfrac{n}{a}\right)$, for n

67. $d = 10 \log \dfrac{I}{I_0}$, for I_0 **68.** $D = 200 + 100 \log x$, for x

69. Use a graphing calculator to solve the equation $e^x = 4 - \ln x$. Give solution(s) to the nearest thousandth.

Solve each problem.

70. ***(Modeling) Decibel Levels*** Decibel rating of the loudness of a sound is modeled by

$$d = 10 \log \frac{I}{I_0},$$

where I is the intensity of a particular sound, and I_0 is the intensity of a very faint threshold sound. A few years ago, there was a controversy about a proposed government limit on factory noise. One group wanted a maximum of 89 decibels, while another group wanted 86. Find the percent by which the 89-decibel intensity exceeds that for 86 decibels.

71. ***Earthquake Intensity*** The magnitude of an earthquake, measured on the Richter scale, is $\log \frac{I}{I_0}$, where I is the amplitude registered on a seismograph 100 km from the epicenter of the earthquake, and I_0 is the amplitude of an earthquake of a certain (small) size. On August 24, 2014, the Napa Valley in California was shaken by an earthquake that measured 6.0 on the Richter scale.

(a) Express this reading in terms of I_0.

(b) On April 1, 2014, a quake measuring 8.2 on the Richter scale struck off the coast of Chile. It was the largest earthquake in 2014. Express the magnitude of an 8.2 reading in terms of I_0 to the nearest hundred thousand.

(c) How much greater than the force of the 6.0 earthquake was the force of the earthquake that measured 8.2?

72. ***Earthquake Intensity*** The San Francisco earthquake of 1906 had a Richter scale rating of 8.3.

(a) Express the magnitude of this earthquake in terms of I_0 to the nearest hundred thousand.

(b) In 1989, the San Francisco region experienced an earthquake with a Richter scale rating of 7.1. Express the magnitude of this earthquake in terms of I_0 to the nearest hundred thousand.

(c) Compare the magnitudes of the two San Francisco earthquakes discussed in parts (a) and (b).

73. ***Interest Rate*** What annual interest rate, to the nearest tenth, will produce \$4700 if \$3500 is left at interest compounded annually for 10 yr?

74. ***Growth of an Account*** Find the number of years (to the nearest tenth) needed for \$48,000 to become \$53,647 at 2.8% interest compounded semiannually.

75. ***Growth of an Account*** Manuel deposits \$10,000 for 12 yr in an account paying 3% interest compounded annually. He then puts this total amount on deposit in another account paying 4% interest compounded semiannually for another 9 yr. Find the total amount on deposit after the entire 21-yr period.

76. ***Growth of an Account*** Anne deposits \$12,000 for 8 yr in an account paying 2.5% interest compounded annually. She then leaves the money alone with no further deposits at 3% interest compounded annually for an additional 6 yr. Find the total amount on deposit after the entire 14-yr period.

77. ***Cost from Inflation*** Suppose the inflation rate is 4%. Use the formula for continuous compounding to find the number of years, to the nearest tenth, for a \$1 item to cost \$2.

78. ***(Modeling) Drug Level in the Bloodstream*** After a medical drug is injected directly into the bloodstream, it is gradually eliminated from the body. Graph the following functions on the interval $[0, 10]$. Use $[0, 500]$ for the range of $A(t)$. Determine the function that best models the amount $A(t)$ (in milligrams) of a drug remaining in the body after t hours if 350 mg were initially injected.

(a) $A(t) = t^2 - t + 350$ **(b)** $A(t) = 350 \log(t + 1)$

(c) $A(t) = 350(0.75)^t$ **(d)** $A(t) = 100(0.95)^t$

79. ***(Modeling) Chicago Cubs' Payroll*** The table shows the total payroll (in millions of dollars) of the Chicago Cubs baseball team for the years 2010–2014.

Year	Total Payroll (millions of dollars)
2010	145.4
2011	134.3
2012	111.0
2013	107.4
2014	92.7

Source: www.baseballprospectus.com/compensation

Letting $f(x)$ represent the total payroll and x represent the number of years since 2010, we find that the function

$$f(x) = 146.02e^{-0.112x}$$

models the data quite well. According to this function, when will the total payroll halve its 2010 value?

80. ***(Modeling) Transistors on Computer Chips*** Computing power has increased dramatically as a result of the ability to place an increasing number of transistors on a single processor chip. The table lists the number of transistors on some popular computer chips made by Intel.

Year	Chip	Transistors
1989	486DX	1,200,000
1994	Pentium	3,300,000
2000	Pentium 4	42,000,000
2006	Core 2 Duo	291,000,000
2008	Core 2 Quad	820,000,000
2010	Core (2nd gen.)	1,160,000,000
2012	Core (3rd gen.)	1,400,000,000

Source: Intel.

(a) Make a scatter diagram of the data. Let the x-axis represent the year, where $x = 0$ corresponds to 1989, and let the y-axis represent the number of transistors.

(b) Decide whether a linear, a logarithmic, or an exponential function best describes the data.

(c) Determine a function f that approximates these data. Plot f and the data on the same coordinate axes.

(d) Assuming that this trend continues, use f to estimate the number of transistors on a chip, to the nearest million, in the year 2016.

