Although each of these planes is rather large, from a distance their motion can be analysed as if each plane were a particle.
Kinematics of a Particle

CHAPTER OBJECTIVES

• To introduce the concepts of position, displacement, velocity, and acceleration.
• To study particle motion along a straight line and represent this motion graphically.
• To investigate particle motion along a curved path using different coordinate systems.
• To present an analysis of dependent motion of two particles.
• To examine the principles of relative motion of two particles using translating axes.

12.1 Introduction

Mechanics is a branch of the physical sciences that is concerned with the state of rest or motion of bodies subjected to the action of forces. Engineering mechanics is divided into two areas of study, namely, statics and dynamics. Statics is concerned with the equilibrium of a body that is either at rest or moves with constant velocity. Here we will consider dynamics, which deals with the accelerated motion of a body. The subject of dynamics will be presented in two parts: kinematics, which treats only the geometric aspects of the motion, and kinetics, which is the analysis of the forces causing the motion. To develop these principles, the dynamics of a particle will be discussed first, followed by topics in rigid-body dynamics in two and then three dimensions.
Historically, the principles of dynamics developed when it was possible to make an accurate measurement of time. Galileo Galilei (1564–1642) was one of the first major contributors to this field. His work consisted of experiments using pendulums and falling bodies. The most significant contributions in dynamics, however, were made by Isaac Newton (1642–1727), who is noted for his formulation of the three fundamental laws of motion and the law of universal gravitational attraction. Shortly after these laws were postulated, important techniques for their application were developed by Euler, D’Alembert, Lagrange, and others.

There are many problems in engineering whose solutions require application of the principles of dynamics. Typically the structural design of any vehicle, such as an automobile or airplane, requires consideration of the motion to which it is subjected. This is also true for many mechanical devices, such as motors, pumps, movable tools, industrial manipulators, and machinery. Furthermore, predictions of the motions of artificial satellites, projectiles, and spacecraft are based on the theory of dynamics. With further advances in technology, there will be an even greater need for knowing how to apply the principles of this subject.

**Problem Solving.** Dynamics is considered to be more involved than statics since both the forces applied to a body and its motion must be taken into account. Also, many applications require using calculus, rather than just algebra and trigonometry. In any case, the most effective way of learning the principles of dynamics is to solve problems. To be successful at this, it is necessary to present the work in a logical and orderly manner as suggested by the following sequence of steps:

1. Read the problem carefully and try to correlate the actual physical situation with the theory you have studied.
2. Draw any necessary diagrams and tabulate the problem data.
3. Establish a coordinate system and apply the relevant principles, generally in mathematical form.
4. Solve the necessary equations algebraically as far as practical; then, use a consistent set of units and complete the solution numerically. Report the answer with no more significant figures than the accuracy of the given data.
5. Study the answer using technical judgment and common sense to determine whether or not it seems reasonable.
6. Once the solution has been completed, review the problem. Try to think of other ways of obtaining the same solution.

In applying this general procedure, do the work as neatly as possible. Being neat generally stimulates clear and orderly thinking, and vice versa.
12.2 Rectilinear Kinematics: Continuous Motion

We will begin our study of dynamics by discussing the kinematics of a particle that moves along a rectilinear or straight line path. Recall that a particle has a mass but negligible size and shape. Therefore we must limit application to those objects that have dimensions that are of no consequence in the analysis of the motion. In most problems, we will be interested in bodies of finite size, such as rockets, projectiles, or vehicles. Each of these objects can be considered as a particle, as long as the motion is characterized by the motion of its mass center and any rotation of the body is neglected.

Rectilinear Kinematics. The kinematics of a particle is characterized by specifying, at any given instant, the particle’s position, velocity, and acceleration.

Position. The straight-line path of a particle will be defined using a single coordinate axis \( s \), Fig. 12–1a. The origin \( O \) on the path is a fixed point, and from this point the position coordinate \( s \) is used to specify the location of the particle at any given instant. The magnitude of \( s \) is the distance from \( O \) to the particle, usually measured in meters (m) or feet (ft), and the sense of direction is defined by the algebraic sign on \( s \). Although the choice is arbitrary, in this case \( s \) is positive since the coordinate axis is positive to the right of the origin. Likewise, it is negative if the particle is located to the left of \( O \). Realize that position is a vector quantity since it has both magnitude and direction. Here, however, it is being represented by the algebraic scalar \( s \) since the direction always remains along the coordinate axis.

Displacement. The displacement of the particle is defined as the change in its position. For example, if the particle moves from one point to another, Fig. 12–1b, the displacement is

\[
\Delta s = s' - s
\]

In this case \( \Delta s \) is positive since the particle’s final position is to the right of its initial position, i.e., \( s' > s \). Likewise, if the final position were to the left of its initial position, \( \Delta s \) would be negative.

The displacement of a particle is also a vector quantity, and it should be distinguished from the distance the particle travels. Specifically, the distance traveled is a positive scalar that represents the total length of path over which the particle travels.
Velocity. If the particle moves through a displacement $\Delta s$ during the time interval $\Delta t$, the average velocity of the particle during this time interval is

$$v_{\text{avg}} = \frac{\Delta s}{\Delta t}$$

If we take smaller and smaller values of $\Delta t$, the magnitude of $\Delta s$ becomes smaller and smaller. Consequently, the instantaneous velocity is a vector defined as $v = \lim_{\Delta t \to 0} (\Delta s/\Delta t)$, or

$$v = \frac{ds}{dt}$$  \hspace{1cm} (12–1)

Since $\Delta t$ or $dt$ is always positive, the sign used to define the sense of the velocity is the same as that of $\Delta s$ or $ds$. For example, if the particle is moving to the right, Fig. 12–1c, the velocity is positive; whereas if it is moving to the left, the velocity is negative. (This is emphasized here by the arrow written at the left of Eq. 12–1.) The magnitude of the velocity is known as the speed, and it is generally expressed in units of m/s or ft/s. Occasionally, the term “average speed” is used. The average speed is always a positive scalar and is defined as the total distance traveled by a particle, $s_T$, divided by the elapsed time $\Delta t$; i.e.,

$$(v_{sp})_{\text{avg}} = \frac{s_T}{\Delta t}$$

For example, the particle in Fig. 12–1d travels along the path of length $s_T$ in time $\Delta t$, so its average speed is $(v_{sp})_{\text{avg}} = s_T/\Delta t$, but its average velocity is $v_{\text{avg}} = -\Delta s/\Delta t$. 

