4 Exponential and Logarithmic Functions

Depreciation of Cars

You are ready to buy that first new car. You know that cars lose value over time due to depreciation and that different cars have different rates of depreciation. So you will research the depreciation rates for the cars you are thinking of buying. After all, for cars that sell for about the same price, the lower the depreciation rate is, the more the car will be worth each year.

—See the Internet-based Chapter Project I—

Outline

A Look Back

Until now, our study of functions has concentrated on polynomial and rational functions. These functions belong to the class of **algebraic functions**—that is, functions that can be expressed in terms of sums, differences, products, quotients, powers, or roots of polynomials. Functions that are not algebraic are called **transcendental** (they transcend, or go beyond, algebraic functions).

A Look Ahead

In this chapter, we study two transcendental functions: the exponential function and the logarithmic function. These functions occur frequently in a wide variety of applications, such as biology, chemistry, economics, and psychology.

The chapter begins with a discussion of composite, one-to-one, and inverse functions, concepts that are needed to explain the relationship between exponential and logarithmic functions.

4.1 Composite Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Find the Value of a Function (Section 1.1, pp. 47–49)
- Domain of a Function (Section 1.1, pp. 51–53)

Now Work the **'Are You Prepared?'** problems on page 286.

OBJECTIVES 1 Form a Composite Function (p. 281)
2 Find the Domain of a Composite Function (p. 282)

1 Form a Composite Function

Suppose that an oil tanker is leaking oil and you want to determine the area of the circular oil patch around the ship. See Figure 1. It is determined that the oil is leaking from the tanker in such a way that the radius of the circular patch of oil around the ship is increasing at a rate of 3 feet per minute. Therefore, the radius r of the oil patch at any time t, in minutes, is given by $r(t) = 3t$. So after 20 minutes, the radius of the oil patch is $r(20) = 3(20) = 60$ feet.

The area A of a circle as a function of the radius r is given by $A(r) = \pi r^2$. The area of the circular patch of oil after 20 minutes is $A(60) = \pi(60)^2 = 3600\pi$ square feet. Note that $60 = r(20)$, so $A(60) = A(r(20))$. The argument of the function A is the output of the function r!

In general, the area of the oil patch can be expressed as a function of time t by evaluating $A(r(t))$ and obtaining $A(r(t)) = A(3t) = \pi(3t)^2 = 9\pi t^2$. The function $A(r(t))$ is a special type of function called a *composite function.*

As another example, consider the function $y = (2x + 3)^2$. Let $y = f(u) = u^2$ and $u = g(x) = 2x + 3$. Then by a substitution process, the original function is obtained as follows: $y = f(u) = f(g(x)) = (2x + 3)^2$.

In general, suppose that f and g are two functions and that x is a number in the domain of g. Evaluating g at x yields $g(x)$. If $g(x)$ is in the domain of f, then evaluating f at $g(x)$ yields the expression $f(g(x))$. The correspondence from x to $f(g(x))$ is called a *composite function* $f \circ g$.

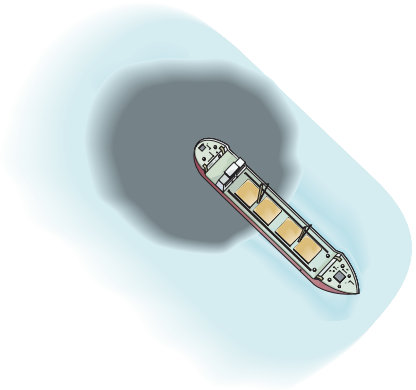


Figure 1

DEFINITION

Given two functions f and g, the **composite function,** denoted by $f \circ g$ (read as "f composed with g"), is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.

Look carefully at Figure 2. Only those values of x in the domain of g for which $g(x)$ is in the domain of f can be in the domain of $f \circ g$. The reason is that if $g(x)$ is not in the domain of f, then $f(g(x))$ is not defined. Because of this, the domain of $f \circ g$ is a subset of the domain of g; the range of $f \circ g$ is a subset of the range of f.

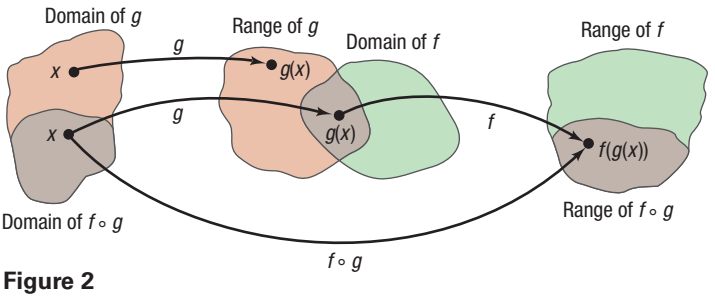


Figure 2

Figure 3 provides a second illustration of the definition. Here x is the input to the function g, yielding $g(x)$. Then $g(x)$ is the input to the function f, yielding $f(g(x))$. Note that the "inside" function g in $f(g(x))$ is "processed" first.

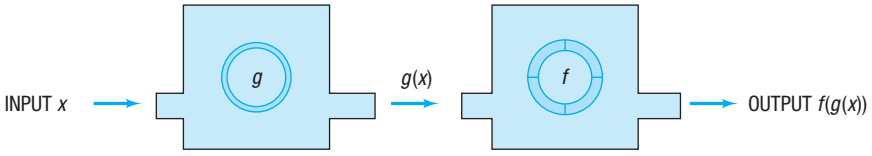


Figure 3

EXAMPLE 1 Evaluating a Composite Function

Suppose that $f(x) = 2x^2 - 3$ and $g(x) = 4x$. Find:

(a) $(f \circ g)(1)$ (b) $(g \circ f)(1)$ (c) $(f \circ f)(-2)$ (d) $(g \circ g)(-1)$

Solution

(a) $(f \circ g)(1) = f(g(1)) = f(4) = 2 \cdot 4^2 - 3 = 29$

$g(x) = 4x$ $f(x) = 2x^2 - 3$
$g(1) = 4$

(b) $(g \circ f)(1) = g(f(1)) = g(-1) = 4 \cdot (-1) = -4$

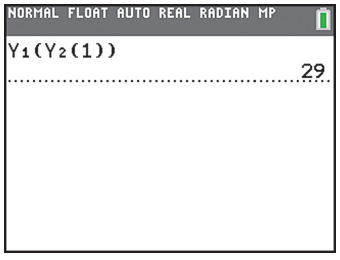
$f(x) = 2x^2 - 3$ $g(x) = 4x$
$f(1) = -1$

(c) $(f \circ f)(-2) = f(f(-2)) = f(5) = 2 \cdot 5^2 - 3 = 47$

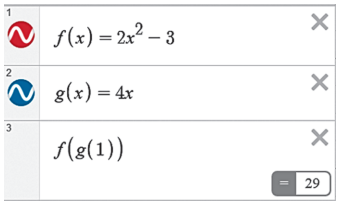
$f(-2) = 2(-2)^2 - 3 = 5$

(d) $(g \circ g)(-1) = g(g(-1)) = g(-4) = 4 \cdot (-4) = -16$

$g(-1) = -4$



(a) TI-84 Plus C



(b) Desmos

Figure 4

Source: *https://www.desmos.com/calculator/du7gj4mczy*

COMMENT Graphing utilities can be used to evaluate composite functions.* Using a TI-84 Plus C graphing calculator, let $Y_1 = f(x) = 2x^2 - 3$ and $Y_2 = g(x) = 4x$, and find $(f \circ g)(1)$ as shown in Figure 4(a). Using Desmos, find $(f \circ g)(1)$ as shown in Figure 4(b). Note that these give the result obtained in Example 1(a). ■

NOW WORK PROBLEM 13

2 Find the Domain of a Composite Function

EXAMPLE 2 Finding a Composite Function and Its Domain

Suppose that $f(x) = x^2 + 3x - 1$ and $g(x) = 2x + 3$.

Find: (a) $f \circ g$ (b) $g \circ f$

Then find the domain of each composite function.

Solution The domain of f and the domain of g are the set of all real numbers.

(a) $(f \circ g)(x) = f(g(x)) = f(2x + 3) = (2x + 3)^2 + 3(2x + 3) - 1$

$f(x) = x^2 + 3x - 1$

$= 4x^2 + 12x + 9 + 6x + 9 - 1 = 4x^2 + 18x + 17$

Because the domains of both f and g are the set of all real numbers, the domain of $f \circ g$ is the set of all real numbers.

*Consult your owner's manual for the appropriate keystrokes.

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 + 3x - 1) = 2(x^2 + 3x - 1) + 3$

$g(x) = 2x + 3$

$= 2x^2 + 6x - 2 + 3 = 2x^2 + 6x + 1$

Because the domains of both f and g are the set of all real numbers, the domain of $g \circ f$ is the set of all real numbers.

Example 2 illustrates that, in general, $f \circ g \neq g \circ f$. Sometimes $f \circ g$ does equal $g \circ f$, as we shall see in Example 5.

Look back at Figure 2 on page 281. In determining the domain of the composite function $(f \circ g)(x) = f(g(x))$, keep the following two thoughts in mind about the input x.

1. Any x not in the domain of g must be excluded.
2. Any x for which $g(x)$ is not in the domain of f must be excluded.

EXAMPLE 3 Finding the Domain of $f \circ g$

Find the domain of $f \circ g$ if $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{4}{x-1}$.

Solution For $(f \circ g)(x) = f(g(x))$, first note that the domain of g is $\{x | x \neq 1\}$, so 1 is excluded from the domain of $f \circ g$. Next note that the domain of f is $\{x | x \neq -2\}$, which means that $g(x)$ cannot equal -2. Solve the equation $g(x) = -2$ to determine what additional value(s) of x to exclude.

$$\frac{4}{x-1} = -2 \qquad g(x) = -2$$

$$4 = -2(x-1) \qquad \text{Multiply both sides by } x - 1.$$

$$4 = -2x + 2 \qquad \text{Apply the Distributive Property.}$$

$$2x = -2 \qquad \text{Add } 2x \text{ to both sides. Subtract 4 from both sides.}$$

$$x = -1 \qquad \text{Divide both sides by 2.}$$

Also exclude -1 from the domain of $f \circ g$.

The domain of $f \circ g$ is $\{x | x \neq -1, x \neq 1\}$.

✓**Check:** For $x = 1$, $g(x) = \dfrac{4}{x-1}$ is not defined, so $(f \circ g)(x) = f(g(x))$ is not defined.

For $x = -1$, $g(-1) = -2$, and $(f \circ g)(-1) = f(g(-1)) = f(-2)$ is not defined.

EXAMPLE 4 Finding a Composite Function and Its Domain

Suppose that $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{4}{x-1}$.

Find: (a) $f \circ g$ (b) $f \circ f$

Then find the domain of each composite function.

Solution The domain of f is $\{x | x \neq -2\}$ and the domain of g is $\{x | x \neq 1\}$.

(a) $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{4}{x-1}\right) = \dfrac{1}{\dfrac{4}{x-1} + 2} = \dfrac{x-1}{4 + 2(x-1)} = \dfrac{x-1}{2x+2} = \dfrac{x-1}{2(x+1)}$

$f(x) = \dfrac{1}{x+2}$ Multiply by $\dfrac{x-1}{x-1}$.

In Example 3, the domain of $f \circ g$ was found to be $\{x | x \neq -1, x \neq 1\}$.

(continued)

The domain of $f \circ g$ also can be found by first looking at the domain of g: $\{x \mid x \neq 1\}$. Exclude 1 from the domain of $f \circ g$ as a result. Then look at $f \circ g$ and note that x cannot equal -1, because $x = -1$ results in division by 0. So exclude -1 from the domain of $f \circ g$. Therefore, the domain of $f \circ g$ is $\{x \mid x \neq -1, x \neq 1\}$.

(b) $$(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x+2}\right) = \frac{1}{\frac{1}{x+2} + 2} = \frac{x+2}{1 + 2(x+2)} = \frac{x+2}{2x+5}$$

$f(x) = \dfrac{1}{x+2}$ Multiply by $\dfrac{x+2}{x+2}$.

The domain of $f \circ f$ consists of all values of x in the domain of f, $\{x \mid x \neq -2\}$, for which

$$f(x) = \frac{1}{x+2} \neq -2$$

$$\begin{aligned} \frac{1}{x+2} &= -2 \\ 1 &= -2(x+2) \\ 1 &= -2x - 4 \\ 2x &= -5 \\ x &= -\frac{5}{2} \end{aligned}$$

or, equivalently,

$$x \neq -\frac{5}{2}$$

The domain of $f \circ f$ is $\left\{x \,\middle|\, x \neq -\frac{5}{2}, x \neq -2\right\}$.

The domain of $f \circ f$ also can be found by recognizing that -2 is not in the domain of f and so should be excluded from the domain of $f \circ f$. Then, looking at $f \circ f$, note that x cannot equal $-\frac{5}{2}$. Do you see why? Therefore, the domain of $f \circ f$ is $\left\{x \,\middle|\, x \neq -\frac{5}{2}, x \neq -2\right\}$.

NOW WORK PROBLEMS 27 AND 29

EXAMPLE 5 Showing That Two Composite Functions Are Equal

If $f(x) = 3x - 4$ and $g(x) = \frac{1}{3}(x+4)$, show that

$$(f \circ g)(x) = (g \circ f)(x) = x$$

for every x in the domain of $f \circ g$ and $g \circ f$.

Solution

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{x+4}{3}\right) && g(x) = \frac{1}{3}(x+4) = \frac{x+4}{3} \\ &= 3\left(\frac{x+4}{3}\right) - 4 && f(x) = 3x - 4 \\ &= x + 4 - 4 = x \end{aligned}$$

Seeing the Concept

Using a graphing calculator, let

$$Y_1 = f(x) = 3x - 4$$

$$Y_2 = g(x) = \frac{1}{3}(x + 4)$$

$$Y_3 = f \circ g, \quad Y_4 = g \circ f$$

Using the viewing window $-3 \le x \le 3$, $-2 \le y \le 2$, graph only Y_3 and Y_4. What do you see? TRACE to verify that $Y_3 = Y_4$.

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x - 4) && f(x) = 3x - 4 \\ &= \frac{1}{3}[(3x - 4) + 4] && g(x) = \frac{1}{3}(x + 4) \\ &= \frac{1}{3}(3x) = x\end{aligned}$$

We conclude that $(f \circ g)(x) = (g \circ f)(x) = x$.

In Section 4.2, we shall see that there is an important relationship between functions f and g for which $(f \circ g)(x) = (g \circ f)(x) = x$.

NOW WORK PROBLEM 39

Calculus Application

Some techniques in calculus require the ability to determine the components of a composite function. For example, the function $H(x) = \sqrt{x + 1}$ is the composition of the functions f and g, where $f(x) = \sqrt{x}$ and $g(x) = x + 1$, because $H(x) = (f \circ g)(x) = f(g(x)) = f(x + 1) = \sqrt{x + 1}$.

EXAMPLE 6 **Finding the Components of a Composite Function**

Find functions f and g such that $f \circ g = H$ if $H(x) = (x^2 + 1)^{50}$.

Solution The function H takes $x^2 + 1$ and raises it to the power 50. A natural way to decompose H is to raise the function $g(x) = x^2 + 1$ to the power 50. Let $f(x) = x^{50}$ and $g(x) = x^2 + 1$. Then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x^2 + 1) \\ &= (x^2 + 1)^{50} = H(x)\end{aligned}$$

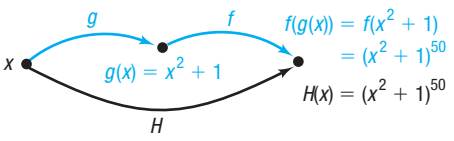


Figure 5

See Figure 5.

Other functions f and g may be found for which $f \circ g = H$ in Example 6. For instance, if $f(x) = x^2$ and $g(x) = (x^2 + 1)^{25}$, then

$$(f \circ g)(x) = f(g(x)) = f((x^2 + 1)^{25}) = [(x^2 + 1)^{25}]^2 = (x^2 + 1)^{50}$$

Although the functions f and g found as a solution to Example 6 are not unique, there is usually a "natural" selection for f and g that comes to mind first.

EXAMPLE 7 **Finding the Components of a Composite Function**

Find functions f and g such that $f \circ g = H$ if $H(x) = \dfrac{1}{x + 1}$.

Solution Here H is the reciprocal of $g(x) = x + 1$. Let $f(x) = \dfrac{1}{x}$ and $g(x) = x + 1$. Then

$$(f \circ g)(x) = f(g(x)) = f(x + 1) = \frac{1}{x + 1} = H(x)$$

NOW WORK PROBLEM 47

4.1 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Find $f(3)$ if $f(x) = -4x^2 + 5x$. (pp. 47–49)

2. Find $f(3x)$ if $f(x) = 4 - 2x^2$. (pp. 47–49)

3. Find the domain of the function $f(x) = \dfrac{x^2 - 1}{x^2 - 25}$. (pp. 51–53)

Concepts and Vocabulary

4. Given two functions f and g, the _____ _____, denoted $f \circ g$, is defined by $(f \circ g)(x) =$ _____.

5. ***True or False*** If $f(x) = x^2$ and $g(x) = \sqrt{x + 9}$, then $(f \circ g)(4) = 5$.

6. If $f(x) = \sqrt{x + 2}$ and $g(x) = \dfrac{3}{x}$, which of the following does $(f \circ g)(x)$ equal?

(a) $\dfrac{3}{\sqrt{x + 2}}$ (b) $\dfrac{3}{\sqrt{x}} + 2$ (c) $\sqrt{\dfrac{3}{x} + 2}$ (d) $\sqrt{\dfrac{3}{x + 2}}$

7. If $H = f \circ g$ and $H(x) = \sqrt{25 - 4x^2}$, which of the following cannot be the component functions f and g?

(a) $f(x) = \sqrt{25 - x^2}$; $g(x) = 4x$
(b) $f(x) = \sqrt{x}$; $g(x) = 25 - 4x^2$
(c) $f(x) = \sqrt{25 - x}$; $g(x) = 4x^2$
(d) $f(x) = \sqrt{25 - 4x}$; $g(x) = x^2$

8. ***True or False*** The domain of the composite function $(f \circ g)(x)$ is the same as the domain of $g(x)$.

Skill Building

In Problems 9 and 10, evaluate each expression using the values given in the table.

9.

x	-3	-2	-1	0	1	2	3
$f(x)$	-7	-5	-3	-1	3	5	7
$g(x)$	8	3	0	-1	0	3	8

(a) $(f \circ g)(1)$ (b) $(f \circ g)(-1)$
(c) $(g \circ f)(-1)$ (d) $(g \circ f)(0)$
(e) $(g \circ g)(-2)$ (f) $(f \circ f)(-1)$

10.

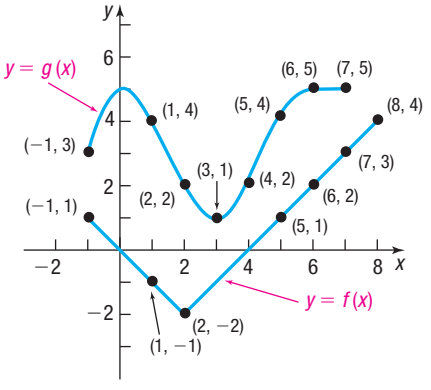
x	-3	-2	-1	0	1	2	3
$f(x)$	11	9	7	5	3	1	-1
$g(x)$	-8	-3	0	1	0	-3	-8

(a) $(f \circ g)(1)$ (b) $(f \circ g)(2)$
(c) $(g \circ f)(2)$ (d) $(g \circ f)(3)$
(e) $(g \circ g)(1)$ (f) $(f \circ f)(3)$

In Problems 11 and 12, evaluate each expression using the graphs of $y = f(x)$ and $y = g(x)$ shown in the figure.

11. (a) $(g \circ f)(-1)$ (b) $(g \circ f)(0)$
(c) $(f \circ g)(-1)$ (d) $(f \circ g)(4)$

12. (a) $(g \circ f)(1)$ (b) $(g \circ f)(5)$
(c) $(f \circ g)(0)$ (d) $(f \circ g)(2)$



In Problems 13–22, for the given functions f and g, find:
(a) $(f \circ g)(4)$ (b) $(g \circ f)(2)$ (c) $(f \circ f)(1)$ (d) $(g \circ g)(0)$

13. $f(x) = 2x$; $g(x) = 3x^2 + 1$

14. $f(x) = 3x + 2$; $g(x) = 2x^2 - 1$

15. $f(x) = 4x^2 - 3$; $g(x) = 3 - \dfrac{1}{2}x^2$

16. $f(x) = 2x^2$; $g(x) = 1 - 3x^2$

17. $f(x) = \sqrt{x}$; $g(x) = 2x$

18. $f(x) = \sqrt{x + 1}$; $g(x) = 3x$

19. $f(x) = |x|$; $g(x) = \dfrac{1}{x^2 + 1}$

20. $f(x) = |x - 2|$; $g(x) = \dfrac{3}{x^2 + 2}$

21. $f(x) = \dfrac{3}{x + 1}$; $g(x) = \sqrt[3]{x}$

22. $f(x) = x^{3/2}$; $g(x) = \dfrac{2}{x + 1}$

In Problems 23–38, for the given functions f and g, find:
(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

State the domain of each composite function.

23. $f(x) = 2x + 3;\quad g(x) = 3x$

24. $f(x) = -x;\quad g(x) = 2x - 4$

25. $f(x) = 3x + 1;\quad g(x) = x^2$

26. $f(x) = x + 1;\quad g(x) = x^2 + 4$

27. $f(x) = x^2;\quad g(x) = x^2 + 4$

28. $f(x) = x^2 + 1;\quad g(x) = 2x^2 + 3$

29. $f(x) = \dfrac{3}{x-1};\quad g(x) = \dfrac{2}{x}$

30. $f(x) = \dfrac{1}{x+3};\quad g(x) = -\dfrac{2}{x}$

31. $f(x) = \dfrac{x}{x-1};\quad g(x) = -\dfrac{4}{x}$

32. $f(x) = \dfrac{x}{x+3};\quad g(x) = \dfrac{2}{x}$

33. $f(x) = \sqrt{x};\quad g(x) = 2x + 3$

34. $f(x) = \sqrt{x-2};\quad g(x) = 1 - 2x$

35. $f(x) = x^2 + 1;\quad g(x) = \sqrt{x-1}$

36. $f(x) = x^2 + 4;\quad g(x) = \sqrt{x-2}$

37. $f(x) = \dfrac{x-5}{x+1};\quad g(x) = \dfrac{x+2}{x-3}$

38. $f(x) = \dfrac{2x-1}{x-2};\quad g(x) = \dfrac{x+4}{2x-5}$

In Problems 39–46, show that $(f \circ g)(x) = (g \circ f)(x) = x$.

39. $f(x) = 2x;\quad g(x) = \dfrac{1}{2}x$

40. $f(x) = 4x;\quad g(x) = \dfrac{1}{4}x$

41. $f(x) = x^3;\quad g(x) = \sqrt[3]{x}$

42. $f(x) = x + 5;\quad g(x) = x - 5$

43. $f(x) = 2x - 6;\quad g(x) = \dfrac{1}{2}(x + 6)$

44. $f(x) = 4 - 3x;\quad g(x) = \dfrac{1}{3}(4 - x)$

45. $f(x) = ax + b;\quad g(x) = \dfrac{1}{a}(x - b)\quad a \neq 0$

46. $f(x) = \dfrac{1}{x};\quad g(x) = \dfrac{1}{x}$

In Problems 47–52, find functions f and g so that $f \circ g = H$.

47. $H(x) = (2x + 3)^4$

48. $H(x) = (1 + x^2)^3$

49. $H(x) = \sqrt{x^2 + 1}$

50. $H(x) = \sqrt{1 - x^2}$

51. $H(x) = |2x + 1|$

52. $H(x) = |2x^2 + 3|$

Applications and Extensions

53. If $f(x) = 2x^3 - 3x^2 + 4x - 1$ and $g(x) = 2$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

54. If $f(x) = \dfrac{x+1}{x-1}$, find $(f \circ f)(x)$.

55. If $f(x) = 2x^2 + 5$ and $g(x) = 3x + a$, find a so that the graph of $f \circ g$ crosses the y-axis at 23.

56. If $f(x) = 3x^2 - 7$ and $g(x) = 2x + a$, find a so that the graph of $f \circ g$ crosses the y-axis at 68.

In Problems 57 and 58, use the functions f and g to find:
(a) $f \circ g$ (b) $g \circ f$
(c) the domain of $f \circ g$ and of $g \circ f$
(d) the conditions for which $f \circ g = g \circ f$

57. $f(x) = ax + b\quad g(x) = cx + d$

58. $f(x) = \dfrac{ax+b}{cx+d}\quad g(x) = mx$

59. Surface Area of a Balloon The surface area S (in square meters) of a hot-air balloon is given by

$$S(r) = 4\pi r^2$$

where r is the radius of the balloon (in meters). If the radius r is increasing with time t (in seconds) according to the formula $r(t) = \dfrac{2}{3}t^3$, $t \geq 0$, find the surface area S of the balloon as a function of the time t.

60. Volume of a Balloon The volume V (in cubic meters) of the hot-air balloon described in Problem 59 is given by $V(r) = \dfrac{4}{3}\pi r^3$. If the radius r is the same function of t as in Problem 59, find the volume V as a function of the time t.

61. Automobile Production The number N of cars produced at a certain factory in one day after t hours of operation is given by $N(t) = 100t - 5t^2$, $0 \leq t \leq 10$. If the cost C (in dollars) of producing N cars is $C(N) = 15{,}000 + 8000N$, find the cost C as a function of the time t of operation of the factory.

62. Environmental Concerns The spread of oil leaking from a tanker is in the shape of a circle. If the radius r (in feet) of the spread after t hours is $r(t) = 200\sqrt{t}$, find the area A of the oil slick as a function of the time t.

63. Production Cost The price p, in dollars, of a certain product and the quantity x sold obey the demand equation

$$p = -\frac{1}{4}x + 100 \quad 0 \leq x \leq 400$$

Suppose that the cost C, in dollars, of producing x units is

$$C = \frac{\sqrt{x}}{25} + 600$$

Assuming that all items produced are sold, find the cost C as a function of the price p.
[**Hint:** Solve for x in the demand equation and then form the composite function.]

64. Cost of a Commodity The price p, in dollars, of a certain commodity and the quantity x sold obey the demand equation

$$p = -\frac{1}{5}x + 200 \quad 0 \le x \le 1000$$

Suppose that the cost C, in dollars, of producing x units is

$$C = \frac{\sqrt{x}}{10} + 400$$

Assuming that all items produced are sold, find the cost C as a function of the price p.

65. Volume of a Cylinder The volume V of a right circular cylinder of height h and radius r is $V = \pi r^2 h$. If the height is twice the radius, express the volume V as a function of r.

66. Volume of a Cone The volume V of a right circular cone is $V = \frac{1}{3}\pi r^2 h$. If the height is twice the radius, express the volume V as a function of r.

67. Foreign Exchange Traders often buy foreign currency in the hope of making money when the currency's value changes. For example, on April 15, 2017, one U.S. dollar could purchase 0.9423 euro, and one euro could purchase 115.238 yen. Let $f(x)$ represent the number of euros you can buy with x dollars, and let $g(x)$ represent the number of yen you can buy with x euros.

(a) Find a function that relates dollars to euros.
(b) Find a function that relates euros to yen.
(c) Use the results of parts (a) and (b) to find a function that relates dollars to yen. That is, find

$$(g \circ f)(x) = g(f(x)).$$

(d) What is $g(f(1000))$?

68. Temperature Conversion The function $C(F) = \frac{5}{9}(F - 32)$ converts a temperature in degrees Fahrenheit, F, to a temperature in degrees Celsius, C. The function $K(C) = C + 273$, converts a temperature in degrees Celsius to a temperature in kelvins, K.

(a) Find a function that converts a temperature in degrees Fahrenheit to a temperature in kelvins.
(b) Determine 80 degrees Fahrenheit in kelvins.

69. Discounts The manufacturer of a computer is offering two discounts on last year's model computer. The first discount is a \$200 rebate and the second discount is 20% off the regular price, p.

(a) Write a function f that represents the sale price if only the rebate applies.
(b) Write a function g that represents the sale price if only the 20% discount applies.
(c) Find $f \circ g$ and $g \circ f$. What does each of these functions represent? Which combination of discounts represents a better deal for the consumer? Why?

70. Taxes Suppose that you work for \$15 per hour. Write a function that represents gross salary G as a function of hours worked h. Your employer is required to withhold taxes (federal income tax, Social Security, Medicare) from your paycheck. Suppose your employer withholds 20% of your income for taxes. Write a function that represents net salary N as a function of gross salary G. Find and interpret $N \circ G$.

71. Suppose that $f(x) = x^3 + x^2 - 16x - 16$ and $g(x) = x^2 - 4$. Find the zeros of $(f \circ g)(x)$.

72. Suppose that $f(x) = 2x^3 - 3x^2 - 8x + 12$ and $g(x) = x + 5$. Find the zeros of $(f \circ g)(x)$.

73. Let $f(x) = ax + b$ and $g(x) = bx + a$, where a and b are integers. If $f(1) = 8$ and $f(g(20)) - g(f(20)) = -14$, find the product of a and b.*

74. If f and g are odd functions, show that the composite function $f \circ g$ is also odd.

75. If f is an odd function and g is an even function, show that the composite functions $f \circ g$ and $g \circ f$ are both even.

*Courtesy of the Joliet Junior College Mathematics Department

Retain Your Knowledge

Problems 76–79 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

76. Given $f(x) = 3x + 8$ and $g(x) = x - 5$, find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$. State the domain of each.

77. Find the real zeros of $f(x) = 2x - 5\sqrt{x} + 2$.

78. Use a graphing utility to graph $f(x) = -x^3 + 4x - 2$ over the interval $[-3, 3]$. Approximate any local maxima and local minima. Determine where the function is increasing and where it is decreasing.

79. Find the domain of $R(x) = \frac{x^2 + 6x + 5}{x - 3}$. Find any horizontal, vertical, or oblique asymptotes.

'Are You Prepared?' Answers

1. -21 **2.** $4 - 18x^2$ **3.** $\{x \mid x \ne -5, x \ne 5\}$

4.2 One-to-One Functions; Inverse Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Functions (Section 1.1, pp. 44–55)
- Increasing/Decreasing Functions (Section 1.3, pp. 70–71)
- Rational Expressions (Appendix A, Section A.6, pp. A47–A54)

Now Work the **'Are You Prepared?'** problems on page 297.

OBJECTIVES
1. Determine Whether a Function Is One-to-One (p. 289)
2. Determine the Inverse of a Function Defined by a Map or a Set of Ordered Pairs (p. 291)
3. Obtain the Graph of the Inverse Function from the Graph of the Function (p. 294)
4. Find the Inverse of a Function Defined by an Equation (p. 295)

1 Determine Whether a Function Is One-to-One

Section 1.1 presented four different ways to represent a function: (1) a map, (2) a set of ordered pairs, (3) a graph, and (4) an equation. For example, Figures 6 and 7 illustrate two different functions represented as mappings. The function in Figure 6 shows the correspondence between states and their populations (in millions). The function in Figure 7 shows a correspondence between animals and life expectancies (in years).

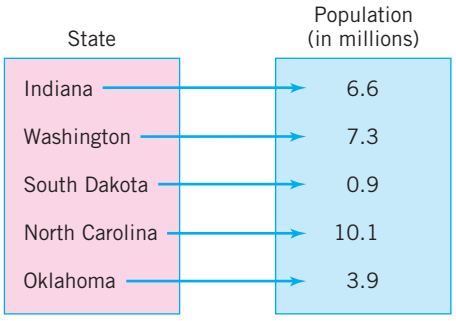


Figure 6

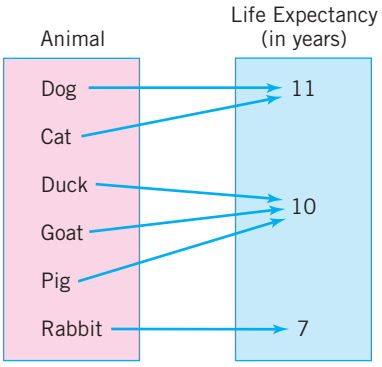


Figure 7

Suppose several people are asked to name a state that has a population of 0.9 million based on the function in Figure 6. Everyone will respond "South Dakota." Now, if the same people are asked to name an animal whose life expectancy is 11 years based on the function in Figure 7, some may respond "dog," while others may respond "cat." What is the difference between the functions in Figures 6 and 7? In Figure 6, no two elements in the domain correspond to the same element in the range. In Figure 7, this is not the case: Different elements in the domain correspond to the same element in the range. Functions such as the one in Figure 6 are given a special name.

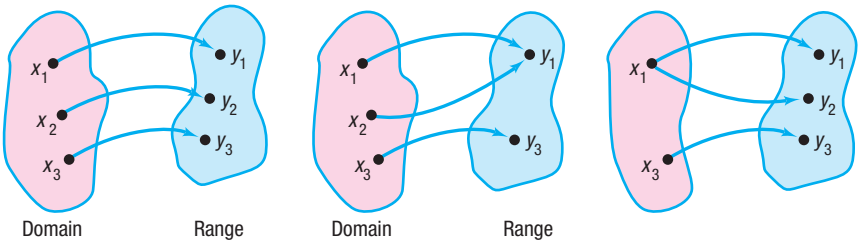
DEFINITION

A function is **one-to-one** if any two different inputs in the domain correspond to two different outputs in the range. That is, if x_1 and x_2 are two different inputs of a function f, then f is one-to-one if $f(x_1) \neq f(x_2)$.

In Words
A function is not one-to-one if two different inputs correspond to the same output.

Put another way, a function f is one-to-one if no y in the range is the image of more than one x in the domain. A function is not one-to-one if any two (or more) different elements in the domain correspond to the same element in the range. So the function in Figure 7 is not one-to-one because two different elements in the

domain, *dog* and *cat*, both correspond to 11 (and also because three different elements in the domain correspond to 10). Figure 8 illustrates the distinction among one-to-one functions, functions that are not one-to-one, and relations that are not functions.



(a) One-to-one function: Each x in the domain has one and only one image in the range.

(b) Function, but not one-to-one: y_1 is the image of both x_1 and x_2.

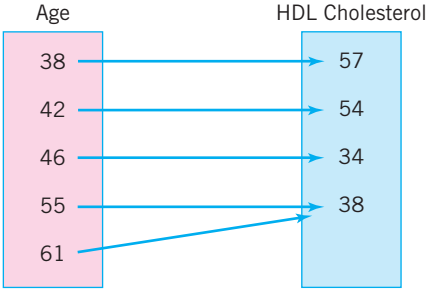
(c) Not a function: x_1 has two images, y_1 and y_2.

Figure 8

EXAMPLE 1 Determining Whether a Function Is One-to-One

Determine whether the following functions are one-to-one.

(a) For the following function, the domain represents the ages of five males, and the range represents their HDL (good) cholesterol scores (mg/dL).



(b) $\{(-2, 6), (-1, 3), (0, 2), (1, 5), (2, 8)\}$

Solution

(a) The function is not one-to-one because there are two different inputs, 55 and 61, that correspond to the same output, 38.

(b) The function is one-to-one because no two distinct inputs correspond to the same output.

NOW WORK PROBLEMS 13 AND 17

For functions defined by an equation $y = f(x)$ and for which the graph of f is known, there is a simple test, called the **horizontal-line test,** to determine whether f is one-to-one.

THEOREM

Horizontal-line Test

If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

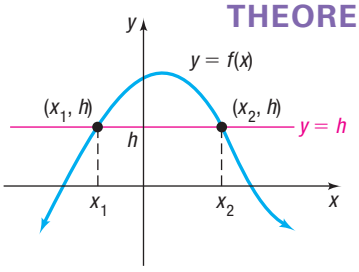


Figure 9 $f(x_1) = f(x_2) = h$ and $x_1 \neq x_2$; f is not a one-to-one function.

The reason why this test works can be seen in Figure 9, where the horizontal line $y = h$ intersects the graph at two distinct points, (x_1, h) and (x_2, h). Since h is the image of both x_1 and x_2 and $x_1 \neq x_2$, f is not one-to-one. Based on Figure 9, the horizontal-line test can be stated in another way: If the graph of any horizontal line intersects the graph of a function f at more than one point, then f is not one-to-one.

EXAMPLE 2 **Using the Horizontal-line Test**

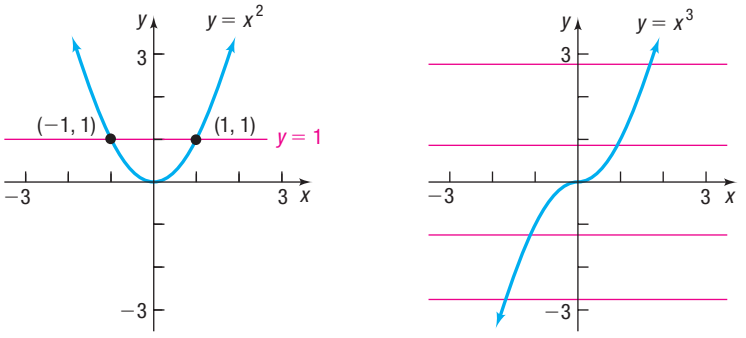
For each function, use its graph to determine whether the function is one-to-one.

(a) $f(x) = x^2$ (b) $g(x) = x^3$

Solution

(a) Figure 10(a) illustrates the horizontal-line test for $f(x) = x^2$. The horizontal line $y = 1$ intersects the graph of f twice, at $(1, 1)$ and at $(-1, 1)$, so f is not one-to-one.

(b) Figure 10(b) illustrates the horizontal-line test for $g(x) = x^3$. Because every horizontal line intersects the graph of g exactly once, it follows that g is one-to-one.



(a) A horizontal line intersects the graph twice; f is not one-to-one.

(b) Every horizontal line intersects the graph exactly once; g is one-to-one.

Figure 10

NOW WORK PROBLEM 21

Look more closely at the one-to-one function $g(x) = x^3$. This function is an increasing function. Because an increasing (or decreasing) function will always have different y-values for unequal x-values, it follows that a function that is increasing (or decreasing) over its domain is also a one-to-one function.

THEOREM

A function that is increasing on an interval I is a one-to-one function on I.
A function that is decreasing on an interval I is a one-to-one function on I.

2 Determine the Inverse of a Function Defined by a Map or a Set of Ordered Pairs

DEFINITION

Suppose that f is a one-to-one function. Then, corresponding to each x in the domain of f, there is exactly one y in the range (because f is a function); and corresponding to each y in the range of f, there is exactly one x in the domain (because f is one-to-one). The correspondence from the range of f back to the domain of f is called the **inverse function of f.** The symbol f^{-1} is used to denote the inverse function of f.

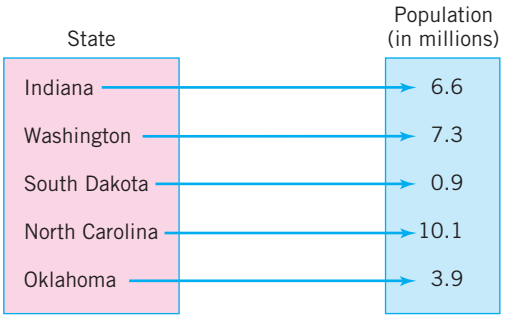
In Words
Suppose that f is a one-to-one function so that the input 5 corresponds to the output 10. In the inverse function f^{-1}, the input 10 will correspond to the output 5.

We will discuss how to find inverses for all four representations of functions: (1) maps, (2) sets of ordered pairs, (3) equations, and (4) graphs. We begin with finding inverses of functions represented by maps or sets of ordered pairs.

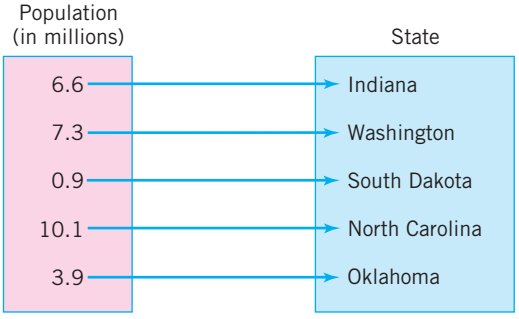
EXAMPLE 3 **Finding the Inverse of a Function Defined by a Map**

Find the inverse of the function defined by the map at the top of the next page. Let the domain of the function represent certain states, and let the range represent the states' populations (in millions). Find the domain and the range of the inverse function.

(*continued*)



Solution The function is one-to-one. To find the inverse function, interchange the elements in the domain with the elements in the range. For example, the function receives as input Indiana and outputs 6.6 million. So the inverse receives as input 6.6 million and outputs Indiana. The inverse function is shown next.



The domain of the inverse function is $\{6.6, 7.3, 0.9, 10.1, 3.9\}$. The range of the inverse function is {Indiana, Washington, South Dakota, North Carolina, Oklahoma}. ●

If the function f is a set of ordered pairs (x, y), then the inverse function of f, denoted f^{-1}, is the set of ordered pairs (y, x).

EXAMPLE 4 Finding the Inverse of a Function Defined by a Set of Ordered Pairs

Find the inverse of the following one-to-one function:

$$\{(-3, -27), (-2, -8), (-1, -1), (0, 0), (1, 1), (2, 8), (3, 27)\}$$

State the domain and the range of the function and its inverse.

Solution The inverse of the given function is found by interchanging the entries in each ordered pair and so is given by

$$\{(-27, -3), (-8, -2), (-1, -1), (0, 0), (1, 1), (8, 2), (27, 3)\}$$

The domain of the function is $\{-3, -2, -1, 0, 1, 2, 3\}$. The range of the function is $\{-27, -8, -1, 0, 1, 8, 27\}$.
The domain of the inverse function is $\{-27, -8, -1, 0, 1, 8, 27\}$. The range of the inverse function is $\{-3, -2, -1, 0, 1, 2, 3\}$. ●

Now Work PROBLEMS 27 AND 31

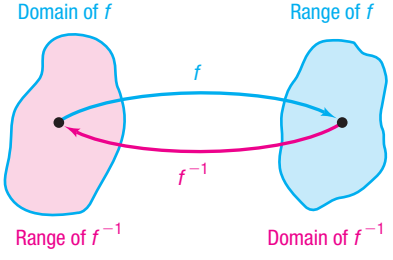


Figure 11

Remember, if f is a one-to-one function, it has an inverse function, f^{-1}. See Figure 11.

The results of Example 4 and Figure 11 suggest two facts about a one-to-one function f and its inverse f^{-1}.

$$\text{Domain of } f = \text{Range of } f^{-1} \qquad \text{Range of } f = \text{Domain of } f^{-1}$$

WARNING Be careful! f^{-1} is a symbol for the inverse function of f. The -1 used in f^{-1} is not an exponent. That is, f^{-1} does *not* mean the reciprocal of f; $f^{-1}(x)$ is not equal to $\dfrac{1}{f(x)}$. ■

Look again at Figure 11 to visualize the relationship. Starting with x, applying f, and then applying f^{-1} gets x back again. Starting with x, applying f^{-1}, and then applying f gets x back again. To put it simply, what f does, f^{-1} undoes, and vice versa. See the illustration that follows.

$$\boxed{\text{Input } x \text{ from domain of } f} \xrightarrow{\text{Apply } f} \boxed{f(x)} \xrightarrow{\text{Apply } f^{-1}} \boxed{f^{-1}(f(x)) = x}$$

$$\boxed{\text{Input } x \text{ from domain of } f^{-1}} \xrightarrow{\text{Apply } f^{-1}} \boxed{f^{-1}(x)} \xrightarrow{\text{Apply } f} \boxed{f(f^{-1}(x)) = x}$$

In other words,

$$\boxed{\begin{aligned} f^{-1}(f(x)) &= x \text{ where } x \text{ is in the domain of } f \\ f(f^{-1}(x)) &= x \text{ where } x \text{ is in the domain of } f^{-1} \end{aligned}}$$

Consider the function $f(x) = 2x$, which multiplies the argument x by 2. The inverse function f^{-1} undoes whatever f does. So the inverse function of f is $f^{-1}(x) = \dfrac{1}{2}x$, which divides the argument by 2. For example, $f(3) = 2(3) = 6$ and $f^{-1}(6) = \dfrac{1}{2}(6) = 3$, so f^{-1} undoes what f did. This is verified by showing that

$$f^{-1}(f(x)) = f^{-1}(2x) = \frac{1}{2}(2x) = x \quad \text{and} \quad f(f^{-1}(x)) = f\left(\frac{1}{2}x\right) = 2\left(\frac{1}{2}x\right) = x.$$

See Figure 12.

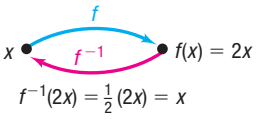


Figure 12

EXAMPLE 5 Verifying Inverse Functions

(a) Verify that the inverse of $g(x) = x^3$ is $g^{-1}(x) = \sqrt[3]{x}$.

(b) Verify that the inverse of $f(x) = 2x + 3$ is $f^{-1}(x) = \dfrac{1}{2}(x - 3)$.

Solution

(a) $g^{-1}(g(x)) = g^{-1}(x^3) = \sqrt[3]{x^3} = x$ for all x in the domain of g

$g(g^{-1}(x)) = g(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$ for all x in the domain of g^{-1}

(b) $f^{-1}(f(x)) = f^{-1}(2x + 3) = \dfrac{1}{2}[(2x + 3) - 3] = \dfrac{1}{2}(2x) = x$ for all x in the domain of f

$f(f^{-1}(x)) = f\left(\dfrac{1}{2}(x - 3)\right) = 2\left[\dfrac{1}{2}(x - 3)\right] + 3 = (x - 3) + 3 = x$ for all x in the domain of f^{-1}

EXAMPLE 6 Verifying Inverse Functions

Verify that the inverse of $f(x) = \dfrac{1}{x - 1}$ is $f^{-1}(x) = \dfrac{1}{x} + 1$. For what values of x is $f^{-1}(f(x)) = x$? For what values of x is $f(f^{-1}(x)) = x$?