81. ***Financial Planning*** The traditional IRA (individual retirement account) is a common tax-deferred saving plan in the United States. Earned income deposited into an IRA is not taxed in the current year, and no taxes are incurred on the interest paid in subsequent years. However, when the money is withdrawn from the account after age $59\frac{1}{2}$, taxes must be paid on the entire amount withdrawn.

Suppose we deposited \$5000 of earned income into an IRA, we can earn an annual interest rate of 4%, and we are in a 25% tax bracket. (*Note:* Interest rates and tax brackets are subject to change over time, but some assumptions must be made to evaluate the investment.) Also, suppose that we deposit the \$5000 at age 25 and withdraw it at age 60, and that interest is compounded continuously.

(a) How much money will remain after we pay the taxes at age 60?

(b) Suppose that instead of depositing the money into an IRA, we pay taxes on the money and the annual interest. How much money will we have at age 60? (*Note:* We effectively start with \$3750 (75% of \$5000), and the money earns 3% (75% of 4%) interest after taxes.)

(c) To the nearest dollar, how much additional money will we earn with the IRA?

(d) Suppose we pay taxes on the original \$5000 but are then able to earn 4% in a tax-free investment. Compare the balance at age 60 with the IRA balance.

82. Consider $f(x) = \log_4(2x^2 - x)$.

(a) Use the change-of-base theorem with base e to write $\log_4(2x^2 - x)$ in a suitable form to graph with a calculator.

(b) Graph the function using a graphing calculator. Use the window $[-2.5, 2.5]$ by $[-5, 2.5]$.

(c) What are the x-intercepts?

(d) Give the equations of the vertical asymptotes.

(e) Why is there no y-intercept?

Chapter 4 Test

1. Consider the function $f(x) = \sqrt[3]{2x - 7}$.

(a) What are the domain and range of f?

(b) Explain why f^{-1} exists.

(c) Write an equation for $f^{-1}(x)$.

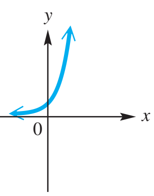
(d) What are the domain and range of f^{-1}?

(e) Graph both f and f^{-1}. How are the two graphs related with respect to the line $y = x$?

2. Match each equation with its graph.

(a) $y = \log_{1/3} x$ **(b)** $y = e^x$ **(c)** $y = \ln x$ **(d)** $y = \left(\frac{1}{3}\right)^x$

A.



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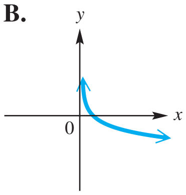

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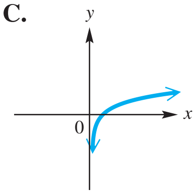

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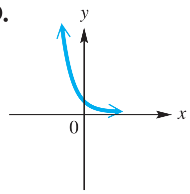

3. Solve $\left(\frac{1}{8}\right)^{2x-3} = 16^{x+1}$.

4. (a) Write $4^{3/2} = 8$ in logarithmic form.

(b) Write $\log_8 4 = \frac{2}{3}$ in exponential form.

5. Graph $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \log_{1/2} x$ on the same axes. What is their relationship?

6. Use properties of logarithms to rewrite the expression. Assume all variables represent positive real numbers.

$$\log_7 \frac{x^2\sqrt[4]{y}}{z^3}$$

Use a calculator to find an approximation to four decimal places for each logarithm.

7. $\log 2388$ **8.** $\ln 2388$ **9.** $\log_9 13$

10. Solve $x^{2/3} = 25$.

Solve each equation. Give irrational solutions as decimals correct to the nearest thousandth.

11. $12^x = 1$ **12.** $9^x = 4$ **13.** $16^{2x+1} = 8^{3x}$

14. $2^{x+1} = 3^{x-4}$ **15.** $e^{0.4x} = 4^{x-2}$

16. $2e^{2x} - 5e^x + 3 = 0$ (Give both exact and approximate values.)

Solve each equation. Give solutions in exact form.

17. $\log_x \frac{9}{16} = 2$ **18.** $\log_2 [(x-4)(x-2)] = 3$

19. $\log_2 x + \log_2 (x+2) = 3$ **20.** $\ln x - 4 \ln 3 = \ln \frac{1}{5}x$

21. $\log_3 (x+1) - \log_3 (x-3) = 2$

22. A friend is taking another mathematics course and says, "I have no idea what an expression like $\log_5 27$ really means." Write an explanation of what it means, and tell how we can find an approximation for it with a calculator.

Solve each problem.

23. ***(Modeling) Skydiver Fall Speed*** A skydiver in free fall travels at a speed modeled by

$$v(t) = 176(1 - e^{-0.18t})$$

feet per second after t seconds. How long, to the nearest second, will it take for the skydiver to attain a speed of 147 ft per sec (100 mph)?

24. ***Growth of an Account*** How many years, to the nearest tenth, will be needed for \$5000 to increase to \$18,000 at 3.0% annual interest compounded **(a)** monthly **(b)** continuously?

25. ***Tripling Time*** For *any* amount of money invested at 2.8% annual interest compounded continuously, how long, to the nearest tenth of a year, will it take to triple?

26. ***(Modeling) Radioactive Decay*** The amount of a certain radioactive material, in grams, present after t days is modeled by

$$A(t) = 600e^{-0.05t}.$$

(a) Find the amount present after 12 days, to the nearest tenth of a gram.

(b) Find the half-life of the material, to the nearest tenth of a day.