Fig. 12–1 (cont.)
**Acceleration.** Provided the velocity of the particle is known at two points, the *average acceleration* of the particle during the time interval $\Delta t$ is defined as

\[ a_{\text{avg}} = \frac{\Delta v}{\Delta t} \]

Here $\Delta v$ represents the difference in the velocity during the time interval $\Delta t$, i.e., $\Delta v = v' - v$, Fig. 12–1e.

The *instantaneous acceleration* at time $t$ is a vector that is found by taking smaller and smaller values of $\Delta t$ and corresponding smaller and smaller values of $\Delta v$, so that $a = \lim_{\Delta t \to 0} (\Delta v/\Delta t)$, or

(12–2)

\[ a = \frac{dv}{dt} \]

Substituting Eq. 12–1 into this result, we can also write

(12–2)

\[ a = \frac{d^2s}{dt^2} \]

Both the average and instantaneous acceleration can be either positive or negative. In particular, when the particle is *slowing down*, or its speed is decreasing, the particle is said to be *decelerating*. In this case, $v'$ in Fig. 12–1f is less than $v$, and so $\Delta v = v' - v$ will be negative. Consequently, $a$ will also be negative, and therefore it will act to the *left*, in the *opposite sense* to $v$. Also, note that when the *velocity is constant*, the *acceleration is zero* since $\Delta v = v - v = 0$. Units commonly used to express the magnitude of acceleration are $\text{m/s}^2$ or $\text{ft/s}^2$.

Finally, an important differential relation involving the displacement, velocity, and acceleration along the path may be obtained by eliminating the time differential $dt$ between Eqs. 12–1 and 12–2, which gives

(12–3)

\[ a \, ds = v \, dv \]

Although we have now produced three important kinematic equations, realize that the above equation is not independent of Eqs. 12–1 and 12–2.
Constant Acceleration, \( a = a_c \). When the acceleration is constant, each of the three kinematic equations \( a_c = \frac{dv}{dt}, v = \frac{ds}{dt}, \) and \( a_c \, ds = v \, dv \) can be integrated to obtain formulas that relate \( a_c, v, s, \) and \( t \).

**Velocity as a Function of Time.** Integrate \( a_c = \frac{dv}{dt} \), assuming that initially \( v = v_0 \) when \( t = 0 \).

\[
\int_{v_0}^{v} dv = \int_{0}^{t} a_c \, dt
\]

\[
(\uparrow)
\]

\[
v = v_0 + a_c t
\]

\[
(12-4)
\]

**Position as a Function of Time.** Integrate \( v = \frac{ds}{dt} = v_0 + a_c t \), assuming that initially \( s = s_0 \) when \( t = 0 \).

\[
\int_{s_0}^{s} ds = \int_{0}^{t} (v_0 + a_c t) \, dt
\]

\[
(\uparrow)
\]

\[
s = s_0 + v_0 t + \frac{1}{2} a_c t^2
\]

\[
(12-5)
\]

**Velocity as a Function of Position.** Either solve for \( t \) in Eq. 12–4 and substitute into Eq. 12–5, or integrate \( v \, dv = a_c \, ds \), assuming that initially \( v = v_0 \) at \( s = s_0 \).

\[
\int_{v_0}^{v} v \, dv = \int_{s_0}^{s} a_c \, ds
\]

\[
(\uparrow)
\]

\[
v^2 = v_0^2 + 2a_c(s - s_0)
\]

\[
(12-6)
\]

The algebraic signs of \( s_0, v_0, \) and \( a_c \), used in the above three equations, are determined from the positive direction of the \( s \) axis as indicated by the arrow written at the left of each equation. Remember that these equations are useful only when the acceleration is constant and when \( t = 0, s = s_0, v = v_0 \). A typical example of constant accelerated motion occurs when a body falls freely toward the earth. If air resistance is neglected and the distance of fall is short, then the downward acceleration of the body when it is close to the earth is constant and approximately \( 9.81 \, m/s^2 \) or \( 32.2 \, ft/s^2 \). The proof of this is given in Example 13.2.
12.2 RECTILINEAR KINEMATICS: CONTINUOUS MOTION

**Important Points**

- Dynamics is concerned with bodies that have accelerated motion.
- Kinematics is a study of the geometry of the motion.
- Kinetics is a study of the forces that cause the motion.
- Rectilinear kinematics refers to straight-line motion.
- Speed refers to the magnitude of velocity.
- Average speed is the total distance traveled divided by the total time. This is different from the average velocity, which is the displacement divided by the time.
- A particle that is slowing down is decelerating.
- A particle can have an acceleration and yet have zero velocity.
- The relationship \( a \, ds = v \, dv \) is derived from \( a = \frac{dv}{dt} \) and \( v = \frac{ds}{dt} \), by eliminating \( dt \).

**Procedure for Analysis**

**Coordinate System.**

- Establish a position coordinate \( s \) along the path and specify its *fixed origin* and positive direction.
- Since motion is along a straight line, the vector quantities position, velocity, and acceleration can be represented as algebraic scalars. For analytical work the sense of \( s, v, \) and \( a \) is then defined by their *algebraic signs*.
- The positive sense for each of these scalars can be indicated by an arrow shown alongside each kinematic equation as it is applied.

**Kinematic Equations.**

- If a relation is known between any *two* of the four variables \( a, v, s \) and \( t \), then a third variable can be obtained by using one of the kinematic equations, \( a = \frac{dv}{dt}, v = \frac{ds}{dt} \) or \( a \, ds = v \, dv \), since each equation relates all three variables.*
- Whenever integration is performed, it is important that the position and velocity be known at a given instant in order to evaluate either the constant of integration if an indefinite integral is used, or the limits of integration if a definite integral is used.
- Remember that Eqs. 12–4 through 12–6 have only limited use. These equations apply *only* when the *acceleration is constant* and the initial conditions are \( s = s_0 \) and \( v = v_0 \) when \( t = 0 \).