Solution The domain of f is $\{x | x \neq 1\}$ and the domain of f^{-1} is $\{x | x \neq 0\}$. Now

$$f^{-1}(f(x)) = f^{-1}\left(\frac{1}{x - 1}\right) = \frac{1}{\dfrac{1}{x - 1}} + 1 = x - 1 + 1 = x \quad \text{provided } x \neq 1$$

(continued)

$$f(f^{-1}(x)) = f\left(\frac{1}{x} + 1\right) = \frac{1}{\left(\frac{1}{x} + 1\right) - 1} = \frac{1}{\frac{1}{x}} = x \quad \text{provided } x \neq 0$$

NOW WORK PROBLEMS 35 AND 39

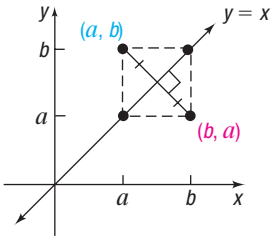


Figure 13

3 Obtain the Graph of the Inverse Function from the Graph of the Function

Suppose that (a, b) is a point on the graph of a one-to-one function f defined by $y = f(x)$. Then $b = f(a)$. This means that $a = f^{-1}(b)$, so (b, a) is a point on the graph of the inverse function f^{-1}. The relationship between the point (a, b) on f and the point (b, a) on f^{-1} is shown in Figure 13. The line segment with endpoints (a, b) and (b, a) is perpendicular to the line $y = x$ and is bisected by the line $y = x$. (Do you see why?) It follows that the point (b, a) on f^{-1} is the reflection about the line $y = x$ of the point (a, b) on f.

THEOREM

The graph of a one-to-one function f and the graph of its inverse function f^{-1} are symmetric with respect to the line $y = x$.

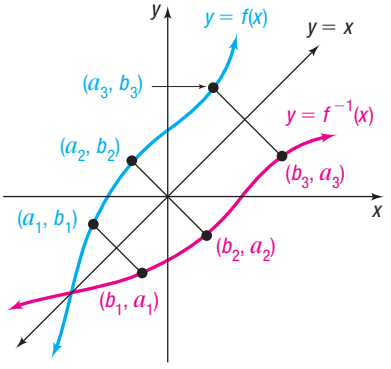


Figure 14

Figure 14 illustrates this result. Once the graph of f is known, the graph of f^{-1} may be obtained by reflecting the graph of f about the line $y = x$.

Exploration

Simultaneously graph $Y_1 = x$, $Y_2 = x^3$, and $Y_3 = \sqrt[3]{x}$ on a square screen with $-3 \le x \le 3$. What do you observe about the graphs of $Y_2 = x^3$, its inverse $Y_3 = \sqrt[3]{x}$, and the line $Y_1 = x$?

Repeat this experiment by simultaneously graphing $Y_1 = x$, $Y_2 = 2x + 3$, and $Y_3 = \frac{1}{2}(x - 3)$ on a square screen with $-6 \le x \le 3$. Do you see the symmetry of the graph of Y_2 and its inverse Y_3 with respect to the line $Y_1 = x$?

EXAMPLE 7 **Graphing the Inverse Function**

The graph in Figure 15(a) is that of a one-to-one function $y = f(x)$. Draw the graph of its inverse.

Solution Begin by adding the graph of $y = x$ to Figure 15(a). Since the points $(-2, -1)$, $(-1, 0)$, and $(2, 1)$ are on the graph of f, the points $(-1, -2)$, $(0, -1)$, and $(1, 2)$ must be on the graph of f^{-1}. Keeping in mind that the graph of f^{-1} is the reflection about the line $y = x$ of the graph of f, draw the graph of f^{-1}. See Figure 15(b).

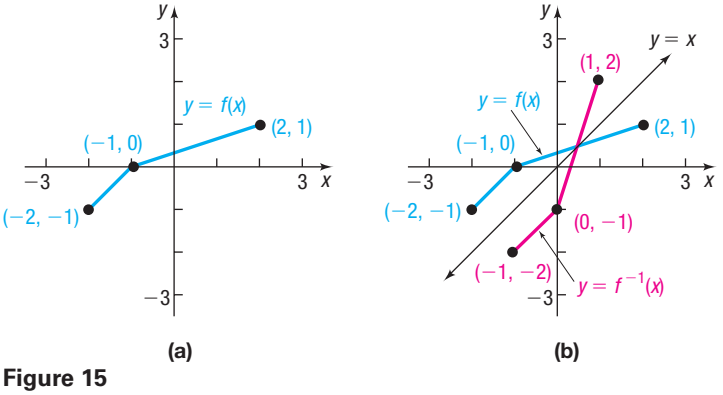


Figure 15

NOW WORK PROBLEM 45

4 Find the Inverse of a Function Defined by an Equation

The fact that the graphs of a one-to-one function f and its inverse function f^{-1} are symmetric with respect to the line $y = x$ tells us more. It says that we can obtain f^{-1} by interchanging the roles of x and y in f. Look again at Figure 14. If f is defined by the equation

$$y = f(x)$$

then f^{-1} is defined by the equation

$$x = f(y)$$

The equation $x = f(y)$ defines f^{-1} *implicitly*. If we can solve this equation for y, we will have the *explicit* form of f^{-1}, that is,

$$y = f^{-1}(x)$$

Let's use this procedure to find the inverse of $f(x) = 2x + 3$. (Because f is a linear function and is increasing, f is one-to-one and so has an inverse function.)

EXAMPLE 8 **How to Find the Inverse Function**

Find the inverse of $f(x) = 2x + 3$. Graph f and f^{-1} on the same coordinate axes.

Step-by-Step Solution

Step 1 Replace $f(x)$ with y. In $y = f(x)$, interchange the variables x and y to obtain $x = f(y)$. This equation defines the inverse function f^{-1} implicitly.

Replace $f(x)$ with y in $f(x) = 2x + 3$ and obtain $y = 2x + 3$. Now interchange the variables x and y to obtain

$$x = 2y + 3$$

This equation defines the inverse function f^{-1} implicitly.

Step 2 If possible, solve the implicit equation for y in terms of x to obtain the explicit form of f^{-1}, $y = f^{-1}(x)$.

To find the explicit form of the inverse, solve $x = 2y + 3$ for y.

$$\begin{aligned} x &= 2y + 3 & \\ 2y + 3 &= x & \text{Reflexive Property; If } a = b\text{, then } b = a. \\ 2y &= x - 3 & \text{Subtract 3 from both sides.} \\ y &= \frac{1}{2}(x - 3) & \text{Multiply both sides by } \tfrac{1}{2}. \end{aligned}$$

The explicit form of the inverse function f^{-1} is

$$f^{-1}(x) = \frac{1}{2}(x - 3)$$

Step 3 Check the result by showing that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

See Example 5(b) for verification that f and f^{-1} are inverses.

The graphs of $f(x) = 2x + 3$ and its inverse $f^{-1}(x) = \frac{1}{2}(x - 3)$ are shown in Figure 16. Note the symmetry of the graphs with respect to the line $y = x$.

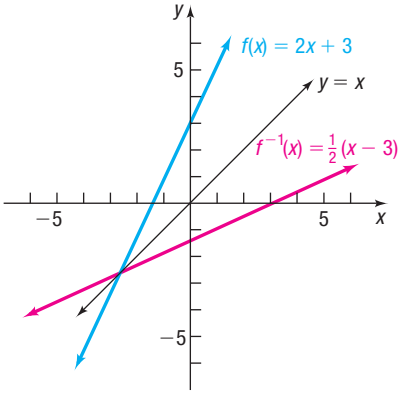


Figure 16

Procedure for Finding the Inverse of a One-to-One Function

STEP 1: In $y = f(x)$, interchange the variables x and y to obtain

$$x = f(y)$$

This equation defines the inverse function f^{-1} implicitly.

STEP 2: If possible, solve the implicit equation for y in terms of x to obtain the explicit form of f^{-1}:

$$y = f^{-1}(x)$$

STEP 3: Check the result by showing that

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

EXAMPLE 9 Finding the Inverse Function

The function

$$f(x) = \frac{2x+1}{x-1} \qquad x \neq 1$$

is one-to-one. Find its inverse function and check the result.

Solution **STEP 1:** Replace $f(x)$ with y and interchange the variables x and y in

$$y = \frac{2x+1}{x-1}$$

to obtain

$$x = \frac{2y+1}{y-1}$$

STEP 2: Solve for y.

$$\begin{aligned}
x &= \frac{2y+1}{y-1} \\
x(y-1) &= 2y+1 && \text{Multiply both sides by } y-1. \\
xy - x &= 2y+1 && \text{Apply the Distributive Property.} \\
xy - 2y &= x+1 && \text{Subtract } 2y \text{ from both sides; add } x \text{ to both sides.} \\
(x-2)y &= x+1 && \text{Factor.} \\
y &= \frac{x+1}{x-2} && \text{Divide by } x-2.
\end{aligned}$$

The inverse function is

$$f^{-1}(x) = \frac{x+1}{x-2} \quad x \neq 2 \qquad \text{Replace } y \text{ by } f^{-1}(x).$$

STEP 3: ✓**Check:**

$$f^{-1}(f(x)) = f^{-1}\left(\frac{2x+1}{x-1}\right) = \frac{\dfrac{2x+1}{x-1}+1}{\dfrac{2x+1}{x-1}-2} = \frac{2x+1+x-1}{2x+1-2(x-1)} = \frac{3x}{3} = x, \quad x \neq 1$$

$$f(f^{-1}(x)) = f\left(\frac{x+1}{x-2}\right) = \frac{2\left(\dfrac{x+1}{x-2}\right)+1}{\dfrac{x+1}{x-2}-1} = \frac{2(x+1)+x-2}{x+1-(x-2)} = \frac{3x}{3} = x, \quad x \neq 2$$

Exploration

In Example 9, we found that if $f(x) = \dfrac{2x+1}{x-1}$, then $f^{-1}(x) = \dfrac{x+1}{x-2}$. Compare the vertical and horizontal asymptotes of f and f^{-1}.

Result The vertical asymptote of f is $x = 1$, and the horizontal asymptote is $y = 2$. The vertical asymptote of f^{-1} is $x = 2$, and the horizontal asymptote is $y = 1$.

NOW WORK PROBLEMS 53 AND 67

If a function is not one-to-one, it has no inverse function. Sometimes, though, an appropriate restriction on the domain of such a function will yield a new function that *is* one-to-one. Then the function defined on the restricted domain has an inverse function. Let's look at an example of this common practice.

EXAMPLE 10 Finding the Inverse of a Domain-restricted Function

Find the inverse of $y = f(x) = x^2$ if $x \geq 0$. Graph f and f^{-1}.

Solution The function $y = x^2$ is not one-to-one. [Refer to Example 2(a).] However, restricting the domain of this function to $x \geq 0$, as indicated, results in a new function that is increasing and therefore is one-to-one. Consequently, the function defined by $y = f(x) = x^2, x \geq 0$, has an inverse function, f^{-1}.

Follow the steps given previously to find f^{-1}.

STEP 1: In the equation $y = x^2, x \geq 0$, interchange the variables x and y. The result is

$$x = y^2 \qquad y \geq 0$$

This equation defines the inverse function implicitly.

STEP 2: Solve for y to get the explicit form of the inverse. Because $y \geq 0$, only one solution for y is obtained: $y = \sqrt{x}$. So $f^{-1}(x) = \sqrt{x}$.

STEP 3: ✓**Check:** $f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = |x| = x$ because $x \geq 0$

$$f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$

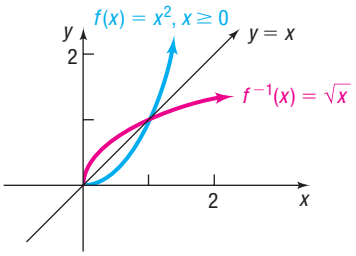


Figure 17

Figure 17 illustrates the graphs of $f(x) = x^2, x \geq 0$, and $f^{-1}(x) = \sqrt{x}$. Note that the domain of f = range of $f^{-1} = [0, \infty)$, and the domain of f^{-1} = range of $f = [0, \infty)$.

SUMMARY

1. If a function f is one-to-one, then it has an inverse function f^{-1}.
2. Domain of f = Range of f^{-1}; Range of f = Domain of f^{-1}.
3. To verify that f^{-1} is the inverse of f, show that $f^{-1}(f(x)) = x$ for every x in the domain of f and that $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}.
4. The graphs of f and f^{-1} are symmetric with respect to the line $y = x$.

4.2 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Is the set of ordered pairs $\{(1,3), (2,3), (-1,2)\}$ a function? Why or why not? (pp. 44–47)
2. Where is the function $f(x) = x^2$ increasing? Where is it decreasing? (pp. 70–71)
3. What is the domain of $f(x) = \dfrac{x+5}{x^2+3x-18}$? (pp. 51–53)
4. Simplify: $\dfrac{\frac{1}{x}+1}{\frac{1}{x^2}-1}$ (pp. A52–A54)

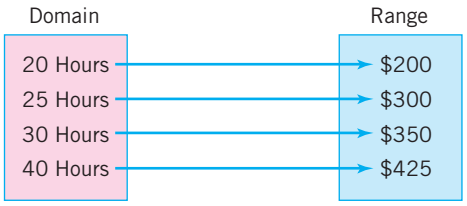
Concepts and Vocabulary

5. If x_1 and x_2 are two different inputs of a function f, then f is one-to-one if _____.
6. If every horizontal line intersects the graph of a function f at no more than one point, then f is a(n) _____ function.
7. If f is a one-to-one function and $f(3) = 8$, then $f^{-1}(8) =$ _____.
8. If f^{-1} denotes the inverse of a function f, then the graphs of f and f^{-1} are symmetric with respect to the line _____.
9. If the domain of a one-to-one function f is $[4, \infty)$, then the range of its inverse function f^{-1} is _____.
10. ***True or False*** If f and g are inverse functions, then the domain of f is the same as the range of g.
11. If $(-2, 3)$ is a point on the graph of a one-to-one function f, which of the following points is on the graph of f^{-1}?
    (a) $(3, -2)$ (b) $(2, -3)$ (c) $(-3, 2)$ (d) $(-2, -3)$
12. Suppose f is a one-to-one function with a domain of $\{x | x \neq 3\}$ and a range of $\left\{x \middle| x \neq \frac{2}{3}\right\}$. Which of the following is the domain of f^{-1}?
    (a) $\{x | x \neq 3\}$ (b) All real numbers
    (c) $\left\{x \middle| x \neq \frac{2}{3}, x \neq 3\right\}$ (d) $\left\{x \middle| x \neq \frac{2}{3}\right\}$

Skill Building

In Problems 13–20, determine whether the function is one-to-one.

13.



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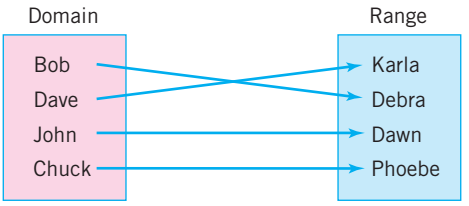

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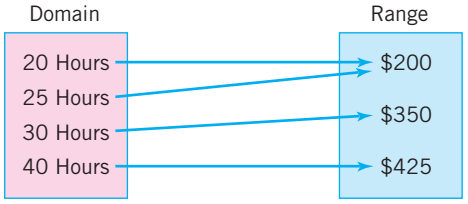


16.

17. $\{(2,6), (-3,6), (4,9), (1,10)\}$

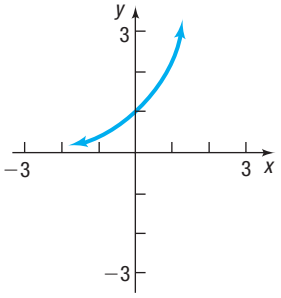
18. $\{(-2,5), (-1,3), (3,7), (4,12)\}$

19. $\{(0,0), (1,1), (2,16), (3,81)\}$

20. $\{(1,2), (2,8), (3,18), (4,32)\}$

In Problems 21–26, the graph of a function f is given. Use the horizontal-line test to determine whether f is one-to-one.

21.



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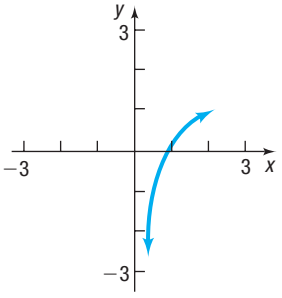

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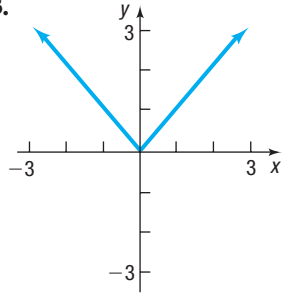

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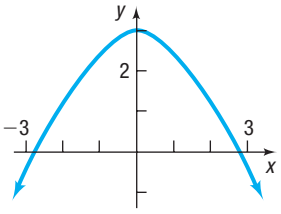

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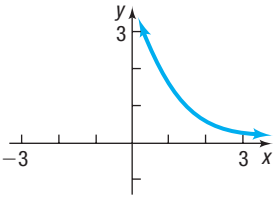

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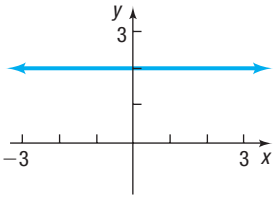
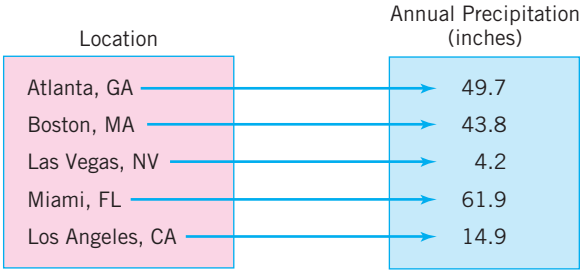


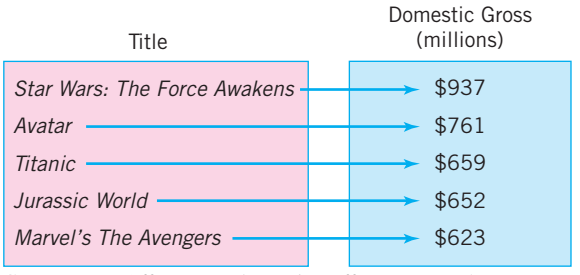
In Problems 27–34, find the inverse of each one-to-one function. State the domain and the range of each inverse function.

27.



Source: *Current Results (www.currentresults.com)*

28.



Source: *Box Office Mojo (www.boxofficemojo.com)*

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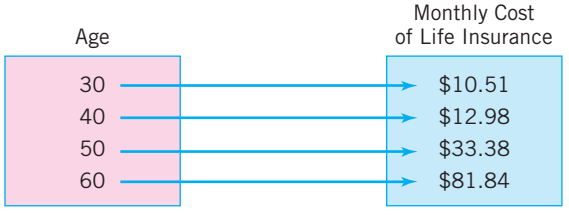
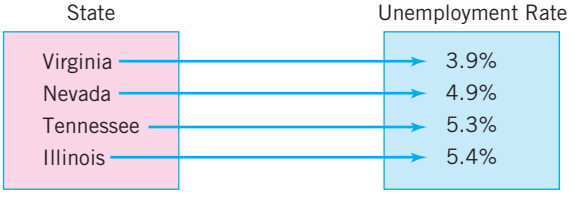


Source: *termlife2go.com*

30.



Source: *United States Bureau of Labor Statistics, February 2017 (www.bls.gov)*

31. $\{(-3,5), (-2,9), (-1,2), (0,11), (1,-5)\}$

32. $\{(-2,2), (-1,6), (0,8), (1,-3), (2,9)\}$

33. $\{(-2,1), (-3,2), (-10,0), (1,9), (2,4)\}$

34. $\{(-2,-8), (-1,-1), (0,0), (1,1), (2,8)\}$

In Problems 35–44, verify that the functions f and g are inverses of each other by showing that $f(g(x)) = x$ and $g(f(x)) = x$. Give any values of x that need to be excluded from the domain of f and the domain of g.

35. $f(x) = 3x + 4;\quad g(x) = \frac{1}{3}(x - 4)$

36. $f(x) = 3 - 2x;\quad g(x) = -\frac{1}{2}(x - 3)$

37. $f(x) = 4x - 8;\quad g(x) = \frac{x}{4} + 2$

38. $f(x) = 2x + 6;\quad g(x) = \frac{1}{2}x - 3$

39. $f(x) = x^3 - 8;\quad g(x) = \sqrt[3]{x + 8}$

40. $f(x) = (x - 2)^2, x \geq 2;\quad g(x) = \sqrt{x} + 2$

41. $f(x) = \frac{1}{x};\quad g(x) = \frac{1}{x}$

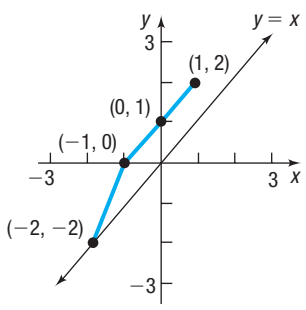
42. $f(x) = x;\quad g(x) = x$

43. $f(x) = \frac{2x + 3}{x + 4};\quad g(x) = \frac{4x - 3}{2 - x}$

44. $f(x) = \frac{x - 5}{2x + 3};\quad g(x) = \frac{3x + 5}{1 - 2x}$

In Problems 45–50, the graph of a one-to-one function f is given. Draw the graph of the inverse function f^{-1}.

45.



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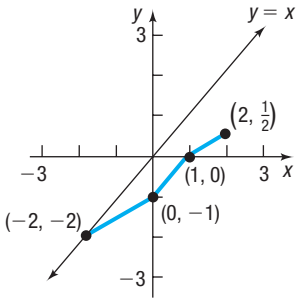

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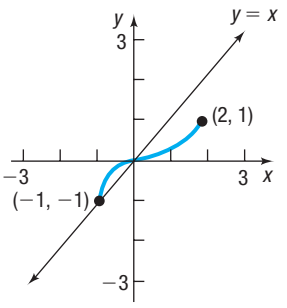

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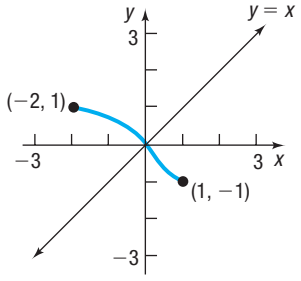

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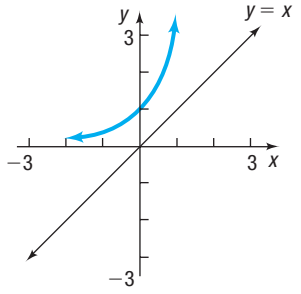

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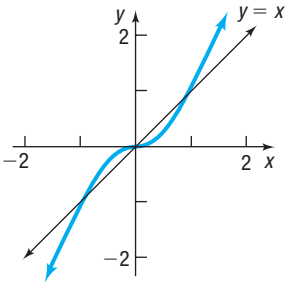


In Problems 51–62, the function f is one-to-one. (a) Find its inverse function f^{-1} and check your answer. (b) Find the domain and the range of f and f^{-1}. (c) Graph f, f^{-1}, and $y = x$ on the same coordinate axes.

51. $f(x) = 3x$

52. $f(x) = -4x$

53. $f(x) = 4x + 2$

54. $f(x) = 1 - 3x$

55. $f(x) = x^3 - 1$

56. $f(x) = x^3 + 1$

57. $f(x) = x^2 + 4,\quad x \geq 0$

58. $f(x) = x^2 + 9,\quad x \geq 0$

59. $f(x) = \frac{4}{x}$

60. $f(x) = -\frac{3}{x}$

61. $f(x) = \frac{1}{x - 2}$

62. $f(x) = \frac{4}{x + 2}$

In Problems 63–80, the function f is one-to-one. (a) Find its inverse function f^{-1} and check your answer. (b) Find the domain and the range of f and f^{-1}.

63. $f(x) = \frac{2}{3 + x}$

64. $f(x) = \frac{4}{2 - x}$

65. $f(x) = \frac{3x}{x + 2}$

66. $f(x) = -\frac{2x}{x - 1}$

67. $f(x) = \frac{2x}{3x - 1}$

68. $f(x) = -\frac{3x + 1}{x}$

69. $f(x) = \frac{3x + 4}{2x - 3}$

70. $f(x) = \frac{2x - 3}{x + 4}$

71. $f(x) = \frac{2x + 3}{x + 2}$

72. $f(x) = \frac{-3x - 4}{x - 2}$

73. $f(x) = \frac{x^2 - 4}{2x^2},\quad x > 0$

74. $f(x) = \frac{x^2 + 3}{3x^2},\quad x > 0$

75. $f(x) = x^{\frac{2}{3}} - 4,\quad x \geq 0$

76. $f(x) = x^{\frac{3}{2}} + 5$

77. $f(x) = \sqrt[3]{x^5 - 2}$

78. $f(x) = \sqrt[5]{x^3 + 13}$

79. $f(x) = \frac{1}{9}(x - 1)^2 + 2,\quad x \geq 1$

80. $f(x) = 2\sqrt{x + 3} - 5$

Applications and Extensions

81. Use the graph of $y = f(x)$ given in Problem 45 to evaluate the following:
(a) $f(-1)$ (b) $f(1)$ (c) $f^{-1}(1)$ (d) $f^{-1}(2)$

82. Use the graph of $y = f(x)$ given in Problem 46 to evaluate the following:
(a) $f(2)$ (b) $f(1)$ (c) $f^{-1}(0)$ (d) $f^{-1}(-1)$

83. If $f(7) = 13$ and f is one-to-one, what is $f^{-1}(13)$?

84. If $g(-5) = 3$ and g is one-to-one, what is $g^{-1}(3)$?

85. The domain of a one-to-one function f is $[5, \infty)$, and its range is $[-2, \infty)$. State the domain and the range of f^{-1}.

86. The domain of a one-to-one function f is $[0, \infty)$, and its range is $[5, \infty)$. State the domain and the range of f^{-1}.

87. The domain of a one-to-one function g is $(-\infty, 0]$, and its range is $[0, \infty)$. State the domain and the range of g^{-1}.

88. The domain of a one-to-one function g is $[0, 15]$, and its range is $(0, 8)$. State the domain and the range of g^{-1}.

89. A function $y = f(x)$ is increasing on the interval $[0, 5]$. What conclusions can you draw about the graph of $y = f^{-1}(x)$?

90. A function $y = f(x)$ is decreasing on the interval $[0, 5]$. What conclusions can you draw about the graph of $y = f^{-1}(x)$?

91. Find the inverse of the linear function

$$f(x) = mx + b, \quad m \neq 0$$

92. Find the inverse of the function

$$f(x) = \sqrt{r^2 - x^2}, \quad 0 \leq x \leq r$$

93. A function f has an inverse function f^{-1}. If the graph of f lies in quadrant I, in which quadrant does the graph of f^{-1} lie?

94. A function f has an inverse function f^{-1}. If the graph of f lies in quadrant II, in which quadrant does the graph of f^{-1} lie?

95. The function $f(x) = |x|$ is not one-to-one. Find a suitable restriction on the domain of f so that the new function that results is one-to-one. Then find the inverse of the new function.

96. The function $f(x) = x^4$ is not one-to-one. Find a suitable restriction on the domain of f so that the new function that results is one-to-one. Then find the inverse of the new function.

In applications, the symbols used for the independent and dependent variables are often based on common usage. So, rather than using $y = f(x)$ to represent a function, an applied problem might use $C = C(q)$ to represent the cost C of manufacturing q units of a good. Because of this, the inverse notation f^{-1} used in a pure mathematics problem is not used when finding inverses of applied problems. Rather, the inverse of a function such as $C = C(q)$ will be $q = q(C)$. So $C = C(q)$ is a function that represents the cost C as a function of the number q of units manufactured, and $q = q(C)$ is a function that represents the number q as a function of the cost C. Problems 97–100 illustrate this idea.

97. Vehicle Stopping Distance Taking into account reaction time, the distance d (in feet) that a car requires to come to a complete stop while traveling r miles per hour is given by the function

$$d(r) = 6.97r - 90.39$$

(a) Express the speed r at which the car is traveling as a function of the distance d required to come to a complete stop.
(b) Verify that $r = r(d)$ is the inverse of $d = d(r)$ by showing that $r(d(r)) = r$ and $d(r(d)) = d$.
(c) Predict the speed that a car was traveling if the distance required to stop was 300 feet.

98. Height and Head Circumference The head circumference C of a child is related to the height H of the child (both in inches) through the function

$$H(C) = 2.15C - 10.53$$

(a) Express the head circumference C as a function of height H.
(b) Verify that $C = C(H)$ is the inverse of $H = H(C)$ by showing that $H(C(H)) = H$ and $C(H(C)) = C$.
(c) Predict the head circumference of a child who is 26 inches tall.

99. Ideal Body Weight One model for the ideal body weight W for men (in kilograms) as a function of height h (in inches) is given by the function

$$W(h) = 50 + 2.3(h - 60)$$

(a) What is the ideal weight of a 6-foot male?
(b) Express the height h as a function of weight W.
(c) Verify that $h = h(W)$ is the inverse of $W = W(h)$ by showing that $h(W(h)) = h$ and $W(h(W)) = W$.
(d) What is the height of a male who is at his ideal weight of 80 kilograms?

[**Note:** The ideal body weight W for women (in kilograms) as a function of height h (in inches) is given by $W(h) = 45.5 + 2.3(h - 60)$.]

100. Temperature Conversion The function $F(C) = \frac{9}{5}C + 32$ converts a temperature from C degrees Celsius to F degrees Fahrenheit.
(a) Express the temperature in degrees Celsius C as a function of the temperature in degrees Fahrenheit F.
(b) Verify that $C = C(F)$ is the inverse of $F = F(C)$ by showing that $C(F(C)) = C$ and $F(C(F)) = F$.
(c) What is the temperature in degrees Celsius if it is 70 degrees Fahrenheit?

101. Income Taxes The function

$$T(g) = 5226.25 + 0.25(g - 37{,}950)$$

represents the 2017 federal income tax T (in dollars) due for a "single" filer whose modified adjusted gross income is g dollars, where $37{,}950 \leq g \leq 91{,}900$.
(a) What is the domain of the function T?
(b) Given that the tax due T is an increasing linear function of modified adjusted gross income g, find the range of the function T.
(c) Find adjusted gross income g as a function of federal income tax T. What are the domain and the range of this function?

102. Income Taxes The function

$$T(g) = 1865 + 0.15(g - 18{,}650)$$

represents the 2017 federal income tax T (in dollars) due for a "married filing jointly" filer whose modified adjusted gross income is g dollars, where $18{,}650 \leq g \leq 75{,}900$.

(a) What is the domain of the function T?
(b) Given that the tax due T is an increasing linear function of modified adjusted gross income g, find the range of the function T.
(c) Find adjusted gross income g as a function of federal income tax T. What are the domain and the range of this function?

103. Gravity on Earth If a rock falls from a height of 100 meters on Earth, the height H (in meters) after t seconds is approximately

$$H(t) = 100 - 4.9t^2$$

(a) In general, quadratic functions are not one-to-one. However, the function H is one-to-one. Why?
(b) Find the inverse of H and verify your result.
(c) How long will it take a rock to fall 80 meters?

104. Period of a Pendulum The period T (in seconds) of a simple pendulum as a function of its length l (in feet) is given by

$$T(l) = 2\pi\sqrt{\frac{l}{32.2}}$$

(a) Express the length l as a function of the period T.
(b) How long is a pendulum whose period is 3 seconds?

105. Given

$$f(x) = \frac{ax + b}{cx + d}$$

find $f^{-1}(x)$. If $c \neq 0$, under what conditions on a, b, c, and d is $f = f^{-1}$?

Explaining Concepts: Discussion and Writing

106. Can a one-to-one function and its inverse be equal? What must be true about the graph of f for this to happen? Give some examples to support your conclusion.

107. Draw the graph of a one-to-one function that contains the points $(-2, -3)$, $(0, 0)$, and $(1, 5)$. Now draw the graph of its inverse. Compare your graph to those of other students. Discuss any similarities. What differences do you see?

108. Give an example of a function whose domain is the set of real numbers and that is neither increasing nor decreasing on its domain, but is one-to-one.
[**Hint:** Use a piecewise-defined function.]

109. Is every odd function one-to-one? Explain.

110. Suppose that $C(g)$ represents the cost C, in dollars, of manufacturing g cars. Explain what $C^{-1}(800{,}000)$ represents.

111. Explain why the horizontal-line test can be used to identify one-to-one functions from a graph.

112. Explain why a function must be one-to-one in order to have an inverse that is a function. Use the function $y = x^2$ to support your explanation.

Retain Your Knowledge

Problems 113–116 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

113. Use the techniques of shifting, compressing or stretching, and reflections to graph $f(x) = -|x + 2| + 3$.

114. Find the zeros of the quadratic function $f(x) = 3x^2 + 5x + 1$. What are the x-intercepts, if any, of the graph of the function?

115. Find the domain of $R(x) = \dfrac{6x^2 - 11x - 2}{2x^2 - x - 6}$. Find any horizontal, vertical, or oblique asymptotes.

116. If $f(x) = 3x^2 - 7x$, find $f(x + h) - f(x)$.

'Are You Prepared?' Answers

1. Yes; for each input x there is one output y.

2. Increasing on $[0, \infty)$; decreasing on $(-\infty, 0]$

3. $\{x \mid x \neq -6, x \neq 3\}$

4. $\dfrac{x}{1 - x}, x \neq 0, x \neq -1$

4.3 Exponential Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Exponents (Appendix A, Section A.1, pp. A7–A9, and Section A.7, pp. A60–A62)
- Graphing Techniques: Transformations (Section 1.5, pp. 93–102)
- Solving Linear Equations (Appendix A, Section A.8, pp. A67–A69)
- Solve Equations by Factoring (Appendix A, Section A.8, pp. A70–A71)
- Average Rate of Change (Section 1.3, pp. 75–76)
- Quadratic Functions (Section 2.3, pp. 144–152)
- Linear Functions (Section 2.1, pp. 125–132)
- Horizontal Asymptotes (Section 3.4, pp. 245–248)

Now Work the **'Are You Prepared?'** problems on page 313.

OBJECTIVES
1 Evaluate Exponential Functions (p. 302)
2 Graph Exponential Functions (p. 305)
3 Define the Number e (p. 309)
4 Solve Exponential Equations (p. 310)

1 Evaluate Exponential Functions

Appendix A, Section A.7, gives a definition for raising a real number a to a rational power. That discussion provides meaning to expressions of the form

$$a^r$$

where the base a is a positive real number and the exponent r is a rational number.

But what is the meaning of a^x, where the base a is a positive real number and the exponent x is an irrational number? Although a rigorous definition requires methods discussed in calculus, the basis for the definition is easy to follow: Select a rational number r that is formed by truncating (removing) all but a finite number of digits from the irrational number x. Then it is reasonable to expect that

$$a^x \approx a^r$$

For example, take the irrational number $\pi = 3.14159\ldots$. Then an approximation to a^π is

$$a^\pi \approx a^{3.14}$$

where the digits after the hundredths position have been removed from the value for π. A better approximation would be

$$a^\pi \approx a^{3.14159}$$

where the digits after the hundred-thousandths position have been removed. Continuing in this way, we can obtain approximations to a^π to any desired degree of accuracy.

Most calculators have an $\boxed{x^y}$ key or a caret key $\boxed{\wedge}$ for working with exponents. To evaluate expressions of the form a^x, enter the base a, then press the $\boxed{x^y}$ key (or the $\boxed{\wedge}$ key), enter the exponent x, and press $\boxed{=}$ (or $\boxed{\text{ENTER}}$).

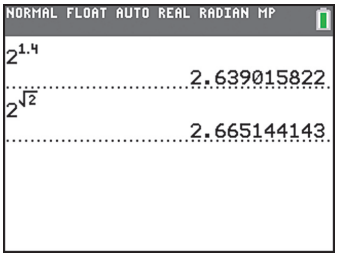


Figure 18 TI-84 Plus C

EXAMPLE 1 Using a Calculator to Evaluate Powers of 2

Using a calculator, evaluate:

(a) $2^{1.4}$ (b) $2^{1.41}$ (c) $2^{1.414}$ (d) $2^{1.4142}$ (e) $2^{\sqrt{2}}$

Solution Figure 18 shows the solution to parts (a) and (e) using a TI-84 Plus C graphing calculator.

(a) $2^{1.4} \approx 2.639015822$
(b) $2^{1.41} \approx 2.657371628$
(c) $2^{1.414} \approx 2.66474965$
(d) $2^{1.4142} \approx 2.665119089$
(e) $2^{\sqrt{2}} \approx 2.665144143$

Now Work PROBLEM 15

It can be shown that the familiar laws for rational exponents hold for real exponents.

THEOREM

Laws of Exponents

If s, t, a, and b are real numbers with $a > 0$ and $b > 0$, then

$$a^s \cdot a^t = a^{s+t} \qquad (a^s)^t = a^{st} \qquad (ab)^s = a^s \cdot b^s$$

$$1^s = 1 \qquad a^{-s} = \frac{1}{a^s} = \left(\frac{1}{a}\right)^s \qquad a^0 = 1 \qquad \textbf{(1)}$$

Introduction to Exponential Growth

Suppose a function f has the following two properties:

1. The value of f doubles with every 1-unit increase in the independent variable x.
2. The value of f at $x = 0$ is 5, so $f(0) = 5$.

Table 1

x	$f(x)$
0	5
1	10
2	20
3	40
4	80

Table 1 shows values of the function f for $x = 0, 1, 2, 3$, and 4.

Let's find an equation $y = f(x)$ that describes this function f. The key fact is that the value of f doubles for every 1-unit increase in x.

$f(0) = 5$

$f(1) = 2f(0) = 2 \cdot 5 = 5 \cdot 2^1$ Double the value of f at 0 to get the value at 1.

$f(2) = 2f(1) = 2(5 \cdot 2) = 5 \cdot 2^2$ Double the value of f at 1 to get the value at 2.

$f(3) = 2f(2) = 2(5 \cdot 2^2) = 5 \cdot 2^3$

$f(4) = 2f(3) = 2(5 \cdot 2^3) = 5 \cdot 2^4$

The pattern leads to

$$f(x) = 2f(x-1) = 2(5 \cdot 2^{x-1}) = 5 \cdot 2^x$$

DEFINITION

An **exponential function** is a function of the form

$$f(x) = Ca^x$$

where a is a positive real number $(a > 0)$, $a \neq 1$, and $C \neq 0$ is a real number. The domain of f is the set of all real numbers. The base a is the **growth factor,** and because $f(0) = Ca^0 = C$, C is called the **initial value.**

WARNING It is important to distinguish a power function, $g(x) = ax^n, n \geq 2$, an integer, from an exponential function, $f(x) = C \cdot a^x, a \neq 1, a > 0$. In a power function, the base is a variable and the exponent is a constant. In an exponential function, the base is a constant and the exponent is a variable. ■

In the definition of an exponential function, the base $a = 1$ is excluded because this function is simply the constant function $f(x) = C \cdot 1^x = C$. Bases that are negative are also excluded, otherwise, many values of x would have to be excluded from the domain, such as $x = \frac{1}{2}$ and $x = \frac{3}{4}$. [Recall that $(-2)^{1/2} = \sqrt{-2}, (-3)^{3/4} = \sqrt[4]{(-3)^3} = \sqrt[4]{-27}$, and so on, are not defined in the set of real numbers.]

Transformations (vertical shifts, horizontal shifts, reflections, and so on) of a function of the form $f(x) = Ca^x$ also represent exponential functions. Some examples of exponential functions are

$$f(x) = 2^x \qquad F(x) = \left(\frac{1}{3}\right)^x + 5 \qquad G(x) = 2 \cdot 3^{x-3}$$

For each function, note that the base of the exponential expression is a constant and the exponent contains a variable.

In the function $f(x) = 5 \cdot 2^x$, notice that the ratio of consecutive outputs is constant for 1-unit increases in the input. This ratio equals the constant 2, the base of the exponential function. In other words,

$$\frac{f(1)}{f(0)} = \frac{5 \cdot 2^1}{5} = 2 \quad \frac{f(2)}{f(1)} = \frac{5 \cdot 2^2}{5 \cdot 2^1} = 2 \quad \frac{f(3)}{f(2)} = \frac{5 \cdot 2^3}{5 \cdot 2^2} = 2 \quad \text{and so on}$$

This leads to the following result.

THEOREM

For an exponential function $f(x) = Ca^x$, where $a > 0$ and $a \neq 1$, if x is any real number, then

$$\boxed{\frac{f(x+1)}{f(x)} = a \quad \text{or} \quad f(x+1) = af(x)}$$

In Words
For 1-unit changes in the input x of an exponential function $f(x) = C \cdot a^x$, the ratio of consecutive outputs is the constant a.

Proof

$$\frac{f(x+1)}{f(x)} = \frac{Ca^{x+1}}{Ca^x} = a^{x+1-x} = a^1 = a \qquad \blacksquare$$

EXAMPLE 2 Identifying Linear or Exponential Functions

Determine whether the given function is linear, exponential, or neither. For those that are linear, find a linear function that models the data. For those that are exponential, find an exponential function that models the data.

(a)

x	y
-1	5
0	2
1	-1
2	-4
3	-7

(b)

x	y
-1	32
0	16
1	8
2	4
3	2

(c)

x	y
-1	2
0	4
1	7
2	11
3	16

Solution For each function, compute the average rate of change of y with respect to x and the ratio of consecutive outputs. If the average rate of change is constant, then the function is linear. If the ratio of consecutive outputs is constant, then the function is exponential.

(a) See Table 2(a). The average rate of change for every 1-unit increase in x is -3. Therefore, the function is a linear function. In a linear function the average rate of change is the slope m, so $m = -3$. The y-intercept b is the value of the function at $x = 0$, so $b = 2$. The linear function that models the data is $f(x) = mx + b = -3x + 2$.

(b) See Table 2(b). For this function, the average rate of change from -1 to 0 is -16, and the average rate of change from 0 to 1 is -8. Because the average rate of change is not constant, the function is not a linear function. The ratio of consecutive outputs for a 1-unit increase in the inputs is a constant, $\frac{1}{2}$. Because the ratio of consecutive outputs is constant, the function is an exponential function with growth factor $a = \frac{1}{2}$. The initial value of the exponential function is $C = 16$. Therefore, the exponential function that models the data is $g(x) = Ca^x = 16 \cdot \left(\frac{1}{2}\right)^x$.

Table 2 **(a)**

x	y	Average Rate of Change	Ratio of Consecutive Outputs
-1	5		
		$\frac{\Delta y}{\Delta x} = \frac{2-5}{0-(-1)} = -3$	$\frac{2}{5}$
0	2		
		$\frac{-1-2}{1-0} = -3$	$\frac{-1}{2} = -\frac{1}{2}$
1	-1		
		$\frac{-4-(-1)}{2-1} = -3$	$\frac{-4}{-1} = 4$
2	-4		
		$\frac{-7-(-4)}{3-2} = -3$	$\frac{-7}{-4} = \frac{7}{4}$
3	-7		

(b)

x	y	Average Rate of Change	Ratio of Consecutive Outputs
-1	32		
		$\frac{\Delta y}{\Delta x} = \frac{16-32}{0-(-1)} = -16$	$\frac{16}{32} = \frac{1}{2}$
0	16		
		-8	$\frac{8}{16} = \frac{1}{2}$
1	8		
		-4	$\frac{4}{8} = \frac{1}{2}$
2	4		
		-2	$\frac{2}{4} = \frac{1}{2}$
3	2		

(c)

x	y	Average Rate of Change	Ratio of Consecutive Outputs
-1	2		
		$\frac{\Delta y}{\Delta x} = \frac{4-2}{0-(-1)} = 2$	2
0	4		
		3	$\frac{7}{4}$
1	7		
		4	$\frac{11}{7}$
2	11		
		5	$\frac{16}{11}$
3	16		

(c) See Table 2(c). For this function, the average rate of change from -1 to 0 is 2, and the average rate of change from 0 to 1 is 3. Because the average rate of change is not constant, the function is not a linear function. The ratio of consecutive outputs from -1 to 0 is 2, and the ratio of consecutive outputs from 0 to 1 is $\frac{7}{4}$. Because the ratio of consecutive outputs is not a constant, the function is not an exponential function.

NOW WORK PROBLEM 27

2 Graph Exponential Functions

If we know how to graph an exponential function of the form $f(x) = a^x$, then we can use transformations (shifting, stretching, and so on) to obtain the graph of any exponential function.

First, let's graph the exponential function $f(x) = 2^x$.

EXAMPLE 3 Graphing an Exponential Function

Graph the exponential function: $f(x) = 2^x$

Solution

The domain of $f(x) = 2^x$ is the set of all real numbers. Begin by locating some points on the graph of $f(x) = 2^x$, as listed in Table 3.

Table 3

x	$f(x) = 2^x$
-10	$2^{-10} \approx 0.00098$
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
10	$2^{10} = 1024$

Because $2^x > 0$ for all x, the range of f is $(0, \infty)$. Therefore, the graph has no x-intercepts, and in fact, the graph will lie above the x-axis for all x. As Table 3 indicates, the y-intercept is 1. Table 3 also indicates that as $x \to -\infty$, the value of $f(x) = 2^x$ gets closer and closer to 0. Thus, the x-axis $(y = 0)$ is a horizontal asymptote to the graph as $x \to -\infty$. This provides the end behavior for x large and negative.

To determine the end behavior for x large and positive, look again at Table 3. As $x \to \infty$, $f(x) = 2^x$ grows very quickly, causing the graph of $f(x) = 2^x$ to rise very rapidly. It is apparent that f is an increasing function and, hence, is one-to-one.

Using all this information, plot some of the points from Table 3 and connect them with a smooth, continuous curve, as shown in Figure 19.

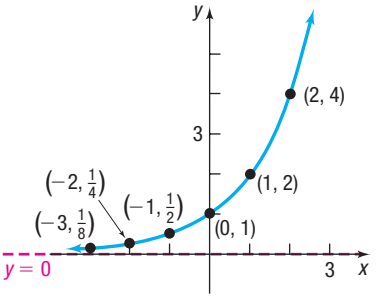
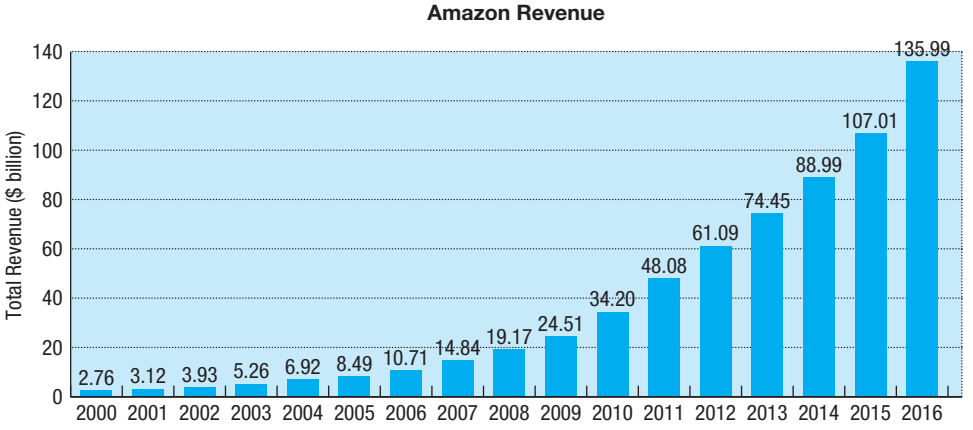


Figure 19 $f(x) = 2^x$

Graphs that look like the one in Figure 19 occur very frequently in a variety of situations. For example, the graph in Figure 20 shows the annual revenue of Amazon from 2000 to 2016. One might conclude from this graph that Amazon's revenue is growing *exponentially*.



Source: *Amazon, Inc.*

Figure 20

Later in this chapter, more will be said about situations that lead to exponential growth. For now, let's continue to explore properties of exponential functions.

The graph of $f(x) = 2^x$ in Figure 19 is typical of all exponential functions of the form $f(x) = a^x$ with $a > 1$. Such functions are increasing functions and hence are one-to-one. Their graphs lie above the x-axis, pass through the point $(0, 1)$, and thereafter rise rapidly as $x \to \infty$. As $x \to -\infty$, the x-axis $(y = 0)$ is a horizontal asymptote. There are no vertical asymptotes. Finally, the graphs are smooth and continuous with no corners or gaps.

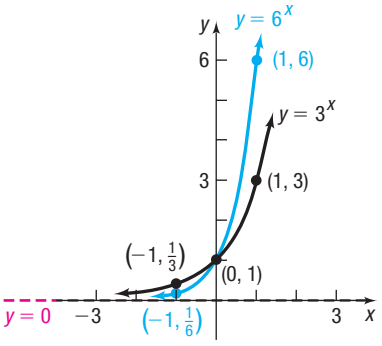


Figure 21

Figure 21 illustrates the graphs of two more exponential functions whose bases are larger than 1. Notice that the larger the base, the steeper the graph is when $x > 0$, and when $x < 0$, the larger the base, the closer the graph of the equation is to the x-axis.

Seeing the Concept

Graph $Y_1 = 2^x$ and compare what you see to Figure 19. Clear the screen, graph $Y_1 = 3^x$ and $Y_2 = 6^x$, and compare what you see to Figure 21. Clear the screen and graph $Y_1 = 10^x$ and $Y_2 = 100^x$.

The following display summarizes information about $f(x) = a^x$, $a > 1$.

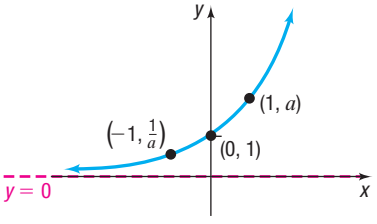


Figure 22 $f(x) = a^x$, $a > 1$

Properties of the Exponential Function $f(x) = a^x$, $a > 1$

1. The domain is the set of all real numbers, or $(-\infty, \infty)$ using interval notation; the range is the set of positive real numbers, or $(0, \infty)$ using interval notation.
2. There are no x-intercepts; the y-intercept is 1.
3. The x-axis $(y = 0)$ is a horizontal asymptote as $x \to -\infty$ $\left[\lim_{x \to -\infty} a^x = 0\right]$.
4. $f(x) = a^x$, $a > 1$, is an increasing function and is one-to-one.
5. The graph of f contains the points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.
6. The graph of f is smooth and continuous, with no corners or gaps. See Figure 22.

Now consider $f(x) = a^x$ when $0 < a < 1$.

EXAMPLE 4 Graphing an Exponential Function

Graph the exponential function: $f(x) = \left(\frac{1}{2}\right)^x$

Solution The domain of $f(x) = \left(\frac{1}{2}\right)^x$ consists of all real numbers. As before, locate some points on the graph as shown in Table 4. Because, $\left(\frac{1}{2}\right)^x > 0$ for all x, the range of f is the interval $(0, \infty)$. The graph lies above the x-axis and has no x-intercepts. The y-intercept is 1. As $x \to -\infty$, $f(x) = \left(\frac{1}{2}\right)^x$ grows very quickly. As $x \to \infty$, the values of $f(x)$ approach 0. The x-axis $(y = 0)$ is a horizontal asymptote as $x \to \infty$. It is apparent that f is a decreasing function and hence is one-to-one. Figure 23 illustrates the graph.

Table 4

x	$f(x) = \left(\frac{1}{2}\right)^x$
−10	$\left(\frac{1}{2}\right)^{-10} = 1024$
−3	$\left(\frac{1}{2}\right)^{-3} = 8$
−2	$\left(\frac{1}{2}\right)^{-2} = 4$
−1	$\left(\frac{1}{2}\right)^{-1} = 2$
0	$\left(\frac{1}{2}\right)^{0} = 1$
1	$\left(\frac{1}{2}\right)^{1} = \frac{1}{2}$
2	$\left(\frac{1}{2}\right)^{2} = \frac{1}{4}$
3	$\left(\frac{1}{2}\right)^{3} = \frac{1}{8}$
10	$\left(\frac{1}{2}\right)^{10} \approx 0.00098$

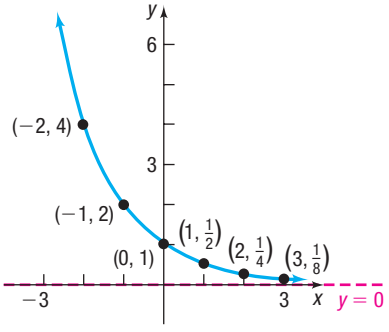


Figure 23 $f(x) = \left(\frac{1}{2}\right)^x$

The graph of $y = \left(\frac{1}{2}\right)^x$ also can be obtained from the graph of $y = 2^x$ using transformations. The graph of $y = \left(\frac{1}{2}\right)^x = 2^{-x}$ is a reflection about the y-axis of the graph of $y = 2^x$ (replace x by $-x$). See Figures 24(a) and (b) on the next page.

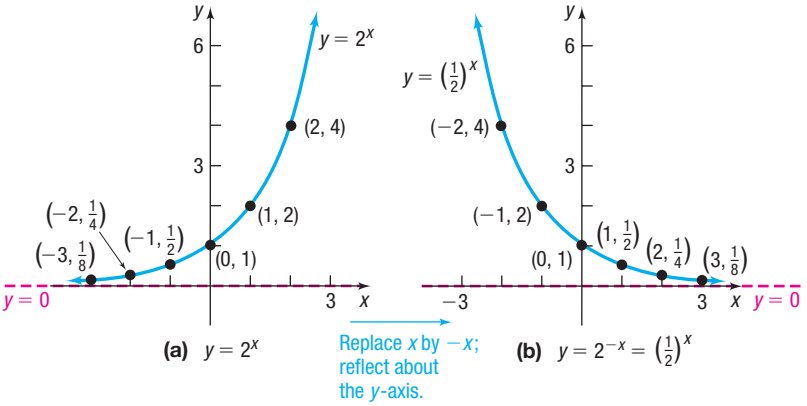


Figure 24

Seeing the Concept

Using a graphing utility, simultaneously graph:

(a) $Y_1 = 3^x$, $Y_2 = \left(\frac{1}{3}\right)^x$ (b) $Y_1 = 6^x$, $Y_2 = \left(\frac{1}{6}\right)^x$

Conclude that the graph of $Y_2 = \left(\frac{1}{a}\right)^x$, for $a > 0$, is the reflection about the y-axis of the graph of $Y_1 = a^x$.

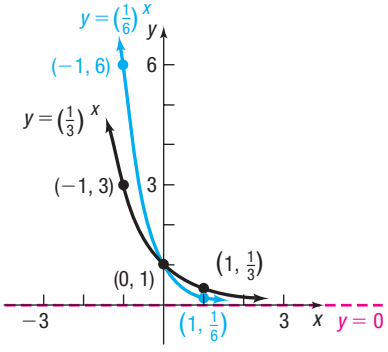


Figure 25

The graph of $f(x) = \left(\frac{1}{2}\right)^x$ in Figure 23 is typical of all exponential functions of the form $f(x) = a^x$ with $0 < a < 1$. Such functions are decreasing and one-to-one. Their graphs lie above the x-axis and pass through the point $(0, 1)$. The graphs rise rapidly as $x \to -\infty$. As $x \to \infty$, the x-axis $(y = 0)$ is a horizontal asymptote. There are no vertical asymptotes. Finally, the graphs are smooth and continuous, with no corners or gaps.

Figure 25 illustrates the graphs of two more exponential functions whose bases are between 0 and 1. Notice that the smaller base results in a graph that is steeper when $x < 0$. When $x > 0$, the graph of the equation with the smaller base is closer to the x-axis.

The following display summarizes information about the function $f(x) = a^x$, $0 < a < 1$.

Properties of the Exponential Function $f(x) = a^x$, $0 < a < 1$

1. The domain is the set of all real numbers, or $(-\infty, \infty)$ using interval notation; the range is the set of positive real numbers, or $(0, \infty)$ using interval notation.
2. There are no x-intercepts; the y-intercept is 1.
3. The x-axis $(y = 0)$ is a horizontal asymptote as $x \to \infty$ $\left[\lim_{x\to\infty} a^x = 0\right]$.
4. $f(x) = a^x$, $0 < a < 1$, is a decreasing function and is one-to-one.
5. The graph of f contains thea points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.
6. The graph of f is smooth and continuous, with no corners or gaps. See Figure 26.

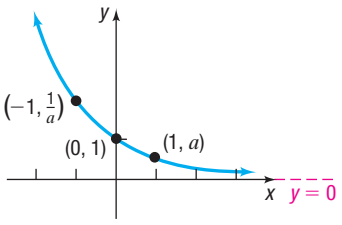


Figure 26 $f(x) = a^x$, $0 < a < 1$

EXAMPLE 5 Graphing Exponential Functions Using Transformations

Graph $f(x) = 2^{-x} - 3$ and determine the domain, range, and horizontal asymptote of f.

Solution Begin with the graph of $y = 2^x$. Figure 27 shows the stages.

As Figure 27(c) illustrates, the domain of $f(x) = 2^{-x} - 3$ is the interval $(-\infty, \infty)$ and the range is the interval $(-3, \infty)$. The horizontal asymptote of f is the line $y = -3$.

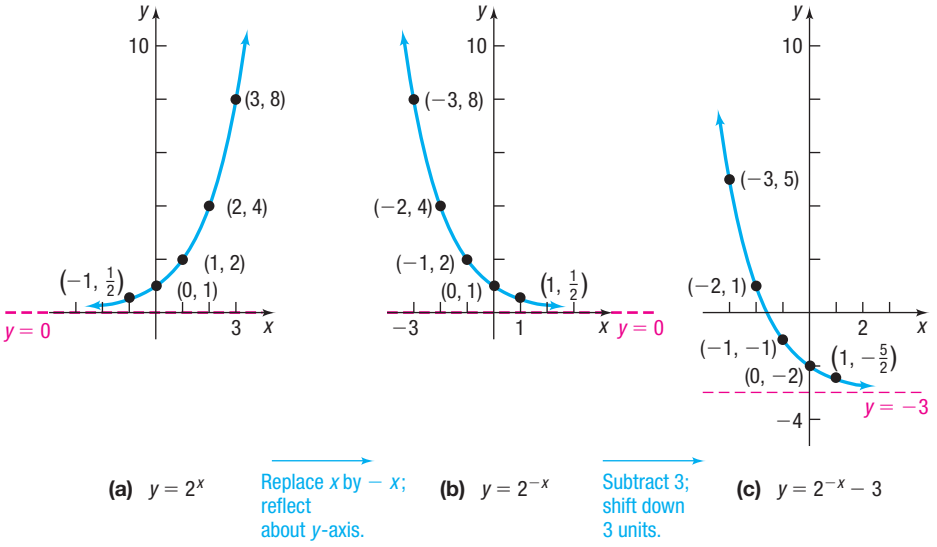


Figure 27

NOW WORK PROBLEM 43

3 Define the Number *e*

Many problems that occur in nature require the use of an exponential function whose base is a certain irrational number, symbolized by the letter e.

One way of arriving at this important number e is given next.

DEFINITION

The **number *e*** is defined as the number that the expression

$$\left(1 + \frac{1}{n}\right)^n \tag{2}$$

approaches as $n \to \infty$. In calculus, this is expressed, using limit notation, as

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Table 5 illustrates what happens to the defining expression (2) as n takes on increasingly large values.

Historical Feature

The number e is named in honor of the Swiss mathematician Leonhard Euler (1707–1783).

Table 5

n	$\frac{1}{n}$	$1 + \frac{1}{n}$	$\left(1 + \frac{1}{n}\right)^n$
1	1	2	2
2	0.5	1.5	2.25
5	0.2	1.2	2.48832
10	0.1	1.1	2.59374246
100	0.01	1.01	2.704813829
1,000	0.001	1.001	2.716923932
10,000	0.0001	1.0001	2.718145927
100,000	0.00001	1.00001	2.718268237
1,000,000	0.000001	1.000001	2.718280469
10,000,000,000	10^{-10}	$1 + 10^{-10}$	2.718281828

Table 6

x	e^x
-2	$e^{-2} \approx 0.14$
-1	$e^{-1} \approx 0.37$
0	$e^0 \approx 1$
1	$e^1 \approx 2.72$
2	$e^2 \approx 7.39$

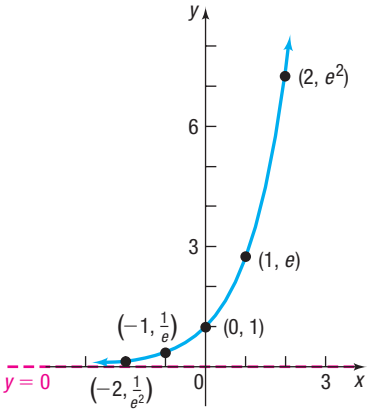


Figure 28 $y = e^x$

The last number in the right column of Table 5 on the previous page is correct to nine decimal places. Therefore, $e = 2.718281828\ldots$. Remember, the three dots indicate that the decimal places continue. Because these decimal places continue but do not repeat, e is an irrational number. The number e is often expressed as a decimal rounded to a specific number of places. For example, $e \approx 2.71828$ is rounded to five decimal places.

The exponential function $f(x) = e^x$, whose base is the number e, occurs with such frequency in applications that it is usually referred to as *the* exponential function. Indeed, most calculators have the key $\boxed{e^x}$ or $\boxed{\exp(x)}$, which may be used to evaluate the exponential function for a given value of x.*

Now use your calculator to approximate e^x for $x = -2, x = -1, x = 0, x = 1$, and $x = 2$. See Table 6. The graph of the exponential function $f(x) = e^x$ is given in Figure 28. Since $2 < e < 3$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$. Do you see why? (Refer to Figures 19 and 21.)

Seeing the Concept

Graph $Y_1 = e^x$ and compare what you see to Figure 28. Use eVALUEate or TABLE to verify the points on the graph shown in Figure 28. Now graph $Y_2 = 2^x$ and $Y_3 = 3^x$ on the same screen as $Y_1 = e^x$. Notice that the graph of $Y_1 = e^x$ lies between these two graphs.

EXAMPLE 6 Graphing Exponential Functions Using Transformations

Graph $f(x) = -e^{x-3}$ and determine the domain, range, and horizontal asymptote of f.

Solution Begin with the graph of $y = e^x$. Figure 29 shows the stages.

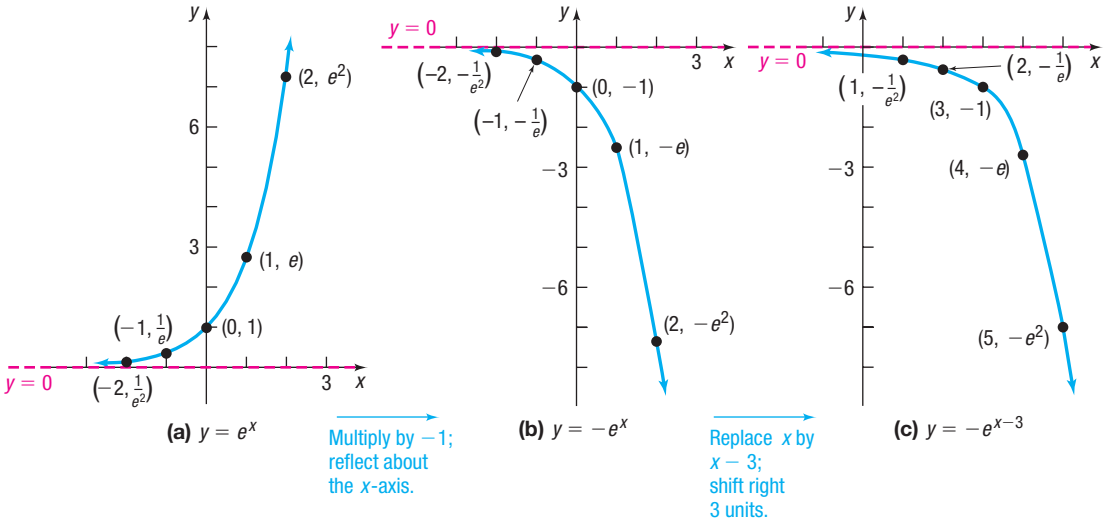


Figure 29

As Figure 29(c) illustrates, the domain of $f(x) = -e^{x-3}$ is the interval $(-\infty, \infty)$, and the range is the interval $(-\infty, 0)$. The horizontal asymptote is the line $y = 0$.

NOW WORK PROBLEM 55

4 Solve Exponential Equations

Equations that involve terms of the form a^x, $a > 0, a \neq 1$, are referred to as **exponential equations.** Such equations can sometimes be solved by appropriately applying the Laws of Exponents and property (3):

*If your calculator does not have one of these keys, refer to your owner's manual.

In Words
When two exponential expressions with the same base are equal, then their exponents are equal.

$$\text{If} \quad a^u = a^v, \quad \text{then} \quad u = v. \qquad \textbf{(3)}$$

Property (3) is a consequence of the fact that exponential functions are one-to-one. To use property (3), each side of the equality must be written with the same base.

EXAMPLE 7 Solving Exponential Equations

Solve each exponential equation.

(a) $3^{x+1} = 81$ (b) $4^{2x-1} = 8^{x+3}$

Solution (a) Because $81 = 3^4$, write the equation as

$$3^{x+1} = 81 = 3^4$$

Now the expressions on each side of the equation have the same base, 3. Set the exponents equal to each other to obtain

$$x + 1 = 4$$
$$x = 3$$

The solution set is $\{3\}$.

(b)

$$4^{2x-1} = 8^{x+3}$$
$$(2^2)^{(2x-1)} = (2^3)^{(x+3)} \qquad 4 = 2^2;\ 8 = 2^3$$
$$2^{2(2x-1)} = 2^{3(x+3)} \qquad (a^r)^s = a^{rs}$$
$$2(2x-1) = 3(x+3) \qquad \text{If } a^u = a^v, \text{ then } u = v.$$
$$4x - 2 = 3x + 9$$
$$x = 11$$

The solution set is $\{11\}$.

Now Work PROBLEMS 65 AND 75

EXAMPLE 8 Solving an Exponential Equation

Solve: $e^{-x^2} = (e^x)^2 \cdot \dfrac{1}{e^3}$

Solution Use the Laws of Exponents first to get a single expression with the base e on the right side.

$$(e^x)^2 \cdot \frac{1}{e^3} = e^{2x} \cdot e^{-3} = e^{2x-3}$$

As a result,

$$e^{-x^2} = e^{2x-3}$$
$$-x^2 = 2x - 3 \qquad \text{Apply property (3).}$$
$$x^2 + 2x - 3 = 0 \qquad \text{Place the quadratic equation in standard form.}$$
$$(x+3)(x-1) = 0 \qquad \text{Factor.}$$
$$x = -3 \quad \text{or} \quad x = 1 \qquad \text{Use the Zero-Product Property.}$$

The solution set is $\{-3, 1\}$.

Now Work PROBLEM 81

EXAMPLE 9 Exponential Probability

Between 9:00 PM and 10:00 PM cars arrive at Burger King's drive-thru at the rate of 12 cars per hour (0.2 car per minute). The following formula from statistics can be used to determine the probability that a car will arrive within t minutes of 9:00 PM.

$$F(t) = 1 - e^{-0.2t}$$

(a) Determine the probability that a car will arrive within 5 minutes of 9 PM (that is, before 9:05 PM).

(b) Determine the probability that a car will arrive within 30 minutes of 9 PM (before 9:30 PM).

(c) Graph F using your graphing utility.

(d) What value does F approach as t increases without bound in the positive direction?

Solution (a) The probability that a car will arrive within 5 minutes is found by evaluating $F(t)$ at $t = 5$.

$$F(5) = 1 - e^{-0.2(5)} \approx 0.63212$$

Use a calculator.

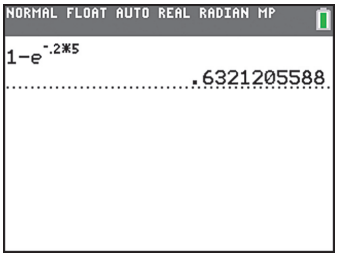


Figure 30 $F(5)$

Figure 30 shows this calculation on a TI-84 Plus C graphing calculator. There is a 63% probability that a car will arrive within 5 minutes.

(b) The probability that a car will arrive within 30 minutes is found by evaluating $F(t)$ at $t = 30$.

$$F(30) = 1 - e^{-0.2(30)} \approx 0.9975$$

Use a calculator.

There is a 99.75% probability that a car will arrive within 30 minutes.

(c) See Figure 31 for the graph of F.

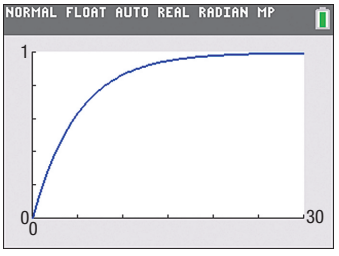


Figure 31 $F(t) = 1 - e^{-0.2t}$

(d) As time passes, the probability that a car will arrive increases. The value that F approaches can be found by letting $t \to \infty$. Since $e^{-0.2t} = \dfrac{1}{e^{0.2t}}$, it follows that $e^{-0.2t} \to 0$ as $t \to \infty$. Therefore, F approaches 1 as t gets large. The algebraic analysis is confirmed by Figure 31.

NOW WORK PROBLEM 113

SUMMARY Properties of the Exponential Function

$f(x) = a^x, \quad a > 1$	Domain: the interval $(-\infty, \infty)$; range: the interval $(0, \infty)$ x-intercepts: none; y-intercept: 1 Horizontal asymptote: x-axis $(y = 0)$ as $x \to -\infty$ Increasing; one-to-one; smooth; continuous See Figure 22 for a typical graph.
$f(x) = a^x, \quad 0 < a < 1$	Domain: the interval $(-\infty, \infty)$; range: the interval $(0, \infty)$ x-intercepts: none; y-intercept: 1 Horizontal asymptote: x-axis $(y = 0)$ as $x \to \infty$ Decreasing; one-to-one; smooth; continuous See Figure 26 for a typical graph.

If $a^u = a^v$, then $u = v$.

4.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. $4^3 =$ ____; $8^{2/3} =$ ___; $3^{-2} =$ ___. (pp. A7–A9; A60–A62)

2. Solve: $x^2 + 3x = 4$ (pp. A70–A71; 145–150)

3. ***True or False*** To graph $y = (x - 2)^3$, shift the graph of $y = x^3$ to the left 2 units. (pp. 93–96)

4. Find the average rate of change of $f(x) = 3x - 5$ from $x = 0$ to $x = 4$. (pp. 75–76; 125–128)

5. ***True or False*** The function $f(x) = \dfrac{2x}{x - 3}$ has $y = 2$ as a horizontal asymptote. (pp. 245–248)

Concepts and Vocabulary

6. A(n) _____ _____ is a function of the form $f(x) = Ca^x$, where $a > 0$, $a \neq 1$, and $C \neq 0$ are real numbers. The base a is the _____ _____ and C is the _____ _____.

7. For an exponential function $f(x) = Ca^x$, $\dfrac{f(x+1)}{f(x)} =$ ___.

8. ***True or False*** The domain of the exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is the set of all real numbers.

9. ***True or False*** The graph of the exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, has no x-intercept.

10. The graph of every exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, passes through three points: ____, ____, and ____.

11. If $3^x = 3^4$, then $x =$ ___.

12. ***True or False*** The graphs of $y = 3^x$ and $y = \left(\dfrac{1}{3}\right)^x$ are identical.

13. Which of the following exponential functions is an increasing function?

(a) $f(x) = 0.5^x$ (b) $f(x) = \left(\dfrac{5}{2}\right)^x$

(c) $f(x) = \left(\dfrac{2}{3}\right)^x$ (d) $f(x) = 0.9^x$

14. Which of the following is the range of the exponential function $f(x) = a^x, a > 0$ and $a \neq 1$?

(a) $(-\infty, \infty)$ (b) $(-\infty, 0)$

(c) $(0, \infty)$ (d) $(-\infty, 0) \cup (0, \infty)$

Skill Building

In Problems 15–26, approximate each number using a calculator. Express your answer rounded to three decimal places.

15. (a) $2^{3.14}$ (b) $2^{3.141}$ (c) $2^{3.1415}$ (d) 2^{π}

16. (a) $2^{2.7}$ (b) $2^{2.71}$ (c) $2^{2.718}$ (d) 2^{e}

17. (a) $3.1^{2.7}$ (b) $3.14^{2.71}$ (c) $3.141^{2.718}$ (d) π^{e}

18. (a) $2.7^{3.1}$ (b) $2.71^{3.14}$ (c) $2.718^{3.141}$ (d) e^{π}

19. $(1 + 0.04)^6$

20. $\left(1 + \dfrac{0.09}{12}\right)^{24}$

21. $8.4\left(\dfrac{1}{3}\right)^{2.9}$

22. $158\left(\dfrac{5}{6}\right)^{8.63}$

23. $e^{1.2}$

24. $e^{-1.3}$

25. $125e^{0.026(7)}$

26. $83.6e^{-0.157(9.5)}$

In Problems 27–34, determine whether the given function is linear, exponential, or neither. For those that are linear functions, find a linear function that models the data; for those that are exponential, find an exponential function that models the data.

27.

x	$f(x)$
−1	3
0	6
1	12
2	18
3	30

28.

x	$g(x)$
−1	2
0	5
1	8
2	11
3	14

29.

x	$H(x)$
−1	$\frac{1}{4}$
0	1
1	4
2	16
3	64

30.

x	$f(x)$
−1	$\frac{3}{2}$
0	3
1	6
2	12
3	24

31.

x	$H(x)$
-1	2
0	4
1	6
2	8
3	10

32.

x	$g(x)$
-1	6
0	1
1	0
2	3
3	10

33.

x	$F(x)$
-1	$\frac{2}{3}$
0	1
1	$\frac{3}{2}$
2	$\frac{9}{4}$
3	$\frac{27}{8}$

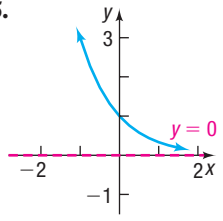
34.

x	$F(x)$
-1	$\frac{1}{2}$
0	$\frac{1}{4}$
1	$\frac{1}{8}$
2	$\frac{1}{16}$
3	$\frac{1}{32}$

In Problems 35–42, the graph of an exponential function is given. Match each graph to one of the following functions:

(A) $y = 3^x$ (B) $y = 3^{-x}$ (C) $y = -3^x$ (D) $y = -3^{-x}$
(E) $y = 3^x - 1$ (F) $y = 3^{x-1}$ (G) $y = 3^{1-x}$ (H) $y = 1 - 3^x$

35.



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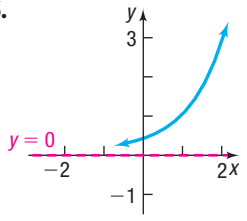

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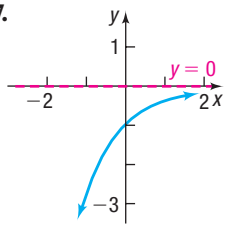

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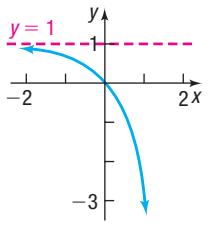

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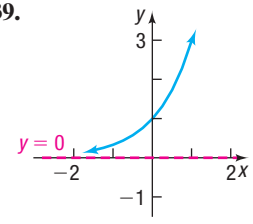

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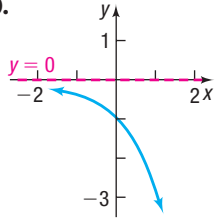

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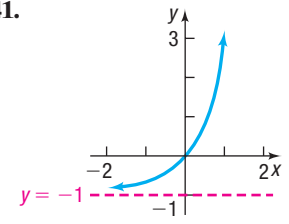

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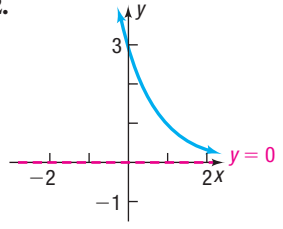


In Problems 43–54, use transformations to graph each function. Determine the domain, range, and horizontal asymptote of each function.

43. $f(x) = 2^x + 1$ **44.** $f(x) = 3^x - 2$ **45.** $f(x) = 3^{x-1}$ **46.** $f(x) = 2^{x+2}$

47. $f(x) = 3 \cdot \left(\frac{1}{2}\right)^x$ **48.** $f(x) = 4 \cdot \left(\frac{1}{3}\right)^x$ **49.** $f(x) = 3^{-x} - 2$ **50.** $f(x) = -3^x + 1$

51. $f(x) = 2 + 4^{x-1}$ **52.** $f(x) = 1 - 2^{x+3}$ **53.** $f(x) = 2 + 3^{x/2}$ **54.** $f(x) = 1 - 2^{-x/3}$

In Problems 55–62, begin with the graph of $y = e^x$ (Figure 28) and use transformations to graph each function. Determine the domain, range, and horizontal asymptote of each function.

55. $f(x) = e^{-x}$ **56.** $f(x) = -e^x$ **57.** $f(x) = e^{x+2}$ **58.** $f(x) = e^x - 1$

59. $f(x) = 5 - e^{-x}$ **60.** $f(x) = 9 - 3e^{-x}$ **61.** $f(x) = 2 - e^{-x/2}$ **62.** $f(x) = 7 - 3e^{2x}$

In Problems 63–82, solve each equation.

63. $7^x = 7^3$ **64.** $5^x = 5^{-6}$ **65.** $2^{-x} = 16$ **66.** $3^{-x} = 81$

67. $\left(\frac{1}{5}\right)^x = \frac{1}{25}$ **68.** $\left(\frac{1}{4}\right)^x = \frac{1}{64}$ **69.** $2^{2x-1} = 4$ **70.** $5^{x+3} = \frac{1}{5}$

71. $3^{x^3} = 9^x$ **72.** $4^{x^2} = 2^x$ **73.** $8^{-x+14} = 16^x$ **74.** $9^{-x+15} = 27^x$

75. $3^{x^2-7} = 27^{2x}$ **76.** $5^{x^2+8} = 125^{2x}$ **77.** $4^x \cdot 2^{x^2} = 16^2$ **78.** $9^{2x} \cdot 27^{x^2} = 3^{-1}$

79. $e^x = e^{3x+8}$ **80.** $e^{3x} = e^{2-x}$ **81.** $e^{x^2} = e^{3x} \cdot \frac{1}{e^2}$ **82.** $(e^4)^x \cdot e^{x^2} = e^{12}$

83. If $4^x = 7$, what does 4^{-2x} equal?

84. If $2^x = 3$, what does 4^{-x} equal?

85. If $3^{-x} = 2$, what does 3^{2x} equal?

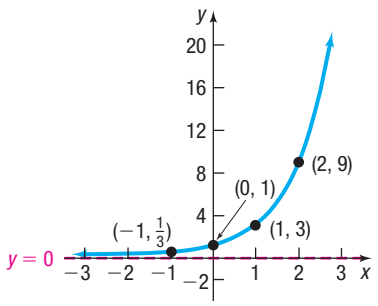
86. If $5^{-x} = 3$, what does 5^{3x} equal?

87. If $9^x = 25$, what does 3^x equal?

88. If $2^{-3x} = \frac{1}{1000}$, what does 2^x equal?

In Problems 89–92, determine the exponential function whose graph is given.

89.



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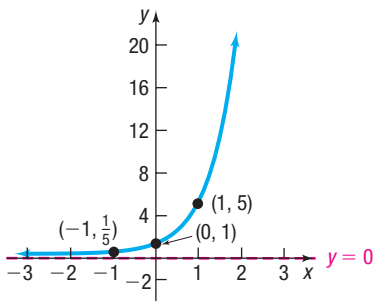

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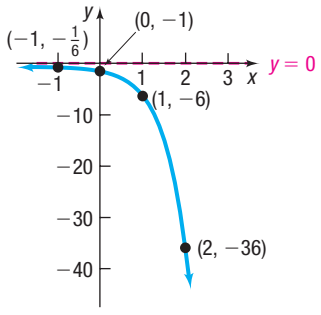

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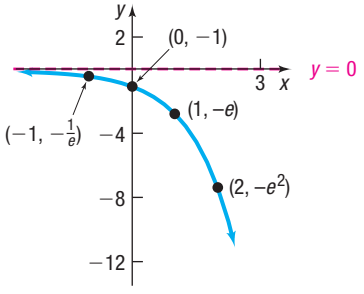


93. Find an exponential function with horizontal asymptote $y = 2$ whose graph contains the points $(0, 3)$ and $(1, 5)$.

94. Find an exponential function with horizontal asymptote $y = -3$ whose graph contains the points $(0, -2)$ and $(-2, 1)$.

Mixed Practice

95. Suppose that $f(x) = 2^x$.
(a) What is $f(4)$? What point is on the graph of f?
(b) If $f(x) = \frac{1}{16}$, what is x? What point is on the graph of f?

96. Suppose that $f(x) = 3^x$.
(a) What is $f(4)$? What point is on the graph of f?
(b) If $f(x) = \frac{1}{9}$, what is x? What point is on the graph of f?

97. Suppose that $g(x) = 4^x + 2$.
(a) What is $g(-1)$? What point is on the graph of g?
(b) If $g(x) = 66$, what is x? What point is on the graph of g?

98. Suppose that $g(x) = 5^x - 3$.
(a) What is $g(-1)$? What point is on the graph of g?
(b) If $g(x) = 122$, what is x? What point is on the graph of g?

99. Suppose that $H(x) = \left(\frac{1}{2}\right)^x - 4$.
(a) What is $H(-6)$? What point is on the graph of H?
(b) If $H(x) = 12$, what is x? What point is on the graph of H?
(c) Find the zero of H.

100. Suppose that $F(x) = \left(\frac{1}{3}\right)^x - 3$.
(a) What is $F(-5)$? What point is on the graph of F?
(b) If $F(x) = 24$, what is x? What point is on the graph of F?
(c) Find the zero of F.

In Problems 101–104, graph each function. Based on the graph, state the domain and the range, and find any intercepts.

101. $f(x) = \begin{cases} e^{-x} & \text{if } x < 0 \\ e^{x} & \text{if } x \geq 0 \end{cases}$

102. $f(x) = \begin{cases} e^{x} & \text{if } x < 0 \\ e^{-x} & \text{if } x \geq 0 \end{cases}$

103. $f(x) = \begin{cases} -e^{x} & \text{if } x < 0 \\ -e^{-x} & \text{if } x \geq 0 \end{cases}$

104. $f(x) = \begin{cases} -e^{-x} & \text{if } x < 0 \\ -e^{x} & \text{if } x \geq 0 \end{cases}$

Applications and Extensions

105. Optics If a single pane of glass obliterates 3% of the light passing through it, the percent p of light that passes through n successive panes is given approximately by the function

$$p(n) = 100(0.97)^n$$

(a) What percent of light will pass through 10 panes?
(b) What percent of light will pass through 25 panes?
(c) Explain the meaning of the base 0.97 in this problem.

106. Atmospheric Pressure The atmospheric pressure p on a balloon or airplane decreases with increasing height. This pressure, measured in millimeters of mercury, is related to the height h (in kilometers) above sea level by the function

$$p(h) = 760e^{-0.145h}$$

(a) Find the atmospheric pressure at a height of 2 km (over a mile).
(b) What is it at a height of 10 kilometers (over 30,000 feet)?

107. Depreciation The price p, in dollars, of a Honda Civic EX-L sedan that is x years old is modeled by

$$p(x) = 22{,}265(0.90)^x$$

(a) How much should a 3-year-old Civic EX-L sedan cost?
(b) How much should a 9-year-old Civic EX-L sedan cost?
(c) Explain the meaning of the base 0.90 in this problem.

108. Healing of Wounds The normal healing of wounds can be modeled by an exponential function. If A_0 represents the original area of the wound and if A equals the area of the wound, then the function

$$A(n) = A_0 e^{-0.35n}$$

describes the area of a wound after n days following an injury when no infection is present to retard the healing. Suppose that a wound initially had an area of 100 square millimeters.

(a) If healing is taking place, how large will the area of the wound be after 3 days?

(b) How large will it be after 10 days?

109. Advanced-Stage Pancreatic Cancer The percentage of patients P who have survived t years after initial diagnosis of advanced-stage pancreatic cancer is modeled by the function

$$P(t) = 100(0.3)^t$$

Source: *Cancer Treatment Centers of America*

(a) According to the model, what percent of patients survive 1 year after initial diagnosis?

(b) What percent of patients survive 2 years after initial diagnosis?

(c) Explain the meaning of the base 0.3 in the context of this problem.

110. Endangered Species In a protected environment, the population P of a certain endangered species recovers over time t (in years) according to the model

$$P(t) = 30(1.149)^t$$

(a) What is the size of the initial population of the species?

(b) According to the model, what will be the population of the species in 5 years?

(c) According to the model, what will be the population of the species in 10 years?

(d) According to the model, what will be the population of the species in 15 years?

(e) What is happening to the population every 5 years?

111. Drug Medication The function

$$D(h) = 5e^{-0.4h}$$

can be used to find the number of milligrams D of a certain drug that is in a patient's bloodstream h hours after the drug has been administered. How many milligrams will be present after 1 hour? After 6 hours?

112. Spreading of Rumors A model for the number N of people in a college community who have heard a certain rumor is

$$N = P(1 - e^{-0.15d})$$

where P is the total population of the community and d is the number of days that have elapsed since the rumor began. In a community of 1000 students, how many students will have heard the rumor after 3 days?

113. Exponential Probability Between 12:00 PM and 1:00 PM, cars arrive at Citibank's drive-thru at the rate of 6 cars per hour (0.1 car per minute). The following formula from statistics can be used to determine the probability that a car will arrive within t minutes of 12:00 PM.

$$F(t) = 1 - e^{-0.1t}$$

(a) Determine the probability that a car will arrive within 10 minutes of 12:00 PM (that is, before 12:10 PM).

(b) Determine the probability that a car will arrive within 40 minutes of 12:00 PM (before 12:40 PM).

(c) What value does F approach as t becomes unbounded in the positive direction?

(d) Graph F using a graphing utility.

(e) Using INTERSECT, determine how many minutes are needed for the probability to reach 50%.

114. Exponential Probability Between 5:00 PM and 6:00 PM, cars arrive at Jiffy Lube at the rate of 9 cars per hour (0.15 car per minute). This formula from statistics can be used to determine the probability that a car will arrive within t minutes of 5:00 PM:

$$F(t) = 1 - e^{-0.15t}$$

(a) Determine the probability that a car will arrive within 15 minutes of 5:00 PM (that is, before 5:15 PM).

(b) Determine the probability that a car will arrive within 30 minutes of 5:00 PM (before 5:30 PM).

(c) What value does F approach as t becomes unbounded in the positive direction?

(d) Graph F using a graphing utility.

(e) Using INTERSECT, determine how many minutes are needed for the probability to reach 60%.

115. Poisson Probability Between 5:00 PM and 6:00 PM, cars arrive at a McDonald's drive-thru at the rate of 20 cars per hour. The following formula from statistics can be used to determine the probability that x cars will arrive between 5:00 PM and 6:00 PM.

$$P(x) = \frac{20^x e^{-20}}{x!}$$

where

$$x! = x \cdot (x-1) \cdot (x-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

(a) Determine the probability that $x = 15$ cars will arrive between 5:00 PM and 6:00 PM.

(b) Determine the probability that $x = 20$ cars will arrive between 5:00 PM and 6:00 PM.

116. Poisson Probability People enter a line for the *Demon Roller Coaster* at the rate of 4 per minute. The following formula from statistics can be used to determine the probability that x people will arrive within the next minute.

$$P(x) = \frac{4^x e^{-4}}{x!}$$

where

$$x! = x \cdot (x-1) \cdot (x-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

(a) Determine the probability that $x = 5$ people will arrive within the next minute.

(b) Determine the probability that $x = 8$ people will arrive within the next minute.

117. Relative Humidity The relative humidity is the ratio (expressed as a percent) of the amount of water vapor in the air to the maximum amount that the air can hold at a specific temperature. The relative humidity, R, is found using the following formula:

$$R = 10^{\left(\frac{4221}{T+459.4} - \frac{4221}{D+459.4} + 2\right)}$$

where T is the air temperature (in °F) and D is the dew point temperature (in °F).

(a) Determine the relative humidity if the air temperature is 50° Fahrenheit and the dew point temperature is 41° Fahrenheit.

(b) Determine the relative humidity if the air temperature is 68° Fahrenheit and the dew point temperature is 59° Fahrenheit.
(c) What is the relative humidity if the air temperature and the dew point temperature are the same?

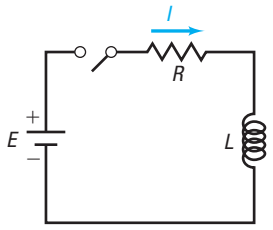
118. Learning Curve Suppose that a student has 500 vocabulary words to learn. If the student learns 15 words after 5 minutes, the function

$$L(t) = 500(1 - e^{-0.0061t})$$

approximates the number of words L that the student will have learned after t minutes.
(a) How many words will the student have learned after 30 minutes?
(b) How many words will the student have learned after 60 minutes?

119. Current in an RL Circuit The equation governing the amount of current I (in amperes) after time t (in seconds) in a single RL circuit consisting of a resistance R (in ohms), an inductance L (in henrys), and an electromotive force E (in volts) is

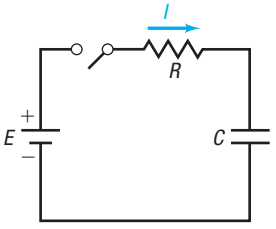
$$I = \frac{E}{R}[1 - e^{-(R/L)t}]$$



(a) If $E = 120$ volts, $R = 10$ ohms, and $L = 5$ henrys, how much current I_1 is flowing after 0.3 second? After 0.5 second? After 1 second?
(b) What is the maximum current?
(c) Graph this function $I = I_1(t)$, measuring I along the y-axis and t along the x-axis.
(d) If $E = 120$ volts, $R = 5$ ohms, and $L = 10$ henrys, how much current I_2 is flowing after 0.3 second? After 0.5 second? After 1 second?
(e) What is the maximum current?
(f) Graph the function $I = I_2(t)$ on the same coordinate axes as $I_1(t)$.

120. Current in an RC Circuit The equation governing the amount of current I (in amperes) after time t (in microseconds) in a single RC circuit consisting of a resistance R (in ohms), a capacitance C (in microfarads), and an electromotive force E (in volts) is

$$I = \frac{E}{R}e^{-t/(RC)}$$



(a) If $E = 120$ volts, $R = 2000$ ohms, and $C = 1.0$ microfarad, how much current I_1 is flowing initially $(t = 0)$? After 1000 microseconds? After 3000 microseconds?
(b) What is the maximum current?
(c) Graph the function $I = I_1(t)$, measuring I along the y-axis and t along the x-axis.
(d) If $E = 120$ volts, $R = 1000$ ohms, and $C = 2.0$ microfarads, how much current I_2 is flowing initially? After 1000 microseconds? After 3000 microseconds?
(e) What is the maximum current?
(f) Graph the function $I = I_2(t)$ on the same coordinate axes as $I_1(t)$.