*Some standard differentiation and integration formulas are given in Appendix A.

During the time this rocket undergoes rectilinear motion, its altitude as a function of time can be measured and expressed as \( s = s(t) \). Its velocity can then be found using \( v = \frac{ds}{dt} \), and its acceleration can be determined from \( a = \frac{dv}{dt} \).
EXAMPLE 12.1

The car in Fig. 12–2 moves in a straight line such that for a short time its velocity is defined by \( v = (3t^2 + 2t) \text{ ft/s} \), where \( t \) is in seconds. Determine its position and acceleration when \( t = 3 \text{ s} \). When \( t = 0 \), \( s = 0 \).

\( \text{Fig. 12–2} \)

**SOLUTION**

**Coordinate System.** The position coordinate extends from the fixed origin \( O \) to the car, positive to the right.

**Position.** Since \( v = f(t) \), the car’s position can be determined from \( v = ds/dt \), since this equation relates \( v, s, \) and \( t \). Noting that \( s = 0 \) when \( t = 0 \), we have

\[
( \Rightarrow ) \quad v = \frac{ds}{dt} = (3t^2 + 2t)
\]

\[
\int_0^s ds = \int_0^t (3t^2 + 2t) dt
\]

\[
\left. s \right|_0^t = \left. t^3 + t^2 \right|_0
\]

\[
s = t^3 + t^2
\]

When \( t = 3 \text{ s} \),

\[
s = (3)^3 + (3)^2 = 36 \text{ ft} \quad \text{Ans.}
\]

**Acceleration.** Since \( v = f(t) \), the acceleration is determined from \( a = dv/dt \), since this equation relates \( a, v, \) and \( t \).

\[
( \Rightarrow ) \quad a = \frac{dv}{dt} = \frac{d}{dt} (3t^2 + 2t)
\]

\[
a = 6t + 2
\]

When \( t = 3 \text{ s} \),

\[
a = 6(3) + 2 = 20 \text{ ft/s}^2 \quad \text{Ans.}
\]

**NOTE:** The formulas for constant acceleration *cannot* be used to solve this problem, because the acceleration is a function of time.

*The same result* can be obtained by evaluating a constant of integration \( C \) rather than using definite limits on the integral. For example, integrating \( ds = (3t^2 + 2t) dt \) yields \( s = t^3 + t^2 + C \). Using the condition that at \( t = 0, s = 0 \), then \( C = 0 \).
EXAMPLE 12.2

A small projectile is fired vertically downward into a fluid medium with an initial velocity of 60 m/s. Due to the drag resistance of the fluid the projectile experiences a deceleration of \( a = (-0.4v^3) \text{ m/s}^2 \), where \( v \) is in m/s. Determine the projectile’s velocity and position 4 s after it is fired.

SOLUTION

Coordinate System. Since the motion is downward, the position coordinate is positive downward, with origin located at \( O \), Fig. 12–3.

Velocity. Here \( a = f(v) \) and so we must determine the velocity as a function of time using \( a = \frac{dv}{dt} \), since this equation relates \( v \), \( a \), and \( t \). (Why not use \( v = v_0 + at \)?) Separating the variables and integrating, with \( v_0 = 60 \text{ m/s} \) when \( t = 0 \), yields

\[
(+ \downarrow)
\]

\[
\int_{60 \text{ m/s}}^{v} \frac{dv}{-0.4v^3} = \int_{0}^{t} \, dt
\]

\[
\frac{1}{-0.4} \left. \left( \frac{1}{-2} \frac{1}{v^2} \right) \right|_{60}^{v} = t - 0
\]

\[
\frac{1}{0.8} \left[ \frac{1}{v^2} - \frac{1}{(60)^2} \right] = t
\]

\[
v = \left\{ \left[ \frac{1}{(60)^2} + 0.8t \right]^{-1/2} \right\} \text{ m/s}
\]

Here the positive root is taken, since the projectile will continue to move downward. When \( t = 4 \text{ s} \),

\[
v = 0.559 \text{ m/s} \downarrow \quad \text{Ans.}
\]

Position. Knowing \( v = f(t) \), we can obtain the projectile’s position from \( v = \frac{ds}{dt} \), since this equation relates \( s \), \( v \), and \( t \). Using the initial condition \( s = 0 \), when \( t = 0 \), we have

\[
(+ \downarrow)
\]

\[
\int_{0}^{s} ds = \int_{0}^{t} \left[ \frac{1}{(60)^2} + 0.8t \right]^{-1/2} \, dt
\]

\[
s = \frac{2}{0.8} \left[ \frac{1}{(60)^2} + 0.8t \right]^{1/2} \bigg|_{0}^{t}
\]

\[
s = \frac{1}{0.4} \left\{ \left[ \frac{1}{(60)^2} + 0.8t \right]^{1/2} - \frac{1}{60} \right\} \text{ m}
\]

When \( t = 4 \text{ s} \),

\[
s = 4.43 \text{ m} \quad \text{Ans.}
\]
During a test a rocket travels upward at 75 m/s, and when it is 40 m from the ground its engine fails. Determine the maximum height reached by the rocket and its speed just before it hits the ground. While in motion the rocket is subjected to a constant downward acceleration of 9.81 m/s² due to gravity. Neglect the effect of air resistance.

**SOLUTION**

**Coordinate System.** The origin \( O \) for the position coordinate \( s \) is taken at ground level with positive upward, Fig. 12–4.