121. If f is an exponential function of the form $f(x) = Ca^x$ with growth factor 3, and if $f(6) = 12$, what is $f(7)$?

122. Another Formula for *e* Use a calculator to compute the values of

$$2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

for $n = 4, 6, 8$, and 10. Compare each result with e.
[**Hint:** $1! = 1, 2! = 2 \cdot 1, 3! = 3 \cdot 2 \cdot 1,$
$n! = n(n-1) \cdot \cdots \cdot (3)(2)(1)$.]

123. Another Formula for *e* Use a calculator to compute the various values of the expression. Compare the values to e.

$$2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{\text{etc.}}}}}}$$

124. Difference Quotient If $f(x) = a^x$, show that

$$\frac{f(x+h) - f(x)}{h} = a^x \cdot \frac{a^h - 1}{h} \quad h \neq 0$$

125. If $f(x) = a^x$, show that $f(A + B) = f(A) \cdot f(B)$.

126. If $f(x) = a^x$, show that $f(-x) = \dfrac{1}{f(x)}$.

127. If $f(x) = a^x$, show that $f(\alpha x) = [f(x)]^\alpha$.

Problems 128 and 129 provide definitions for two other transcendental functions.

128. The **hyperbolic sine function,** designated by $\sinh x$, is defined as

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

(a) Show that $f(x) = \sinh x$ is an odd function.
(b) Graph $f(x) = \sinh x$ using a graphing utility.

129. The **hyperbolic cosine function,** designated by $\cosh x$, is defined as

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

(a) Show that $f(x) = \cosh x$ is an even function.
(b) Graph $f(x) = \cosh x$ using a graphing utility.
(c) Refer to Problem 128. Show that, for every x,

$$(\cosh x)^2 - (\sinh x)^2 = 1$$

130. Historical Problem Pierre de Fermat (1601–1665) conjectured that the function

$$f(x) = 2^{(2^x)} + 1$$

for $x = 1, 2, 3, \ldots$, would always have a value equal to a prime number. But Leonhard Euler (1707–1783) showed that this formula fails for $x = 5$. Use a calculator to determine the prime numbers produced by f for $x = 1, 2, 3, 4$. Then show that $f(5) = 641 \times 6{,}700{,}417$, which is not prime.

Explaining Concepts: Discussion and Writing

131. The number of bacteria in a 4-liter container doubles every minute. After 60 minutes the container is full. How long did it take to fill half the container?

132. Explain in your own words what the number e is. Provide at least two applications that use this number.

133. Do you think that there is a power function that increases more rapidly than an exponential function whose base is greater than 1? Explain.

134. As the base a of an exponential function $f(x) = a^x$, where $a > 1$, increases, what happens to the behavior of its graph for $x > 0$? What happens to the behavior of its graph for $x < 0$?

135. The graphs of $y = a^{-x}$ and $y = \left(\frac{1}{a}\right)^x$ are identical. Why?

Retain Your Knowledge

Problems 136–139 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

136. Solve the inequality: $x^3 + 5x^2 \le 4x + 20$.

137. Solve the inequality: $\frac{x+1}{x-2} \ge 1$.

138. Find the equation of the quadratic function f that has its vertex at $(3, 5)$ and contains the point $(2, 3)$.

139. Consider the quadratic function $f(x) = x^2 + 2x - 3$.
(a) Graph f by determining whether its graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any.
(b) Determine the domain and range of f.
(c) Determine where f is increasing and where it is decreasing.

'Are You Prepared?' Answers

1. 64; 4; $\frac{1}{9}$ **2.** $\{-4, 1\}$ **3.** False **4.** 3 **5.** True

4.4 Logarithmic Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solve Linear Inequalities (Appendix A, Section A.10, pp. A87–A88)
- Solve Quadratic Inequalities (Section 2.5, pp. 167–169)
- Polynomial and Rational Inequalities (Section 3.6, pp. 266–270)
- Solve Linear Equations (Appendix A, Section A.8, pp. A67–A69)

Now Work the **'Are You Prepared?'** problems on page 327.

OBJECTIVES

Recall that a one-to-one function $y = f(x)$ has an inverse function that is defined (implicitly) by the equation $x = f(y)$. In particular, the exponential function $y = f(x) = a^x$, where $a > 0$ and $a \ne 1$, is one-to-one and, hence, has an inverse function that is defined implicitly by the equation

$$x = a^y, \quad a > 0, \quad a \ne 1$$

This inverse function is so important that it is given a name, the *logarithmic function*.

DEFINITION

The **logarithmic function to the base *a*,** where $a > 0$ and $a \neq 1$, is denoted by $y = \log_a x$ (read as "y is the logarithm to the base a of x") and is defined by

$$y = \log_a x \quad \text{if and only if} \quad x = a^y$$

The domain of the logarithmic function $y = \log_a x$ is $x > 0$.

In Words
When you read $\log_a x$, think to yourself "*a* raised to what power gives me *x*?"

As this definition illustrates, **a logarithm is a name for a certain exponent.** So $\log_a x$ represents the exponent to which a must be raised to obtain x.

EXAMPLE 1 **Relating Logarithms to Exponents**

(a) If $y = \log_3 x$, then $x = 3^y$. For example, the logarithmic statement $4 = \log_3 81$ is equivalent to the exponential statement $81 = 3^4$.

(b) If $y = \log_5 x$, then $x = 5^y$. For example, $-1 = \log_5\left(\frac{1}{5}\right)$ is equivalent to $\frac{1}{5} = 5^{-1}$.

1 Change Exponential Statements to Logarithmic Statements and Logarithmic Statements to Exponential Statements

The definition of a logarithm can be used to convert from exponential form to logarithmic form, and vice versa, as the following two examples illustrate.

EXAMPLE 2 **Changing Exponential Statements to Logarithmic Statements**

Change each exponential statement to an equivalent statement involving a logarithm.

(a) $1.2^3 = m$ (b) $e^b = 9$ (c) $a^4 = 24$

Solution Use the fact that $y = \log_a x$ and $x = a^y$, where $a > 0$ and $a \neq 1$, are equivalent.

(a) If $1.2^3 = m$, then $3 = \log_{1.2} m$.
(b) If $e^b = 9$, then $b = \log_e 9$.
(c) If $a^4 = 24$, then $4 = \log_a 24$.

NOW WORK PROBLEM 11

EXAMPLE 3 **Changing Logarithmic Statements to Exponential Statements**

Change each logarithmic statement to an equivalent statement involving an exponent.

(a) $\log_a 4 = 5$ (b) $\log_e b = -3$ (c) $\log_3 5 = c$

Solution
(a) If $\log_a 4 = 5$, then $a^5 = 4$.
(b) If $\log_e b = -3$, then $e^{-3} = b$.
(c) If $\log_3 5 = c$, then $3^c = 5$.

NOW WORK PROBLEM 19

2 Evaluate Logarithmic Expressions

To find the exact value of a logarithm, write the logarithm in exponential notation using the fact that $y = \log_a x$ is equivalent to $a^y = x$, and use the fact that if $a^u = a^v$, then $u = v$.

EXAMPLE 4 **Finding the Exact Value of a Logarithmic Expression**

Find the exact value of:

(a) $\log_2 16$ (b) $\log_3 \frac{1}{27}$

(*continued*)

Solution (a) To evaluate $\log_2 16$, think "2 raised to what power yields 16?"

$$y = \log_2 16$$
$$2^y = 16 \quad \text{Change to exponential form.}$$
$$2^y = 2^4 \quad 16 = 2^4$$
$$y = 4 \quad \text{Equate exponents.}$$

Therefore, $\log_2 16 = 4$.

(b) To evaluate $\log_3 \frac{1}{27}$, think "3 raised to what power yields $\frac{1}{27}$?"

$$y = \log_3 \frac{1}{27}$$
$$3^y = \frac{1}{27} \quad \text{Change to exponential form.}$$
$$3^y = 3^{-3} \quad \frac{1}{27} = \frac{1}{3^3} = 3^{-3}$$
$$y = -3 \quad \text{Equate exponents.}$$

Therefore, $\log_3 \frac{1}{27} = -3$.

NOW WORK PROBLEM 27

3 Determine the Domain of a Logarithmic Function

The logarithmic function $y = \log_a x$ has been defined as the inverse of the exponential function $y = a^x$. That is, if $f(x) = a^x$, then $f^{-1}(x) = \log_a x$. Based on the discussion given in Section 4.2 on inverse functions, for a function f and its inverse f^{-1},

$$\text{Domain of } f^{-1} = \text{Range of } f \quad \text{and} \quad \text{Range of } f^{-1} = \text{Domain of } f$$

Consequently, it follows that

$$\boxed{\begin{array}{l}\text{Domain of the logarithmic function} = \text{Range of the exponential function} = (0, \infty) \\ \text{Range of the logarithmic function} = \text{Domain of the exponential function} = (-\infty, \infty)\end{array}}$$

The next box summarizes some properties of the logarithmic function:

$$\boxed{\begin{array}{c} y = \log_a x \quad (\text{defining equation: } x = a^y) \\ \text{Domain: } (0, \infty) \quad \text{Range: } (-\infty, \infty)\end{array}}$$

The domain of a logarithmic function consists of the *positive* real numbers, so the argument of a logarithmic function must be greater than zero.

EXAMPLE 5 **Finding the Domain of a Logarithmic Function**

Find the domain of each logarithmic function.

(a) $F(x) = \log_2(x + 3)$

(b) $g(x) = \log_5\left(\frac{1 + x}{1 - x}\right)$

(c) $h(x) = \log_{1/2}|x|$

Solution (a) The domain of F consists of all x for which $x + 3 > 0$, that is, $x > -3$. Using interval notation, the domain of F is $(-3, \infty)$.

(b) The domain of g is restricted to

$$\frac{1 + x}{1 - x} > 0$$

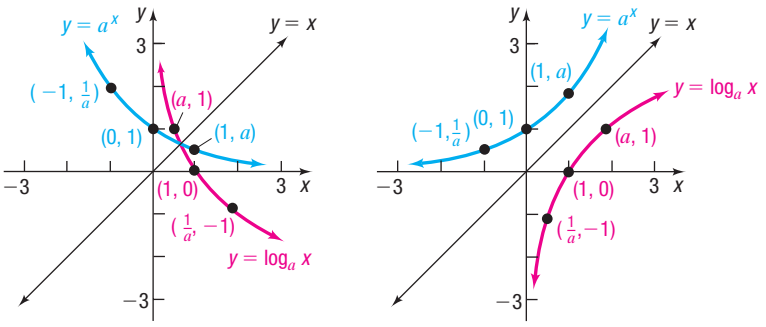
Solve this inequality to find that the domain of g consists of all x between -1 and 1, that is, $-1 < x < 1$ or, using interval notation, $(-1, 1)$.

(c) Since $|x| > 0$, provided that $x \neq 0$, the domain of h consists of all real numbers except zero or, using interval notation, $(-\infty, 0) \cup (0, \infty)$.

NOW WORK PROBLEMS 41 AND 47

4 Graph Logarithmic Functions

Because exponential functions and logarithmic functions are inverses of each other, the graph of the logarithmic function $y = \log_a x$ is the reflection about the line $y = x$ of the graph of the exponential function $y = a^x$, as shown in Figure 32.



(a) $0 < a < 1$ **(b)** $a > 1$

Figure 32

For example, to graph $y = \log_2 x$, graph $y = 2^x$ and reflect it about the line $y = x$. See Figure 33. To graph $y = \log_{1/3} x$, graph $y = \left(\frac{1}{3}\right)^x$ and reflect it about the line $y = x$. See Figure 34.

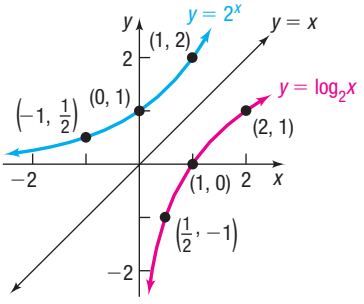


Figure 33

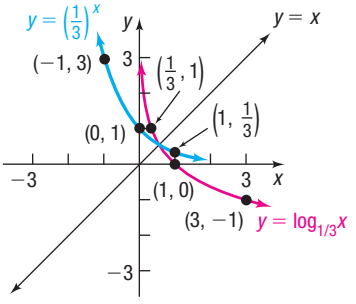


Figure 34

NOW WORK PROBLEM 61

The graphs of $y = \log_a x$ in Figures 32(a) and (b) lead to the following properties.

Properties of the Logarithmic Function $f(x) = \log_a x$

1. The domain is the set of positive real numbers, or $(0, \infty)$ using interval notation; the range is the set of all real numbers, or $(-\infty, \infty)$ using interval notation.
2. The x-intercept of the graph is 1. There is no y-intercept.
3. The y-axis $(x = 0)$ is a vertical asymptote of the graph.
4. A logarithmic function is decreasing if $0 < a < 1$ and is increasing if $a > 1$.
5. The graph of f contains the points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$.
6. The graph is smooth and continuous, with no corners or gaps.

In Words
$y = \log_e x$ is written $y = \ln x$.

If the base of a logarithmic function is the number e, the result is the **natural logarithm function.** This function occurs so frequently in applications that it is given a special symbol, **ln** (from the Latin, *logarithmus naturalis*). That is,

$$y = \ln x \quad \text{if and only if} \quad x = e^y \tag{1}$$

Because $y = \ln x$ and the exponential function $y = e^x$ are inverse functions, the graph of $y = \ln x$ can be obtained by reflecting the graph of $y = e^x$ about the line $y = x$. See Figure 35.

Other points on the graph of $f(x) = \ln x$ can be found by using a calculator with an $\boxed{\ln}$ key. See Table 7.

Seeing the Concept

Graph $Y_1 = e^x$ and $Y_2 = \ln x$ on the same square screen. Use eVALUEate to verify the points on the graph given in Figure 35. Do you see the symmetry of the two graphs with respect to the line $y = x$?

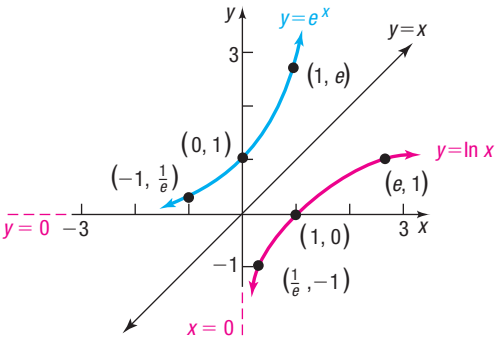


Figure 35

Table 7

x	$\ln x$
$\frac{1}{2}$	-0.69
2	0.69
3	1.10

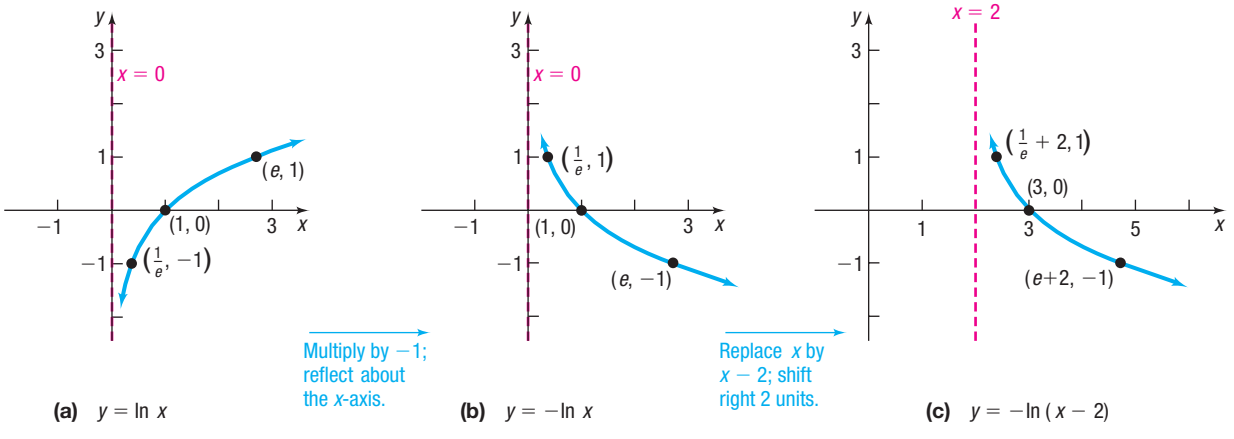
EXAMPLE 6 Graphing a Logarithmic Function and Its Inverse

(a) Find the domain of the logarithmic function $f(x) = -\ln(x - 2)$.
(b) Graph f.
(c) From the graph, determine the range and vertical asymptote of f.
(d) Find f^{-1}, the inverse of f.
(e) Use f^{-1} to confirm the range of f found in part (c). From the domain of f, find the range of f^{-1}.
(f) Graph f^{-1}.

Solution

(a) The domain of f consists of all x for which $x - 2 > 0$, or, equivalently, $x > 2$. The domain of f is $\{x|x > 2\}$ or $(2, \infty)$ in interval notation.

(b) To obtain the graph of $y = -\ln(x - 2)$, begin with the graph of $y = \ln x$ and use transformations. See Figure 36.



(a) $y = \ln x$ **(b)** $y = -\ln x$ **(c)** $y = -\ln(x - 2)$

Figure 36

(c) The range of $f(x) = -\ln(x - 2)$ is the set of all real numbers. The vertical asymptote is $x = 2$. [Do you see why? The original asymptote $(x = 0)$ is shifted to the right 2 units.]

(d) To find f^{-1}, begin with $y = -\ln(x - 2)$. The inverse function is defined (implicitly) by the equation

$$x = -\ln(y - 2)$$

Proceed to solve for y.

$$-x = \ln(y - 2) \quad \text{Isolate the logarithm.}$$
$$e^{-x} = y - 2 \quad \text{Change to an exponential statement.}$$
$$y = e^{-x} + 2 \quad \text{Solve for y.}$$

The inverse of f is $f^{-1}(x) = e^{-x} + 2$.

(e) The domain of f^{-1} equals the range of f, which is the set of all real numbers, from part (c). The range of f^{-1} is the domain of f, which is $(2, \infty)$ in interval notation.

(f) To graph f^{-1}, use the graph of f in Figure 36(c) and reflect it about the line $y = x$. See Figure 37. The graph of $f^{-1}(x) = e^{-x} + 2$ also could be obtained by using transformations.

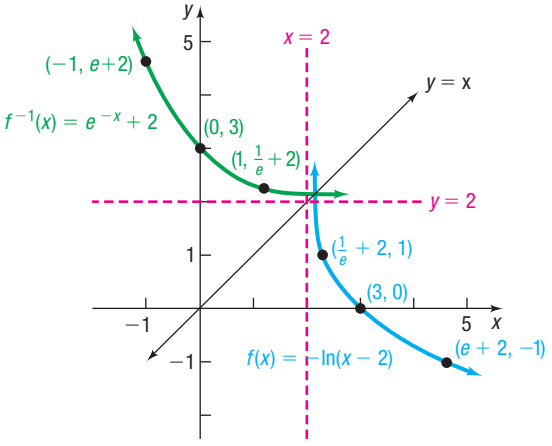


Figure 37

NOW WORK PROBLEM 73

If the base of a logarithmic function is the number 10, the result is the **common logarithm function.** If the base a of the logarithmic function is not indicated, it is understood to be 10. That is,

$$y = \log x \quad \text{if and only if} \quad x = 10^y$$

Because $y = \log x$ and the exponential function $y = 10^x$ are inverse functions, the graph of $y = \log x$ can be obtained by reflecting the graph of $y = 10^x$ about the line $y = x$. See Figure 38.

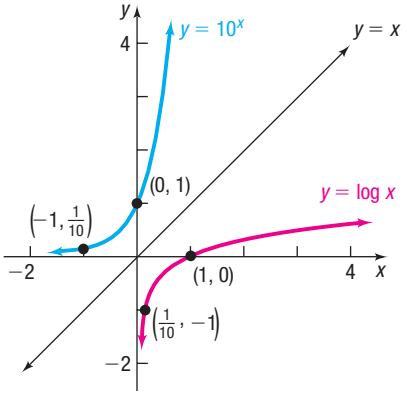


Figure 38

EXAMPLE 7 Graphing a Logarithmic Function and Its Inverse

(a) Find the domain of the logarithmic function $f(x) = 3\log(x - 1)$.

(b) Graph f.

(c) From the graph, determine the range and vertical asymptote of f.

(d) Find f^{-1}, the inverse of f.

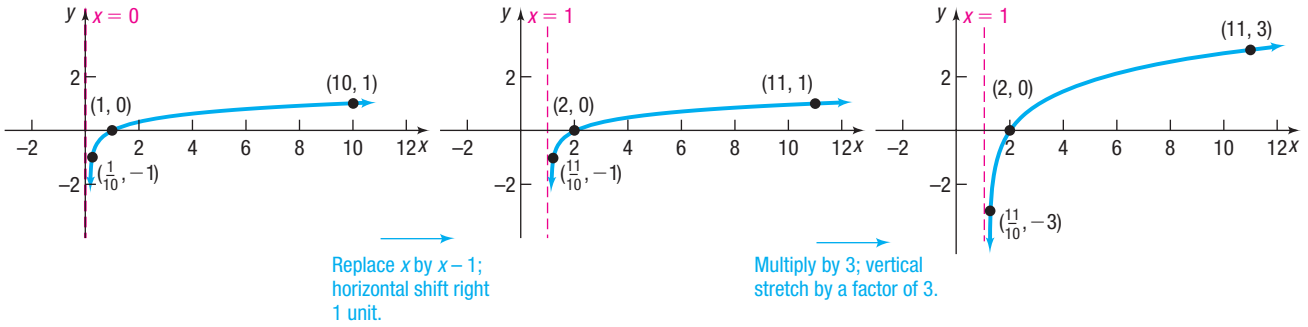
(e) Find the domain and the range of f^{-1}.

(f) Graph f^{-1}.

(continued)

Solution (a) The domain of f consists of all x for which $x - 1 > 0$, or, equivalently, $x > 1$. The domain of f is $\{x \mid x > 1\}$, or $(1, \infty)$ in interval notation.

(b) To obtain the graph of $y = 3\log(x - 1)$, begin with the graph of $y = \log x$ and use transformations. See Figure 39.



(a) $y = \log x$ **(b)** $y = \log(x - 1)$ **(c)** $y = 3\log(x - 1)$

Figure 39

(c) The range of $f(x) = 3\log(x - 1)$ is the set of all real numbers. The vertical asymptote is $x = 1$.

(d) Begin with $y = 3\log(x - 1)$. The inverse function is defined (implicitly) by the equation

$$x = 3\log(y - 1)$$

Proceed to solve for y.

$$\frac{x}{3} = \log(y - 1) \quad \text{Isolate the logarithm.}$$

$$10^{x/3} = y - 1 \quad \text{Change to an exponential statement.}$$

$$y = 10^{x/3} + 1 \quad \text{Solve for y.}$$

The inverse of f is $f^{-1}(x) = 10^{x/3} + 1$.

(e) The domain of f^{-1} is the range of f, which is the set of all real numbers, from part (c). The range of f^{-1} is the domain of f, which is $(1, \infty)$ in interval notation.

(f) To graph f^{-1}, use the graph of f in Figure 39(c) and reflect it about the line $y = x$. See Figure 40. The graph of $f^{-1}(x) = 10^{x/3} + 1$ also could be obtained by using transformations.

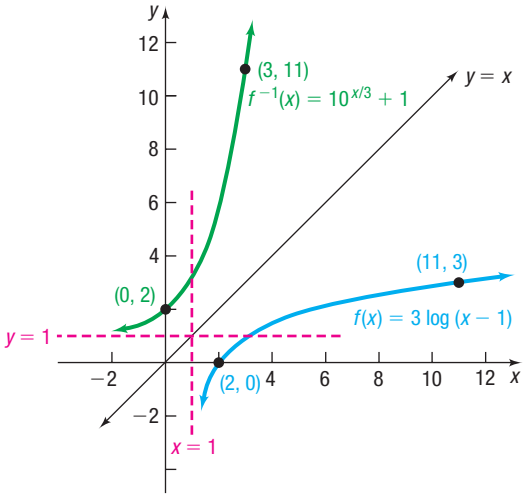


Figure 40

NOW WORK PROBLEM 81

5 Solve Logarithmic Equations

Equations that contain logarithms are called **logarithmic equations.** Care must be taken when solving logarithmic equations algebraically. In the expression $\log_a M$, remember that a and M are positive and $a \neq 1$. Be sure to check each apparent solution in the original equation and discard any that are extraneous.

Some logarithmic equations can be solved by changing the logarithmic equation to exponential form using the fact that $y = \log_a x$ means $a^y = x$.

EXAMPLE 8 Solving Logarithmic Equations

Solve:

(a) $\log_3(4x - 7) = 2$ (b) $\log_x 64 = 2$

Solution (a) To solve, change the logarithmic equation to exponential form.

$$\begin{aligned} \log_3(4x - 7) &= 2 && \text{Change to exponential form using the fact that } y = \log_a x \text{ means } a^y = x. \\ 4x - 7 &= 3^2 \\ 4x - 7 &= 9 \\ 4x &= 16 \\ x &= 4 \end{aligned}$$

✓**Check:** $\log_3(4x - 7) = \log_3(4 \cdot 4 - 7) = \log_3 9 = 2$ $3^2 = 9$

The solution set is $\{4\}$.

(b) To solve, change the logarithmic equation to exponential form.

$$\begin{aligned} \log_x 64 &= 2 \\ x^2 &= 64 && \text{Change to exponential form.} \\ x &= \pm\sqrt{64} = \pm 8 && \text{Square Root Method} \end{aligned}$$

Because the base of a logarithm must be positive, discard -8. Check the potential solution 8.

✓**Check:** $\log_8 64 = 2$ $8^2 = 64$

The solution set is $\{8\}$.

EXAMPLE 9 Using Logarithms to Solve an Exponential Equation

Solve: $e^{2x} = 5$

Solution To solve, change the exponential equation to logarithmic form.

$$\begin{aligned} e^{2x} &= 5 \\ 2x &= \ln 5 && \text{Change to a logarithmic equation using the fact that if } e^y = x, \text{ then } y = \ln x. \\ x &= \frac{\ln 5}{2} && \text{Exact solution} \\ &\approx 0.805 && \text{Approximate solution} \end{aligned}$$

The solution set is $\left\{\frac{\ln 5}{2}\right\}$.

NOW WORK PROBLEMS 89 AND 101

EXAMPLE 10 Alcohol and Driving

Blood alcohol concentration (BAC) is a measure of the amount of alcohol in a person's bloodstream. A BAC of 0.04% means that a person has 4 parts alcohol per 10,000 parts blood in the body. Relative risk is defined as the likelihood of

(*continued*)

one event occurring divided by the likelihood of a second event occurring. For example, if an individual with a BAC of 0.02% is 1.4 times as likely to have a car accident as an individual who has not been drinking, the relative risk of an accident with a BAC of 0.02% is 1.4. Recent medical research suggests that the relative risk R of having an accident while driving a car can be modeled by an equation of the form

$$R = e^{kx}$$

where x is the percent of concentration of alcohol in the bloodstream and k is a constant.

(a) Research indicates that the relative risk of a person having an accident with a BAC of 0.02% is 1.4. Find the constant k in the equation.

(b) Using this value of k, what is the relative risk if the concentration is 0.17%?

(c) Using this same value of k, what BAC corresponds to a relative risk of 100?

(d) If the law asserts that anyone with a relative risk of 4 or more should not have driving privileges, at what concentration of alcohol in the bloodstream should a driver be arrested and charged with a DUI (driving under the influence)?

Solution (a) For a concentration of alcohol in the blood of 0.02% and a relative risk of 1.4, let $x = 0.02$ and $R = 1.4$ in the equation and solve for k.

$$\begin{aligned} R &= e^{kx} \\ 1.4 &= e^{k(0.02)} && R = 1.4;\ x = 0.02 \\ 0.02k &= \ln 1.4 && \text{Change to a logarithmic statement.} \\ k &= \frac{\ln 1.4}{0.02} \approx 16.82 && \text{Solve for } k. \end{aligned}$$

(b) A concentration of 0.17% means $x = 0.17$. Use $k = 16.82$ in the equation to find the relative risk R:

$$R = e^{kx} = e^{(16.82)(0.17)} \approx 17.5$$

For a concentration of alcohol in the blood of 0.17%, the relative risk of an accident is about 17.5. That is, a person with a BAC of 0.17% is 17.5 times as likely to have a car accident as a person with no alcohol in the bloodstream.

(c) A relative risk of 100 means $R = 100$. Use $k = 16.82$ in the equation $R = e^{kx}$. The concentration x of alcohol in the blood obeys

$$\begin{aligned} 100 &= e^{16.82x} && R = e^{kx};\ R = 100;\ k = 16.82 \\ 16.82x &= \ln 100 && \text{Change to a logarithmic statement.} \\ x &= \frac{\ln 100}{16.82} \approx 0.27 && \text{Solve for } x. \end{aligned}$$

NOTE A BAC of 0.30% results in loss of consciousness in most people. ■

For a concentration of alcohol in the blood of 0.27%, the relative risk of an accident is 100.

(d) A relative risk of 4 means $R = 4$. Use $k = 16.82$ in the equation $R = e^{kx}$. The concentration x of alcohol in the bloodstream obeys

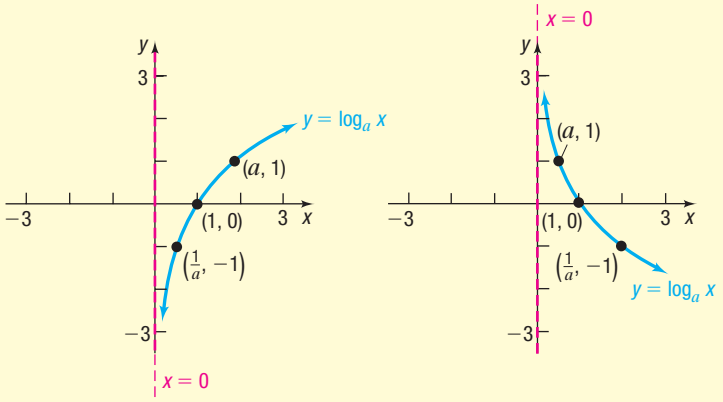
$$\begin{aligned} 4 &= e^{16.82x} \\ 16.82x &= \ln 4 \\ x &= \frac{\ln 4}{16.82} \approx 0.082 \end{aligned}$$

NOTE The blood alcohol content at which a DUI citation is given is 0.08%. ■

A driver with a BAC of 0.082% or more should be arrested and charged with DUI.

SUMMARY **Properties of the Logarithmic Function**

$f(x) = \log_a x, \quad a > 1$ $(y = \log_a x$ means $x = a^y)$	Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$ x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); increasing; one-to-one See Figure 41(a) for a typical graph.
$f(x) = \log_a x, \quad 0 < a < 1$ $(y = \log_a x$ means $x = a^y)$	Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$ x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); decreasing; one-to-one See Figure 41(b) for a typical graph.



(a) $a > 1$ **(b)** $0 < a < 1$

Figure 41

4.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Solve each inequality:
(a) $3x - 7 \le 8 - 2x$ (pp. A87–A88)
(b) $x^2 - x - 6 > 0$ (pp. 167–169)

2. Solve the inequality: $\dfrac{x-1}{x+4} > 0$ (pp. 268–270)

3. Solve: $2x + 3 = 9$ (pp. A67–A69)

Concepts and Vocabulary

4. The domain of the logarithmic function $f(x) = \log_a x$ is _____.

5. The graph of every logarithmic function $f(x) = \log_a x$, where $a > 0$ and $a \ne 1$, passes through three points: _____, _____, and _____.

6. If the graph of a logarithmic function $f(x) = \log_a x$, where $a > 0$ and $a \ne 1$, is increasing, then its base must be larger than _____.

7. ***True or False*** If $y = \log_a x$, then $y = a^x$.

8. ***True or False*** The graph of $f(x) = \log_a x$, where $a > 0$ and $a \ne 1$, has an x-intercept equal to 1 and no y-intercept.

9. Select the answer that completes the statement: $y = \ln x$ if and only if _____.
(a) $x = e^y$ (b) $y = e^x$ (c) $x = 10^y$ (d) $y = 10^x$

10. Choose the domain of $f(x) = \log_3(x + 2)$.
(a) $(-\infty, \infty)$ (b) $(2, \infty)$ (c) $(-2, \infty)$ (d) $(0, \infty)$

Skill Building

In Problems 11–18, change each exponential statement to an equivalent statement involving a logarithm.

11. $9 = 3^2$ **12.** $16 = 4^2$ **13.** $a^2 = 1.6$ **14.** $a^3 = 2.1$

15. $2^x = 7.2$ **16.** $3^x = 4.6$ **17.** $e^x = 8$ **18.** $e^{2.2} = M$

In Problems 19–26, change each logarithmic statement to an equivalent statement involving an exponent.

19. $\log_2 8 = 3$ **20.** $\log_3\left(\frac{1}{9}\right) = -2$ **21.** $\log_a 3 = 6$ **22.** $\log_b 4 = 2$

23. $\log_3 2 = x$ **24.** $\log_2 6 = x$ **25.** $\ln 4 = x$ **26.** $\ln x = 4$

In Problems 27–38, find the exact value of each logarithm without using a calculator.

27. $\log_2 1$ **28.** $\log_8 8$ **29.** $\log_5 25$ **30.** $\log_3\left(\frac{1}{9}\right)$

31. $\log_{1/2} 16$ **32.** $\log_{1/3} 9$ **33.** $\log_{10}\sqrt{10}$ **34.** $\log_5 \sqrt[3]{25}$

35. $\log_{\sqrt{2}} 4$ **36.** $\log_{\sqrt{3}} 9$ **37.** $\ln\sqrt{e}$ **38.** $\ln e^3$

In Problems 39–50, find the domain of each function.

39. $f(x) = \ln(x - 3)$ **40.** $g(x) = \ln(x - 1)$ **41.** $F(x) = \log_2 x^2$

42. $H(x) = \log_5 x^3$ **43.** $f(x) = 3 - 2\log_4\left(\frac{x}{2} - 5\right)$ **44.** $g(x) = 8 + 5\ln(2x + 3)$

45. $f(x) = \ln\left(\frac{1}{x + 1}\right)$ **46.** $g(x) = \ln\left(\frac{1}{x - 5}\right)$ **47.** $g(x) = \log_5\left(\frac{x + 1}{x}\right)$

48. $h(x) = \log_3\left(\frac{x}{x - 1}\right)$ **49.** $f(x) = \sqrt{\ln x}$ **50.** $g(x) = \frac{1}{\ln x}$

In Problems 51–58, use a calculator to evaluate each expression. Round your answer to three decimal places.

51. $\ln\frac{5}{3}$ **52.** $\frac{\ln 5}{3}$ **53.** $\frac{\ln\frac{10}{3}}{0.04}$ **54.** $\frac{\ln\frac{2}{3}}{-0.1}$

55. $\frac{\ln 4 + \ln 2}{\log 4 + \log 2}$ **56.** $\frac{\log 15 + \log 20}{\ln 15 + \ln 20}$ **57.** $\frac{2\ln 5 + \log 50}{\log 4 - \ln 2}$ **58.** $\frac{3\log 80 - \ln 5}{\log 5 + \ln 20}$

59. Find a so that the graph of $f(x) = \log_a x$ contains the point $(2, 2)$.

60. Find a so that the graph of $f(x) = \log_a x$ contains the point $\left(\frac{1}{2}, -4\right)$.

In Problems 61–64, graph each function and its inverse on the same set of axes.

61. $f(x) = 3^x; f^{-1}(x) = \log_3 x$ **62.** $f(x) = 4^x; f^{-1}(x) = \log_4 x$

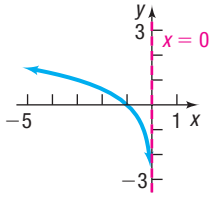
63. $f(x) = \left(\frac{1}{2}\right)^x; f^{-1}(x) = \log_{1/2} x$ **64.** $f(x) = \left(\frac{1}{3}\right)^x; f^{-1}(x) = \log_{1/3} x$

In Problems 65–72, the graph of a logarithmic function is given. Match each graph to one of the following functions.

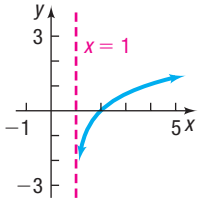
(A) $y = \log_3 x$ (B) $y = \log_3(-x)$ (C) $y = -\log_3 x$ (D) $y = -\log_3(-x)$

(E) $y = \log_3 x - 1$ (F) $y = \log_3(x - 1)$ (G) $y = \log_3(1 - x)$ (H) $y = 1 - \log_3 x$

65.



66.



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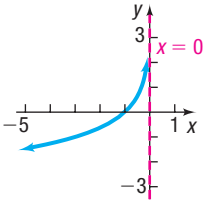

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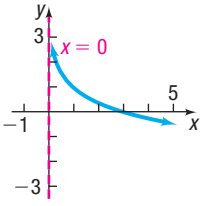

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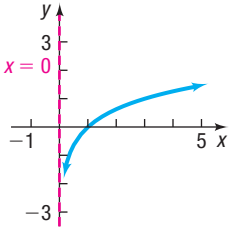

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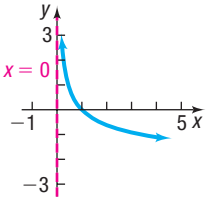

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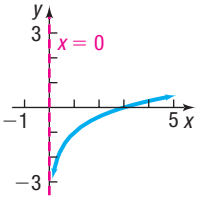

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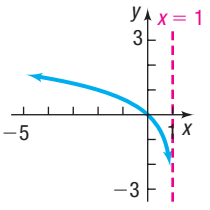


In Problems 73–88, use the given function f.

(a) Find the domain of f. (b) Graph f. (c) From the graph, determine the range and any asymptotes of f. (d) Find f^{-1}, the inverse of f. (e) Find the domain and the range of f^{-1}. (f) Graph f^{-1}.

73. $f(x) = \ln(x + 4)$ **74.** $f(x) = \ln(x - 3)$ **75.** $f(x) = 2 + \ln x$ **76.** $f(x) = -\ln(-x)$

77. $f(x) = \ln(2x) - 3$ **78.** $f(x) = -2\ln(x + 1)$ **79.** $f(x) = \log(x - 4) + 2$ **80.** $f(x) = \frac{1}{2}\log x - 5$

81. $f(x) = \frac{1}{2}\log(2x)$ **82.** $f(x) = \log(-2x)$ **83.** $f(x) = 3 + \log_3(x + 2)$ **84.** $f(x) = 2 - \log_3(x + 1)$

85. $f(x) = e^{x+2} - 3$ **86.** $f(x) = 3e^x + 2$ **87.** $f(x) = 2^{x/3} + 4$ **88.** $f(x) = -3^{x+1}$

In Problems 89–112, solve each equation.

89. $\log_3 x = 2$ **90.** $\log_5 x = 3$ **91.** $\log_2(2x + 1) = 3$ **92.** $\log_3(3x - 2) = 2$

93. $\log_x 4 = 2$ **94.** $\log_x\left(\frac{1}{8}\right) = 3$ **95.** $\ln e^x = 5$ **96.** $\ln e^{-2x} = 8$

97. $\log_4 64 = x$ **98.** $\log_5 625 = x$ **99.** $\log_3 243 = 2x + 1$ **100.** $\log_6 36 = 5x + 3$

101. $e^{3x} = 10$ **102.** $e^{-2x} = \frac{1}{3}$ **103.** $e^{2x+5} = 8$ **104.** $e^{-2x+1} = 13$

105. $\log_3(x^2 + 1) = 2$ **106.** $\log_5(x^2 + x + 4) = 2$ **107.** $\log_2 8^x = -3$ **108.** $\log_3 3^x = -1$

109. $5e^{0.2x} = 7$ **110.** $8 \cdot 10^{2x-7} = 3$ **111.** $2 \cdot 10^{2-x} = 5$ **112.** $4e^{x+1} = 5$

Mixed Practice

113. Suppose that $G(x) = \log_3(2x + 1) - 2$.

(a) What is the domain of G?
(b) What is $G(40)$? What point is on the graph of G?
(c) If $G(x) = 3$, what is x? What point is on the graph of G?
(d) What is the zero of G?

114. Suppose that $F(x) = \log_2(x + 1) - 3$.

(a) What is the domain of F?
(b) What is $F(7)$? What point is on the graph of F?
(c) If $F(x) = -1$, what is x? What point is on the graph of F?
(d) What is the zero of F?

In Problems 115–118, graph each function. Based on the graph, state the domain and the range, and find any intercepts.

115. $f(x) = \begin{cases} \ln(-x) & \text{if } x < 0 \\ \ln x & \text{if } x > 0 \end{cases}$

116. $f(x) = \begin{cases} \ln(-x) & \text{if } x \le -1 \\ -\ln(-x) & \text{if } -1 < x < 0 \end{cases}$

117. $f(x) = \begin{cases} -\ln x & \text{if } 0 < x < 1 \\ \ln x & \text{if } x \ge 1 \end{cases}$

118. $f(x) = \begin{cases} \ln x & \text{if } 0 < x < 1 \\ -\ln x & \text{if } x \ge 1 \end{cases}$

Applications and Extensions

119. Chemistry The pH of a chemical solution is given by the formula

$$\text{pH} = -\log_{10}[\text{H}^+]$$

where $[\text{H}^+]$ is the concentration of hydrogen ions in moles per liter. Values of pH range from 0 (acidic) to 14 (alkaline).

(a) What is the pH of a solution for which $[\text{H}^+]$ is 0.1?
(b) What is the pH of a solution for which $[\text{H}^+]$ is 0.01?
(c) What is the pH of a solution for which $[\text{H}^+]$ is 0.001?
(d) What happens to pH as the hydrogen ion concentration decreases?
(e) Determine the hydrogen ion concentration of an orange (pH = 3.5).
(f) Determine the hydrogen ion concentration of human blood (pH = 7.4).

120. Diversity Index Shannon's diversity index is a measure of the diversity of a population. The diversity index is given by the formula

$$H = -(p_1 \log p_1 + p_2 \log p_2 + \cdots + p_n \log p_n)$$

where p_1 is the proportion of the population that is species 1, p_2 is the proportion of the population that is species 2, and so on. In this problem, the population is people in the United States and the species is race.

(a) According to the U.S. Census Bureau, the distribution of race in the United States in 2015 was as follows:

Race	Proportion
White	0.617
Black or African American	0.124
American Indian and Alaskan Native	0.007
Asian	0.053
Native Hawaiian and Other Pacific Islander	0.002
Hispanic	0.177
Two or More Races	0.020

Source: *U.S. Census Bureau (www.census.gov)*

Compute the diversity index of the United States in 2015.

(*continued*)

(b) The largest value of the diversity index is given by $H_{\max} = \log(S)$, where S is the number of categories of race. Compute $H_{\max}$.
(c) The **evenness ratio** is given by $E_H = \dfrac{H}{H_{\max}}$, where $0 \le E_H \le 1$. If $E_H = 1$, there is complete evenness. Compute the evenness ratio for the United States.
(d) Obtain the distribution of race for the United States in 2010 from the Census Bureau. Compute Shannon's diversity index. Is the United States becoming more diverse? Why?

121. Atmospheric Pressure The atmospheric pressure p on an object decreases with increasing height. This pressure, measured in millimeters of mercury, is related to the height h (in kilometers) above sea level by the function

$$p(h) = 760e^{-0.145h}$$

(a) Find the height of an aircraft if the atmospheric pressure is 320 millimeters of mercury.
(b) Find the height of a mountain if the atmospheric pressure is 667 millimeters of mercury.

122. Healing of Wounds The normal healing of wounds can be modeled by an exponential function. If A_0 represents the original area of the wound, and if A equals the area of the wound, then the function

$$A(n) = A_0 e^{-0.35n}$$

describes the area of a wound after n days following an injury when no infection is present to retard the healing. Suppose that a wound initially had an area of 100 square millimeters.
(a) If healing is taking place, after how many days will the wound be one-half its original size?
(b) How long before the wound is 10% of its original size?

123. Exponential Probability Between 12:00 PM and 1:00 PM, cars arrive at Citibank's drive-thru at the rate of 6 cars per hour (0.1 car per minute). The following formula from statistics can be used to determine the probability that a car will arrive within t minutes of 12:00 PM.

$$F(t) = 1 - e^{-0.1t}$$

(a) Determine how many minutes are needed for the probability to reach 50%.
(b) Determine how many minutes are needed for the probability to reach 80%.
(c) Is it possible for the probability to equal 100%? Explain.

124. Exponential Probability Between 5:00 PM and 6:00 PM, cars arrive at Jiffy Lube at the rate of 9 cars per hour (0.15 car per minute). The following formula from statistics can be used to determine the probability that a car will arrive within t minutes of 5:00 PM.

$$F(t) = 1 - e^{-0.15t}$$

(a) Determine how many minutes are needed for the probability to reach 50%.
(b) Determine how many minutes are needed for the probability to reach 80%.

125. Drug Medication The formula

$$D = 5e^{-0.4h}$$

can be used to find the number of milligrams D of a certain drug that is in a patient's bloodstream h hours after the drug was administered. When the number of milligrams reaches 2, the drug is to be administered again. What is the time between injections?

126. Spreading of Rumors A model for the number N of people in a college community who have heard a certain rumor is

$$N = P(1 - e^{-0.15d})$$

where P is the total population of the community and d is the number of days that have elapsed since the rumor began. In a community of 1000 students, how many days will elapse before 450 students have heard the rumor?

127. Current in an RL Circuit The equation governing the amount of current I (in amperes) after time t (in seconds) in a simple RL circuit consisting of a resistance R (in ohms), an inductance L (in henrys), and an electromotive force E (in volts) is

$$I = \frac{E}{R}\left[1 - e^{-(R/L)t}\right]$$

If $E = 12$ volts, $R = 10$ ohms, and $L = 5$ henrys, how long does it take to obtain a current of 0.5 ampere? Of 1.0 ampere? Graph the equation.

128. Learning Curve Psychologists sometimes use the function

$$L(t) = A(1 - e^{-kt})$$

to measure the amount L learned at time t. Here A represents the amount to be learned, and the number k measures the rate of learning. Suppose that a student has an amount A of 200 vocabulary words to learn. A psychologist determines that the student has learned 20 vocabulary words after 5 minutes.
(a) Determine the rate of learning k.
(b) Approximately how many words will the student have learned after 10 minutes?
(c) After 15 minutes?
(d) How long does it take for the student to learn 180 words?

Loudness of Sound *Problems 129–132 use the following discussion: The* ***loudness*** $L(x)$, *measured in decibels (dB), of a sound of intensity* x, *measured in watts per square meter, is defined as* $L(x) = 10 \log \dfrac{x}{I_0}$, *where* $I_0 = 10^{-12}$ *watt per square meter is the least intense sound that a human ear can detect. Determine the loudness, in decibels, of each of the following sounds.*

129. Normal conversation: intensity of $x = 10^{-7}$ watt per square meter.

130. Amplified rock music: intensity of 10^{-1} watt per square meter.

131. Heavy city traffic: intensity of $x = 10^{-3}$ watt per square meter.

132. Diesel truck traveling 40 miles per hour 50 feet away: intensity 10 times that of a passenger car traveling 50 miles per hour 50 feet away whose loudness is 70 decibels.

The Richter Scale *Problems 133 and 134 use the following discussion: The* ***Richter scale*** *is one way of converting seismographic readings into numbers that provide an easy reference for measuring the magnitude M of an earthquake. All earthquakes are compared to a* ***zero-level earthquake*** *whose seismographic reading measures 0.001 millimeter at a distance of 100 kilometers from the epicenter. An earthquake whose seismographic reading measures x millimeters has* ***magnitude*** $M(x)$, *given by*

$$M(x) = \log\left(\frac{x}{x_0}\right)$$

where $x_0 = 10^{-3}$ *is the reading of a zero-level earthquake the same distance from its epicenter. In Problems 133 and 134, determine the magnitude of each earthquake.*

133. Magnitude of an Earthquake Mexico City in 1985: seismographic reading of 125,892 millimeters 100 kilometers from the center

134. Magnitude of an Earthquake San Francisco in 1906: seismographic reading of 50,119 millimeters 100 kilometers from the center

135. Alcohol and Driving The concentration of alcohol in a person's bloodstream is measurable. Suppose that the relative risk R of having an accident while driving a car can be modeled by an equation of the form

$$R = e^{kx}$$

where x is the percent concentration of alcohol in the bloodstream and k is a constant.

(a) Suppose that a concentration of alcohol in the bloodstream of 0.03 percent results in a relative risk of an accident of 1.4. Find the constant k in the equation.
(b) Using this value of k, what is the relative risk if the concentration is 0.17 percent?
(c) Using the same value of k, what concentration of alcohol corresponds to a relative risk of 100?
(d) If the law asserts that anyone with a relative risk of having an accident of 5 or more should not have driving privileges, at what concentration of alcohol in the bloodstream should a driver be arrested and charged with a DUI?
(e) Compare this situation with that of Example 10. If you were a lawmaker, which situation would you support? Give your reasons.

136. The Marriage Problem There is an infamous problem from mathematics that attempts to quantify the number of potential mates one should date before settling on one's "true love." The function

$$L(x) = -x \ln x$$

represents the probability of finding the ideal mate after rejecting the first x proportion of potential mates. For example, if you reject the first 20% = 0.20 of individuals you date, the probability of finding the ideal mate is $L(0.2) \approx 0.322$. So, if you are willing to date only up to 20 individuals, you should reject the first $0.2(20) = 4$ individuals before attempting to decide on the ideal mate. Presumably, you are using those first 4 individuals to help you decide which traits you value in a mate.

(a) Determine and interpret $L(0.1)$.
(b) Determine and interpret $L(0.6)$.
(c) What is the implied domain of L?
(d) Graph $L = L(x)$ over the domain using a graphing utility.
(e) Judging on the basis of the approach suggested by the model, what is the value of x that maximizes L? What is the highest probability of finding the ideal mate?

Explaining Concepts: Discussion and Writing

137. Is there any function of the form $y = x^{\alpha}, 0 < \alpha < 1$, that increases more slowly than a logarithmic function whose base is greater than 1? Explain.

138. In the definition of the logarithmic function, the base a is not allowed to equal 1. Why?

139. Critical Thinking In buying a new car, one consideration might be how well the value of the car holds up over time. Different makes of cars have different depreciation rates. One way to compute a depreciation rate for a car is given here. Suppose that the current prices of a certain automobile are as shown in the table.

	Age in Years				
New	1	2	3	4	5
\$38,000	\$36,600	\$32,400	\$28,750	\$25,400	\$21,200

Use the formula New = Old(e^{Rt}) to find R, the annual depreciation rate, for a specific time t. When might be the best time to trade in the car? Consult the NADA ("blue") book and compare two like models that you are interested in. Which has the better depreciation rate?

Retain Your Knowledge

Problems 140–143 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

140. Find the real zeros of $g(x) = 4x^4 - 37x^2 + 9$. What are the x-intercepts of the graph of g?

141. Use the Intermediate Value Theorem to show that the function $f(x) = 4x^3 - 2x^2 - 7$ has a real zero in the interval $[1, 2]$.

142. Find the average rate of change of $f(x) = 9^x$ from $\frac{1}{2}$ to 1.

143. A complex polynomial function f of degree 4 with real coefficients has the zeros $-1, 2$, and $3 - i$. Find the remaining zero(s) of f. Then find a polynomial function that has the zeros.

'Are You Prepared?' Answers

1. (a) $x \le 3$ (b) $x < -2$ or $x > 3$ **2.** $x < -4$ or $x > 1$ **3.** $\{3\}$

4.5 Properties of Logarithms

OBJECTIVES

1 Work with the Properties of Logarithms

Logarithms have some very useful properties that can be derived directly from the definition and the laws of exponents.

EXAMPLE 1 **Establishing Properties of Logarithms**

(a) Show that $\log_a 1 = 0$. (b) Show that $\log_a a = 1$.

Solution (a) We established this fact earlier while graphing $y = \log_a x$ (see Figure 32 on page 321). To show the result algebraically, let $y = \log_a 1$. Then

$$\begin{aligned} y &= \log_a 1 \\ a^y &= 1 && \text{Change to an exponential form.} \\ a^y &= a^0 && a^0 = 1 \text{ since } a > 0, a \neq 1 \\ y &= 0 && \text{Solve for } y. \\ \log_a 1 &= 0 && y = \log_a 1 \end{aligned}$$

(b) Let $y = \log_a a$. Then

$$\begin{aligned} y &= \log_a a \\ a^y &= a && \text{Change to an exponential form.} \\ a^y &= a^1 && a = a^1 \\ y &= 1 && \text{Solve for } y. \\ \log_a a &= 1 && y = \log_a a \end{aligned}$$

To summarize:

$$\log_a 1 = 0 \qquad \log_a a = 1$$

THEOREM **Properties of Logarithms**

In the properties given next, M and a are positive real numbers, $a \neq 1$, and r is any real number.

The number $\log_a M$ is the exponent to which a must be raised to obtain M. That is,

$$a^{\log_a M} = M \qquad \textbf{(1)}$$

The logarithm to the base a of a raised to a power equals that power. That is,

$$\log_a a^r = r \qquad \textbf{(2)}$$

The proofs use the fact that $y = a^x$ and $y = \log_a x$ are inverses.

Proof of Property (1) For inverse functions,

$$f(f^{-1}(x)) = x \quad \text{for all } x \text{ in the domain of } f^{-1}$$

Use $f(x) = a^x$ and $f^{-1}(x) = \log_a x$ to find

$$f(f^{-1}(x)) = a^{\log_a x} = x \quad \text{for } x > 0$$

Now let $x = M$ to obtain $a^{\log_a M} = M$, where $M > 0$. ■

Proof of Property (2) For inverse functions,

$$f^{-1}(f(x)) = x \quad \text{for all } x \text{ in the domain of } f$$

Use $f(x) = a^x$ and $f^{-1}(x) = \log_a x$ to find

$$f^{-1}(f(x)) = \log_a a^x = x \quad \text{for all real numbers } x$$

Now let $x = r$ to obtain $\log_a a^r = r$, where r is any real number. ■

EXAMPLE 2 Using Properties (1) and (2)

(a) $2^{\log_2 \pi} = \pi$ (b) $\log_{0.2} 0.2^{-\sqrt{2}} = -\sqrt{2}$ (c) $\ln e^{kt} = kt$

NOW WORK PROBLEM 15

Other useful properties of logarithms are given next.

THEOREM

Properties of Logarithms

In the following properties, M, N, and a are positive real numbers, $a \neq 1$, and r and x are any real number.

The Log of a Product Equals the Sum of the Logs

$$\log_a(MN) = \log_a M + \log_a N \tag{3}$$

The Log of a Quotient Equals the Difference of the Logs

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N \tag{4}$$

The Log of a Power Equals the Product of the Power and the Log

$$\log_a M^r = r \log_a M \tag{5}$$

$$a^x = e^{x \ln a} \tag{6}$$

We shall derive properties (3), (5), and (6) and leave the derivation of property (4) as an exercise (see Problem 109).

Proof of Property (3) Let $A = \log_a M$ and let $B = \log_a N$. These expressions are equivalent to the exponential expressions

$$a^A = M \quad \text{and} \quad a^B = N$$

Now

$$\begin{aligned} \log_a(MN) = \log_a(a^A a^B) &= \log_a a^{A+B} && \text{Law of Exponents} \\ &= A + B && \text{Property (2) of logarithms} \\ &= \log_a M + \log_a N \end{aligned}$$ ■

Proof of Property (5) Let $A = \log_a M$. This expression is equivalent to

$$a^A = M$$

Now

$$\begin{aligned} \log_a M^r = \log_a (a^A)^r &= \log_a a^{rA} && \text{Law of Exponents} \\ &= rA && \text{Property (2) of logarithms} \\ &= r \log_a M \end{aligned}$$

■

Proof of Property (6) Property (1), with $a = e$, yields

$$e^{\ln M} = M$$

Now let $M = a^x$ and apply property (5).

$$e^{\ln a^x} = e^{x \ln a} = (e^{\ln a})^x = a^x$$

■

NOW WORK PROBLEM 19

2 Write a Logarithmic Expression as a Sum or Difference of Logarithms

Logarithms can be used to transform products into sums, quotients into differences, and powers into factors. Such transformations prove useful in certain types of calculus problems.

EXAMPLE 3 Writing a Logarithmic Expression as a Sum of Logarithms

Write $\log_a (x\sqrt{x^2 + 1})$, $x > 0$, as a sum of logarithms. Express all powers as factors.

Solution

$$\begin{aligned} \log_a (x\sqrt{x^2 + 1}) &= \log_a x + \log_a \sqrt{x^2 + 1} && \log_a (M \cdot N) = \log_a M + \log_a N \\ &= \log_a x + \log_a (x^2 + 1)^{1/2} \\ &= \log_a x + \frac{1}{2} \log_a (x^2 + 1) && \log_a M^r = r \log_a M \end{aligned}$$

●

EXAMPLE 4 Writing a Logarithmic Expression as a Difference of Logarithms

Write

$$\ln \frac{x^2}{(x - 1)^3} \qquad x > 1$$

as a difference of logarithms. Express all powers as factors.

Solution

$$\ln \frac{x^2}{(x - 1)^3} = \ln x^2 - \ln (x - 1)^3 = 2 \ln x - 3 \ln (x - 1)$$

$$\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N \qquad \log_a M^r = r \log_a M$$

●

EXAMPLE 5 Writing a Logarithmic Expression as a Sum and Difference of Logarithms

Write

$$\log_a \frac{\sqrt{x^2 + 1}}{x^3 (x + 1)^4} \qquad x > 0$$

as a sum and difference of logarithms. Express all powers as factors.

Solution

$$\log_a \frac{\sqrt{x^2+1}}{x^3(x+1)^4} = \log_a \sqrt{x^2+1} - \log_a[x^3(x+1)^4] \qquad \text{Property (4)}$$

$$= \log_a \sqrt{x^2+1} - [\log_a x^3 + \log_a (x+1)^4] \qquad \text{Property (3)}$$

$$= \log_a (x^2+1)^{1/2} - \log_a x^3 - \log_a (x+1)^4$$

$$= \frac{1}{2}\log_a (x^2+1) - 3\log_a x - 4\log_a (x+1) \qquad \text{Property (5)}$$

WARNING In using properties (3) through (5), be careful about the values that the variable may assume. For example, the domain of the variable for $\log_a x$ is $x > 0$ and for $\log_a (x-1)$ is $x > 1$. If these functions are added, the domain is $x > 1$. That is, the equality $\log_a x + \log_a (x-1) = \log_a [x(x-1)]$ is true only for $x > 1$. ■

NOW WORK PROBLEM 51

3 Write a Logarithmic Expression as a Single Logarithm

Another use of properties (3) through (5) is to write sums and/or differences of logarithms with the same base as a single logarithm. This skill will be needed to solve certain logarithmic equations discussed in the next section.

EXAMPLE 6 Writing Expressions as a Single Logarithm

Write each of the following as a single logarithm.

(a) $\log_a 7 + 4\log_a 3$ (b) $\frac{2}{3}\ln 8 - \ln(5^2 - 1)$

(c) $\log_a x + \log_a 9 + \log_a (x^2+1) - \log_a 5$

Solution

(a)
$$\log_a 7 + 4\log_a 3 = \log_a 7 + \log_a 3^4 \qquad r\log_a M = \log_a M^r$$
$$= \log_a 7 + \log_a 81$$
$$= \log_a (7 \cdot 81) \qquad \log_a M + \log_a N = \log_a (M \cdot N)$$
$$= \log_a 567$$

(b)
$$\frac{2}{3}\ln 8 - \ln(5^2 - 1) = \ln 8^{2/3} - \ln(25 - 1) \qquad r\log_a M = \log_a M^r$$
$$= \ln 4 - \ln 24 \qquad 8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$$
$$= \ln\left(\frac{4}{24}\right) \qquad \log_a M - \log_a N = \log_a\left(\frac{M}{N}\right)$$
$$= \ln\left(\frac{1}{6}\right)$$
$$= \ln 1 - \ln 6$$
$$= -\ln 6 \qquad \ln 1 = 0$$

(c)
$$\log_a x + \log_a 9 + \log_a (x^2+1) - \log_a 5 = \log_a (9x) + \log_a (x^2+1) - \log_a 5$$
$$= \log_a [9x(x^2+1)] - \log_a 5$$
$$= \log_a \left[\frac{9x(x^2+1)}{5}\right]$$

WARNING A common error made by some students is to express the logarithm of a sum as the sum of logarithms.

$\log_a (M+N)$ is not equal to $\log_a M + \log_a N$

Correct statement $\log_a (MN) = \log_a M + \log_a N$ Property (3)

Another common error is to express the difference of logarithms as the quotient of logarithms.

$\log_a M - \log_a N$ is not equal to $\dfrac{\log_a M}{\log_a N}$

Correct statement $\log_a M - \log_a N = \log_a\left(\dfrac{M}{N}\right)$ Property (4)

A third common error is to express a logarithm raised to a power as the product of the power times the logarithm.

$$(\log_a M)^r \quad \text{is not equal to} \quad r \log_a M$$

Correct statement $\quad \log_a M^r = r \log_a M \quad$ Property (5) ■

NOW WORK PROBLEMS 57 AND 63

Two other important properties of logarithms are consequences of the fact that the logarithmic function $y = \log_a x$ is a one-to-one function.

THEOREM

Properties of Logarithms

In the following properties, M, N, and a are positive real numbers, $a \neq 1$.

If $M = N$, then $\log_a M = \log_a N$. **(7)**

If $\log_a M = \log_a N$, then $M = N$. **(8)**

Property (7) is used as follows: Starting with the equation $M = N$, "take the logarithm of both sides" to obtain $\log_a M = \log_a N$.

Properties (7) and (8) are useful for solving *exponential and logarithmic equations*, a topic discussed in the next section.

4 Evaluate a Logarithm Whose Base Is Neither 10 Nor *e*

Logarithms to the base 10—common logarithms—were used to facilitate arithmetic computations before the widespread use of calculators. (See the Historical Feature at the end of this section.) Natural logarithms—that is, logarithms whose base is the number e—remain very important because they arise frequently in the study of natural phenomena.

Common logarithms are usually abbreviated by writing **log,** with the base understood to be 10, just as natural logarithms are abbreviated by **ln,** with the base understood to be e.

Most calculators have both $\boxed{\log}$ and $\boxed{\ln}$ keys to calculate the common logarithm and the natural logarithm of a number, respectively. Let's look at an example to see how to approximate logarithms having a base other than 10 or e.

EXAMPLE 7 **Approximating a Logarithm Whose Base Is Neither 10 Nor *e***

Approximate $\log_2 7$. Round the answer to four decimal places.

Solution Remember, $\log_2 7$ means "2 raised to what exponent equals 7?" Let $y = \log_2 7$. Then $2^y = 7$. Because $2^2 = 4$ and $2^3 = 8$, the value of $\log_2 7$ is between 2 and 3.

$$2^y = 7$$

$$\ln 2^y = \ln 7 \quad \text{Property (7)}$$

$$y \ln 2 = \ln 7 \quad \text{Property (5)}$$

$$y = \frac{\ln 7}{\ln 2} \quad \text{Exact value}$$

$$y \approx 2.8074 \quad \text{Approximate value rounded to four decimal places}$$ ●

Example 7 shows how to approximate a logarithm whose base is 2 by changing to logarithms involving the base e. In general, the **Change-of-Base Formula** is used.

THEOREM

Change-of-Base Formula

If $a \neq 1$, $b \neq 1$, and M are positive real numbers, then

$$\log_a M = \frac{\log_b M}{\log_b a} \tag{9}$$

Proof Let $y = \log_a M$. Then

$$\begin{aligned} a^y &= M \\ \log_b a^y &= \log_b M && \text{Property (7)} \\ y \log_b a &= \log_b M && \text{Property (5)} \\ y &= \frac{\log_b M}{\log_b a} && \text{Solve for } y. \\ \log_a M &= \frac{\log_b M}{\log_b a} && y = \log_a M \end{aligned}$$

■

TECHNOLOGY NOTE Some calculators have features for evaluating logarithms with bases other than 10 or e. For example, the TI-84 Plus C has the logBASE function (under Math > Math > A: logBASE). Consult the user's manual for your calculator. ■

Because most calculators have keys only for $\boxed{\log}$ and $\boxed{\ln}$, in practice, the Change-of-Base Formula uses either $b = 10$ or $b = e$. That is,

$$\log_a M = \frac{\log M}{\log a} \quad \text{and} \quad \log_a M = \frac{\ln M}{\ln a} \tag{10}$$

EXAMPLE 8 **Using the Change-of-Base Formula**

Approximate:

(a) $\log_5 89$ (b) $\log_{\sqrt{2}} \sqrt{5}$

Round answers to four decimal places.

Solution

(a) $\log_5 89 = \dfrac{\log 89}{\log 5} \approx \dfrac{1.949390007}{0.6989700043} \approx 2.7889$ or $\log_5 89 = \dfrac{\ln 89}{\ln 5} \approx \dfrac{4.48863637}{1.609437912} \approx 2.7889$

(b) $\log_{\sqrt{2}} \sqrt{5} = \dfrac{\log \sqrt{5}}{\log \sqrt{2}} = \dfrac{\frac{1}{2}\log 5}{\frac{1}{2}\log 2} = \dfrac{\log 5}{\log 2} \approx 2.3219$ or $\log_{\sqrt{2}} \sqrt{5} = \dfrac{\ln \sqrt{5}}{\ln \sqrt{2}} = \dfrac{\frac{1}{2}\ln 5}{\frac{1}{2}\ln 2} = \dfrac{\ln 5}{\ln 2} \approx 2.3219$

●

NOW WORK PROBLEMS 23 AND 71

5 Graph a Logarithmic Function Whose Base Is Neither 10 Nor e

The Change-of-Base Formula also can be used to graph logarithmic functions whose base is neither 10 nor e.

EXAMPLE 9 **Graphing a Logarithmic Function Whose Base Is Neither 10 Nor *e***

Use a graphing utility to graph $y = \log_2 x$.

Solution Let's use the Change-of-Base Formula to express $y = \log_2 x$ in terms of logarithms with base 10 or base e. Graph either $y = \dfrac{\ln x}{\ln 2}$ or $y = \dfrac{\log x}{\log 2}$ to obtain the graph of $y = \log_2 x$. Figure 42 shows the graph using a TI-84 Plus C.

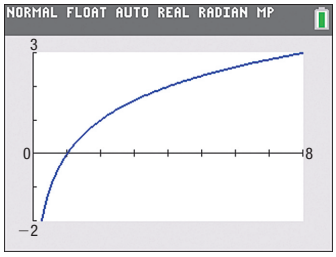


Figure 42 $y = \log_2 x$

✓**Check:** Verify that $y = \dfrac{\ln x}{\ln 2}$ and $y = \dfrac{\log x}{\log 2}$ result in the same graph by graphing each on the same screen.

NOW WORK PROBLEM 79

SUMMARY **Properties of Logarithms**

In the list that follows, a, b, M, N, and r are real numbers. Also, $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$, $M > 0$, and $N > 0$.

Definition	$y = \log_a x$ means $x = a^y$	
Properties of logarithms	$\log_a 1 = 0$; $\log_a a = 1$	$\log_a M^r = r \log_a M$
	$a^{\log_a M} = M$; $\log_a a^r = r$	$a^r = e^{r \ln a}$
	$\log_a (MN) = \log_a M + \log_a N$	If $M = N$, then $\log_a M = \log_a N$.
	$\log_a \left(\dfrac{M}{N}\right) = \log_a M - \log_a N$	If $\log_a M = \log_a N$, then $M = N$.
Change-of-Base Formula	$\log_a M = \dfrac{\log_b M}{\log_b a}$	

Historical Feature

John Napier (1550–1617)

Logarithms were invented about 1590 by John Napier (1550–1617) and Joost Bürgi (1552–1632), working independently. Napier, whose work had the greater influence, was a Scottish lord, a secretive man whose neighbors were inclined to believe him to be in league with the devil. His approach to logarithms was very different from ours; it was based on the relationship between arithmetic and geometric sequences—discussed in a later chapter—and not on the inverse function relationship of logarithms to exponential functions (described in Section 4.4). Napier's tables, published in 1614, listed what would now be called *natural logarithms* of sines and were rather difficult to use. A London professor, Henry Briggs, became interested in the tables and visited Napier. In their conversations, they developed the idea of common logarithms, which were published in 1617. Their importance for calculation was immediately recognized, and by 1650 they were being printed as far away as China. They remained an important calculation tool until the advent of the inexpensive handheld calculator about 1972, which has decreased their calculation—but not their theoretical—importance.

A side effect of the invention of logarithms was the popularization of the decimal system of notation for real numbers.

4.5 Assess Your Understanding

Concepts and Vocabulary

1. $\log_a 1 =$ _____
2. $a^{\log_a M} =$ _____
3. $\log_a a^r =$ _____
4. $\log_a (MN) =$ _____ + _____
5. $\log_a \left(\dfrac{M}{N}\right) =$ _____ − _____
6. $\log_a M^r =$ _____
7. If $\log_8 M = \dfrac{\log_5 7}{\log_5 8}$, then $M =$ _____.

8. ***True or False*** $\ln(x+3) - \ln(2x) = \dfrac{\ln(x+3)}{\ln(2x)}$

9. ***True or False*** $\log_2(3x^4) = 4\log_2(3x)$

10. ***True or False*** $\log\left(\dfrac{2}{3}\right) = \dfrac{\log 2}{\log 3}$

11. Choose the expression equivalent to 2^x.
(a) e^{2x} (b) $e^{x\ln 2}$ (c) $e^{\log_2 x}$ (d) $e^{2\ln x}$

12. Writing $\log_a x - \log_a y + 2\log_a z$ as a single logarithm results in which of the following?
(a) $\log_a(x - y + 2z)$ (b) $\log_a\left(\dfrac{xz^2}{y}\right)$
(c) $\log_a\left(\dfrac{2xz}{y}\right)$ (d) $\log_a\left(\dfrac{x}{yz^2}\right)$

Skill Building

In Problems 13–28, use properties of logarithms to find the exact value of each expression. Do not use a calculator.

13. $\log_3 3^{71}$ **14.** $\log_2 2^{-13}$ **15.** $\ln e^{-4}$ **16.** $\ln e^{\sqrt{2}}$

17. $2^{\log_2 7}$ **18.** $e^{\ln 8}$ **19.** $\log_8 2 + \log_8 4$ **20.** $\log_6 9 + \log_6 4$

21. $\log_6 18 - \log_6 3$ **22.** $\log_8 16 - \log_8 2$ **23.** $\log_2 6 \cdot \log_6 8$ **24.** $\log_3 8 \cdot \log_8 9$

25. $3^{\log_3 5 - \log_3 4}$ **26.** $5^{\log_5 6 + \log_5 7}$ **27.** $e^{\log_{e^2} 16}$ **28.** $e^{\log_{e^2} 9}$

In Problems 29–36, suppose that $\ln 2 = a$ *and* $\ln 3 = b$. *Use properties of logarithms to write each logarithm in terms of a and b.*

29. $\ln 6$ **30.** $\ln \dfrac{2}{3}$ **31.** $\ln 1.5$ **32.** $\ln 0.5$

33. $\ln 8$ **34.** $\ln 27$ **35.** $\ln \sqrt[5]{6}$ **36.** $\ln \sqrt[4]{\dfrac{2}{3}}$

In Problems 37–56, write each expression as a sum and/or difference of logarithms. Express powers as factors.

37. $\log_5(25x)$ **38.** $\log_3 \dfrac{x}{9}$ **39.** $\log_2 z^3$ **40.** $\log_7 x^5$

41. $\ln(ex)$ **42.** $\ln \dfrac{e}{x}$ **43.** $\ln \dfrac{x}{e^x}$ **44.** $\ln(xe^x)$

45. $\log_a(u^2v^3)$ $u > 0, v > 0$ **46.** $\log_2\left(\dfrac{a}{b^2}\right)$ $a > 0, b > 0$ **47.** $\ln(x^2\sqrt{1-x})$ $0 < x < 1$

48. $\ln(x\sqrt{1+x^2})$ $x > 0$ **49.** $\log_2\left(\dfrac{x^3}{x-3}\right)$ $x > 3$ **50.** $\log_5\left(\dfrac{\sqrt[3]{x^2+1}}{x^2-1}\right)$ $x > 1$

51. $\log\left[\dfrac{x(x+2)}{(x+3)^2}\right]$ $x > 0$ **52.** $\log\left[\dfrac{x^3\sqrt{x+1}}{(x-2)^2}\right]$ $x > 2$ **53.** $\ln\left[\dfrac{x^2-x-2}{(x+4)^2}\right]^{1/3}$ $x > 2$

54. $\ln\left[\dfrac{(x-4)^2}{x^2-1}\right]^{2/3}$ $x > 4$ **55.** $\ln \dfrac{5x\sqrt{1+3x}}{(x-4)^3}$ $x > 4$ **56.** $\ln\left[\dfrac{5x^2\sqrt[3]{1-x}}{4(x+1)^2}\right]$ $0 < x < 1$

In Problems 57–70, write each expression as a single logarithm.

57. $3\log_5 u + 4\log_5 v$ **58.** $2\log_3 u - \log_3 v$ **59.** $\log_3 \sqrt{x} - \log_3 x^3$

60. $\log_2\left(\dfrac{1}{x}\right) + \log_2\left(\dfrac{1}{x^2}\right)$ **61.** $\log_4(x^2-1) - 5\log_4(x+1)$ **62.** $\log(x^2+3x+2) - 2\log(x+1)$

63. $\ln\left(\dfrac{x}{x-1}\right) + \ln\left(\dfrac{x+1}{x}\right) - \ln(x^2-1)$ **64.** $\log\left(\dfrac{x^2+2x-3}{x^2-4}\right) - \log\left(\dfrac{x^2+7x+6}{x+2}\right)$ **65.** $8\log_2\sqrt{3x-2} - \log_2\left(\dfrac{4}{x}\right) + \log_2 4$

66. $21\log_3\sqrt[3]{x} + \log_3(9x^2) - \log_3 9$ **67.** $2\log_a(5x^3) - \dfrac{1}{2}\log_a(2x+3)$ **68.** $\dfrac{1}{3}\log(x^3+1) + \dfrac{1}{2}\log(x^2+1)$

69. $2\log_2(x+1) - \log_2(x+3) - \log_2(x-1)$ **70.** $3\log_5(3x+1) - 2\log_5(2x-1) - \log_5 x$

In Problems 71–78, use the Change-of-Base Formula and a calculator to evaluate each logarithm. Round your answer to three decimal places.

71. $\log_3 21$ **72.** $\log_5 18$ **73.** $\log_{1/3} 71$ **74.** $\log_{1/2} 15$

75. $\log_{\sqrt{2}} 7$ **76.** $\log_{\sqrt{5}} 8$ **77.** $\log_{\pi} e$ **78.** $\log_{\pi} \sqrt{2}$

In Problems 79–84, graph each function using a graphing utility and the Change-of-Base Formula.

79. $y = \log_4 x$ **80.** $y = \log_5 x$ **81.** $y = \log_2(x + 2)$

82. $y = \log_4(x - 3)$ **83.** $y = \log_{x-1}(x + 1)$ **84.** $y = \log_{x+2}(x - 2)$

Mixed Practice

85. If $f(x) = \ln x$, $g(x) = e^x$, and $h(x) = x^2$, find:
(a) $(f \circ g)(x)$. What is the domain of $f \circ g$?
(b) $(g \circ f)(x)$. What is the domain of $g \circ f$?
(c) $(f \circ g)(5)$
(d) $(f \circ h)(x)$. What is the domain of $f \circ h$?
(e) $(f \circ h)(e)$

86. If $f(x) = \log_2 x$, $g(x) = 2^x$, and $h(x) = 4x$, find:
(a) $(f \circ g)(x)$. What is the domain of $f \circ g$?
(b) $(g \circ f)(x)$. What is the domain of $g \circ f$?
(c) $(f \circ g)(3)$
(d) $(f \circ h)(x)$. What is the domain of $f \circ h$?
(e) $(f \circ h)(8)$

Applications and Extensions

In Problems 87–96, express y as a function of x. The constant C is a positive number.

87. $\ln y = \ln x + \ln C$

88. $\ln y = \ln(x + C)$

89. $\ln y = \ln x + \ln(x + 1) + \ln C$

90. $\ln y = 2 \ln x - \ln(x + 1) + \ln C$

91. $\ln y = 3x + \ln C$

92. $\ln y = -2x + \ln C$

93. $\ln(y - 3) = -4x + \ln C$

94. $\ln(y + 4) = 5x + \ln C$

95. $3 \ln y = \frac{1}{2}\ln(2x + 1) - \frac{1}{3}\ln(x + 4) + \ln C$

96. $2 \ln y = -\frac{1}{2}\ln x + \frac{1}{3}\ln(x^2 + 1) + \ln C$

97. Find the value of $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8$.

98. Find the value of $\log_2 4 \cdot \log_4 6 \cdot \log_6 8$.

99. Find the value of $\log_2 3 \cdot \log_3 4 \cdot \cdots \cdot \log_n(n + 1) \cdot \log_{n+1} 2$.

100. Find the value of $\log_2 2 \cdot \log_2 4 \cdot \cdots \cdot \log_2 2^n$.

101. Show that $\log_a(x + \sqrt{x^2 - 1}) + \log_a(x - \sqrt{x^2 - 1}) = 0$.

102. Show that $\log_a(\sqrt{x} + \sqrt{x - 1}) + \log_a(\sqrt{x} - \sqrt{x - 1}) = 0$.

103. Show that $\ln(1 + e^{2x}) = 2x + \ln(1 + e^{-2x})$.

104. Difference Quotient If $f(x) = \log_a x$, show that $\dfrac{f(x + h) - f(x)}{h} = \log_a\left(1 + \dfrac{h}{x}\right)^{1/h}$, $h \neq 0$.

105. If $f(x) = \log_a x$, show that $-f(x) = \log_{1/a} x$.

106. If $f(x) = \log_a x$, show that $f(AB) = f(A) + f(B)$.

107. If $f(x) = \log_a x$, show that $f\left(\dfrac{1}{x}\right) = -f(x)$.

108. If $f(x) = \log_a x$, show that $f(x^\alpha) = \alpha f(x)$.

109. Show that $\log_a\left(\dfrac{M}{N}\right) = \log_a M - \log_a N$, where a, M, and N are positive real numbers and $a \neq 1$.

110. Show that $\log_a\left(\dfrac{1}{N}\right) = -\log_a N$, where a and N are positive real numbers and $a \neq 1$.

Explaining Concepts: Discussion and Writing

111. Graph $Y_1 = \log(x^2)$ and $Y_2 = 2\log(x)$ using a graphing utility. Are they equivalent? What might account for any differences in the two functions?

112. Write an example that illustrates why $(\log_a x)^r \neq r \log_a x$.

113. Write an example that illustrates why

$$\log_2(x + y) \neq \log_2 x + \log_2 y.$$

114. Does $3^{\log_3(-5)} = -5$? Why or why not?

Retain Your Knowledge

Problems 115–118 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

115. Use a graphing utility to solve $x^3 - 3x^2 - 4x + 8 = 0$. Round answers to two decimal places.

116. Without solving, determine the character of the solution of the quadratic equation $4x^2 - 28x + 49 = 0$ in the complex number system.

117. Find the real zeros of

$$f(x) = 5x^5 + 44x^4 + 116x^3 + 95x^2 - 4x - 4$$

118. Graph $f(x) = \sqrt{2 - x}$ using the techniques of shifting, compressing or stretching, and reflecting. State the domain and the range of f.

4.6 Logarithmic and Exponential Equations

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solving Equations Using a Graphing Utility (Appendix B, Section B.4, pp. B6–B7)
- Solving Quadratic Equations (Section 2.3, pp. 145–150)
- Solving Equations Quadratic in Form (Section 2.3, pp. 151–152)

Now Work the **'Are You Prepared?'** problems on page 346.

OBJECTIVES
1. Solve Logarithmic Equations (p. 341)
2. Solve Exponential Equations (p. 343)
3. Solve Logarithmic and Exponential Equations Using a Graphing Utility (p. 345)

1 Solve Logarithmic Equations

In Section 4.4 logarithmic equations were solved by changing a logarithmic expression to an exponential expression. That is, they were solved by using the definition of a logarithm:

$$y = \log_a x \quad \text{is equivalent to} \quad x = a^y \qquad a > 0, a \neq 1$$

For example, to solve the equation $\log_2(1 - 2x) = 3$, write the logarithmic equation as an equivalent exponential equation $1 - 2x = 2^3$ and solve for x.

$$\log_2(1 - 2x) = 3$$

$$1 - 2x = 2^3 \qquad \text{Change to an exponential equation.}$$

$$-2x = 7 \qquad \text{Simplify.}$$

$$x = -\frac{7}{2} \qquad \text{Solve.}$$

Check: $\log_2(1 - 2x) = \log_2\left(1 - 2\left(-\frac{7}{2}\right)\right) = \log_2(1 + 7) = \log_2 8 = 3 \qquad 2^3 = 8$

For most logarithmic equations, some manipulation of the equation (usually using properties of logarithms) is required to obtain a solution. Also, to avoid extraneous solutions with logarithmic equations, it is wise to determine the domain of the variable first.

Let's begin with an example of a logarithmic equation that requires using the fact that a logarithmic function is a one-to-one function.

If $\log_a M = \log_a N$, then $M = N$ $\quad M, N$, and a are positive and $a \neq 1$.

EXAMPLE 1 Solving a Logarithmic Equation

Solve: $2\log_5 x = \log_5 9$

Solution The domain of the variable in this equation is $x > 0$. Note that each logarithm is to the same base, 5. Then find the exact solution as follows:

$$2\log_5 x = \log_5 9$$

$$\log_5 x^2 = \log_5 9 \qquad \log_a M^r = r\log_a M$$

$$x^2 = 9 \qquad \text{If } \log_a M = \log_a N, \text{ then } M = N.$$

$$x = 3 \quad \text{or} \quad x = -3$$

Recall that the domain of the variable is $x > 0$. Therefore, -3 is extraneous and must be discarded.

✓**Check:** $2\log_5 3 \stackrel{?}{=} \log_5 9$

$$\log_5 3^2 \stackrel{?}{=} \log_5 9 \qquad r\log_a M = \log_a M^r$$

$$\log_5 9 = \log_5 9$$

The solution set is $\{3\}$.

Now Work PROBLEM 13

Often one or more properties of logarithms are needed to rewrite the equation as a single logarithm. In the next example, the log of a product property is used.

EXAMPLE 2 Solving a Logarithmic Equation

Solve: $\log_5(x+6) + \log_5(x+2) = 1$

Solution The domain of the variable requires that $x + 6 > 0$ and $x + 2 > 0$, so $x > -6$ and $x > -2$. This means any solution must satisfy $x > -2$. To obtain an exact solution, express the left side as a single logarithm. Then change the equation to an equivalent exponential equation.

$$\log_5(x+6) + \log_5(x+2) = 1$$

$$\log_5[(x+6)(x+2)] = 1 \qquad \log_a M + \log_a N = \log_a(MN)$$

$$(x+6)(x+2) = 5^1 \qquad \text{Change to an exponential equation.}$$

$$x^2 + 8x + 12 = 5 \qquad \text{Simplify.}$$

$$x^2 + 8x + 7 = 0 \qquad \text{Place the quadratic equation in standard form.}$$

$$(x+7)(x+1) = 0 \qquad \text{Factor.}$$

$$x = -7 \quad \text{or} \quad x = -1 \qquad \text{Zero-Product Property}$$

Only $x = -1$ satisfies the restriction that $x > -2$, so $x = -7$ is extraneous. The solution set is $\{-1\}$, which you should check.

WARNING A negative solution is not automatically extraneous. Check whether the potential solution causes the argument of any logarithmic expression in the equation to be negative.

Now Work PROBLEM 21

EXAMPLE 3 **Solving a Logarithmic Equation**

Solve: $\ln x = \ln(x + 6) - \ln(x - 4)$

Solution The domain of the variable requires that $x > 0$, $x + 6 > 0$, and $x - 4 > 0$. As a result, the domain of the variable here is $x > 4$. Begin the solution using the log of a difference property.

$$\ln x = \ln(x + 6) - \ln(x - 4)$$

$$\ln x = \ln\left(\frac{x + 6}{x - 4}\right) \qquad \ln M - \ln N = \ln\left(\frac{M}{N}\right)$$

$$x = \frac{x + 6}{x - 4} \qquad \text{If } \ln M = \ln N, \text{ then } M = N.$$

$$x(x - 4) = x + 6 \qquad \text{Multiply both sides by } x - 4.$$

$$x^2 - 4x = x + 6 \qquad \text{Simplify.}$$

$$x^2 - 5x - 6 = 0 \qquad \text{Place the quadratic equation in standard form.}$$

$$(x - 6)(x + 1) = 0 \qquad \text{Factor.}$$

$$x = 6 \quad \text{or} \quad x = -1 \qquad \text{Zero-Product Property}$$

Because the domain of the variable is $x > 4$, discard -1 as extraneous. The solution set is $\{6\}$, which you should check.

WARNING In using properties of logarithms to solve logarithmic equations, avoid using the property $\log_a x^r = r \log_a x$, when r is even. The reason can be seen in this example:

Solve: $\log_3 x^2 = 4$

Solution: The domain of the variable x is all real numbers except 0.

(a) $\log_3 x^2 = 4$

$$x^2 = 3^4 = 81 \qquad \text{Change to exponential form.}$$

$$x = -9 \text{ or } x = 9$$

(b) $\log_3 x^2 = 4 \qquad \log_a x^r = r \log_a x$

$$2 \log_3 x = 4 \qquad \text{Domain of variable is } x > 0.$$

$$\log_3 x = 2$$

$$x = 9$$

Both -9 and 9 are solutions of $\log_3 x^2 = 4$ (as you can verify). The process in part (b) does not find the solution -9 because the domain of the variable was further restricted to $x > 0$ due to the application of the property $\log_a x^r = r \log_a x$.

NOW WORK PROBLEM 31

2 Solve Exponential Equations

In Sections 4.3 and 4.4, exponential equations were solved algebraically by expressing each side of the equation using the same base. That is, they were solved by using the one-to-one property of the exponential function:

$$\boxed{\text{If } a^u = a^v, \text{ then } u = v \qquad a > 0, a \neq 1}$$

For example, to solve the exponential equation $4^{2x+1} = 16$, notice that $16 = 4^2$ and apply the property above to obtain the equation $2x + 1 = 2$, and the solution is $x = \frac{1}{2}$.

Not all exponential equations can be readily expressed so that each side of the equation has the same base. For such equations, algebraic techniques often can be used to obtain exact solutions.

In the next example two exponential equations are solved by changing the exponential expression to a logarithmic expression.

EXAMPLE 4 Solving Exponential Equations

Solve: (a) $2^x = 5$ (b) $8 \cdot 3^x = 5$

Solution (a) Because 5 cannot be written as an integer power of 2 ($2^2 = 4$ and $2^3 = 8$), write the exponential equation as the equivalent logarithmic equation.

$$2^x = 5$$

$$x = \log_2 5 = \frac{\ln 5}{\ln 2}$$

Change-of-Base Formula (9), Section 4.5

Alternatively, the equation $2^x = 5$ can be solved by taking the natural logarithm (or common logarithm) of each side. Taking the natural logarithm yields

$$2^x = 5$$

$$\ln 2^x = \ln 5 \qquad \text{If } M = N, \text{ then } \ln M = \ln N.$$

$$x \ln 2 = \ln 5 \qquad \ln M^r = r \ln M$$

$$x = \frac{\ln 5}{\ln 2} \qquad \text{Exact solution}$$

$$\approx 2.322 \qquad \text{Approximate solution}$$

The solution set is $\left\{\frac{\ln 5}{\ln 2}\right\}$.

(b) $8 \cdot 3^x = 5$

$$3^x = \frac{5}{8} \qquad \text{Solve for } 3^x.$$

$$x = \log_3\left(\frac{5}{8}\right) = \frac{\ln\left(\frac{5}{8}\right)}{\ln 3} \qquad \text{Exact solution}$$

$$\approx -0.428 \qquad \text{Approximate solution}$$

The solution set is $\left\{\frac{\ln\left(\frac{5}{8}\right)}{\ln 3}\right\}$.

NOW WORK PROBLEM 43

EXAMPLE 5 Solving an Exponential Equation

Solve: $5^{x-2} = 3^{3x+2}$

Solution Because the bases are different, first apply property (7), Section 4.5 (take the natural logarithm of each side), and then use a property of logarithms. The result is an equation in x that can be solved.

$$5^{x-2} = 3^{3x+2}$$

$$\ln 5^{x-2} = \ln 3^{3x+2} \qquad \text{If } M = N, \ln M = \ln N.$$

$$(x - 2) \ln 5 = (3x + 2) \ln 3 \qquad \ln M^r = r \ln M$$

$$(\ln 5)x - 2 \ln 5 = (3 \ln 3)x + 2 \ln 3 \qquad \text{Distribute.}$$

NOTE Because of the properties of logarithms, exact solutions involving logarithms often can be expressed in multiple ways. For example, the solution to $5^{x-2} = 3^{3x+2}$ from Example 5 can be expressed equivalently as $\frac{2\ln 15}{\ln 5 - \ln 27}$ or $\frac{\ln 225}{\ln(5/27)}$, among others. Do you see why? ■

$$(\ln 5)x - (3\ln 3)x = 2\ln 3 + 2\ln 5 \qquad \text{Place terms involving } x \text{ on the left.}$$

$$(\ln 5 - 3\ln 3)x = 2(\ln 3 + \ln 5) \qquad \text{Factor.}$$

$$x = \frac{2(\ln 3 + \ln 5)}{\ln 5 - 3\ln 3} \qquad \text{Exact solution}$$

$$\approx -3.212 \qquad \text{Approximate solution}$$

The solution set is $\left\{\frac{2(\ln 3 + \ln 5)}{\ln 5 - 3\ln 3}\right\}$.

NOW WORK PROBLEM 53

The next example deals with an exponential equation that is quadratic in form.

EXAMPLE 6 Solving an Exponential Equation That Is Quadratic in Form

Solve: $4^x - 2^x - 12 = 0$

Solution Note that $4^x = (2^2)^x = 2^{(2x)} = (2^x)^2$, so the equation is quadratic in form and can be written as

$$(2^x)^2 - 2^x - 12 = 0 \qquad \text{Let } u = 2^x\text{; then } u^2 - u - 12 = 0.$$

Now factor as usual.

$$(2^x - 4)(2^x + 3) = 0 \qquad (u - 4)(u + 3) = 0$$

$$2^x - 4 = 0 \quad \text{or} \quad 2^x + 3 = 0 \qquad u - 4 = 0 \quad \text{or} \quad u + 3 = 0$$

$$2^x = 4 \qquad 2^x = -3 \qquad u = 2^x = 4 \qquad u = 2^x = -3$$

The equation on the left has the solution $x = 2$, since $2^x = 4 = 2^2$; the equation on the right has no solution, since $2^x > 0$ for all x. The only solution is 2. The solution set is $\{2\}$.