**Maximum Height.** Since the rocket is traveling upward, \( v_A = +75 \text{ m/s} \) when \( t = 0 \). At the maximum height \( s = s_B \) the velocity \( v_B = 0 \). For the entire motion, the acceleration is \( a_c = -9.81 \text{ m/s}^2 \) (negative since it acts in the opposite sense to positive velocity or positive displacement). Since \( a_c \) is constant the rocket’s position may be related to its velocity at the two points \( A \) and \( B \) on the path by using Eq. 12–6, namely,

\[
(±) \quad v_B^2 = v_A^2 + 2a_c(s_B - s_A)
\]

\[
0 = (75 \text{ m/s})^2 + 2(-9.81 \text{ m/s}^2)(s_B - 40 \text{ m})
\]

\[
s_B = 327 \text{ m} \quad \text{Ans.}
\]

**Velocity.** To obtain the velocity of the rocket just before it hits the ground, we can apply Eq. 12–6 between points \( B \) and \( C \), Fig. 12–4.

\[
(±) \quad v_C^2 = v_B^2 + 2a_c(s_C - s_B)
\]

\[
= 0 + 2(-9.81 \text{ m/s}^2)(0 - 327 \text{ m})
\]

\[
v_C = -80.1 \text{ m/s} = 80.1 \text{ m/s} \downarrow \quad \text{Ans.}
\]

The negative root was chosen since the rocket is moving downward.

Similarly, Eq. 12–6 may also be applied between points \( A \) and \( C \), i.e.,

\[
(±) \quad v_C^2 = v_A^2 + 2a_c(s_C - s_A)
\]

\[
= (75 \text{ m/s})^2 + 2(-9.81 \text{ m/s}^2)(0 - 40 \text{ m})
\]

\[
v_C = -80.1 \text{ m/s} = 80.1 \text{ m/s} \downarrow \quad \text{Ans.}
\]

**NOTE:** It should be realized that the rocket is subjected to a deceleration from \( A \) to \( B \) of 9.81 m/s², and then from \( B \) to \( C \) it is accelerated at this rate. Furthermore, even though the rocket momentarily comes to rest at \( B \) \( (v_B = 0) \) the acceleration at \( B \) is still 9.81 m/s² downward!
A metallic particle is subjected to the influence of a magnetic field as it travels downward through a fluid that extends from plate $A$ to plate $B$, Fig. 12–5. If the particle is released from rest at the midpoint $C$, and the acceleration is $a = (4s)$ m/s$^2$, where $s$ is in meters, determine the velocity of the particle when it reaches plate $B$, $s = 200$ mm, and the time it takes to travel from $C$ to $B$.

**SOLUTION**

**Coordinate System.** As shown in Fig. 12–5, $s$ is positive downward, measured from plate $A$.

**Velocity.** Since $a = f(s)$, the velocity as a function of position can be obtained by using $v \, dv = a \, ds$. Realizing that $v = 0$ at $s = 0.1$ m, we have

$$v \, dv = a \, ds$$

$$\int_0^v v \, dv = \int_{0.1}^s 4s \, ds$$

$$\frac{1}{2} v^2 \bigg|_0^v = \frac{4}{2} \frac{1}{2} s^2 \bigg|_{0.1}^s$$

$$v = \sqrt{2(s^2 - 0.01)} \text{ m/s}$$

At $s = 200$ mm = 0.2 m,

$$v_B = 0.346 \text{ m/s} = 346 \text{ mm/s} \quad \text{Ans.}$$

The positive root is chosen since the particle is traveling downward, i.e., in the $+s$ direction.

**Time.** The time for the particle to travel from $C$ to $B$ can be obtained using $v = ds/dt$ and Eq. 1, where $s = 0.1$ m when $t = 0$. From Appendix A,

$$ds = v \, dt$$

$$\int_{0.1}^s \frac{ds}{(s^2 - 0.01)^{1/2}} = \int_0^t 2 \, dt$$

$$\ln \left( \sqrt{s^2 - 0.01} + s \right) \bigg|_{0.1}^s = 2t \bigg|_0^t$$

$$\ln \left( \sqrt{s^2 - 0.01} + s \right) + 2.303 = 2t$$

At $s = 0.2$ m,

$$t = \frac{\ln \left( \sqrt{(0.2)^2 - 0.01} + 0.2 \right) + 2.303}{2} = 0.658 \text{ s} \quad \text{Ans.}$$

**Note:** The formulas for constant acceleration cannot be used here because the acceleration changes with position, i.e., $a = 4s$. 

A particle moves along a horizontal path with a velocity of $v = (3t^2 - 6t)$ m/s, where $t$ is the time in seconds. If it is initially located at the origin $O$, determine the distance traveled in 3.5 s, and the particle’s average velocity and average speed during the time interval.

**SOLUTION**

**Coordinate System.** Here positive motion is to the right, measured from the origin $O$, Fig. 12–6a.

**Distance Traveled.** Since $v = f(t)$, the position as a function of time may be found by integrating $v = ds/dt$ with $t = 0, s = 0$.

\[
\int_0^s ds = \int_0^t (3t^2 - 6t) \, dt
\]

\[s = (t^3 - 3t^2) \text{ m} \quad (1)
\]

In order to determine the distance traveled in 3.5 s, it is necessary to investigate the path of motion. If we consider a graph of the velocity function, Fig. 12–6b, then it reveals that for $0 < t < 2$ s the velocity is negative, which means the particle is traveling to the left, and for $t > 2$ s the velocity is positive, and hence the particle is traveling to the right.

Also, note that $v = 0$ at $t = 2$ s. The particle’s position when $t = 0$, $t = 2$ s, and $t = 3.5$ s can now be determined from Eq. 1. This yields

\[s|_{t=0} = 0 \quad s|_{t=2} = -4.0 \text{ m} \quad s|_{t=3.5} = 6.125 \text{ m}
\]

The path is shown in Fig. 12–6a. Hence, the distance traveled in 3.5 s is

\[s_T = 4.0 + 4.0 + 6.125 = 14.125 \text{ m} = 14.1 \text{ m} \quad \text{Ans.}
\]

**Velocity.** The displacement from $t = 0$ to $t = 3.5$ s is

\[\Delta s = s|_{t=3.5} - s|_{t=0} = 6.125 \text{ m} - 0 = 6.125 \text{ m}
\]

and so the average velocity is

\[v_{\text{avg}} = \frac{\Delta s}{\Delta t} = \frac{6.125 \text{ m}}{3.5 \text{ s} - 0} = 1.75 \text{ m/s} \quad \text{Ans.}
\]

The average speed is defined in terms of the distance traveled $s_T$. This positive scalar is

\[(v_{\text{sp}})_{\text{avg}} = \frac{s_T}{\Delta t} = \frac{14.125 \text{ m}}{3.5 \text{ s} - 0} = 4.04 \text{ m/s} \quad \text{Ans.}
\]

**Note:** In this problem, the acceleration is $a = dv/dt = (6t - 6)$ m/s$^2$, which is not constant.
F12–1. Initially, the car travels along a straight road with a speed of 35 m/s. If the brakes are applied and the speed of the car is reduced to 10 m/s in 15 s, determine the constant deceleration of the car.