NOW WORK PROBLEM 61

3 Solve Logarithmic and Exponential Equations Using a Graphing Utility

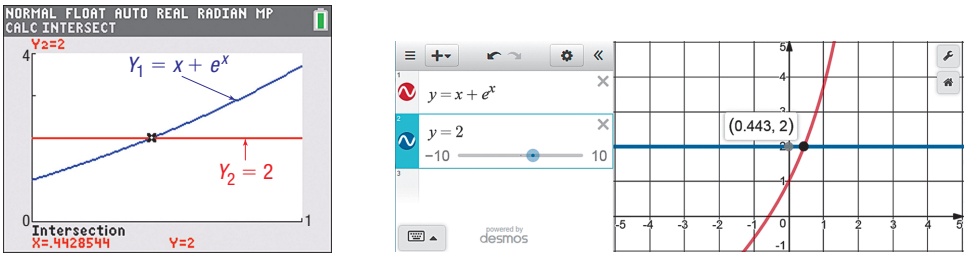
The algebraic techniques introduced in this section to obtain exact solutions apply only to certain types of logarithmic and exponential equations. Solutions for other types are usually studied in calculus, using numerical methods. For such types, a graphing utility can be used to approximate the solution.

EXAMPLE 7 Solving Equations Using a Graphing Utility

Solve: $x + e^x = 2$

Express the solution(s) rounded to two decimal places.

Solution The solution is found using a TI-84 Plus C by graphing $Y_1 = x + e^x$ and $Y_2 = 2$ as shown in Figure 43(a) on the next page. Note that because Y_1 is an increasing function (do you know why?), there is only one point of intersection for Y_1 and Y_2. Using the INTERSECT command reveals that the solution is 0.44, rounded to two decimal places. Figure 43(b) shows the solution using Desmos.



(a) TI-84 Plus C **(b)** Desmos

Figure 43 ***Source:*** *https://www.desmos.com/calculator/chlyfovqs0*

NOW WORK PROBLEM 71

4.6 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Solve $x^2 - 7x - 30 = 0$. (pp. 145–150)
2. Solve $(x + 3)^2 - 4(x + 3) + 3 = 0$. (pp. 151–152)
3. Approximate the solution(s) to $x^3 = x^2 - 5$ using a graphing utility. (pp. B6–B7)
4. Approximate the solution(s) to $x^3 - 2x + 2 = 0$ using a graphing utility. (pp. B6–B7)

Skill Building

In Problems 5–40, solve each logarithmic equation. Express any irrational solution in exact form and as a decimal rounded to three decimal places.

5. $\log_4 x = 2$
6. $\log(x + 6) = 1$
7. $\log_2(5x) = 4$
8. $\log_3(3x - 1) = 2$
9. $\log_4(x + 2) = \log_4 8$
10. $\log_5(2x + 3) = \log_5 3$
11. $\frac{1}{2}\log_3 x = 2\log_3 2$
12. $-2\log_4 x = \log_4 9$
13. $3\log_2 x = -\log_2 27$
14. $2\log_5 x = 3\log_5 4$
15. $3\log_2(x - 1) + \log_2 4 = 5$
16. $2\log_3(x + 4) - \log_3 9 = 2$
17. $\log x + \log(x + 15) = 2$
18. $\log x + \log(x - 21) = 2$
19. $\log(2x + 1) = 1 + \log(x - 2)$
20. $\log(2x) - \log(x - 3) = 1$
21. $\log_2(x + 7) + \log_2(x + 8) = 1$
22. $\log_6(x + 4) + \log_6(x + 3) = 1$
23. $\log_8(x + 6) = 1 - \log_8(x + 4)$
24. $\log_5(x + 3) = 1 - \log_5(x - 1)$
25. $\ln x + \ln(x + 2) = 4$
26. $\ln(x + 1) - \ln x = 2$
27. $\log_3(x + 1) + \log_3(x + 4) = 2$
28. $\log_2(x + 1) + \log_2(x + 7) = 3$
29. $\log_{1/3}(x^2 + x) - \log_{1/3}(x^2 - x) = -1$
30. $\log_4(x^2 - 9) - \log_4(x + 3) = 3$
31. $\log_a(x - 1) - \log_a(x + 6) = \log_a(x - 2) - \log_a(x + 3)$
32. $\log_a x + \log_a(x - 2) = \log_a(x + 4)$
33. $2\log_5(x - 3) - \log_5 8 = \log_5 2$
34. $\log_3 x - 2\log_3 5 = \log_3(x + 1) - 2\log_3 10$
35. $2\log_6(x + 2) = 3\log_6 2 + \log_6 4$
36. $3(\log_7 x - \log_7 2) = 2\log_7 4$
37. $2\log_{13}(x + 2) = \log_{13}(4x + 7)$
38. $\log(x - 1) = \frac{1}{3}\log 2$
39. $(\log_3 x)^2 - 5(\log_3 x) = 6$
40. $\ln x - 3\sqrt{\ln x} + 2 = 0$

In Problems 41–68, solve each exponential equation. Express any irrational solution in exact form and as a decimal rounded to three decimal places.

41. $2^{x-5} = 8$
42. $5^{-x} = 25$
43. $2^x = 10$
44. $3^x = 14$
45. $8^{-x} = 1.2$
46. $2^{-x} = 1.5$
47. $5(2^{3x}) = 8$
48. $0.3(4^{0.2x}) = 0.2$
49. $3^{1-2x} = 4^x$
50. $2^{x+1} = 5^{1-2x}$
51. $\left(\frac{3}{5}\right)^x = 7^{1-x}$
52. $\left(\frac{4}{3}\right)^{1-x} = 5^x$

53. $1.2^x = (0.5)^{-x}$ **54.** $0.3^{1+x} = 1.7^{2x-1}$ **55.** $\pi^{1-x} = e^x$ **56.** $e^{x+3} = \pi^x$

57. $2^{2x} + 2^x - 12 = 0$ **58.** $3^{2x} + 3^x - 2 = 0$ **59.** $3^{2x} + 3^{x+1} - 4 = 0$ **60.** $2^{2x} + 2^{x+2} - 12 = 0$

61. $16^x + 4^{x+1} - 3 = 0$ **62.** $9^x - 3^{x+1} + 1 = 0$ **63.** $25^x - 8 \cdot 5^x = -16$ **64.** $36^x - 6 \cdot 6^x = -9$

65. $3 \cdot 4^x + 4 \cdot 2^x + 8 = 0$ **66.** $2 \cdot 49^x + 11 \cdot 7^x + 5 = 0$ **67.** $4^x - 10 \cdot 4^{-x} = 3$ **68.** $3^x - 14 \cdot 3^{-x} = 5$

In Problems 69–82, use a graphing utility to solve each equation. Express your answer rounded to two decimal places.

69. $\log_5(x + 1) - \log_4(x - 2) = 1$ **70.** $\log_2(x - 1) - \log_6(x + 2) = 2$

71. $e^x = -x$ **72.** $e^{2x} = x + 2$ **73.** $e^x = x^2$ **74.** $e^x = x^3$

75. $\ln x = -x$ **76.** $\ln(2x) = -x + 2$ **77.** $\ln x = x^3 - 1$ **78.** $\ln x = -x^2$

79. $e^x + \ln x = 4$ **80.** $e^x - \ln x = 4$ **81.** $e^{-x} = \ln x$ **82.** $e^{-x} = -\ln x$

Mixed Practice

In Problems 83–96, solve each equation. Express irrational solutions in exact form and as a decimal rounded to three decimal places.

83. $\log_9(7x - 5) = \log_3(x + 1)$
[**Hint:** Change $\log_9(7x - 5)$ to base 3.]

84. $\log_2(x + 1) - \log_4 x = 1$

85. $\log_2(3x + 2) - \log_4 x = 3$

86. $\log_{16} x + \log_4 x + \log_2 x = 7$

87. $\log_9 x + 3\log_3 x = 14$

88. $2(\log_4 x)^2 + 3\log_8 x = \log_2 16$

89. $(\sqrt[3]{2})^{2-x} = 2^{x^2}$

90. $\log_2 x^{\log_2 x} = 4$

91. $\dfrac{e^x + e^{-x}}{2} = 1$
[**Hint:** Multiply each side by e^x.]

92. $\dfrac{e^x + e^{-x}}{2} = 3$

93. $\dfrac{e^x - e^{-x}}{2} = 2$

94. $\dfrac{e^x - e^{-x}}{2} = -2$

95. $\log_5 x + \log_3 x = 1$
[**Hint:** Use the Change-of-Base Formula.]

96. $\log_2 x + \log_6 x = 3$

97. $f(x) = \log_2(x + 3)$ and $g(x) = \log_2(3x + 1)$.
(a) Solve $f(x) = 3$. What point is on the graph of f?
(b) Solve $g(x) = 4$. What point is on the graph of g?
(c) Solve $f(x) = g(x)$. Do the graphs of f and g intersect? If so, where?
(d) Solve $(f + g)(x) = 7$.
(e) Solve $(f - g)(x) = 2$.

98. $f(x) = \log_3(x + 5)$ and $g(x) = \log_3(x - 1)$.
(a) Solve $f(x) = 2$. What point is on the graph of f?
(b) Solve $g(x) = 3$. What point is on the graph of g?
(c) Solve $f(x) = g(x)$. Do the graphs of f and g intersect? If so, where?
(d) Solve $(f + g)(x) = 3$.
(e) Solve $(f - g)(x) = 2$.

99. (a) Graph $f(x) = 3^{x+1}$ and $g(x) = 2^{x+2}$, on the same Cartesian plane.
(b) Find the point(s) of intersection of the graphs of f and g by solving $f(x) = g(x)$. Round answers to three decimal places. Label any intersection points on the graph drawn in part (a).
(c) Based on the graph, solve $f(x) > g(x)$.

100. (a) Graph $f(x) = 5^{x-1}$ and $g(x) = 2^{x+1}$, on the same Cartesian plane.
(b) Find the point(s) of intersection of the graphs of f and g by solving $f(x) = g(x)$. Label any intersection points on the graph drawn in part (a).
(c) Based on the graph, solve $f(x) > g(x)$.

101. (a) Graph $f(x) = 3^x$ and $g(x) = 10$ on the same Cartesian plane.
(b) Shade the region bounded by the y-axis, $f(x) = 3^x$, and $g(x) = 10$ on the graph drawn in part (a).
(c) Solve $f(x) = g(x)$ and label the point of intersection on the graph drawn in part (a).

102. (a) Graph $f(x) = 2^x$ and $g(x) = 12$ on the same Cartesian plane.
(b) Shade the region bounded by the y-axis, $f(x) = 2^x$, and $g(x) = 12$ on the graph drawn in part (a).
(c) Solve $f(x) = g(x)$ and label the point of intersection on the graph drawn in part (a).

103. (a) Graph $f(x) = 2^{x+1}$ and $g(x) = 2^{-x+2}$ on the same Cartesian plane.
(b) Shade the region bounded by the y-axis, $f(x) = 2^{x+1}$, and $g(x) = 2^{-x+2}$ on the graph drawn in part (a).
(c) Solve $f(x) = g(x)$ and label the point of intersection on the graph drawn in part (a).

104. (a) Graph $f(x) = 3^{-x+1}$ and $g(x) = 3^{x-2}$ on the same Cartesian plane.
(b) Shade the region bounded by the y-axis, $f(x) = 3^{-x+1}$, and $g(x) = 3^{x-2}$ on the graph drawn in part (a).
(c) Solve $f(x) = g(x)$ and label the point of intersection on the graph drawn in part (a).

105. (a) Graph $f(x) = 2^x - 4$.
(b) Find the zero of f.
(c) Based on the graph, solve $f(x) < 0$.

106. (a) Graph $g(x) = 3^x - 9$.
(b) Find the zero of g.
(c) Based on the graph, solve $g(x) > 0$.

Applications and Extensions

107. A Population Model The resident population of the United States in 2017 was 325 million people and was growing at a rate of 0.7% per year. Assuming that this growth rate continues, the model $P(t) = 325(1.007)^{t-2017}$ represents the population P (in millions of people) in year t.

(a) According to this model, when will the population of the United States be 415 million people?
(b) According to this model, when will the population of the United States be 448 million people?

Source: *U.S. Census Bureau*

108. A Population Model The population of the world in 2017 was 7.39 billion people and was growing at a rate of 1.1% per year. Assuming that this growth rate continues, the model $P(t) = 7.39(1.011)^{t-2017}$ represents the population P (in billions of people) in year t.
(a) According to this model, when will the population of the world be 9 billion people?
(b) According to this model, when will the population of the world be 12.5 billion people?

Source: *U.S. Census Bureau*

109. Depreciation The value V of a Chevy Cruze LT that is t years old can be modeled by $V(t) = 19{,}200(0.82)^t$.
(a) According to the model, when will the car be worth \$12,000?
(b) According to the model, when will the car be worth \$9000?
(c) According to the model, when will the car be worth \$3000?

Source: *Kelley Blue Book*

110. Depreciation The value V of a Honda Civic LX that is t years old can be modeled by $V(t) = 19{,}705(0.848)^t$.

(a) According to the model, when will the car be worth \$14,000?
(b) According to the model, when will the car be worth \$10,000?
(c) According to the model, when will the car be worth \$7500?

Source: *Kelley Blue Book*

Discussion and Writing

111. Fill in a reason for each step in the following two solutions.

Solve: $\log_3(x-1)^2 = 2$

Solution A	**Solution B**
$\log_3(x-1)^2 = 2$	$\log_3(x-1)^2 = 2$
$(x-1)^2 = 3^2 = 9$ ___	$2\log_3(x-1) = 2$ ___
$(x-1) = \pm 3$ ___	$\log_3(x-1) = 1$ ___
$x-1 = -3$ or $x-1 = 3$ ___	$x-1 = 3^1 = 3$ ___
$x = -2$ or $x = 4$ ___	$x = 4$ ___

Both solutions given in Solution A check. Explain what caused the solution $x = -2$ to be lost in Solution B.

Retain Your Knowledge

Problems 112–115 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

112. Solve: $4x^3 + 3x^2 - 25x + 6 = 0$.

113. Find the domain of $f(x) = \sqrt{x+3} + \sqrt{x-1}$.

114. For $f(x) = \dfrac{x}{x-2}$ and $g(x) = \dfrac{x+5}{x-3}$, find $f \circ g$. Then find the domain of $f \circ g$.

115. Determine whether the function $\{(0,-4), (2,-2), (4,0), (6,2)\}$ is one-to-one.

'Are You Prepared?' Answers

1. $\{-3, 10\}$ **2.** $\{-2, 0\}$ **3.** $\{-1.43\}$ **4.** $\{-1.77\}$

4.7 Financial Models

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Simple Interest (Appendix A, Section A.9, pp. A76–A77)

Now Work the **'Are You Prepared?'** problems on page 355.

OBJECTIVES
1. Determine the Future Value of a Lump Sum of Money (p. 349)
2. Calculate Effective Rates of Return (p. 352)
3. Determine the Present Value of a Lump Sum of Money (p. 353)
4. Determine the Rate of Interest or the Time Required to Double a Lump Sum of Money (p. 354)

1 Determine the Future Value of a Lump Sum of Money

Interest is money paid for the use of money. The total amount borrowed (whether by an individual from a bank in the form of a loan or by a bank from an individual in the form of a savings account) is called the **principal.** The **rate of interest,** expressed as a percent, is the amount charged for the use of the principal for a given period of time, usually on a yearly (that is, per annum) basis.

THEOREM

Simple Interest Formula

If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged is

$$I = Prt \qquad (1)$$

Interest charged according to formula (1) is called **simple interest.**

In problems involving interest, the term **payment period** is defined as follows:

Annually:	Once per year	**Monthly:**	12 times per year
Semiannually:	Twice per year	**Daily:**	365 times per year*
Quarterly:	Four times per year		

When the interest due at the end of a payment period is added to the principal so that the interest computed at the end of the next payment period is based on this new principal amount (old principal + interest), the interest is said to have been **compounded. Compound interest** is interest paid on the principal and on previously earned interest.

EXAMPLE 1 **Computing Compound Interest**

A credit union pays interest of 2% per annum compounded quarterly on a certain savings plan. If \$1000 is deposited in such a plan and the interest is left to accumulate, how much is in the account after 1 year?

Solution Use the simple interest formula, $I = Prt$. The principal P is \$1000 and the rate of interest is $2\% = 0.02$. After the first quarter of a year, the time t is $\frac{1}{4}$ year, so the interest earned is

$$I = Prt = (\$1000)(0.02)\left(\frac{1}{4}\right) = \$5$$

The new principal is $P + I = \$1000 + \$5 = \$1005$. At the end of the second quarter, the interest on this principal is

$$I = (\$1005)(0.02)\left(\frac{1}{4}\right) = \$5.03$$

*Most banks use a 360-day "year." Why do you think they do?

At the end of the third quarter, the interest on the new principal of \$1005 + \$5.03 = \$1010.03 is

$$I = (\$1010.03)(0.02)\left(\frac{1}{4}\right) = \$5.05$$

Finally, after the fourth quarter, the interest is

$$I = (\$1015.08)(0.02)\left(\frac{1}{4}\right) = \$5.08$$

After 1 year the account contains \$1015.08 + \$5.08 = \$1020.16.

The pattern of the calculations performed in Example 1 leads to a general formula for compound interest. For this purpose, let P represent the principal to be invested at a per annum interest rate r that is compounded n times per year, so the time of each compounding period is $\frac{1}{n}$ years. (For computing purposes, r is expressed as a decimal.) The interest earned after each compounding period is given by formula (1).

$$\text{Interest} = \text{principal} \times \text{rate} \times \text{time} = P \cdot r \cdot \frac{1}{n} = P \cdot \left(\frac{r}{n}\right)$$

The amount A after one compounding period is

$$A = P + P \cdot \left(\frac{r}{n}\right) = P \cdot \left(1 + \frac{r}{n}\right)$$

After two compounding periods, the amount A, based on the new principal $P \cdot \left(1 + \frac{r}{n}\right)$, is

$$A = \underbrace{P \cdot \left(1 + \frac{r}{n}\right)}_{\text{New principal}} + \underbrace{P \cdot \left(1 + \frac{r}{n}\right)\left(\frac{r}{n}\right)}_{\text{Interest on new principal}} \underset{\text{Factor out } P \cdot \left(1 + \frac{r}{n}\right)}{=} P \cdot \left(1 + \frac{r}{n}\right)\left(1 + \frac{r}{n}\right) = P \cdot \left(1 + \frac{r}{n}\right)^2$$

After three compounding periods, the amount A is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^2 + P \cdot \left(1 + \frac{r}{n}\right)^2\left(\frac{r}{n}\right) = P \cdot \left(1 + \frac{r}{n}\right)^2 \cdot \left(1 + \frac{r}{n}\right) = P \cdot \left(1 + \frac{r}{n}\right)^3$$

Continuing this way, after n compounding periods (1 year), the amount A is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^n$$

Because t years will contain $n \cdot t$ compounding periods, the amount after t years is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

THEOREM

Compound Interest Formula

The amount A after t years due to a principal P invested at an annual interest rate r, expressed as a decimal, compounded n times per year is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt} \qquad \textbf{(2)}$$

Exploration

To see the effects of compounding interest monthly on an initial deposit of \$1, graph $Y_1 = \left(1 + \frac{r}{12}\right)^{12x}$ with $r = 0.06$ and $r = 0.12$ for $0 \le x \le 30$. What is the future value of \$1 in 30 years when the interest rate per annum is $r = 0.06$ (6%)? What is the future value of \$1 in 30 years when the interest rate per annum is $r = 0.12$ (12%)? Does doubling the interest rate double the future value?

For example, to rework Example 1, use $P = \$1000$, $r = 0.02$, $n = 4$ (quarterly compounding), and $t = 1$ year to obtain

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt} = 1000\left(1 + \frac{0.02}{4}\right)^{4 \cdot 1} = \$1020.15^*$$

*The result shown here differs from Example 1 due to rounding.

In equation (2), the amount A is typically referred to as the **future value** of the account, and P is called the **present value.**

NOW WORK PROBLEM 7

EXAMPLE 2 Comparing Investments Using Different Compounding Periods

Investing \$1000 at an annual rate of 10% compounded annually, semiannually, quarterly, monthly, and daily will yield the following amounts after 1 year:

Annual compounding $(n = 1)$:
$$A = P \cdot (1 + r) = (\$1000)(1 + 0.10) = \$1100.00$$

Semiannual compounding $(n = 2)$:
$$A = P \cdot \left(1 + \frac{r}{2}\right)^2 = (\$1000)(1 + 0.05)^2 = \$1102.50$$

Quarterly compounding $(n = 4)$:
$$A = P \cdot \left(1 + \frac{r}{4}\right)^4 = (\$1000)(1 + 0.025)^4 = \$1103.81$$

Monthly compounding $(n = 12)$:
$$A = P \cdot \left(1 + \frac{r}{12}\right)^{12} = (\$1000)\left(1 + \frac{0.10}{12}\right)^{12} = \$1104.71$$

Daily compounding $(n = 365)$:
$$A = P \cdot \left(1 + \frac{r}{365}\right)^{365} = (\$1000)\left(1 + \frac{0.10}{365}\right)^{365} = \$1105.16$$

From Example 2, note that the effect of compounding more frequently is that the amount after 1 year is higher: \$1000 compounded 4 times a year at 10% results in \$1103.81, \$1000 compounded 12 times a year at 10% results in \$1104.71, and \$1000 compounded 365 times a year at 10% results in \$1105.16. This leads to the following question: What would happen to the amount after 1 year if the number of times that the interest is compounded were increased without bound?

Let's find the answer. Suppose that P is the principal, r is the per annum interest rate, and n is the number of times that the interest is compounded each year. The amount after 1 year is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^n$$

Rewrite this expression as follows:

$$A = P \cdot \left(1 + \frac{r}{n}\right)^n = P \cdot \left(1 + \frac{1}{\frac{n}{r}}\right)^n = P \cdot \left[\left(1 + \frac{1}{\frac{n}{r}}\right)^{n/r}\right]^r \underset{h = \frac{n}{r}}{=} P \cdot \left[\left(1 + \frac{1}{h}\right)^h\right]^r \qquad \textbf{(3)}$$

Now suppose that the number n of times that the interest is compounded per year gets larger and larger; that is, suppose that $n \to \infty$. Then $h = \frac{n}{r} \to \infty$, and the expression in brackets in equation (3) equals e. That is, $A \to Pe^r$.

Table 8 compares $\left(1 + \frac{r}{n}\right)^n$, for large values of n, to e^r for $r = 0.05$, $r = 0.10$, $r = 0.15$, and $r = 1$. The larger that n gets, the closer $\left(1 + \frac{r}{n}\right)^n$ gets to e^r. No matter how frequent the compounding, the amount after 1 year has the definite ceiling Pe^r.

Table 8

	$\left(1 + \frac{r}{n}\right)^n$			
	$n = 100$	$n = 1000$	$n = 10{,}000$	e^r
$r = 0.05$	1.0512580	1.0512698	1.051271	1.0512711
$r = 0.10$	1.1051157	1.1051654	1.1051704	1.1051709
$r = 0.15$	1.1617037	1.1618212	1.1618329	1.1618342
$r = 1$	2.7048138	2.7169239	2.7181459	2.7182818

When interest is compounded so that the amount after 1 year is Pe^r, the interest is said to be **compounded continuously.**

THEOREM

Continuous Compounding

The amount A after t years due to a principal P invested at an annual interest rate r compounded continuously is

$$A = Pe^{rt} \tag{4}$$

EXAMPLE 3 **Using Continuous Compounding**

The amount A that results from investing a principal P of \$1000 at an annual rate r of 10% compounded continuously for a time t of 1 year is

$$A = \$1000e^{0.10} = (\$1000)(1.10517) = \$1105.17$$

NOW WORK PROBLEM 13

2 Calculate Effective Rates of Return

Suppose that you have \$1000 and a bank offers to pay 3% annual interest on a savings account with interest compounded monthly. What annual interest rate must be earned for you to have the same amount at the end of the year as if the interest had been compounded annually (once per year)? To answer this question, first determine the value of the \$1000 in the account that earns 3% compounded monthly.

$$A = \$1000\left(1 + \frac{0.03}{12}\right)^{12} \qquad \text{Use } A = P\left(1 + \frac{r}{n}\right)^n \text{ with } P = \$1000,\ r = 0.03,\ n = 12.$$

$$= \$1030.42$$

So the interest earned is \$30.42. Using $I = Prt$ with $t = 1$, $I = \$30.42$, and $P = \$1000$, the annual simple interest rate is $0.03042 = 3.042\%$. This interest rate is known as the *effective rate of interest*.

The **effective rate of interest** is the equivalent annual simple interest rate that would yield the same amount as compounding n times per year, or continuously, after 1 year.

THEOREM

Effective Rate of Interest

The effective rate of interest r_e of an investment earning an annual interest rate r is given by

$$\text{Compounding } n \text{ times per year:} \quad r_e = \left(1 + \frac{r}{n}\right)^n - 1$$

$$\text{Continuous compounding:} \quad r_e = e^r - 1$$

EXAMPLE 4 **Computing the Effective Rate of Interest—Which Is the Best Deal?**

Suppose you want to buy a 5-year certificate of deposit (CD). You visit three banks to determine their CD rates. American Express offers you 2.15% annual interest compounded monthly, and First Internet Bank offers you 2.20% compounded quarterly. Discover offers 2.12% compounded daily. Determine which bank is offering the best deal.

Solution The bank that offers the best deal is the one with the highest effective interest rate.

American Express	First Internet Bank	Discover
$r_e = \left(1 + \frac{0.0215}{12}\right)^{12} - 1$	$r_e = \left(1 + \frac{0.022}{4}\right)^{4} - 1$	$r_e = \left(1 + \frac{0.0212}{365}\right)^{365} - 1$
$\approx 1.02171 - 1$	$\approx 1.02218 - 1$	$\approx 1.02143 - 1$
$= 0.02171$	$= 0.02218$	$= 0.02143$
$= 2.171\%$	$= 2.218\%$	$= 2.143\%$

The effective rate of interest is highest for First Internet Bank, so First Internet Bank is offering the best deal.

NOW WORK PROBLEM 23

3 Determine the Present Value of a Lump Sum of Money

When people in finance speak of the "time value of money," they are usually referring to the *present value* of money. The **present value** of A dollars to be received at a future date is the principal that you would need to invest now so that it will grow to A dollars in the specified time period. The present value of money to be received at a future date is always less than the amount to be received, since the amount to be received will equal the present value (money invested now) *plus* the interest accrued over the time period.

The compound interest formula (2) is used to develop a formula for present value. If P is the present value of A dollars to be received after t years at a per annum interest rate r compounded n times per year, then, by formula (2),

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

To solve for P, divide both sides by $\left(1 + \frac{r}{n}\right)^{nt}$. The result is

$$\frac{A}{\left(1 + \frac{r}{n}\right)^{nt}} = P \quad \text{or} \quad P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt}$$

THEOREM

Present Value Formulas

The present value P of A dollars to be received after t years, assuming a per annum interest rate r compounded n times per year, is

$$P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt} \tag{5}$$

If the interest is compounded continuously, then

$$P = Ae^{-rt} \tag{6}$$

To derive formula (6), solve formula (4) for P.

EXAMPLE 5 **Computing the Value of a Zero-Coupon Bond**

A zero-coupon (noninterest-bearing) bond can be redeemed in 10 years for \$1000. How much should you be willing to pay for it now if you want a return of

(a) 8% compounded monthly? (b) 7% compounded continuously?

Solution (a) To find the present value of \$1000, use formula (5) with $A = \$1000$, $n = 12$, $r = 0.08$, and $t = 10$.

$$P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt} = \$1000\left(1 + \frac{0.08}{12}\right)^{-12(10)} = \$450.52$$

For a return of 8% compounded monthly, pay \$450.52 for the bond.

(b) Here use formula (6) with $A = \$1000$, $r = 0.07$, and $t = 10$.

$$P = Ae^{-rt} = \$1000e^{-(0.07)(10)} = \$496.59$$

For a return of 7% compounded continuously, pay \$496.59 for the bond.

NOW WORK PROBLEM 15

4 Determine the Rate of Interest or the Time Required to Double a Lump Sum of Money

EXAMPLE 6 **Rate of Interest Required to Double an Investment**

What annual rate of interest compounded annually is needed in order to double an investment in 5 years?

Solution If P is the principal and P is to double, the amount A will be $2P$. Use the compound interest formula with $n = 1$ and $t = 5$ to find r.

$$\begin{aligned} A &= P \cdot \left(1 + \frac{r}{n}\right)^{nt} \\ 2P &= P \cdot (1 + r)^5 && A = 2P,\ n = 1,\ t = 5 \\ 2 &= (1 + r)^5 && \text{Divide both sides by } P. \\ 1 + r &= \sqrt[5]{2} && \text{Take the fifth root of each side.} \\ r &= \sqrt[5]{2} - 1 \approx 1.148698 - 1 = 0.148698 \end{aligned}$$

The annual rate of interest needed to double the principal in 5 years is 14.87%.

NOW WORK PROBLEM 31

EXAMPLE 7 **Time Required to Double or Triple an Investment**

(a) How long will it take for an investment to double in value if it earns 5% compounded continuously?

(b) How long will it take to triple at this rate?

Solution (a) If P is the initial investment and P is to double, the amount A will be $2P$. Use formula (4) for continuously compounded interest with $r = 0.05$. Then

$$\begin{aligned} A &= Pe^{rt} \\ 2P &= Pe^{0.05t} && A = 2P,\ r = 0.05 \\ 2 &= e^{0.05t} && \text{Divide out the } P\text{'s.} \\ 0.05t &= \ln 2 && \text{Rewrite as a logarithm.} \\ t &= \frac{\ln 2}{0.05} \approx 13.86 && \text{Solve for } t. \end{aligned}$$

It will take about 14 years to double the investment.

(b) To triple the investment, let $A = 3P$ in formula (4).

$$\begin{aligned} A &= Pe^{rt} \\ 3P &= Pe^{0.05t} && A = 3P,\ r = 0.05 \\ 3 &= e^{0.05t} && \text{Divide out the } P\text{'s.} \\ 0.05t &= \ln 3 && \text{Rewrite as a logarithm.} \\ t &= \frac{\ln 3}{0.05} \approx 21.97 && \text{Solve for } t. \end{aligned}$$

It will take about 22 years to triple the investment.

NOW WORK PROBLEM 35

4.7 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. What is the interest due if \$500 is borrowed for 6 months at a simple interest rate of 6% per annum? (pp. A76–A77)

2. If you borrow \$5000 and, after 9 months, pay off the loan in the amount of \$5500, what per annum rate of interest was charged? (pp. A76–A77)

Concepts and Vocabulary

3. The total amount borrowed (whether by an individual from a bank in the form of a loan or by a bank from an individual in the form of a savings account) is called the _____.

4. If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged is ____ = _____. Interest charged according to this formula is called _____ _____.

5. In working problems involving interest, if the payment period of the interest is quarterly, then interest is paid _____ times per year.

6. The _____ ____ __ _____ is the equivalent annual simple interest rate that would yield the same amount as compounding n times per year, or continuously, after 1 year.

Skill Building

In Problems 7–14, find the amount that results from each investment.

7. \$100 invested at 4% compounded quarterly after a period of 2 years

8. \$50 invested at 6% compounded monthly after a period of 3 years

9. \$500 invested at 8% compounded quarterly after a period of $2\frac{1}{2}$ years
10. \$300 invested at 12% compounded monthly after a period of $1\frac{1}{2}$ years
11. \$600 invested at 5% compounded daily after a period of 3 years
12. \$700 invested at 6% compounded daily after a period of 2 years
13. \$1000 invested at 11% compounded continuously after a period of 2 years
14. \$400 invested at 7% compounded continuously after a period of 3 years

In Problems 15–22, find the principal needed now to get each amount; that is, find the present value.

15. To get \$100 after 2 years at 6% compounded monthly
16. To get \$75 after 3 years at 8% compounded quarterly
17. To get \$1000 after $2\frac{1}{2}$ years at 6% compounded daily
18. To get \$800 after $3\frac{1}{2}$ years at 7% compounded monthly
19. To get \$600 after 2 years at 4% compounded quarterly
20. To get \$300 after 4 years at 3% compounded daily
21. To get \$80 after $3\frac{1}{4}$ years at 9% compounded continuously
22. To get \$800 after $2\frac{1}{2}$ years at 8% compounded continuously

In Problems 23–26, find the effective rate of interest.

23. For 5% compounded quarterly
24. For 6% compounded monthly
25. For 5% compounded continuously
26. For 6% compounded continuously

In Problems 27–30, determine the rate that represents the better deal.

27. 6% compounded quarterly or $6\frac{1}{4}$% compounded annually
28. 9% compounded quarterly or $9\frac{1}{4}$% compounded annually
29. 9% compounded monthly or 8.8% compounded daily
30. 8% compounded semiannually or 7.9% compounded daily
31. What rate of interest compounded annually is required to double an investment in 3 years?
32. What rate of interest compounded annually is required to double an investment in 6 years?
33. What rate of interest compounded annually is required to triple an investment in 5 years?
34. What rate of interest compounded annually is required to triple an investment in 10 years?
35. (a) How long does it take for an investment to double in value if it is invested at 8% compounded monthly?
    (b) How long does it take if the interest is compounded continuously?
36. (a) How long does it take for an investment to triple in value if it is invested at 6% compounded monthly?
    (b) How long does it take if the interest is compounded continuously?
37. What rate of interest compounded quarterly will yield an effective interest rate of 7%?
38. What rate of interest compounded continuously will yield an effective interest rate of 6%?

Applications and Extensions

39. **Time Required to Reach a Goal** If Tanisha has \$100 to invest at 4% per annum compounded monthly, how long will it be before she has \$150? If the compounding is continuous, how long will it be?
40. **Time Required to Reach a Goal** If Angela has \$100 to invest at 2.5% per annum compounded monthly, how long will it be before she has \$175? If the compounding is continuous, how long will it be?
41. **Time Required to Reach a Goal** How many years will it take for an initial investment of \$10,000 to grow to \$25,000? Assume a rate of interest of 6% compounded continuously.
42. **Time Required to Reach a Goal** How many years will it take for an initial investment of \$25,000 to grow to \$80,000? Assume a rate of interest of 7% compounded continuously.
43. **Price Appreciation of Homes** What will a \$90,000 condominium cost 5 years from now if the price appreciation for condos over that period averages 3% compounded annually?
44. **Credit Card Interest** A department store charges 1.25% per month on the unpaid balance for customers with charge accounts (interest is compounded monthly). A customer charges \$200 and does not pay her bill for 6 months. What is the bill at that time?
45. **Saving for a Car** Jerome will be buying a used car for \$15,000 in 3 years. How much money should he ask his parents for now so that, if he invests it at 5% compounded continuously, he will have enough to buy the car?
46. **Paying off a Loan** John requires \$3000 in 6 months to pay off a loan that has no prepayment privileges. If he has the \$3000 now, how much of it should he save in an account paying 3% compounded monthly so that in 6 months he will have exactly \$3000?

47. Return on a Stock George contemplates the purchase of 100 shares of a stock selling for $15 per share. The stock pays no dividends. The history of the stock indicates that it should grow at an annual rate of 15% per year. How much should the 100 shares of stock be worth in 5 years?

48. Return on an Investment A business purchased for $650,000 in 2012 is sold in 2015 for $850,000. What is the annual rate of return for this investment?

49. Comparing Savings Plans Jim places $1000 in a bank account that pays 5.6% compounded continuously. After 1 year, will he have enough money to buy a computer system that costs $1060? If another bank will pay Jim 5.9% compounded monthly, is this a better deal?

50. Savings Plans On January 1, Kim places $1000 in a certificate of deposit that pays 6.8% compounded continuously and matures in 3 months. Then Kim places the $1000 and the interest in a passbook account that pays 5.25% compounded monthly. How much does Kim have in the passbook account on May 1?

51. Comparing IRA Investments Will invests $2000 in his IRA in a bond trust that pays 9% interest compounded semiannually. His friend Henry invests $2000 in his IRA in a certificate of deposit that pays $8\frac{1}{2}\%$ compounded continuously. Who has more money after 20 years, Will or Henry?

52. Comparing Two Alternatives Suppose that April has access to an investment that will pay 10% interest compounded continuously. Which is better: to be given $1000 now so that she can take advantage of this investment opportunity or to be given $1325 after 3 years?

53. College Costs The average annual cost of college at 4-year private colleges was $33,480 in the 2016–2017 academic year. This was a 3.6% increase from the previous year.

Source: *The College Board*

(a) If the cost of college increases by 3.6% each year, what will be the average cost of college at a 4-year private college for the 2036–2037 academic year?

(b) College savings plans, such as a 529 plan, allow individuals to put money aside now to help pay for college later. If one such plan offers a rate of 2% compounded continuously, how much should be put in a college savings plan in 2018 to pay for 1 year of the cost of college at a 4-year private college for an incoming freshman in 2036?

54. Analyzing Interest Rates on a Mortgage Colleen and Bill have just purchased a house for $650,000, with the seller holding a second mortgage of $100,000. They promise to pay the seller $100,000 plus all accrued interest 5 years from now. The seller offers them three interest options on the second mortgage:

(a) Simple interest at 6% per annum
(b) 5.5% interest compounded monthly
(c) 5.25% interest compounded continuously

Which option is best? That is, which results in paying the least interest on the loan?

55. Comparing Bank Accounts Two bank accounts are opened at the same time. The first has a principal of $1000 in an account earning 5% compounded monthly. The second has a principal of $2000 in an account earning 4% interest compounded annually. Determine the number of years, to the nearest tenth, at which the account balances will be equal.

56. Per Capita Federal Debt In 2017, the federal debt was about $20 trillion. In 2017, the U.S. population was about 325 million. Assuming that the federal debt is increasing about 5.5% per year and the U.S. population is increasing about 0.7% per year, determine the per capita debt (total debt divided by population) in 2030.

Inflation *Problems 57–62 require the following discussion.* ***Inflation*** *is a term used to describe the erosion of the purchasing power of money. For example, if the annual inflation rate is 3%, then $1000 worth of purchasing power now will have only $970 worth of purchasing power in 1 year because 3% of the original $1000 (0.03 × 1000 = 30) has been eroded due to inflation. In general, if the rate of inflation averages r per annum over n years, the amount A that $P will purchase after n years is*

$$A = P \cdot (1 - r)^n$$

where r is expressed as a decimal.

57. Inflation If the inflation rate averages 3%, how much will $1000 purchase in 2 years?

58. Inflation If the inflation rate averages 2%, how much will $1000 purchase in 3 years?

59. Inflation If the amount that $1000 will purchase is only $950 after 2 years, what was the average inflation rate?

60. Inflation If the amount that $1000 will purchase is only $930 after 2 years, what was the average inflation rate?

61. Inflation If the average inflation rate is 2%, how long is it until purchasing power is cut in half?

62. Inflation If the average inflation rate is 4%, how long is it until purchasing power is cut in half?

Problems 63–66 involve zero-coupon bonds. A ***zero-coupon bond*** *is a bond that is sold now at a discount and will pay its face value at the time when it matures; no interest payments are made.*

63. Zero-Coupon Bonds A zero-coupon bond can be redeemed in 20 years for $10,000. How much should you be willing to pay for it now if you want a return of:

(a) 5% compounded monthly?
(b) 5% compounded continuously?

64. Zero-Coupon Bonds A child's grandparents are considering buying a $80,000 face-value, zero-coupon bond at birth so that she will have money for her college education 17 years later. If they want a rate of return of 6% compounded annually, what should they pay for the bond?

65. Zero-Coupon Bonds How much should a $10,000 face-value, zero-coupon bond, maturing in 10 years, be sold for now if its rate of return is to be 4.5% compounded annually?

66. Zero-Coupon Bonds If Pat pays $15,334.65 for a $25,000 face-value, zero-coupon bond that matures in 8 years, what is his annual rate of return?

67. Time to Double or Triple an Investment The formula

$$t = \frac{\ln m}{n \ln\left(1 + \frac{r}{n}\right)}$$

can be used to find the number of years t required to multiply an investment m times when r is the per annum interest rate compounded n times a year.

(a) How many years will it take to double the value of an IRA that compounds annually at the rate of 6%?

(b) How many years will it take to triple the value of a savings account that compounds quarterly at an annual rate of 5%?

(c) Give a derivation of this formula.

68. Time to Reach an Investment Goal The formula

$$t = \frac{\ln A - \ln P}{r}$$

can be used to find the number of years t required for an investment P to grow to a value A when compounded continuously at an annual rate r.

(a) How long will it take to increase an initial investment of \$1000 to \$4500 at an annual rate of 5.75%?

(b) What annual rate is required to increase the value of a \$2000 IRA to \$30,000 in 35 years?

(c) Give a derivation of this formula.

Problems 69–72 require the following discussion. The ***consumer price index (CPI)*** *indicates the relative change in price over time for a fixed basket of goods and services. It is a cost of living index that helps measure the effect of inflation on the cost of goods and services. The CPI uses the base period 1982–1984 for comparison (the CPI for this period is 100). The CPI for March 2017 was \$243.80. This means that \$100 in the period 1982–1984 had the same purchasing power as \$243.80 in March 2017. In general, if the rate of inflation averages r percent per annum over n years, then the CPI index after n years is*

$$\text{CPI} = \text{CPI}_0\left(1 + \frac{r}{100}\right)^n$$

where CPI_0 is the CPI index at the beginning of the n-year period.

Source: *U.S. Bureau of Labor Statistics*

69. Consumer Price Index

(a) The CPI was 224.9 for 2011 and 240.0 for 2016. Assuming that annual inflation remained constant for this time period, determine the average annual inflation rate.

(b) Using the inflation rate from part (a), in what year will the CPI reach 300?

70. Consumer Price Index If the current CPI is 234.2 and the average annual inflation rate is 2.8%, what will be the CPI in 5 years?

71. Consumer Price Index If the average annual inflation rate is 3.1%, how long will it take for the CPI index to double? (A doubling of the CPI index means purchasing power is cut in half.)

72. Consumer Price Index The base period for the CPI changed in 1998. Under the previous weight and item structure, the CPI for 1995 was 456.5. If the average annual inflation rate was 5.57%, what year was used as the base period for the CPI?

Explaining Concepts: Discussion and Writing

73. Explain in your own words what the term *compound interest* means. What does *continuous compounding* mean?

74. Explain in your own words the meaning of *present value*.

75. Critical Thinking You have just contracted to buy a house and will seek financing in the amount of \$100,000. You go to several banks. Bank 1 will lend you \$100,000 at the rate of 4.125% amortized over 30 years with a loan origination fee of 0.45%. Bank 2 will lend you \$100,000 at the rate of 3.375% amortized over 15 years with a loan origination fee of 0.95%. Bank 3 will lend you \$100,000 at the rate of 4.25% amortized over 30 years with no loan origination fee. Bank 4 will lend you \$100,000 at the rate of 3.625% amortized over 15 years with no loan origination fee. Which loan would you take? Why? Be sure to have sound reasons for your choice. Use the information in the table to assist you. If the amount of the monthly payment does not matter to you, which loan would you take? Again, have sound reasons for your choice. Compare your final decision with others in the class. Discuss.

	Monthly Payment	Loan Origination Fee
Bank 1	\$485.00	\$450.00
Bank 2	\$709.00	\$950.00
Bank 3	\$492.00	\$0.00
Bank 4	\$721.00	\$0.00

Retain Your Knowledge

Problems 76–79 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

76. Find the remainder R when $f(x) = 6x^3 + 3x^2 + 2x - 11$ is divided by $g(x) = x - 1$. Is g a factor of f?

77. The function $f(x) = \dfrac{x}{x-2}$ is one-to-one. Find f^{-1}.

78. Find the real zeros of

$$f(x) = x^5 - x^4 - 15x^3 - 21x^2 - 16x - 20.$$

Then write f in factored form.

79. Solve: $\log_2(x+3) = 2\log_2(x-3)$

'Are You Prepared?' Answers

1. \$15 **2.** $13\frac{1}{3}\%$

4.8 Exponential Growth and Decay Models; Newton's Law; Logistic Growth and Decay Models

OBJECTIVES
1. Find Equations of Populations That Obey the Law of Uninhibited Growth (p. 359)
2. Find Equations of Populations That Obey the Law of Decay (p. 361)
3. Use Newton's Law of Cooling (p. 362)
4. Use Logistic Models (p. 364)

1 Find Equations of Populations That Obey the Law of Uninhibited Growth

Many natural phenomena have been found to follow the law that an amount A varies with time t according to the function

$$A(t) = A_0 e^{kt} \qquad (1)$$

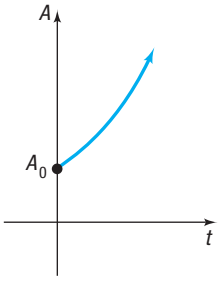
Here A_0 is the original amount $(t = 0)$ and $k \neq 0$ is a constant.

If $k > 0$, then equation (1) states that the amount A is increasing over time; if $k < 0$, the amount A is decreasing over time. In either case, when an amount A varies over time according to equation (1), it is said to follow the **exponential law,** or the **law of uninhibited growth** $(k > 0)$ **or decay** $(k < 0)$. See Figure 44.

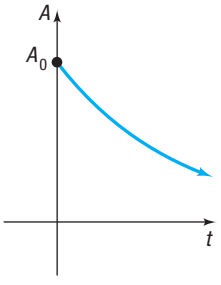
For example, as seen in Section 4.7, continuously compounded interest was shown to follow the law of uninhibited growth. In this section, additional phenomena that follow the exponential law will be studied.

Cell division is the growth process of many living organisms, such as amoebas, plants, and human skin cells. Based on an ideal situation in which no cells die and no by-products are produced, the number of cells present at a given time follows the law of uninhibited growth. Actually, however, after enough time has passed, growth at an exponential rate will cease due to the influence of factors such as lack of living space and dwindling food supply. The law of uninhibited growth accurately models only the early stages of the cell division process.

The cell division process begins with a culture containing N_0 cells. Each cell in the culture grows for a certain period of time and then divides into two identical



(a) $A(t) = A_0 e^{kt}$, $k > 0$
Exponential growth



(b) $A(t) = A_0 e^{kt}$, $k < 0$
Exponential decay

Figure 44

cells. Assume that the time needed for each cell to divide in two is constant and does not change as the number of cells increases. These new cells then grow, and eventually each divides in two, and so on.