F12–2. A ball is thrown vertically upward with a speed of 15 m/s. Determine the time of flight when it returns to its original position.

F12–3. A particle travels along a straight line with a velocity of \( v = (4t^2 - 3t) \) m/s, where \( t \) is in seconds. Determine the position of the particle when \( t = 4 \) s. \( s = 0 \) when \( t = 0 \).

F12–4. A particle travels along a straight line with a speed \( v = (0.5t^3 - 8t) \) m/s, where \( t \) is in seconds. Determine the acceleration of the particle when \( t = 2 \) s.

F12–5. The position of the particle is given by \( s = (2t^2 - 8t + 6) \) m, where \( t \) is in seconds. Determine the time when the velocity of the particle is zero, and the total distance traveled by the particle when \( t = 3 \) s.

F12–6. A particle travels along a straight line with an acceleration of \( a = (10 - 0.2t) \) m/s², where \( s \) is measured in meters. Determine the velocity of the particle when \( s = 10 \) m if \( v = 5 \) m/s at \( s = 0 \).

F12–7. A particle moves along a straight line such that its acceleration is \( a = (4t^2 - 2) \) m/s², where \( t \) is in seconds. When \( t = 0 \), the particle is located 2 m to the left of the origin, and when \( t = 2 \) s, it is 20 m to the left of the origin. Determine the position of the particle when \( t = 4 \) s.

F12–8. A particle travels along a straight line with a velocity of \( v = (20 - 0.05t^2) \) m/s, where \( s \) is in meters. Determine the acceleration of the particle at \( s = 15 \) m.
12–1. A car starts from rest and with constant acceleration achieves a velocity of 15 m/s when it travels a distance of 200 m. Determine the acceleration of the car and the time required.

12–2. A train starts from rest at a station and travels with a constant acceleration of \( \alpha \). Determine the velocity of the train when \( t = 30 \) s and the distance traveled during this time.

12–3. An elevator descends from rest with an acceleration of 5 ft/s\(^2\) until it achieves a velocity of 15 ft/s. Determine the time required and the distance traveled.

*12–4. A car is traveling at 15 m/s, when the traffic light 50 m ahead turns yellow. Determine the required constant deceleration of the car and the time needed to stop the car at the light.

•12–5. A particle is moving along a straight line with the acceleration \( a = (12t - 3t^{1/2}) \) ft/s\(^2\), where \( t \) is in seconds. Determine the velocity and the position of the particle as a function of time. When \( t = 0 \), \( v = 0 \) and \( s = 15 \) ft.

12–6. A ball is released from the bottom of an elevator which is traveling upward with a velocity of 6 ft/s. If the ball strikes the bottom of the elevator shaft in 3 s, determine the height of the elevator from the bottom of the shaft at the instant the ball is released. Also, find the velocity of the ball when it strikes the bottom of the shaft.

12–7. A car has an initial speed of 25 m/s and a constant deceleration of 3 m/s\(^2\). Determine the velocity of the car when \( t = 4 \) s. What is the displacement of the car during the 4-s time interval? How much time is needed to stop the car?

*12–8. If a particle has an initial velocity of \( v_0 = 12 \) ft/s to the right, at \( s_0 = 0 \), determine its position when \( t = 10 \) s, if \( a = 2 \) ft/s\(^2\) to the left.

•12–9. The acceleration of a particle traveling along a straight line is \( a = k/v \), where \( k \) is a constant. If \( s = 0, v = v_0 \) when \( t = 0 \), determine the velocity of the particle as a function of time \( t \).

12–10. Car \( A \) starts from rest at \( t = 0 \) and travels along a straight road with a constant acceleration of 6 ft/s\(^2\) until it reaches a speed of 80 ft/s. Afterwards it maintains this speed. Also, when \( t = 0 \), car \( B \) located 6000 ft down the road is traveling towards \( A \) at a constant speed of 60 ft/s. Determine the distance traveled by car \( A \) when they pass each other.

12–11. A particle travels along a straight line with a velocity \( v = (12 - 3t^2) \) m/s, where \( t \) is in seconds. When \( t = 1 \) s, the particle is located 10 m to the left of the origin. Determine the acceleration when \( t = 4 \) s, the displacement from \( t = 0 \) to \( t = 10 \) s, and the distance the particle travels during this time period.

*12–12. A sphere is fired downwards into a medium with an initial speed of 27 m/s. If it experiences a deceleration of \( a = (-6t) \) m/s\(^2\), where \( t \) is in seconds, determine the distance traveled before it stops.

•12–13. A particle travels along a straight line such that in 2 s it moves from an initial position \( s_A = +0.5 \) m to a position \( s_B = -1.5 \) m. Then in another 4 s it moves from \( s_B \) to \( s_C = +2.5 \) m. Determine the particle’s average velocity and average speed during the 6-s time interval.

12–14. A particle travels along a straight-line path such that in 4 s it moves from an initial position \( s_A = -8 \) m to a position \( s_B = +3 \) m. Then in another 5 s it moves from \( s_B \) to \( s_C = -6 \) m. Determine the particle’s average velocity and average speed during the 9-s time interval.
12-15. Tests reveal that a normal driver takes about 0.75 s before he or she can react to a situation to avoid a collision. It takes about 3 s for a driver having 0.1% alcohol in his system to do the same. If such drivers are traveling on a straight road at 30 mph (44 ft/s) and their cars can decelerate at 2 ft/s², determine the shortest stopping distance d for each from the moment they see the pedestrians. *Moral:* If you must drink, please don’t drive!

\[ v_i = 44 \text{ ft/s} \]

**Prob. 12-15**

12-16. As a train accelerates uniformly it passes successive kilometer marks while traveling at velocities of 2 m/s and then 10 m/s. Determine the train’s velocity when it passes the next kilometer mark and the time it takes to travel the 2-km distance.