Uninhibited Growth of Cells

A model that gives the number N of cells in a culture after a time t has passed (in the early stages of growth) is

$$N(t) = N_0 e^{kt} \qquad k > 0 \tag{2}$$

where N_0 is the initial number of cells and k is a positive constant that represents the growth rate of the cells.

Using formula (2) to model the growth of cells employs a function that yields positive real numbers, even though the number of cells being counted must be an integer. This is a common practice in many applications.

EXAMPLE 1 Bacterial Growth

A colony of bacteria grows according to the law of uninhibited growth according to the function $N(t) = 100e^{0.045t}$, where N is measured in grams and t is measured in days.

(a) Determine the initial amount of bacteria.
(b) What is the growth rate of the bacteria?
(c) What is the population after 5 days?
(d) How long will it take for the population to reach 140 grams?
(e) What is the doubling time for the population?

Solution (a) The initial amount of bacteria, N_0, is obtained when $t = 0$, so

$$N_0 = N(0) = 100e^{0.045(0)} = 100 \text{ grams}$$

(b) Compare $N(t) = 100e^{0.045t}$ to $N(t) = N_0e^{kt}$. The value of k, 0.045, indicates a growth rate of 4.5%.

(c) The population after 5 days is $N(5) = 100e^{0.045(5)} \approx 125.2$ grams.

(d) To find how long it takes for the population to reach 140 grams, solve the equation $N(t) = 140$.

$$\begin{aligned} 100e^{0.045t} &= 140 \\ e^{0.045t} &= 1.4 && \text{Divide both sides of the equation by 100.} \\ 0.045t &= \ln 1.4 && \text{Rewrite as a logarithm.} \\ t &= \frac{\ln 1.4}{0.045} && \text{Divide both sides of the equation by 0.045.} \\ &\approx 7.5 \text{ days} \end{aligned}$$

The population reaches 140 grams in about 7.5 days.

(e) The population doubles when $N(t) = 200$ grams, so the doubling time can be found by solving the equation $200 = 100e^{0.045t}$ for t.

$$\begin{aligned} 200 &= 100e^{0.045t} \\ 2 &= e^{0.045t} && \text{Divide both sides of the equation by 100.} \\ \ln 2 &= 0.045t && \text{Rewrite as a logarithm.} \\ t &= \frac{\ln 2}{0.045} && \text{Divide both sides of the equation by 0.045.} \\ &\approx 15.4 \text{ days} \end{aligned}$$

The population doubles approximately every 15.4 days.

NOW WORK PROBLEM 1

EXAMPLE 2 **Bacterial Growth**

A colony of bacteria increases according to the law of uninhibited growth.

(a) If N is the number of cells and t is the time in hours, express N as a function of t.
(b) If the number of bacteria doubles in 3 hours, find the function that gives the number of cells in the culture.
(c) How long will it take for the size of the colony to triple?
(d) How long will it take for the population to double a second time (that is, increase four times)?

Solution (a) Using formula (2), the number N of cells at time t is

$$N(t) = N_0 e^{kt}$$

where N_0 is the initial number of bacteria present and k is a positive number.

(b) To find the growth rate k, note that the number of cells doubles in 3 hours. Hence

$$N(3) = 2N_0$$

But $N(3) = N_0 e^{k(3)}$, so

$$\begin{aligned} N_0 e^{k(3)} &= 2N_0 \\ e^{3k} &= 2 && \text{Divide both sides by } N_0. \\ 3k &= \ln 2 && \text{Write the exponential equation as a logarithm.} \\ k &= \frac{1}{3}\ln 2 \approx 0.23105 \end{aligned}$$

The function that models this growth process is therefore

$$N(t) = N_0 e^{0.23105t}$$

(c) The time t needed for the size of the colony to triple requires that $N = 3N_0$. Substitute $3N_0$ for N to get

$$\begin{aligned} 3N_0 &= N_0 e^{0.23105t} \\ 3 &= e^{0.23105t} && \text{Divide both sides by } N_0. \\ 0.23105t &= \ln 3 && \text{Write the exponential equation as a logarithm.} \\ t &= \frac{\ln 3}{0.23105} \approx 4.755 \text{ hours} \end{aligned}$$

It will take about 4.755 hours or 4 hours, 45 minutes for the size of the colony to triple.

(d) If a population doubles in 3 hours, it will double a second time in 3 more hours, for a total time of 6 hours.

2 Find Equations of Populations That Obey the Law of Decay

Radioactive materials follow the law of uninhibited decay.

Uninhibited Radioactive Decay

The amount A of a radioactive material present at time t is given by

$$A(t) = A_0 e^{kt} \qquad k < 0 \tag{3}$$

where A_0 is the original amount of radioactive material and k is a negative number that represents the rate of decay.

All radioactive substances have a specific **half-life,** which is the time required for half of the radioactive substance to decay. **Carbon dating** uses the fact that all living organisms contain two kinds of carbon, carbon-12 (a stable carbon) and carbon-14

(a radioactive carbon with a half-life of 5730 years). While an organism is living, the ratio of carbon-12 to carbon-14 is constant. But when an organism dies, the original amount of carbon-12 present remains unchanged, whereas the amount of carbon-14 begins to decrease. This change in the amount of carbon-14 present relative to the amount of carbon-12 present makes it possible to calculate when the organism died.

EXAMPLE 3 **Estimating the Age of Ancient Tools**

Traces of burned wood along with ancient stone tools in an archeological dig in Chile were found to contain approximately 1.67% of the original amount of carbon-14. If the half-life of carbon-14 is 5730 years, approximately when was the tree cut and burned?

Solution Using formula (3), the amount A of carbon-14 present at time t is

$$A(t) = A_0 e^{kt}$$

where A_0 is the original amount of carbon-14 present and k is a negative number. We first seek the number k. To find it, we use the fact that after 5730 years, half of the original amount of carbon-14 remains, so $A(5730) = \frac{1}{2}A_0$. Then

$$\frac{1}{2}A_0 = A_0 e^{k(5730)}$$

$$\frac{1}{2} = e^{5730k} \quad \text{Divide both sides of the equation by } A_0.$$

$$5730k = \ln\frac{1}{2} \quad \text{Rewrite as a logarithm.}$$

$$k = \frac{1}{5730}\ln\frac{1}{2} \approx -0.000120968$$

Formula (3), therefore, becomes

$$A(t) = A_0 e^{-0.000120968t}$$

If the amount A of carbon-14 now present is 1.67% of the original amount, it follows that

$$0.0167A_0 = A_0 e^{-0.000120968t}$$

$$0.0167 = e^{-0.000120968t} \quad \text{Divide both sides of the equation by } A_0.$$

$$-0.000120968t = \ln 0.0167 \quad \text{Rewrite as a logarithm.}$$

$$t = \frac{\ln 0.0167}{-0.000120968} \approx 33{,}830 \text{ years}$$

The tree was cut and burned about 33,830 years ago. Some archeologists use this conclusion to argue that humans lived in the Americas nearly 34,000 years ago, much earlier than is generally accepted.

NOW WORK PROBLEM 3

3 Use Newton's Law of Cooling

Newton's Law of Cooling* states that the temperature of a heated object decreases exponentially over time toward the temperature of the surrounding medium.

Newton's Law of Cooling

The temperature u of a heated object at a given time t can be modeled by the following function:

$$u(t) = T + (u_0 - T)e^{kt} \qquad k < 0 \qquad \textbf{(4)}$$

where T is the constant temperature of the surrounding medium, u_0 is the initial temperature of the heated object, and k is a negative constant.

*Named after Sir Isaac Newton (1643–1727), one of the cofounders of calculus.

EXAMPLE 4 Using Newton's Law of Cooling

An object is heated to 100°C (degrees Celsius) and is then allowed to cool in a room whose air temperature is 30°C.

(a) If the temperature of the object is 80°C after 5 minutes, when will its temperature be 50°C?

(b) Determine the elapsed time before the temperature of the object is 35°C.

(c) What do you notice about the temperature as time passes?

Solution (a) Using formula (4) with $T = 30$ and $u_0 = 100$, the temperature $u(t)$ (in degrees Celsius) of the object at time t (in minutes) is

$$u(t) = 30 + (100 - 30)e^{kt} = 30 + 70e^{kt}$$

where k is a negative constant. To find k, use the fact that $u = 80$ when $t = 5$. Then

$$\begin{aligned} u(t) &= 30 + 70e^{kt} \\ 80 &= 30 + 70e^{k(5)} && u(5) = 80 \\ 50 &= 70e^{5k} && \text{Simplify.} \\ e^{5k} &= \frac{50}{70} && \text{Solve for } e^{5k}. \\ 5k &= \ln\frac{5}{7} && \text{Take ln of both sides.} \\ k &= \frac{1}{5}\ln\frac{5}{7} \approx -0.0673 && \text{Solve for } k. \end{aligned}$$

Formula (4), therefore, becomes

$$u(t) = 30 + 70e^{-0.0673t} \tag{5}$$

Find t when $u = 50$°C.

$$\begin{aligned} 50 &= 30 + 70e^{-0.0673t} \\ 20 &= 70e^{-0.0673t} && \text{Simplify.} \\ e^{-0.0673t} &= \frac{20}{70} \\ -0.0673t &= \ln\frac{2}{7} && \text{Take ln of both sides.} \\ t &= \frac{\ln\frac{2}{7}}{-0.0673} \approx 18.6 \text{ minutes} && \text{Solve for } t. \end{aligned}$$

The temperature of the object will be 50°C after about 18.6 minutes, or 18 minutes, 36 seconds.

(b) Use equation (5) to find t when $u = 35$°C.

$$\begin{aligned} 35 &= 30 + 70e^{-0.0673t} \\ 5 &= 70e^{-0.0673t} && \text{Simplify.} \\ e^{-0.0673t} &= \frac{5}{70} \\ -0.0673t &= \ln\frac{5}{70} && \text{Take ln of both sides.} \\ t &= \frac{\ln\frac{5}{70}}{-0.0673} \approx 39.2 \text{ minutes} && \text{Solve for } t. \end{aligned}$$

The object will reach a temperature of 35°C after about 39.2 minutes.

(*continued*)

(c) Look at equation (5). As t increases, the exponent $-0.0673t$ becomes unbounded in the negative direction. As a result, the value of $e^{-0.0673t}$ approaches zero, so the value of u, the temperature of the object, approaches 30°C, the air temperature of the room.

NOW WORK PROBLEM 13

4 Use Logistic Models

The exponential growth model $A(t) = A_0e^{kt}$, $k > 0$, assumes uninhibited growth, meaning that the value of the function grows without limit. Recall that cell division could be modeled using this function, assuming that no cells die and no by-products are produced. However, cell division eventually is limited by factors such as living space and food supply. The **logistic model,** given next, can describe situations where the growth or decay of the dependent variable is limited.

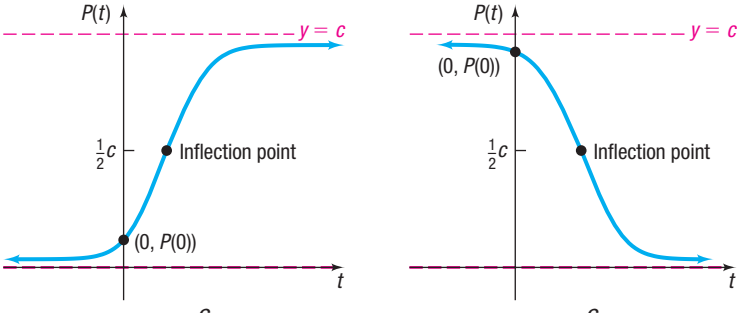
Logistic Model

In a logistic model, the population P after time t is given by the function

$$P(t) = \frac{c}{1 + ae^{-bt}} \tag{6}$$

where a, b, and c are constants with $a > 0$ and $c > 0$. The model is a growth model if $b > 0$; the model is a decay model if $b < 0$.

The number c is called the **carrying capacity** (for growth models) because the value $P(t)$ approaches c as t approaches infinity; that is, $\lim_{t\to\infty} P(t) = c$. The number $|b|$ is the growth rate for $b > 0$ and the decay rate for $b < 0$. Figure 45(a) shows the graph of a typical logistic growth function, and Figure 45(b) shows the graph of a typical logistic decay function.



(a) $P(t) = \dfrac{c}{1 + ae^{-bt}}$, $b > 0$
Logistic growth

(b) $P(t) = \dfrac{c}{1 + ae^{-bt}}$, $b < 0$
Logistic decay

Figure 45

From the figures, the following properties of logistic growth functions emerge.

Properties of the Logistic Model, Equation (6)

1. The domain is the set of all real numbers. The range is the interval $(0, c)$, where c is the carrying capacity.
2. There are no x-intercepts; the y-intercept is $P(0)$.
3. There are two horizontal asymptotes: $y = 0$ and $y = c$.
4. $P(t)$ is an increasing function if $b > 0$ and a decreasing function if $b < 0$.
5. There is an **inflection point** where $P(t)$ equals $\frac{1}{2}$ of the carrying capacity. The inflection point is the point on the graph where the graph changes from being curved upward to being curved downward for growth functions, and the point where the graph changes from being curved downward to being curved upward for decay functions.
6. The graph is smooth and continuous, with no corners or gaps.

EXAMPLE 5 Fruit Fly Population

Fruit flies are placed in a half-pint milk bottle with a banana (for food) and yeast plants (for food and to provide a stimulus to lay eggs). Suppose that the fruit fly population after t days is given by

$$P(t) = \frac{230}{1 + 56.5e^{-0.37t}}$$

(a) State the carrying capacity and the growth rate.
(b) Determine the initial population.
(c) What is the population after 5 days?
(d) How long does it take for the population to reach 180?
(e) Use a graphing utility to determine how long it takes for the population to reach one-half of the carrying capacity.

Solution

(a) As $t \to \infty$, $e^{-0.37t} \to 0$ and $P(t) \to \frac{230}{1}$. The carrying capacity of the half-pint bottle is 230 fruit flies. The growth rate is $|b| = |0.37| = 37\%$ per day.

(b) To find the initial number of fruit flies in the half-pint bottle, evaluate $P(0)$.

$$\begin{aligned} P(0) &= \frac{230}{1 + 56.5e^{-0.37(0)}} \\ &= \frac{230}{1 + 56.5} \\ &= 4 \end{aligned}$$

Thus, initially, there were 4 fruit flies in the half-pint bottle.

(c) To find the number of fruit flies in the half-pint bottle after 5 days, evaluate $P(5)$.

$$P(5) = \frac{230}{1 + 56.5e^{-0.37(5)}} \approx 23 \text{ fruit flies}$$

After 5 days, there are approximately 23 fruit flies in the bottle.

(d) To determine when the population of fruit flies will be 180, solve the equation $P(t) = 180$.

$$\begin{aligned} \frac{230}{1 + 56.5e^{-0.37t}} &= 180 \\ 230 &= 180(1 + 56.5e^{-0.37t}) \\ 1.2778 &= 1 + 56.5e^{-0.37t} && \text{Divide both sides by 180.} \\ 0.2778 &= 56.5e^{-0.37t} && \text{Subtract 1 from both sides.} \\ 0.0049 &= e^{-0.37t} && \text{Divide both sides by 56.5.} \\ \ln(0.0049) &= -0.37t && \text{Rewrite as a logarithmic expression.} \\ t &\approx 14.4 \text{ days} && \text{Divide both sides by } -0.37. \end{aligned}$$

It will take approximately 14.4 days (14 days, 10 hours) for the population to reach 180 fruit flies.

(e) One-half of the carrying capacity is 115 fruit flies. Solve $P(t) = 115$ by graphing $Y_1 = \frac{230}{1 + 56.5e^{-0.37x}}$ and $Y_2 = 115$ and using INTERSECT. See Figure 46. The population will reach one-half of the carrying capacity in about 10.9 days (10 days, 22 hours).

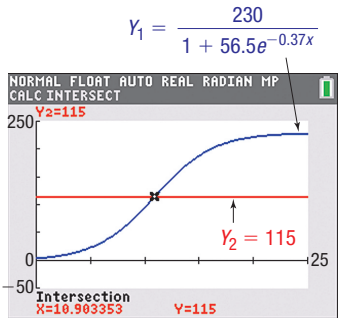


Figure 46 TI-84 Plus C

Look back at Figure 46. Notice the point where the graph reaches 115 fruit flies (one-half of the carrying capacity): The graph changes from being curved upward to being curved downward. Using the language of calculus, the graph changes from increasing at an increasing rate to increasing at a decreasing rate. For any logistic growth function, when the population reaches one-half the carrying capacity, the population growth starts to slow down.

NOW WORK PROBLEM 23

EXAMPLE 6 **Wood Products**

The EFISCEN wood product model classifies wood products according to their life-span. There are four classifications: short (1 year), medium short (4 years), medium long (16 years), and long (50 years). Based on data obtained from the European Forest Institute, the percentage of remaining wood products after t years for wood products with long life-spans (such as those used in the building industry) is given by

$$P(t) = \frac{100.3952}{1 + 0.0316e^{0.0581t}}$$

(a) What is the decay rate?
(b) What is the percentage of remaining wood products after 10 years?
(c) How long does it take for the percentage of remaining wood products to reach 50%?
(d) Explain why the numerator given in the model is reasonable.

Exploration

On the same viewing rectangle, graph $Y_1 = \dfrac{500}{1 + 24e^{-0.03x}}$ and $Y_2 = \dfrac{500}{1 + 24e^{-0.08x}}$. What effect does the growth rate $|b|$ have on the logistic growth function?

Solution

(a) The decay rate is $|b| = |-0.0581| = 5.81\%$.
(b) Evaluate $P(10)$.

$$P(10) = \frac{100.3952}{1 + 0.0316e^{0.0581(10)}} \approx 95.0$$

So 95% of long-life-span wood products remain after 10 years.

(c) Solve the equation $P(t) = 50$.

$$\frac{100.3952}{1 + 0.0316e^{0.0581t}} = 50$$

$$100.3952 = 50(1 + 0.0316e^{0.0581t})$$

$$2.0079 = 1 + 0.0316e^{0.0581t} \quad \text{Divide both sides by 50.}$$

$$1.0079 = 0.0316e^{0.0581t} \quad \text{Subtract 1 from both sides.}$$

$$31.8956 = e^{0.0581t} \quad \text{Divide both sides by 0.0316.}$$

$$\ln(31.8956) = 0.0581t \quad \text{Rewrite as a logarithmic expression.}$$

$$t \approx 59.6 \text{ years} \quad \text{Divide both sides by 0.0581.}$$

It will take approximately 59.6 years for the percentage of long-life-span wood products remaining to reach 50%.

(d) The numerator of 100.3952 is reasonable because the maximum percentage of wood products remaining that is possible is 100%.

NOW WORK PROBLEM 27

4.8 Assess Your Understanding

Applications and Extensions

1. **Growth of an Insect Population** The size P of a certain insect population at time t (in days) obeys the model $P(t) = 500e^{0.02t}$.
(a) Determine the number of insects at $t = 0$ days.
(b) What is the growth rate of the insect population?
(c) What is the population after 10 days?
(d) When will the insect population reach 800?
(e) When will the insect population double?

2. **Growth of Bacteria** The number N of bacteria present in a culture at time t (in hours) obeys the model $N(t) = 1000e^{0.01t}$.
(a) Determine the number of bacteria at $t = 0$ hours.
(b) What is the growth rate of the bacteria?
(c) What is the population after 4 hours?
(d) When will the number of bacteria reach 1700?
(e) When will the number of bacteria double?

3. **Radioactive Decay** Strontium-90 is a radioactive material that decays according to the function $A(t) = A_0e^{-0.0244t}$, where A_0 is the initial amount present and A is the amount present at time t (in years). Assume that a scientist has a sample of 500 grams of strontium-90.
(a) What is the decay rate of strontium-90?
(b) How much strontium-90 is left after 10 years?
(c) When will 400 grams of strontium-90 be left?
(d) What is the half-life of strontium-90?

4. **Radioactive Decay** Iodine-131 is a radioactive material that decays according to the function $A(t) = A_0e^{-0.087t}$, where A_0 is the initial amount present and A is the amount present at time t (in days). Assume that a scientist has a sample of 100 grams of iodine-131.
(a) What is the decay rate of iodine-131?
(b) How much iodine-131 is left after 9 days?
(c) When will 70 grams of iodine-131 be left?
(d) What is the half-life of iodine-131?

5. **Growth of a Colony of Mosquitoes** The population of a colony of mosquitoes obeys the law of uninhibited growth.
   (a) If N is the population of the colony and t is the time in days, express N as a function of t.
   (b) If there are 1000 mosquitoes initially and there are 1800 after 1 day, what is the size of the colony after 3 days?
   (c) How long is it until there are 10,000 mosquitoes?

6. **Bacterial Growth** A culture of bacteria obeys the law of uninhibited growth.
   (a) If N is the number of bacteria in the culture and t is the time in hours, express N as a function of t.
   (b) If 500 bacteria are present initially and there are 800 after 1 hour, how many will be present in the culture after 5 hours?
   (c) How long is it until there are 20,000 bacteria?

7. **Population Growth** The population of a southern city follows the exponential law.
   (a) If N is the population of the city and t is the time in years, express N as a function of t.
   (b) If the population doubled in size over an 18-month period and the current population is 10,000, what will the population be 2 years from now?

8. **Population Decline** The population of a midwestern city follows the exponential law.
   (a) If N is the population of the city and t is the time in years, express N as a function of t.
   (b) If the population decreased from 900,000 to 800,000 from 2005 to 2007, what was the population in 2009?

9. **Radioactive Decay** The half-life of radium is 1690 years. If 10 grams is present now, how much will be present in 50 years?

10. **Radioactive Decay** The half-life of radioactive potassium is 1.3 billion years. If 10 grams is present now, how much will be present in 100 years? In 1000 years?

11. **Estimating the Age of a Tree** A piece of charcoal is found to contain 30% of the carbon-14 that it originally had. When did the tree from which the charcoal came die? Use 5730 years as the half-life of carbon-14.

12. **Estimating the Age of a Fossil** A fossilized leaf contains 70% of its normal amount of carbon-14. How old is the fossil? Use 5730 years as the half-life of carbon-14.

13. **Cooling Time of a Pizza** A pizza baked at 450°F is removed from the oven at 5:00 PM and placed in a room that is a constant 70°F. After 5 minutes, the pizza is at 300°F.
   (a) At what time can you begin eating the pizza if you want its temperature to be 135°F?
   (b) Determine the time that needs to elapse before the pizza is 160°F.
   (c) What do you notice about the temperature as time passes?

14. **Newton's Law of Cooling** A thermometer reading 72°F is placed in a refrigerator where the temperature is a constant 38°F.
   (a) If the thermometer reads 60°F after 2 minutes, what will it read after 7 minutes?
   (b) How long will it take before the thermometer reads 39°F?
   (c) Determine the time that must elapse before the thermometer reads 45°F.
   (d) What do you notice about the temperature as time passes?

15. **Newton's Law of Heating** A thermometer reading 8°C is brought into a room with a constant temperature of 35°C. If the thermometer reads 15°C after 3 minutes, what will it read after being in the room for 5 minutes? For 10 minutes?
   [**Hint:** You need to construct a formula similar to equation (4).]

16. **Warming Time of a Beer Stein** A beer stein has a temperature of 28°F. It is placed in a room with a constant temperature of 70°F. After 10 minutes, the temperature of the stein has risen to 35°F. What will the temperature of the stein be after 30 minutes? How long will it take the stein to reach a temperature of 45°F? (See the hint given for Problem 15.)

17. **Decomposition of Chlorine in a Pool** Under certain water conditions, the free chlorine (hypochlorous acid, HOCl) in a swimming pool decomposes according to the law of uninhibited decay. After shocking his pool, Ben tested the water and found the amount of free chlorine to be 2.5 parts per million (ppm). Twenty-four hours later, Ben tested the water again and found the amount of free chlorine to be 2.2 ppm. What will be the reading after 3 days (that is, 72 hours)? When the chlorine level reaches 1.0 ppm, Ben must shock the pool again. How long can Ben go before he must shock the pool again?

18. **Decomposition of Dinitrogen Pentoxide** At 45°C, dinitrogen pentoxide (N_2O_5) decomposes into nitrous dioxide (NO_2) and oxygen (O_2) according to the law of uninhibited decay. An initial amount of 0.25 mole of dinitrogen pentoxide decomposes to 0.15 mole in 17 minutes. How much dinitrogen pentoxide will remain after 30 minutes? How long will it take until 0.01 mole of dinitrogen pentoxide remains?

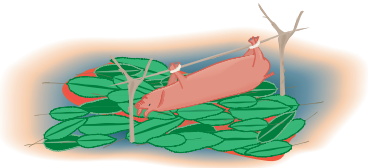
19. **Decomposition of Sucrose** Reacting with water in an acidic solution at 35°C, sucrose ($C_{12}H_{22}O_{11}$) decomposes into glucose ($C_6H_{12}O_6$) and fructose ($C_6H_{12}O_6$)* according to the law of uninhibited decay. An initial amount of 0.40 mole of sucrose decomposes to 0.36 mole in 30 minutes. How much sucrose will remain after 2 hours? How long will it take until 0.10 mole of sucrose remains?

20. **Decomposition of Salt in Water** Salt (NaCl) decomposes in water into sodium (Na^+) and chloride (Cl^-) ions according to the law of uninhibited decay. If the initial amount of salt is 25 kilograms and, after 10 hours, 15 kilograms of salt is left, how much salt is left after 1 day? How long does it take until $\frac{1}{2}$ kilogram of salt is left?

*Author's Note: Surprisingly, the chemical formulas for glucose and fructose are the same: This is not a typo.

21. Radioactivity from Chernobyl After the release of radioactive material into the atmosphere from a nuclear power plant at Chernobyl (Ukraine) in 1986, the hay in Austria was contaminated by iodine-131 (half-life 8 days). If it is safe to feed the hay to cows when 10% of the iodine-131 remains, how long did the farmers need to wait to use this hay?

22. Pig Roasts The hotel Bora-Bora is having a pig roast. At noon, the chef put the pig in a large earthen oven. The pig's original temperature was 75°F. At 2:00 PM the chef checked the pig's temperature and was upset because it had reached only 100°F. If the oven's temperature remains a constant 325°F, at what time may the hotel serve its guests, assuming that pork is done when it reaches 175°F?



23. Population of a Bacteria Culture The logistic growth model

$$P(t) = \frac{1000}{1 + 32.33e^{-0.439t}}$$

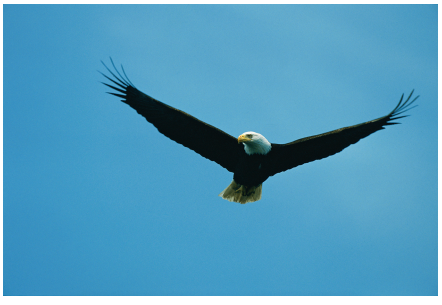
represents the population (in grams) of a bacterium after t hours.

(a) Determine the carrying capacity of the environment.
(b) What is the growth rate of the bacteria?
(c) Determine the initial population size.
(d) What is the population after 9 hours?
(e) When will the population be 700 grams?
(f) How long does it take for the population to reach one-half the carrying capacity?

24. Population of an Endangered Species Often environmentalists capture an endangered species and transport the species to a controlled environment where the species can produce offspring and regenerate its population. Suppose that six American bald eagles are captured, transported to Montana, and set free. Based on experience, the environmentalists expect the population to grow according to the model

$$P(t) = \frac{500}{1 + 82.33e^{-0.162t}}$$

where t is measured in years.



(a) Determine the carrying capacity of the environment.
(b) What is the growth rate of the bald eagle?
(c) What is the population after 3 years?
(d) When will the population be 300 eagles?
(e) How long does it take for the population to reach one-half of the carrying capacity?

25. Invasive Species A habitat can be altered by invasive species that crowd out or replace native species. The logistic model

$$P(t) = \frac{431}{1 + 7.91e^{-0.017t}}$$

represent the number of invasive species present in the Great Lakes t years after 1900.

(a) Evaluate and interpret $P(0)$.
(b) What is the growth rate of invasive species?
(c) Use a graphing utility to graph $P = P(t)$
(d) How many invasive species were present in the Great Lakes in 2000?
(e) In what year was the number of invasive species 175?

Source: *NOAA*

26. Word Users According to a survey by Olsten Staffing Services, the percentage of companies reporting usage of Microsoft Word t years since 1984 is given by

$$P(t) = \frac{99.744}{1 + 3.014e^{-0.799t}}$$

(a) What is the growth rate in the percentage of Microsoft Word users?
(b) Use a graphing utility to graph $P = P(t)$.
(c) What was the percentage of Microsoft Word users in 1990?
(d) During what year did the percentage of Microsoft Word users reach 90%?
(e) Explain why the numerator given in the model is reasonable. What does it imply?

27. Home Computers The logistic model

$$P(t) = \frac{95.4993}{1 + 0.0405e^{0.1968t}}$$

represents the percentage of households that do not own a personal computer t years since 1984.

(a) Evaluate and interpret $P(0)$.
(b) Use a graphing utility to graph $P = P(t)$.
(c) What percentage of households did not own a personal computer in 1995?
(d) In what year did the percentage of households that do not own a personal computer reach 10%?

Source: *U.S. Department of Commerce*

28. Farmers The logistic model

$$W(t) = \frac{14{,}656{,}248}{1 + 0.059e^{0.057t}}$$

represents the number of farm workers in the United States t years after 1910.

(a) Evaluate and interpret $W(0)$.
(b) Use a graphing utility to graph $W = W(t)$.
(c) How many farm workers were there in the United States in 2010?
(d) When did the number of farm workers in the United States reach 10,000,000?
(e) According to this model, what happens to the number of farm workers in the United States as t approaches ∞? Based on this result, do you think that it is reasonable to use this model to predict the number of farm workers in the United States in 2060? Why?

Source: *U.S. Department of Agriculture*

29. Birthdays The logistic model

$$P(n) = \frac{113.3198}{1 + 0.115e^{0.0912n}}$$

gives the probability that in a room of n people, no two people share the same birthday.

(a) Use a graphing utility to graph $P = P(n)$.
(b) In a room of $n = 15$ people, what is the probability that no two share the same birthday?
(c) How many people must be in a room before the probability that no two people share the same birthday falls below 10%?
(d) What happens to the probability as n increases? Explain what this result means.

30. Social Networking The logistic model

$$P(t) = \frac{88.3}{1 + 2.17e^{-0.338t}}$$

gives the percentage of Americans who "use any social media," where t represents the number of years after 2008.

(a) Evaluate and interpret $P(0)$.
(b) What is the growth rate?
(c) Use a graphing utility to graph $P = P(t)$.
(d) During 2015, what percentage of Americans used social media?
(e) In what year did 63.1% of Americans use social media?

Source: *Edison Research, 2017*

Problems 31 and 32 refer to the following discussion: Uninhibited growth can be modeled by exponential functions other than $A(t) = A_0e^{kt}$. For example, if an initial population P_0 requires n units of time to double, then the function $P(t) = P_0 \cdot 2^{t/n}$ models the size of the population at time t. Likewise, a population requiring n units of time to triple can be modeled by $P(t) = P_0 \cdot 3^{t/n}$.

31. Growth of a Human Population The population of a town is growing exponentially.

(a) If its population doubled in size over an 8-year period and the current population is 25,000, write an exponential function of the form $P(t) = P_0 \cdot 2^{t/n}$ that models the population.
(b) Graph the function using a graphing utility.
(c) What will the population be in 3 years?
(d) When will the population reach 80,000?
(e) Express the model from part (a) in the form $A(t) = A_0e^{kt}$.

32. Growth of an Insect Population An insect population grows exponentially.

(a) If the population triples in 20 days, and 50 insects are present initially, write an exponential function of the form $P(t) = P_0 \cdot 3^{t/n}$ that models the population.
(b) Graph the function using a graphing utility.
(c) What will the population be in 47 days?
(d) When will the population reach 700?
(e) Express the model from part (a) in the form $A(t) = A_0e^{kt}$.

Retain Your Knowledge

Problems 33–36 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

33. Find the equation of the linear function f that passes through the points $(4, 1)$ and $(8, -5)$.

34. Determine whether the graphs of the linear functions $f(x) = 5x - 1$ and $g(x) = \frac{1}{5}x + 1$ are parallel, perpendicular, or neither.

35. Write the logarithmic expression $\ln\left(\frac{x^2\sqrt{y}}{z}\right)$ as the sum and/or difference of logarithms. Express powers as factors.

36. Find the domain of $f(x) = \frac{x + 3}{x^2 + 2x - 8}$.

4.9 Building Exponential, Logarithmic, and Logistic Models from Data

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Building Linear Models from Data (Section 2.2, pp. 136–140)
- Building Cubic Models from Data (Section 3.1, pp. 213–214)
- Building Quadratic Models from Data (Section 2.6, pp. 175–177)

OBJECTIVES
1 Build an Exponential Model from Data (p. 370)
2 Build a Logarithmic Model from Data (p. 371)
3 Build a Logistic Model from Data (p. 372)

Finding the linear function of best fit $(y = ax + b)$ for a set of data was discussed in Section 2.2. Likewise, finding the quadratic function of best fit $(y = ax^2 + bx + c)$ and finding the cubic function of best fit $(y = ax^3 + bx^2 + cx + d)$ were discussed in Sections 2.6 and 3.1, respectively.

In this section we discuss how to use a graphing utility to find equations of best fit that describe the relation between two variables when the relation is thought to be exponential $(y = ab^x)$, logarithmic $(y = a + b \ln x)$, or logistic $\left(y = \dfrac{c}{1 + ae^{-bx}}\right)$.

As before, a scatter diagram of the data is drawn to help determine the appropriate model to use.

Figure 47 shows scatter diagrams that will typically be observed for the three models. Below each scatter diagram are any restrictions on the values of the parameters.

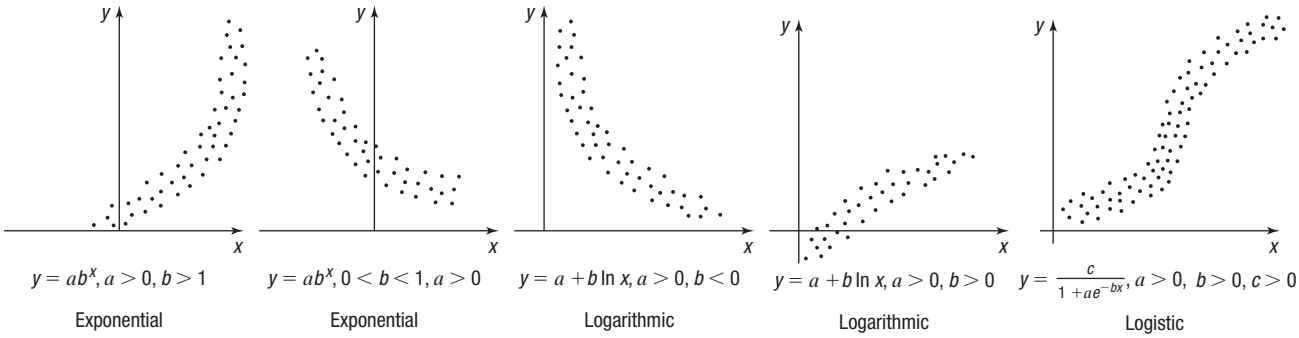


Figure 47

Most graphing utilities have REGression options that fit data to a specific type of curve. Once the data have been entered and a scatter diagram obtained, the type of curve that you want to fit to the data is selected. Then that REGression option is used to obtain the curve of *best fit* of the type selected.

The correlation coefficient r will appear only if the model can be written as a linear expression. As it turns out, r will appear for the linear, power, exponential, and logarithmic models, since these models can be written as a linear expression. Remember, the closer $|r|$ is to 1, the better the fit.

1 Build an Exponential Model from Data

We saw in Section 4.7 that the future value of money behaves exponentially, and we saw in Section 4.8 that growth and decay models also behave exponentially. The next example shows how data can lead to an exponential model.

EXAMPLE 1 Fitting an Exponential Function to Data

Table 9

Year, x	Account Value, y
0	20,000
1	21,516
2	23,355
3	24,885
4	27,484
5	30,053
6	32,622

Mariah deposited \$20,000 into a well-diversified mutual fund 6 years ago. The data in Table 9 represent the value of the account each year for the last 7 years.

(a) Using a graphing utility, draw a scatter diagram with year as the independent variable.

(b) Using a graphing utility, build an exponential model from the data.

(c) Express the function found in part (b) in the form $A = A_0e^{kt}$.

(d) Graph the exponential function found in part (b) or (c) on the scatter diagram.

(e) Using the solution to part (b) or (c), predict the value of the account after 10 years.

(f) Interpret the value of k found in part (c).

Solution

(a) Enter the data into the graphing utility and draw the scatter diagram as shown in Figure 48 on a TI-84 Plus C.

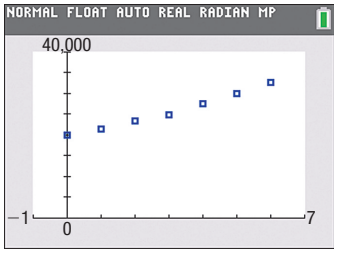


Figure 48 TI-84 Plus C

(b) A graphing utility fits the data in Table 9 to an exponential model of the form $y = ab^x$ using the EXPonential REGression option. Figure 49 shows that $y = ab^x = 19{,}820.43(1.085568)^x$ on a TI-84 Plus C. Notice that $|r| = 0.999$, which is close to 1, indicating a good fit.

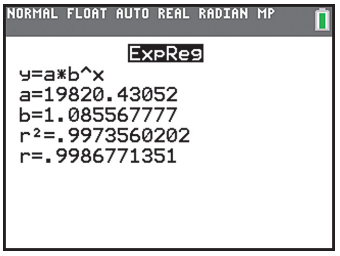


Figure 49 Exponential model using a TI-84 Plus C

(c) To express $y = ab^x$ in the form $A = A_0e^{kt}$, where $x = t$ and $y = A$, proceed as follows:

$$ab^x = A_0e^{kt}$$

If $x = t = 0$, then $a = A_0$. This leads to

$$b^x = e^{kt}$$

$$b^x = (e^k)^t$$

$$b = e^k \qquad x = t$$

Because $y = ab^x = 19{,}820.43(1.085568)^x$, this means that $a = 19{,}820.43$ and $b = 1.085568$.

$$a = A_0 = 19{,}820.43 \quad \text{and} \quad b = e^k = 1.085568$$

To find k, rewrite $e^k = 1.085568$ as a logarithm to obtain

$$k = \ln(1.085568) \approx 0.08210$$

As a result, $A = A_0e^{kt} = 19{,}820.43e^{0.08210t}$.

(d) See Figure 50 for the graph of the exponential function of best fit on a TI-84 Plus C. Figure 51 shows the exponential model using Desmos.

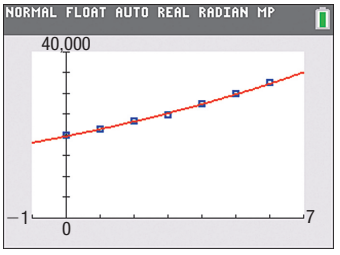


Figure 50

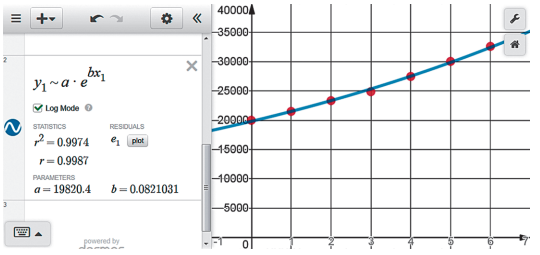


Figure 51 Exponential model using Desmos*

Source: *https://www.desmos.com/calculator/fltmjedgh2*

(e) Let $t = 10$ in the function found in part (c). The predicted value of the account after 10 years is

$$A = A_0e^{kt} = 19{,}820.43e^{0.08210(10)} \approx \$45{,}047$$

(f) The value of $k = 0.08210 = 8.210\%$ represents the annual growth rate of the account. It represents the rate of interest earned, assuming the account is growing continuously.

NOW WORK PROBLEM 1

*For this result in Desmos to agree precisely with the result of a TI-84 Plus C, the "Log Mode" option must be selected. Consult the help feature in Desmos for more information about this option.

2 Build a Logarithmic Model from Data

Some relations between variables follow a logarithmic model.

EXAMPLE 2 Fitting a Logarithmic Function to Data

Table 10

Atmospheric Pressure, p	Height, h
760	0
740	0.184
725	0.328
700	0.565
650	1.079
630	1.291
600	1.634
580	1.862
550	2.235

Jodi, a meteorologist, is interested in finding a function that explains the relation between the height of a weather balloon (in kilometers) and the atmospheric pressure (measured in millimeters of mercury) on the balloon. She collects the data shown in Table 10.

(a) Using a graphing utility, draw a scatter diagram of the data with atmospheric pressure as the independent variable.

(b) It is known that the relation between atmospheric pressure and height follows a logarithmic model. Using a graphing utility, build a logarithmic model from the data.

(c) Draw the logarithmic function found in part (b) on the scatter diagram.

(d) Use the function found in part (b) to predict the height of the weather balloon if the atmospheric pressure is 560 millimeters of mercury.

Solution

(a) Enter the data into the graphing utility, and draw the scatter diagram. See Figure 52.

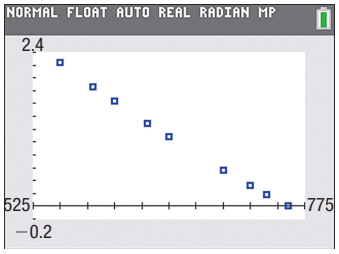


Figure 52 TI-84 Plus C

(b) A graphing utility fits the data in Table 10 to a logarithmic function of the form $y = a + b \ln x$ by using the LOGarithm REGression option. Figure 53 shows the result on a TI-84 Plus C. The logarithmic model from the data is

$$h(p) = 45.7863 - 6.9025 \ln p$$

where h is the height of the weather balloon and p is the atmospheric pressure. Notice that $|r|$ is close to 1, indicating a good fit.

(c) Figure 54 shows the graph of $h(p) = 45.7863 - 6.9025 \ln p$ on the scatter diagram. Figure 55 shows the logarithmic model using Desmos.

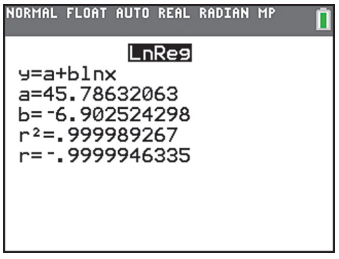


Figure 53 Logarithmic model using a TI-84 Plus C

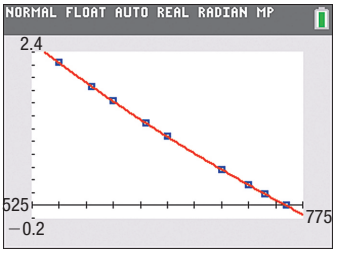


Figure 54

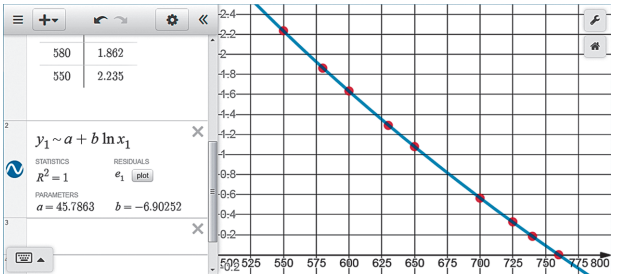


Figure 55 Logarithmic model using Desmos
Source: *https://www.desmos.com/calculator/5vbdrvg998*

(d) Using the function found in part (b), Jodi predicts the height of the weather balloon when the atmospheric pressure is 560 to be

$$\begin{aligned} h(560) &= 45.7863 - 6.9025 \ln 560 \\ &\approx 2.108 \text{ kilometers} \end{aligned}$$

NOW WORK PROBLEM 5

3 Build a Logistic Model from Data

Logistic growth models can be used to model situations for which the value of the dependent variable is limited. Many real-world situations conform to this scenario. For example, the population of the human race is limited by the availability of natural resources such as food and shelter. When the value of the dependent variable is limited, a logistic growth model is often appropriate.

EXAMPLE 3 Fitting a Logistic Function to Data

The data in Table 11 represent the amount of yeast biomass in a culture after t hours.

Table 11

Time (hours)	Yeast Biomass	Time (hours)	Yeast Biomass	Time (hours)	Yeast Biomass
0	9.6	7	257.3	14	640.8
1	18.3	8	350.7	15	651.1
2	29.0	9	441.0	16	655.9
3	47.2	10	513.3	17	659.6
4	71.1	11	559.7	18	661.8
5	119.1	12	594.8		
6	174.6	13	629.4		

Source: *Tor Carlson (Über Geschwindigkeit and Grösse der Hefevermehrung in Würze, Biochemische Zeitschrift, Bd. 57, pp. 313–334, 1913)*

(a) Using a graphing utility, draw a scatter diagram of the data with time as the independent variable.
(b) Using a graphing utility, build a logistic model from the data.
(c) Using a graphing utility, graph the function found in part (b) on the scatter diagram.
(d) What is the predicted carrying capacity of the culture?
(e) Use the function found in part (b) to predict the population of the culture at $t = 19$ hours.

Solution

(a) See Figure 56 for a scatter diagram of the data on a TI-84 Plus C.