12-17. A ball is thrown with an upward velocity of 5 m/s from the top of a 10-m high building. One second later another ball is thrown vertically from the ground with a velocity of 10 m/s. Determine the height from the ground where the two balls pass each other.

12-18. A car starts from rest and moves with a constant acceleration of 1.5 m/s² until it achieves a velocity of 25 m/s. It then travels with constant velocity for 60 seconds. Determine the average speed and the total distance traveled.

12-19. A car is to be hoisted by elevator to the fourth floor of a parking garage, which is 48 ft above the ground. If the elevator can accelerate at 0.6 ft/s², decelerate at 0.3 ft/s², and reach a maximum speed of 8 ft/s, determine the shortest time to make the lift, starting from rest and ending at rest.

12-20. A particle is moving along a straight line such that its speed is defined as \( v = (-4s^2) \) m/s, where s is in meters. If \( s = 2 \) m when \( t = 0 \), determine the velocity and acceleration as functions of time.

12-21. Two particles A and B start from rest at the origin \( s = 0 \) and move along a straight line such that \( a_A = (6t - 3) \) ft/s² and \( a_B = (12t^2 - 8) \) ft/s², where \( t \) is in seconds. Determine the distance between them when \( t = 4 \) s and the total distance each has traveled in \( t = 4 \) s.

12-22. A particle moving along a straight line is subjected to a deceleration \( a = (-2v) \) m/s², where \( v \) is in m/s. If it has a velocity \( v = 8 \) m/s and a position \( s = 10 \) m when \( t = 0 \), determine its velocity and position when \( t = 4 \) s.

12-23. A particle is moving along a straight line such that its acceleration is defined as \( a = (-2v) \) m/s², where \( v \) is in meters per second. If \( v = 20 \) m/s when \( s = 0 \) and \( t = 0 \), determine the particle’s position, velocity, and acceleration as functions of time.

12-24. A particle starts from rest and travels along a straight line with an acceleration \( a = (30 - 0.2v) \) ft/s², where \( v \) is in ft/s. Determine the time when the velocity of the particle is \( v = 30 \) ft/s.

12-25. When a particle is projected vertically upwards with an initial velocity of \( v_0 \), it experiences an acceleration \( a = -(g + kv^2) \), where \( g \) is the acceleration due to gravity, \( k \) is a constant and \( v \) is the velocity of the particle. Determine the maximum height reached by the particle.

12-26. The acceleration of a particle traveling along a straight line is \( a = (0.02v^2) \) m/s², where \( t \) is in seconds. If \( v = 0, s = 0 \) when \( t = 0 \), determine the velocity and acceleration of the particle at \( s = 4 \) m.

12-27. A particle moves along a straight line with an acceleration of \( a = 5/(3s^{1/3} + s^{5/2}) \) m/s², where \( s \) is in meters. Determine the particle’s velocity when \( s = 2 \) m, if it starts from rest when \( s = 1 \) m. Use Simpson’s rule to evaluate the integral.

12-28. If the effects of atmospheric resistance are accounted for, a falling body has an acceleration defined by the equation \( a = 9.81(1 - v^2(10^{-6})) \) m/s², where \( v \) is in m/s and the positive direction is downward. If the body is released from rest at a very high altitude, determine (a) the velocity when \( t = 5 \) s, and (b) the body’s terminal or maximum attainable velocity (as \( t \to \infty \)).
12–29. The position of a particle along a straight line is given by \( s = (1.5t^3 - 13.5t^2 + 22.5t) \) ft, where \( t \) is in seconds. Determine the position of the particle when \( t = 6 \) s and the total distance it travels during the 6-s time interval. *Hint:* Plot the path to determine the total distance traveled.

12–30. The velocity of a particle traveling along a straight line is \( v = v_0 - kt \), where \( k \) is constant. If \( v = 0 \) when \( t = 0 \), determine the position and acceleration of the particle as a function of time.

12–31. The acceleration of a particle as it moves along a straight line is given by \( a = (2t - 1) \) m/s\(^2\), where \( t \) is in seconds. If \( s = 1 \) m and \( v = 2 \) m/s when \( t = 0 \), determine the particle’s velocity and position when \( t = 6 \) s. Also, determine the total distance the particle travels during this time period.

12–32. Ball A is thrown vertically upward from the top of a 30-m-high-building with an initial velocity of 5 m/s. At the same instant another ball B is thrown upward from the ground with an initial velocity of 20 m/s. Determine the height from the ground and the time at which they pass.

12–33. A motorcycle starts from rest at \( t = 0 \) and travels along a straight road with a constant acceleration of 6 ft/s\(^2\) until it reaches a speed of 50 ft/s. Afterwards it maintains this speed. Also, when \( t = 0 \), a car located 6000 ft down the road is traveling toward the motorcycle at a constant speed of 30 ft/s. Determine the time and the distance traveled by the motorcycle when they pass each other.

12–34. A particle moves along a straight line with a velocity \( v = (200s) \) mm/s, where \( s \) is in millimeters. Determine the acceleration of the particle at \( s = 2000 \) mm. How long does the particle take to reach this position if \( s = 500 \) mm when \( t = 0 \)?

12–35. A particle has an initial speed of 27 m/s. If it experiences a deceleration of \( a = (-6t) \) m/s\(^2\), where \( t \) is in seconds, determine its velocity, after it has traveled 10 m. How much time does this take?

12–36. The acceleration of a particle traveling along a straight line is \( a = (8 - 2s) \) m/s\(^2\), where \( s \) is in meters. If \( v = 0 \) at \( s = 0 \), determine the velocity of the particle at \( s = 2 \) m, and the position of the particle when the velocity is maximum.

12–37. Ball A is thrown vertically upwards with a velocity \( v_0 \). Ball B is thrown upwards from the same point with the same velocity \( t \) seconds later. Determine the elapsed time \( t < 2v_0/g \) from the instant ball A is thrown to when the balls pass each other, and find the velocity of each ball at this instant.

12–38. As a body is projected to a high altitude above the earth’s surface, the variation of the acceleration of gravity with respect to altitude \( y \) must be taken into account. Neglecting air resistance, this acceleration is determined from the formula \( a = -g_0 \left( R^2/(R + y)^2 \right) \), where \( g_0 \) is the constant gravitational acceleration at sea level, \( R \) is the radius of the earth, and the positive direction is measured upward. If \( g_0 = 9.81 \) m/s\(^2\) and \( R = 6356 \) km, determine the minimum initial velocity (escape velocity) at which a projectile should be shot vertically from the earth’s surface so that it does not fall back to the earth. *Hint:* This requires that \( v = 0 \) as \( y \to \infty \).

12–39. Accounting for the variation of gravitational acceleration \( a \) with respect to altitude \( y \) (see Prob. 12–38), derive an equation that relates the velocity of a freely falling particle to its altitude. Assume that the particle is released from rest at an altitude \( y_0 \) from the earth’s surface. With what velocity does the particle strike the earth if it is released from rest at an altitude \( y_0 = 500 \) km? Use the numerical data in Prob. 12–38.

12–40. When a particle falls through the air, its initial acceleration \( a = g \) diminishes until it is zero, and thereafter it falls at a constant or terminal velocity \( v_f \). If this variation of the acceleration can be expressed as \( a = (g/v_f^2)(v_f^2 - v^2) \), determine the time needed for the velocity to become \( v = v_f/2 \). Initially the particle falls from rest.

12–41. A particle is moving along a straight line such that its position from a fixed point is \( s = (12 - 15t^2 + 5t^3) \) m, where \( t \) is in seconds. Determine the total distance traveled by the particle from \( t = 1 \) s to \( t = 3 \) s. Also, find the average speed of the particle during this time interval.
12.3 Rectilinear Kinematics: Erratic Motion

When a particle has erratic or changing motion then its position, velocity, and acceleration cannot be described by a single continuous mathematical function along the entire path. Instead, a series of functions will be required to specify the motion at different intervals. For this reason, it is convenient to represent the motion as a graph. If a graph of the motion that relates any two of the variables \( s, v, a, t \) can be drawn, then this graph can be used to construct subsequent graphs relating two other variables since the variables are related by the differential relationships \( v = \frac{ds}{dt} \), \( a = \frac{dv}{dt} \), or \( a ds = v dv \). Several situations occur frequently.

The \( s-t \), \( v-t \), and \( a-t \) Graphs. To construct the \( v-t \) graph given the \( s-t \) graph, Fig. 12–7a, the equation \( v = \frac{ds}{dt} \) should be used, since it relates the variables \( s \) and \( t \) to \( v \). This equation states that

\[
\frac{ds}{dt} = v \\
\text{slope of } s-t \text{ graph} = \text{velocity}
\]

For example, by measuring the slope on the \( s-t \) graph when \( t = t_1 \), the velocity is \( v_1 \), which is plotted in Fig. 12–7b. The \( v-t \) graph can be constructed by plotting this and other values at each instant.

The \( a-t \) graph can be constructed from the \( v-t \) graph in a similar manner, Fig. 12–8, since

\[
\frac{dv}{dt} = a \\
\text{slope of } v-t \text{ graph} = \text{acceleration}
\]

Examples of various measurements are shown in Fig. 12–8a and plotted in Fig. 12–8b.

If the \( s-t \) curve for each interval of motion can be expressed by a mathematical function \( s = s(t) \), then the equation of the \( v-t \) graph for the same interval can be obtained by differentiating this function with respect to time since \( v = \frac{ds}{dt} \). Likewise, the equation of the \( a-t \) graph for the same interval can be determined by differentiating \( v = v(t) \) since \( a = \frac{dv}{dt} \). Since differentiation reduces a polynomial of degree \( n \) to that of degree \( n - 1 \), then if the \( s-t \) graph is parabolic (a second-degree curve), the \( v-t \) graph will be a sloping line (a first-degree curve), and the \( a-t \) graph will be a constant or a horizontal line (a zero-degree curve).
If the \( a-t \) graph is given, Fig. 12–9a, the \( v-t \) graph may be constructed using \( a = \frac{dv}{dt} \), written as

\[
\Delta v = \int_0^t a \, dt
\]

Hence, to construct the \( v-t \) graph, we begin with the particle's initial velocity \( v_0 \) and then add to this small increments of area \((\Delta v)\) determined from the \( a-t \) graph. In this manner successive points, \( v_1 = v_0 + \Delta v \), etc., for the \( v-t \) graph are determined, Fig. 12–9b. Notice that an algebraic addition of the area increments of the \( a-t \) graph is necessary, since areas lying above the \( t \) axis correspond to an increase in \( v \) ("positive" area), whereas those lying below the axis indicate a decrease in \( v \) ("negative" area).

Similarly, if the \( v-t \) graph is given, Fig. 12–10a, it is possible to determine the \( s-t \) graph using \( v = \frac{ds}{dt} \), written as

\[
\Delta s = \int_0^t v \, dt
\]

In the same manner as stated above, we begin with the particle’s initial position \( s_0 \) and add (algebraically) to this small area increments \( \Delta s \) determined from the \( v-t \) graph, Fig. 12–10b.

If segments of the \( a-t \) graph can be described by a series of equations, then each of these equations can be integrated to yield equations describing the corresponding segments of the \( v-t \) graph. In a similar manner, the \( s-t \) graph can be obtained by integrating the equations which describe the segments of the \( v-t \) graph. As a result, if the \( a-t \) graph is linear (a first-degree curve), integration will yield a \( v-t \) graph that is parabolic (a second-degree curve) and an \( s-t \) graph that is cubic (third-degree curve).
The *v*–*s* and *a*–*s* Graphs. If the *a*–*s* graph can be constructed, then points on the *v*–*s* graph can be determined by using \( v \, dv = a \, ds \). Integrating this equation between the limits \( v = v_0 \) at \( s = s_0 \) and \( v = v_1 \) at \( s = s_1 \), we have,

\[
\frac{1}{2}(v_1^2 - v_0^2) = \int_{s_0}^{s_1} a \, ds
\]

area under *a*–*s* graph

Therefore, if the red area in Fig. 12–11a is determined, and the initial velocity \( v_0 \) at \( s_0 = 0 \) is known, then \( v_1 = \left(2 \int_{s_0}^{s_1} a \, ds + v_0^2 \right)^{1/2} \), Fig. 12–11b. Successive points on the *v*–*s* graph can be constructed in this manner.