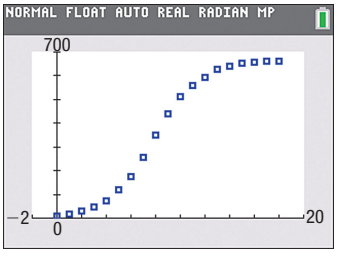


Figure 56 TI-84 Plus C

(b) A graphing utility fits the data in Table 11 to a logistic growth model of the form $y = \dfrac{c}{1 + ae^{-bx}}$ by using the LOGISTIC regression option. Figure 57 shows the result on a TI-84 Plus C. The logistic model from the data is

$$y = \frac{663.0}{1 + 71.6e^{-0.5470x}}$$

where y is the amount of yeast biomass in the culture and x is the time.

(c) See Figure 58 for the graph of the logistic model on a TI-84 Plus C. Figure 59 shows the logistic model using Desmos.

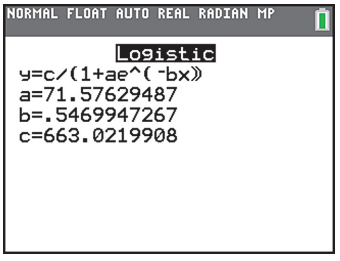


Figure 57 Logistic model using a TI-84 Plus C

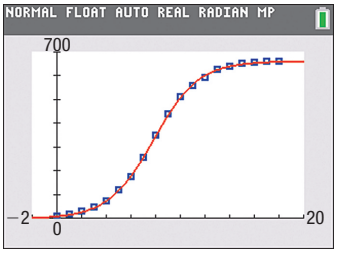


Figure 58

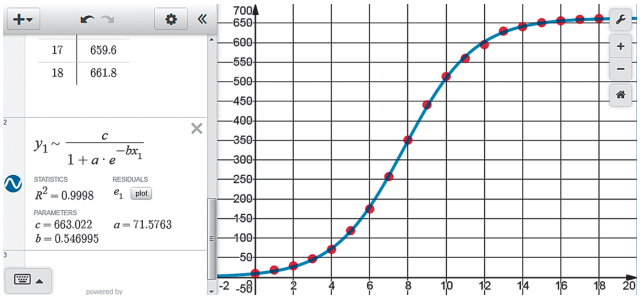


Figure 59 Logistic model using Desmos
Source: *https://www.desmos.com/calculator/36fnqd6psj*

(d) Based on the logistic growth model found in part (b), the carrying capacity of the culture is 663.

(e) Using the logistic growth model found in part (b), the predicted amount of yeast biomass at $t = 19$ hours is

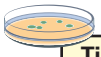
$$y = \frac{663.0}{1 + 71.6e^{-0.5470(19)}} \approx 661.5$$

NOW WORK PROBLEM 7

4.9 Assess Your Understanding

Applications and Extensions

1. **Biology** A strain of *E. coli*, Beu 397-recA441, is placed into a nutrient broth at 30 Celsius and allowed to grow. The following data are collected. Theory states that the number of bacteria in the petri dish will initially grow according to the law of uninhibited growth. The population is measured using an optical device in which the amount of light that passes through the petri dish is measured.



Time (hours), x	Population, y
0	0.09
2.5	0.18
3.5	0.26
4.5	0.35
6	0.50

Source: *Dr. Polly Lavery, Joliet Junior College*

(a) Draw a scatter diagram treating time as the independent variable.
(b) Using a graphing utility, build an exponential model from the data.
(c) Express the function found in part (b) in the form $N(t) = N_0e^{kt}$.
(d) Graph the exponential function found in part (b) or (c) on the scatter diagram.
(e) Use the exponential function from part (b) or (c) to predict the population at $x = 7$ hours.
(f) Use the exponential function from part (b) or (c) to predict when the population will reach 0.75.

2. **Ethanol Production** The data in the table below represent ethanol production (in billions of gallons) in the United States from 2000 to 2016.

Year	Ethanol Produced (billion gallons)	Year	Ethanol Produced (billion gallons)
2000 ($x = 0$)	1.6	2009 ($x = 9$)	10.8
2001 ($x = 1$)	1.8	2010 ($x = 10$)	13.2
2002 ($x = 2$)	2.1	2011 ($x = 11$)	13.9
2003 ($x = 3$)	2.8	2012 ($x = 12$)	13.3
2004 ($x = 4$)	3.4	2013 ($x = 13$)	13.3
2005 ($x = 5$)	3.9	2014 ($x = 14$)	14.3
2006 ($x = 6$)	4.9	2015 ($x = 15$)	14.8
2007 ($x = 7$)	6.5	2016 ($x = 16$)	15.3
2008 ($x = 8$)	9.2		

Source: *Renewable Fuels Association, 2017(www.ethanolrfa.org)*

(a) Using a graphing utility, draw a scatter diagram of the data using 0 for 2000, 1 for 2001, and so on, as the independent variable.
(b) Using a graphing utility, build an exponential model from the data.
(c) Express the function found in part (b) in the form $A(t) = A_0e^{kt}$.
(d) Graph the exponential function found in part (b) or (c) on the scatter diagram.
(e) Use the model to predict the amount of ethanol that will be produced in 2019.
(f) Interpret the meaning of k in the function found in part (c).

3. **Advanced-Stage Breast Cancer** The data in the table below represent the percentage of patients who have survived after diagnosis of advanced-stage breast cancer at 6-month intervals of time.

Time after Diagnosis (years)	Percentage Surviving
0.5	95.7
1	83.6
1.5	74.0
2	58.6
2.5	47.4
3	41.9
3.5	33.6

Source: *Cancer Treatment Centers of America*

(a) Using a graphing utility, draw a scatter diagram of the data with time after diagnosis as the independent variable.
(b) Using a graphing utility, build an exponential model from the data.
(c) Express the function found in part (b) in the form $A(t) = A_0e^{kt}$.
(d) Graph the exponential function found in part (b) or (c) on the scatter diagram.
(e) What percentage of patients diagnosed with advanced-stage cancer are expected to survive for 4 years after initial diagnosis?
(f) Interpret the meaning of k in the function found in part (c).

4. **Chemistry** A chemist has a 100-gram sample of a radioactive material. He records the amount of radioactive material every week for 7 weeks and obtains the following data:

Week	Weight (grams)
0	100.0
1	88.3
2	75.9
3	69.4
4	59.1
5	51.8
6	45.5

(a) Using a graphing utility, draw a scatter diagram with week as the independent variable.

(b) Using a graphing utility, build an exponential model from the data.
(c) Express the function found in part (b) in the form $A(t) = A_0e^{kt}$.
(d) Graph the exponential function found in part (b) or (c) on the scatter diagram.
(e) From the result found in part (b), determine the half-life of the radioactive material.
(f) How much radioactive material will be left after 50 weeks?
(g) When will there be 20 grams of radioactive material?

5. **Milk Production** The data in the table below represent the number of dairy farms (in thousands) and the amount of milk produced (in billions of pounds) in the United States for various years.

Year	Dairy Farms (thousands)	Milk Produced (billion pounds)
1980	334	128
1985	269	143
1990	193	148
1995	140	155
2000	105	167
2005	78	177
2010	63	193
2015	44	209

Source: *National Agricultural Statistics Services, 2016*

(a) Using a graphing utility, draw a scatter diagram of the data with the number of dairy farms as the independent variable.
(b) Using a graphing utility, build a logarithmic model from the data.
(c) Graph the logarithmic function found in part (b) on the scatter diagram.
(d) In 2008, there were 67 thousand dairy farms in the United States. Use the function in part (b) to predict the amount of milk produced in 2008.
(e) The actual amount of milk produced in 2008 was 190 billion pounds. How does your prediction in part (d) compare to this?

6. **Social Networking** The data in the table below represent the percent of U.S. citizens aged 12 and older who have a profile on at least one social network.

Year	Percent on a Social Networking Site
2008 ($x = 8$)	24
2009 ($x = 9$)	34
2010 ($x = 10$)	48
2011 ($x = 11$)	52
2012 ($x = 12$)	56
2013 ($x = 13$)	62
2014 ($x = 14$)	67
2015 ($x = 15$)	73
2016 ($x = 16$)	78
2017 ($x = 17$)	81

Source: *Statista.com*

(a) Using a graphing utility, draw a scatter diagram of the data using 8 for 2008, 9 for 2009, and so on, as the independent variable, and percent on social networking site as the dependent variable.
(b) Using a graphing utility, build a logarithmic model from the data.
(c) Graph the logarithmic function found in part (b) on the scatter diagram.
(d) Use the model to predict the percent of U.S. citizens on social networking sites in 2019.
(e) Use the model to predict the year in which 98% of U.S. citizens will be on social networking sites.

7. **Population Model** The following data represent the population of the United States. An ecologist is interested in building a model that describes the population of the United States.

Year	Population
1900	76,212,168
1910	92,228,496
1920	106,021,537
1930	123,202,624
1940	132,164,569
1950	151,325,798
1960	179,323,175
1970	203,302,031
1980	226,542,203
1990	248,709,873
2000	281,421,906
2010	308,745,538

Source: *U.S. Census Bureau (www.census.gov)*

(a) Using a graphing utility, draw a scatter diagram of the data using years since 1900 as the independent variable and population as the dependent variable.
(b) Using a graphing utility, build a logistic model from the data.
(c) Using a graphing utility, draw the function found in part (b) on the scatter diagram.
(d) Based on the function found in part (b), what is the carrying capacity of the United States?
(e) Use the function found in part (b) to predict the population of the United States in 2012.
(f) When will the United States population be 350,000,000?
(g) Compare actual U.S. Census figures to the predictions found in parts (e) and (f). Discuss any differences.

8. Population Model The data on the right represent the world population. An ecologist is interested in building a model that describes the world population.

(a) Using a graphing utility, draw a scatter diagram of the data using years since 2000 as the independent variable and population as the dependent variable.
(b) Using a graphing utility, build a logistic model from the data.
(c) Using a graphing utility, draw the function found in part (b) on the scatter diagram.
(d) Based on the function found in part (b), what is the carrying capacity of the world?
(e) Use the function found in part (b) to predict the population of the world in 2021.
(f) When will world population be 10 billion?

Year	Population (billions)	Year	Population (billions)
2001	6.20	2010	6.93
2002	6.28	2011	7.01
2003	6.36	2012	7.10
2004	6.44	2013	7.18
2005	6.52	2014	7.27
2006	6.60	2015	7.35
2007	6.68	2016	7.43
2008	6.76	2017	7.52
2009	6.85		

Source: *worldometers.info*

9. Cell Phone Towers The following data represent the number of cell sites in service in the United States from 1985 to 2015 at the end of each year.

Year	Cell Sites (thousands)	Year	Cell Sites (thousands)	Year	Cell Sites (thousands)
1985 ($x = 1$)	0.9	1996 ($x = 12$)	30.0	2006 ($x = 22$)	195.6
1986 ($x = 2$)	1.5	1997 ($x = 13$)	51.6	2007 ($x = 23$)	213.3
1987 ($x = 3$)	2.3	1998 ($x = 14$)	65.9	2008 ($x = 24$)	242.1
1988 ($x = 4$)	3.2	1999 ($x = 15$)	81.7	2009 ($x = 25$)	247.1
1989 ($x = 5$)	4.2	2000 ($x = 16$)	104.3	2010 ($x = 26$)	253.1
1990 ($x = 6$)	5.6	2001 ($x = 17$)	127.5	2011 ($x = 27$)	283.4
1991 ($x = 7$)	7.8	2002 ($x = 18$)	139.3	2012 ($x = 28$)	301.8
1992 ($x = 8$)	10.3	2003 ($x = 19$)	163.0	2013 ($x = 29$)	304.4
1993 ($x = 9$)	12.8	2004 ($x = 20$)	175.7	2014 ($x = 30$)	298.1
1994 ($x = 10$)	17.9	2005 ($x = 21$)	183.7	2015 ($x = 31$)	307.6
1995 ($x = 11$)	22.7				

Source: *©2016 CTIA-The Wireless Association®. All Rights Reserved.*

(a) Using a graphing utility, draw a scatter diagram of the data using 1 for 1985, 2 for 1986, and so on as the independent variable, and number of cell sites as the dependent variable.
(b) Using a graphing utility, build a logistic model from the data.
(c) Graph the logistic function found in part (b) on the scatter diagram.
(d) What is the predicted carrying capacity for cell sites in the United States?
(e) Use the model to predict the number of cell sites in the United States at the end of 2022.

10. Cable Rates The data on the right represent the average monthly rate charged for expanded basic cable television in the United States from 1995 to 2015. A market researcher believes that external factors, such as the growth of satellite television and internet programming, have affected the cost of basic cable. She is interested in building a model that will describe the average monthly cost of basic cable.

(a) Using a graphing utility, draw a scatter diagram of the data using 0 for 1995, 1 for 1996, and so on, as the independent variable and average monthly rate as the dependent variable.
(b) Using a graphing utility, build a logistic model from the data.
(c) Graph the logistic function found in part (b) on the scatter diagram.
(d) Based on the model found in part (b), what is the maximum possible average monthly rate for basic cable?
(e) Use the model to predict the average rate for basic cable in 2021.

Year	Average Monthly Rate (dollars)	Year	Average Monthly Rate (dollars)
1995 ($x = 0$)	22.35	2006 ($x = 11$)	45.26
1996 ($x = 1$)	24.28	2007 ($x = 12$)	47.27
1997 ($x = 2$)	26.31	2008 ($x = 13$)	49.65
1998 ($x = 3$)	27.88	2009 ($x = 14$)	52.37
1999 ($x = 4$)	28.94	2010 ($x = 15$)	54.44
2000 ($x = 5$)	31.22	2011 ($x = 16$)	57.46
2001 ($x = 6$)	33.75	2012 ($x = 17$)	61.63
2002 ($x = 7$)	36.47	2013 ($x = 18$)	64.41
2003 ($x = 8$)	38.95	2014 ($x = 19$)	66.61
2004 ($x = 9$)	41.04	2015 ($x = 20$)	69.03
2005 ($x = 10$)	43.04		

Source: *Federal Communications Commission, 2016*

Mixed Practice

11. Online Advertising Revenue The data in the table below represent the U.S. online advertising revenues for the years 2005–2015.

Year	U.S. Online Advertising Revenue ($ billions)
2005 ($x = 0$)	12.5
2006 ($x = 1$)	16.9
2007 ($x = 2$)	21.2
2008 ($x = 3$)	23.4
2009 ($x = 4$)	22.7
2010 ($x = 5$)	26.0
2011 ($x = 6$)	31.7
2012 ($x = 7$)	36.6
2013 ($x = 8$)	42.8
2014 ($x = 9$)	49.5
2015 ($x = 10$)	59.6

Source: *marketingcharts.com*

(a) Using a graphing utility, draw a scatter diagram of the data using 0 for 2005, 1 for 2006, and so on as the independent variable, and online advertising revenue as the dependent variable.
(b) Based on the scatter diagram drawn in part (a), decide what model (linear, quadratic, cubic, exponential, logarithmic, or logistic) that you think best describes the relation between year and revenue.
(c) Using a graphing utitlity, find the model of best fit.
(d) Using a graphing utility, draw the model of best fit on the scatter diagram drawn in part (a).
(e) Use your model to predict the online advertising revenue in 2021.

12. Age versus Total Cholesterol The following data represent the age and average total cholesterol for adult males at various ages.

Age	Total Cholesterol
27	189
40	205
50	215
60	210
70	210
80	194

(a) Using a graphing utility, draw a scatter diagram of the data using age, x, as the independent variable and total cholesterol, y, as the dependent variable.
(b) Based on the scatter diagram drawn in part (a), decide on a model (linear, quadratic, cubic, exponential, logarithmic, or logistic) that you think best describes the relation between age and total cholesterol. Be sure to justify your choice of model.
(c) Using a graphing utility, find the model of best fit.
(d) Using a graphing utility, draw the model of best fit on the scatter diagram drawn in part (a).
(e) Use your model to predict the total cholesterol of a 35-year-old male.

13. Golfing The data below represent the expected percentage of putts that will be made by professional golfers on the PGA Tour depending on distance. For example, it is expected that 99.3% of 2-foot putts will be made.

Distance (feet)	Expected Percentage	Distance (feet)	Expected Percentage
2	99.3	14	25.0
3	94.8	15	22.0
4	85.8	16	20.0
5	74.7	17	19.0
6	64.7	18	17.0
7	55.6	19	16.0
8	48.5	20	14.0
9	43.4	21	13.0
10	38.3	22	12.0
11	34.2	23	11.0
12	30.1	24	11.0
13	27.0	25	10.0

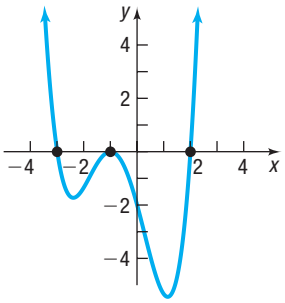
Source: *TheSandTrap.com*

(a) Using a graphing utility, draw a scatter diagram of the data with distance as the independent variable.
(b) Based on the scatter diagram drawn in part (a), decide on a model (linear, quadratic, cubic, exponential, logarithmic, or logistic) that you think best describes the relation between distance and expected percentage. Be sure to justify your choice of model.
(c) Using a graphing utility, find the model of best fit.
(d) Graph the function found in part (c) on the scatter diagram.
(e) Use the function found in part (c) to predict what percentage of 30-foot putts will be made.

Retain Your Knowledge

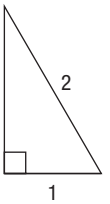
Problems 14–17 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

14. Construct a polynomial function that might have the graph shown. (More than one answer is possible.)



15. Rationalize the denominator of $\frac{3}{\sqrt{2}}$.

16. Use the Pythagorean Theorem to find the exact length of the unlabeled side in the given right triangle.



17. Graph the equation $(x - 3)^2 + y^2 = 25$.

Chapter Review

Things to Know

Composite function (p. 281)

$(f \circ g)(x) = f(g(x))$; The domain of $f \circ g$ is the set of all numbers x in the domain of g for which $g(x)$ is in the domain of f.

One-to-one function (p. 289)

A function for which any two different inputs in the domain correspond to two different outputs in the range

For any choice of elements x_1, x_2 in the domain of f, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Horizontal-line test (p. 290)

If every horizontal line intersects the graph of a function f in at most one point, f is one-to-one.

Inverse function f^{-1} of f (pp. 291–294)

Domain of f = range of f^{-1}; range of f = domain of f^{-1}

$f^{-1}(f(x)) = x$ for all x in the domain of f

$f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}

The graphs of f and f^{-1} are symmetric with respect to the line $y = x$.

Properties of the exponential function (pp. 303, 307, 308)

$f(x) = Ca^x, \quad a > 1, C > 0$

Domain: the interval $(-\infty, \infty)$
Range: the interval $(0, \infty)$
x-intercepts: none; y-intercept: C
Horizontal asymptote: x-axis $(y = 0)$ as $x \to -\infty$
Increasing; one-to-one; smooth; continuous
See Figure 22 for a typical graph.

$f(x) = Ca^x, \quad 0 < a < 1, C > 0$

Domain: the interval $(-\infty, \infty)$
Range: the interval $(0, \infty)$
x-intercepts: none; y-intercept: C
Horizontal asymptote: x-axis $(y = 0)$ as $x \to \infty$
Decreasing; one-to-one; smooth; continuous
See Figure 26 for a typical graph.

Number e (p. 309)

Value approached by the expression $\left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$; that is, $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$.

Property of exponents (p. 311)

If $a^u = a^v$, then $u = v$.

Properties of the logarithmic function (pp. 319–321)

$f(x) = \log_a x, \quad a > 1$
$(y = \log_a x$ means $x = a^y)$

Domain: the interval $(0, \infty)$
Range: the interval $(-\infty, \infty)$
x-intercept: 1; y-intercept: none
Vertical asymptote: $x = 0$ (y-axis)
Increasing; one-to-one; smooth; continuous
See Figure 32(b) for a typical graph.

	$f(x) = \log_a x, \quad 0 < a < 1$ $(y = \log_a x \text{ means } x = a^y)$	Domain: the interval $(0, \infty)$ Range: the interval $(-\infty, \infty)$ x-intercept: 1; y-intercept: none Vertical asymptote: $x = 0$ (y-axis) Decreasing; one-to-one; smooth; continuous See Figure 32(a) for a typical graph.
Natural logarithm (p. 322)	$y = \ln x$ means $x = e^y$.	
Properties of logarithms (pp. 332, 333, 336)	$\log_a 1 = 0 \quad \log_a a = 1 \quad a^{\log_a M} = M \quad \log_a a^r = r$ $\log_a(MN) = \log_a M + \log_a N \quad \log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$ $\log_a M^r = r \log_a M \quad a^x = e^{x \ln a}$ If $M = N$, then $\log_a M = \log_a N$ If $\log_a M = \log_a N$, then $M = N$	

Formulas

Change-of-Base Formula (p. 337)	$\log_a M = \dfrac{\log_b M}{\log_b a}$
Compound Interest Formula (p. 350)	$A = P \cdot \left(1 + \dfrac{r}{n}\right)^{nt}$
Continuous compounding (p. 352)	$A = Pe^{rt}$
Effective rate of interest (p. 353)	Compounding n times per year: $r_e = \left(1 + \dfrac{r}{n}\right)^n - 1$ Continuous compounding: $r_e = e^r - 1$
Present Value Formulas (p. 354)	$P = A \cdot \left(1 + \dfrac{r}{n}\right)^{-nt}$ or $P = Ae^{-rt}$
Growth and decay (pp. 359–361)	$A(t) = A_0 e^{kt}$
Newton's Law of Cooling (p. 362)	$u(t) = T + (u_0 - T)e^{kt} \quad k < 0$
Logistic model (p. 364)	$P(t) = \dfrac{c}{1 + ae^{-bt}}$

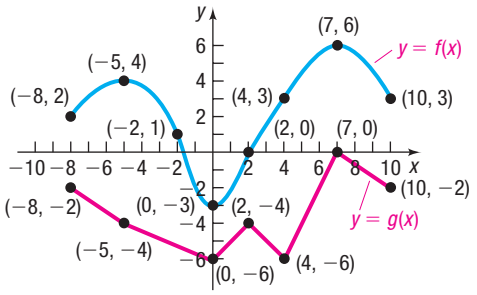
Objectives

Section	You should be able to . . .	Example(s)	Review Exercises
4.1	1 Form a composite function (p. 281)	1, 2, 4, 5	1–7
	2 Find the domain of a composite function (p. 282)	2–4	5–7
4.2	1 Determine whether a function is one-to-one (p. 289)	1, 2	8(a), 9
	2 Determine the inverse of a function defined by a map or a set of ordered pairs (p. 291)	3, 4	8(b)
	3 Obtain the graph of the inverse function from the graph of the function (p. 294)	7	9
	4 Find the inverse of a function defined by an equation (p. 295)	8, 9, 10	10–13
4.3	1 Evaluate exponential functions (p. 302)	1	14(a), (c), 47(a)
	2 Graph exponential functions (p. 305)	3–6	31–33
	3 Define the number e (p. 309)	p. 309	
	4 Solve exponential equations (p. 310)	7, 8	35, 36, 39, 41
4.4	1 Change exponential statements to logarithmic statements and logarithmic statements to exponential statements (p. 319)	2, 3	15, 16
	2 Evaluate logarithmic expressions (p. 319)	4	14(b), (d), 19, 46(b), 48(a), 49
	3 Determine the domain of a logarithmic function (p. 320)	5	17, 18, 34(a)
	4 Graph logarithmic functions (p. 321)	6, 7	34(b), 46(a)
	5 Solve logarithmic equations (p. 325)	8, 9	37, 40, 46(c), 48(b)

Section	You should be able to . . .	Example(s)	Review Exercises
4.5	1 Work with the properties of logarithms (p. 332)	1, 2	20, 21
	2 Write a logarithmic expression as a sum or difference of logarithms (p. 334)	3–5	22–25
	3 Write a logarithmic expression as a single logarithm (p. 335)	6	26–28
	4 Evaluate a logarithm whose base is neither 10 nor e (p. 336)	7, 8	29
	5 Graph a logarithmic function whose base is neither 10 nor e (p. 337)	9	30
4.6	1 Solve logarithmic equations (p. 341)	1–3	37, 43
	2 Solve exponential equations (p. 343)	4–6	38, 42, 44, 45
	3 Solve logarithmic and exponential equations using a graphing utility (p. 345)	7	35–45
4.7	1 Determine the future value of a lump sum of money (p. 349)	1–3	50
	2 Calculate effective rates of return (p. 352)	4	50
	3 Determine the present value of a lump sum of money (p. 353)	5	51
	4 Determine the rate of interest or the time required to double a lump sum of money (p. 354)	6, 7	50
4.8	1 Find equations of populations that obey the law of uninhibited growth (p. 359)	1, 2	54
	2 Find equations of populations that obey the law of decay (p. 361)	3	52, 55
	3 Use Newton's Law of Cooling (p. 362)	4	53
	4 Use logistic models (p. 364)	5, 6	56
4.9	1 Build an exponential model from data (p. 370)	1	57
	2 Build a logarithmic model from data (p. 372)	2	58
	3 Build a logistic model from data (p. 372)	3	59

Review Exercises

1. Evaluate each expression using the graphs of $y = f(x)$ and $y = g(x)$ shown in the figure.



(a) $(g \circ f)(-8)$
(b) $(f \circ g)(-8)$
(c) $(g \circ g)(7)$
(d) $(g \circ f)(-5)$

In Problems 2–4, for the given functions f and g find:

(a) $(f \circ g)(2)$
(b) $(g \circ f)(-2)$
(c) $(f \circ f)(4)$
(d) $(g \circ g)(-1)$

2. $f(x) = 3x - 5;\quad g(x) = 1 - 2x^2$

3. $f(x) = \sqrt{x + 2};\quad g(x) = 2x^2 + 1$

4. $f(x) = e^x;\quad g(x) = 3x - 2$

In Problems 5–8, find $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ for each pair of functions. State the domain of each composite function.

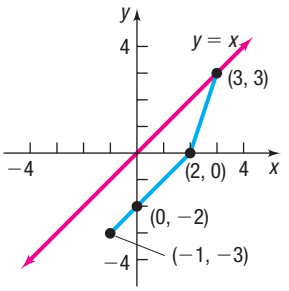
5. $f(x) = 2 - x;\quad g(x) = 3x + 1$

6. $f(x) = \sqrt{3x};\quad g(x) = 1 + x + x^2$

7. $f(x) = \dfrac{x + 1}{x - 1};\quad g(x) = \dfrac{1}{x}$

8. For the function $\{(1, 2), (3, 5), (5, 8), (6, 10)\}$
(a) Verify that the function is one-to-one.
(b) Find the inverse of the given function.

9. State why the graph of the function is one-to-one. Then draw the graph of the inverse function f^{-1}. For convenience (and as a hint), the graph of $y = x$ is also given.



In Problems 10–13, the function f is one-to-one. Find the inverse of each function and check your answer. State the domain and the range of f and f^{-1}.

10. $f(x) = \dfrac{2x + 3}{5x - 2}$

11. $f(x) = \dfrac{1}{x - 1}$

12. $f(x) = \sqrt{x - 2}$

13. $f(x) = x^{1/3} + 1$

14. If $f(x) = 3^x$ and $g(x) = \log_3 x$, evaluate each of the following.

(a) $f(4)$ (b) $g(9)$ (c) $f(-2)$ (d) $g\left(\frac{1}{27}\right)$

15. Convert $5^2 = z$ to an equivalent statement involving a logarithm.

16. Convert $\log_5 u = 13$ to an equivalent statement involving an exponent.

In Problems 17 and 18, find the domain of each logarithmic function.

17. $f(x) = \log(3x - 2)$

18. $H(x) = \log_2(x^2 - 3x + 2)$

In Problems 19–21, evaluate each expression. Do not use a calculator.

19. $\log_2\left(\frac{1}{8}\right)$

20. $\ln e^{\sqrt{2}}$

21. $2^{\log_2 0.4}$

In Problems 22–25, write each expression as the sum and/or difference of logarithms. Express powers as factors.

22. $\log_3\left(\frac{uv^2}{w}\right), \quad u > 0, v > 0, w > 0$

23. $\log_2(a^2\sqrt{b})^4, \quad a > 0, b > 0$

24. $\log(x^2\sqrt{x^3 + 1}), \quad x > 0$

25. $\ln\left(\frac{2x + 3}{x^2 - 3x + 2}\right)^2, \quad x > 2$

In Problems 26–28, write each expression as a single logarithm.

26. $3\log_4 x^2 + \frac{1}{2}\log_4\sqrt{x}$

27. $\ln\left(\frac{x - 1}{x}\right) + \ln\left(\frac{x}{x + 1}\right) - \ln(x^2 - 1)$

28. $\frac{1}{2}\ln(x^2 + 1) - 4\ln\frac{1}{2} - \frac{1}{2}[\ln(x - 4) + \ln x]$

29. Use the Change-of-Base Formula and a calculator to evaluate $\log_4 19$. Round your answer to three decimal places.

30. Graph $y = \log_3 x$ using a graphing utility and the Change-of-Base Formula.

In Problems 31–34, use the given function f to:

(a) Find the domain of f. *(b) Graph f.* *(c) From the graph, determine the range and any asymptotes of f.*
(d) Find f^{-1}. *(e) Find the domain and the range of f^{-1}.* *(f) Graph f^{-1}.*

31. $f(x) = 2^{x-3}$

32. $f(x) = 1 + 3^{-x}$

33. $f(x) = 3e^{x-2}$

34. $f(x) = \frac{1}{2}\ln(x + 3)$

In Problems 35–45, solve each equation. Express any irrational solution in exact form and as a decimal rounded to 3 decimal places.

35. $8^{6+3x} = 4$

36. $3^{x^2+x} = \sqrt{3}$

37. $\log_x 64 = -3$

38. $5^x = 3^{x+2}$

39. $25^{2x} = 5^{x^2-12}$

40. $\log_3\sqrt{x - 2} = 2$

41. $8 = 4^{x^2} \cdot 2^{5x}$

42. $2^x \cdot 5 = 10^x$

43. $\log_6(x + 3) + \log_6(x + 4) = 1$

44. $e^{1-x} = 5$

45. $9^x + 4 \cdot 3^x - 3 = 0$

---

46. Suppose that $f(x) = \log_2(x - 2) + 1$.
(a) Graph *f*.
(b) What is $f(6)$? What point is on the graph of *f*?
(c) Solve $f(x) = 4$. What point is on the graph of *f*?
(d) Based on the graph drawn in part (a), solve $f(x) > 0$.
(e) Find $f^{-1}(x)$. Graph f^{-1} on the same Cartesian plane as *f*.

47. Amplifying Sound An amplifier's power output *P* (in watts) is related to its decibel voltage gain *d* by the formula

$$P = 25e^{0.1d}$$

(a) Find the power output for a decibel voltage gain of 4 decibels.
(b) For a power output of 50 watts, what is the decibel voltage gain?

48. Limiting Magnitude of a Telescope A telescope is limited in its usefulness by the brightness of the star that it is aimed at and by the diameter of its lens. One measure of a star's brightness is its *magnitude;* the dimmer the star, the larger its magnitude. A formula for the limiting magnitude L of a telescope—that is, the magnitude of the dimmest star that it can be used to view—is given by

$$L = 9 + 5.1 \log d$$

where d is the diameter (in inches) of the lens.
(a) What is the limiting magnitude of a 3.5-inch telescope?
(b) What diameter is required to view a star of magnitude 14?

49. Salvage Value The number of years n for a piece of machinery to depreciate to a known salvage value can be found using the formula

$$n = \frac{\log s - \log i}{\log(1 - d)}$$

where s is the salvage value of the machinery, i is its initial value, and d is the annual rate of depreciation.
(a) How many years will it take for a piece of machinery to decline in value from \$90,000 to \$10,000 if the annual rate of depreciation is 0.20 (20%)?
(b) How many years will it take for a piece of machinery to lose half of its value if the annual rate of depreciation is 15%?

50. Funding a College Education A child's grandparents purchase a \$10,000 bond fund that matures in 18 years to be used for her college education. The bond fund pays 4% interest compounded semiannually. How much will the bond fund be worth at maturity? What is the effective rate of interest? How long will it take the bond to double in value under these terms?

51. Funding a College Education A child's grandparents wish to purchase a bond that matures in 18 years to be used for her college education. The bond pays 4% interest compounded semiannually. How much should they pay so that the bond will be worth \$85,000 at maturity?

52. When Did a Prehistoric Man Die? The bones of a prehistoric man found in the desert of New Mexico contain approximately 5% of the original amount of carbon-14. If the half-life of carbon-14 is 5730 years, approximately how long ago did the man die?

53. Temperature of a Skillet A skillet is removed from an oven where the temperature is 450°F and placed in a room whose temperature is 70°F. After 5 minutes, the temperature of the skillet is 400°F. How long will it be until its temperature is 150°F?

54. World Population The annual growth rate of the world's population in 2017 was 1.1% = 0.011. The population of the world in 2017 was 7,362,350,168. Letting $t = 0$ represent 2017, predict the world's population in the year 2022.

Source: *U.S. Census Bureau*

55. Radioactive Decay The half-life of cobalt is 5.27 years. If 100 grams of radioactive cobalt is present now, how much will be present in 20 years? In 40 years?

56. Logistic Growth The logistic growth model

$$P(t) = \frac{0.8}{1 + 1.67e^{-0.16t}}$$

represents the proportion of new cars with a global positioning system (GPS). Let $t = 0$ represent 2006, $t = 1$ represent 2007, and so on.
(a) What proportion of new cars in 2006 had a GPS?
(b) Determine the maximum proportion of new cars that have a GPS.
(c) Using a graphing utility, graph $P = P(t)$.
(d) When will 75% of new cars have a GPS?

57. Rising Tuition The following data represent the average in-state tuition and fees (in 2016 dollars) at public four-year colleges and universities in the United States from the academic year 1980–81 to the academic year 2016–17.

Academic Year	Tuition and Fees (2016 dollars)
1980–81 ($x = 0$)	2340
1990–91 ($x = 10$)	3521
2000–01 ($x = 20$)	4885
2004–05 ($x = 24$)	6513
2008–09 ($x = 28$)	7219
2012–13 ($x = 32$)	9082
2016–17 ($x = 36$)	9648

Source: *The College Board (www.collegeboard.org)*

(a) Using a graphing utility, draw a scatter diagram with academic year as the independent variable.
(b) Using a graphing utility, build an exponential model from the data.
(c) Express the function found in part (b) in the form $A(t) = A_0e^{kt}$.
(d) Graph the exponential function found in part (b) or (c) on the scatter diagram.
(e) Predict the academic year when the average tuition will reach \$12,000.

58. Wind Chill Factor The data below represent the wind speed (mph) and wind chill factor at an air temperature of 15°F.

Wind Speed (mph)	Wind Chill Factor (°F)
5	7
10	3
15	0
20	−2
25	−4
30	−5
35	−7

Source: *U.S. National Weather Service*

(a) Using a graphing utility, draw a scatter diagram with wind speed as the independent variable.
(b) Using a graphing utility, build a logarithmic model from the data.
(c) Using a graphing utility, draw the logarithmic function found in part (b) on the scatter diagram.
(d) Use the function found in part (b) to predict the wind chill factor if the air temperature is 15°F and the wind speed is 23 mph.

59. Spreading of a Disease Jack and Diane live in a small town of 50 people. Unfortunately, both Jack and Diane have a cold. Those who come in contact with someone who has this cold will themselves catch the cold. The following data represent the number of people in the small town who have caught the cold after t days.

Days, t	Number of People with Cold, C
0	2
1	4
2	8
3	14
4	22
5	30
6	37
7	42
8	44

(a) Using a graphing utility, draw a scatter diagram of the data. Comment on the type of relation that appears to exist between the day and the number of people with a cold.
(b) Using a graphing utility, build a logistic model from the data.
(c) Graph the function found in part (b) on the scatter diagram.
(d) According to the function found in part (b), what is the maximum number of people who will catch the cold? In reality, what is the maximum number of people who could catch the cold?
(e) Sometime between the second and third day, 10 people in the town had a cold. According to the model found in part (b), when did 10 people have a cold?
(f) How long will it take for 46 people to catch the cold?

Chapter Test

CHAPTER Test Prep VIDEOS

Chapter Test Prep Videos include step-by-step solutions to all chapter test exercises and can be found in MyLab™ Math or on this text's YouTube Channel. Refer to the Preface for a link to the YouTube channel.

1. Given $f(x) = \dfrac{x+2}{x-2}$ and $g(x) = 2x + 5$, find:
(a) $f \circ g$ and state its domain (b) $(g \circ f)(-2)$
(c) $(f \circ g)(-2)$

2. Determine whether the function is one-to-one.
(a) $y = 4x^2 + 3$
(b) $y = \sqrt{x+3} - 5$

3. Find the inverse of $f(x) = \dfrac{2}{3x-5}$ and check your answer. State the domain and the range of f and f^{-1}.

4. If the point $(3, -5)$ is on the graph of a one-to-one function f, what point must be on the graph of f^{-1}?

In Problems 5–7, solve each equation.

5. $3^x = 243$ **6.** $\log_b 16 = 2$
7. $\log_5 x = 4$

In Problems 8–11, use a calculator to evaluate each expression. Round your answer to three decimal places.

8. $e^3 + 2$ **9.** $\log 20$
10. $\log_3 21$ **11.** $\ln 133$

In Problems 12 and 13, use the given function f to:
(a) Find the domain of f. *(b) Graph f.*
(c) From the graph, determine the range and any asymptotes of f.
(d) Find f^{-1}, the inverse of f.
(e) Find the domain and the range of f^{-1}. *(f) Graph f^{-1}.*

12. $f(x) = 4^{x+1} - 2$ **13.** $f(x) = 1 - \log_5(x-2)$

In Problems 14–19, solve each equation.

14. $5^{x+2} = 125$ **15.** $\log(x+9) = 2$
16. $8 - 2e^{-x} = 4$ **17.** $\log(x^2+3) = \log(x+6)$
18. $7^{x+3} = e^x$ **19.** $\log_2(x-4) + \log_2(x+4) = 3$

20. Write $\log_2\left(\dfrac{4x^3}{x^2-3x-18}\right)$ as the sum and/or difference of logarithms. Express powers as factors.

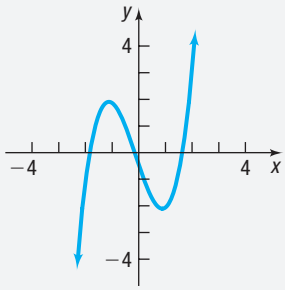
21. A 50-mg sample of a radioactive substance decays to 34 mg after 30 days. How long will it take for there to be 2 mg remaining?

22. (a) If \$1000 is invested at 5% compounded monthly, how much is there after 8 months?
(b) If you want to have \$1000 in 9 months, how much do you need to place in a savings account now that pays 5% compounded quarterly?
(c) How long does it take to double your money if you can invest it at 6% compounded annually?

23. The decibel level, D, of sound is given by the equation $D = 10\log\left(\dfrac{I}{I_0}\right)$, where I is the intensity of the sound and $I_0 = 10^{-12}$ watt per square meter.
(a) If the shout of a single person measures 80 decibels, how loud will the sound be if two people shout at the same time? That is, how loud would the sound be if the intensity doubled?
(b) The pain threshold for sound is 125 decibels. If the Athens Olympic Stadium 2004 (Olympiako Stadio Athinas 'Spyros Louis') can seat 74,400 people, how many people in the crowd need to shout at the same time for the resulting sound level to meet or exceed the pain threshold? (Ignore any possible sound dampening.)

Cumulative Review

1. (a) Is the following graph the graph of a function? If it is, is the function one-to-one?



(b) Assuming the graph is a function, what type of function might it be (polynomial, exponential, and so on)? Why?

2. For the function $f(x) = 2x^2 - 3x + 1$, find the following:
(a) $f(3)$ (b) $f(-x)$ (c) $f(x + h)$

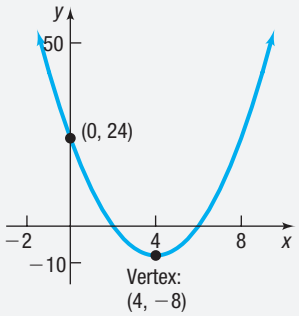
3. Determine which of the following points is or are on the graph of $x^2 + y^2 = 1$.
(a) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

4. Solve the equation $3(x - 2) = 4(x + 5)$.

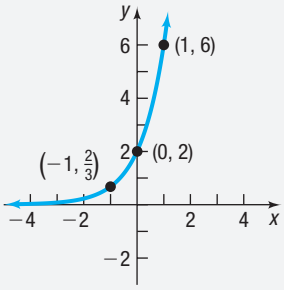
5. Graph the line $2x - 4y = 16$.

6. (a) Graph the quadratic function $f(x) = -x^2 + 2x - 3$ by determining whether its graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercept(s), if any.
(b) Solve $f(x) \le 0$.

7. Determine the quadratic function whose graph is given in the figure.



8. Is the graph that of a polynomial, exponential, or logarithmic function? Determine the function whose graph is given.



9. Graph $f(x) = 3(x + 1)^3 - 2$ using transformations.

10. (a) Given that $f(x) = x^2 + 2$ and $g(x) = \dfrac{2}{x - 3}$, find $f(g(x))$ and state its domain. What is $f(g(5))$?
(b) If $f(x) = x + 2$ and $g(x) = \log_2 x$, find $(f(g(x))$ and state its domain. What is $f(g(14))$?

11. For the polynomial function $f(x) = 4x^3 + 9x^2 - 30x - 8$:
(a) Find the real zeros of f.
(b) Determine the intercepts of the graph of f.
(c) Use a graphing utility to approximate the local maxima and local minima.
(d) Draw a complete graph of f. Be sure to label the intercepts and turning points.

12. For the function $g(x) = 3^x + 2$:
(a) Graph g using transformations. State the domain, range, and horizontal asymptote of g.
(b) Determine the inverse of g. State the domain, range, and vertical asymptote of g^{-1}.
(c) On the same graph as g, graph g^{-1}.

13. Solve the equation: $4^{x-3} = 8^{2x}$

14. Solve the equation: $\log_3(x + 1) + \log_3(2x - 3) = \log_9 9$

15. Suppose that $f(x) = \log_3(x + 2)$. Solve:
(a) $f(x) = 0$ (b) $f(x) > 0$
(c) $f(x) = 3$

16. Data Analysis The following data represent the percent of all drivers by age who have been stopped by the police for any reason within the past year. The median age represents the midpoint of the upper and lower limit for the age range.

Age Range	Median Age, x	Percent Stopped, y
16–19	17.5	18.2
20–29	24.5	16.8
30–39	34.5	11.3
40–49	44.5	9.4
50–59	54.5	7.7
≥ 60	69.5	3.8

(a) Using your graphing utility, draw a scatter diagram of the data treating median age, x, as the independent variable.
(b) Determine a model that you feel best describes the relation between median age and percent stopped. You may choose from among linear, quadratic, cubic, exponential, logarithmic, and logistic models.
(c) Provide a justification for the model that you selected in part (b).

Chapter Projects

Internet-based Project

I. Depreciation of Cars Kelley Blue Book is a guide that provides the current retail price of cars. You can access the Kelley Blue Book online at *www.kbb.com*.

1. Identify three cars that you are considering purchasing, and find the Kelley Blue Book value of the cars for 0 (brand new), 1, 2, 3, 4, and 5 years of age. Online, the value of the car can be found by selecting Price New/Used. Enter the year, make, and model of the new or used car you are selecting. To be consistent, assume the cars will be driven 12,000 miles per year, so a 1-year-old car will have 12,000 miles, a 2-year-old car will have 24,000 miles, and so on. Choose the same options for each year, and select Buy from a Private Party when choosing a price type. Finally, determine the suggested retail price for cars that are in Excellent, Good, and Fair shape. You should have a total of 16 observations (1 for a brand new car, 3 for a 1-year-old car, 3 for a 2-year-old car, and so on).
2. Draw a scatter diagram of the data with age as the independent variable and value as the dependent variable using Excel, a TI-graphing calculator, or some other spreadsheet. The Chapter 2 project describes how to draw a scatter diagram in Excel.
3. Determine the exponential function of best fit. Graph the exponential function of best fit on the scatter diagram. To do this in Excel, right click on any data point in the scatter diagram. Now select Add Trendline. Select the Exponential radio button and select Display Equation on Chart. See Figure 60. Move the Trendline Options window off to the side, if necessary, and you will see the exponential function of best fit displayed on the scatter diagram. Do you think the function accurately describes the relation between age of the car and suggested retail price?

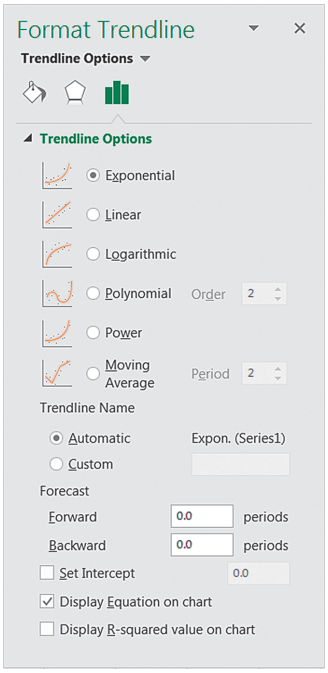


Figure 60

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4. The exponential function of best fit is of the form $y = Ce^{rx}$, where y is the suggested retail value of the car and x is the age of the car (in years). What does the value of C represent? What does the value of r represent? What is the depreciation rate for each car that you are considering?
5. Write a report detailing which car you would purchase based on the depreciation rate you found for each car.

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The following projects are available on the Instructor's Resource Center (IRC):

II. Hot Coffee A fast-food restaurant wants a special container to hold coffee. The restaurant wishes the container to quickly cool the coffee from 200° to 130°F and keep the liquid between 110° and 130°F as long as possible. The restaurant has three containers to select from. Which one should be purchased?

III. Project at Motorola ***Thermal Fatigue of Solder Connections*** Product reliability is a major concern of a manufacturer. Here a logarithmic transformation is used to simplify the analysis of a cell phone's ability to withstand temperature change.