If the *v*–*s* graph is known, the acceleration \( a \) at any position \( s \) can be determined using \( a \, ds = v \, dv \), written as

\[
a = v \left( \frac{dv}{ds} \right)
\]

acceleration = slope of *v*–*s* graph

Thus, at any point \((s, v)\) in Fig. 12–12a, the slope \(dv/ds\) of the *v*–*s* graph is measured. Then with \( v \) and \( dv/ds \) known, the value of \( a \) can be calculated, Fig. 12–12b.

The *v*–*s* graph can also be constructed from the *a*–*s* graph, or vice versa, by approximating the known graph in various intervals with mathematical functions, \( v = f(s) \) or \( a = g(s) \), and then using \( a \, ds = v \, dv \) to obtain the other graph.
EXAMPLE 12.6

A bicycle moves along a straight road such that its position is described by the graph shown in Fig. 12–13a. Construct the \(v-t\) and \(a-t\) graphs for \(0 \leq t \leq 30\) s.

\[ v = 2t \]  
\[ v = 20 \]  
\[ s = 20t - 100 \]  
\[ s = t^2 \]  
\[ 100 \]  
\[ 500 \]  
\[ 10 \]  
\[ 30 \]  
\[ t (s) \]  
\[ (a) \]  

\[ s (ft) \]  

\[ v (ft/s) \]  

\[ a (ft/s^2) \]  

\[ (b) \]  

\[ (c) \]  

**SOLUTION**

**\(v-t\) Graph.** Since \(v = ds/dt\), the \(v-t\) graph can be determined by differentiating the equations defining the \(s-t\) graph, Fig. 12–13a. We have

\[ 0 \leq t < 10\ s; \quad s = (t^2) \text{ ft} \quad v = \frac{ds}{dt} = (2t) \text{ ft/s} \]

\[ 10 \leq t \leq 30\ s; \quad s = (20t - 100) \text{ ft} \quad v = \frac{ds}{dt} = 20 \text{ ft/s} \]

The results are plotted in Fig. 12–13b. We can also obtain specific values of \(v\) by measuring the slope of the \(s-t\) graph at a given instant. For example, at \(t = 20\) s, the slope of the \(s-t\) graph is determined from the straight line from 10 s to 30 s, i.e.,

\[ t = 20\ s; \quad v = \frac{\Delta s}{\Delta t} = \frac{500 \text{ ft} - 100 \text{ ft}}{30 \text{ s} - 10 \text{ s}} = 20 \text{ ft/s} \]

**\(a-t\) Graph.** Since \(a = dv/dt\), the \(a-t\) graph can be determined by differentiating the equations defining the lines of the \(v-t\) graph. This yields

\[ 0 \leq t < 10\ s; \quad v = (2t) \text{ ft/s} \quad a = \frac{dv}{dt} = 2 \text{ ft/s}^2 \]

\[ 10 \leq t \leq 30\ s; \quad v = 20 \text{ ft/s} \quad a = \frac{dv}{dt} = 0 \]

The results are plotted in Fig. 12–13c.

**NOTE:** Show that \(a = 2 \text{ ft/s}^2\) when \(t = 5\) s by measuring the slope of the \(v-t\) graph.
EXAMPLE 12.7

The car in Fig. 12–14a starts from rest and travels along a straight track such that it accelerates at 10 m/s² for 10 s, and then decelerates at 2 m/s². Draw the $v$–$t$ and $s$–$t$ graphs and determine the time $t'$ needed to stop the car. How far has the car traveled?

SOLUTION

$v$–$t$ Graph. Since $dv = a \, dt$, the $v$–$t$ graph is determined by integrating the straight-line segments of the $a$–$t$ graph. Using the initial condition $v = 0$ when $t = 0$, we have

$$0 \leq t < 10 \, s; \quad a = \left( 10 \right) \, \text{m/s}^2; \quad \int_0^v \, dv = \int_0^t \, 10 \, \text{dt}, \quad v = 10t$$

When $t = 10 \, s$, $v = 10(10) = 100 \, \text{m/s}$. Using this as the initial condition for the next time period, we have

$$10 \, s < t \leq t'; \quad a = \left( -2 \right) \, \text{m/s}^2; \quad \int_t^v \, dv = \int_{10}^{t'} \, -2 \, \text{dt}, \quad v = (-2t + 120) \, \text{m/s}$$

When $t = t'$ we require $v = 0$. This yields, Fig. 12–14b

$$t' = 60 \, \text{s} \quad \text{Ans.}$$

A more direct solution for $t'$ is possible by realizing that the area under the $a$–$t$ graph is equal to the change in the car’s velocity. We require $\Delta v = 0 = A_1 + A_2$, Fig. 12–14a. Thus

$$0 = 100 \, \text{m/s}^2(10 \, \text{s}) + (-2 \, \text{m/s}^2)(t' - 10 \, \text{s})$$

$$t' = 60 \, \text{s} \quad \text{Ans.}$$

$s$–$t$ Graph. Since $ds = v \, dt$, integrating the equations of the $v$–$t$ graph yields the corresponding equations of the $s$–$t$ graph. Using the initial condition $s = 0$ when $t = 0$, we have

$$0 \leq t \leq 10 \, \text{s; } \quad v = (10t) \, \text{m/s}; \quad \int_0^s \, ds = \int_0^t \, 10t \, \text{dt}, \quad s = (5t^2) \, \text{m}$$

When $t = 10 \, \text{s}$, $s = 5(10)^2 = 500 \, \text{m}$. Using this initial condition,

$$10 \, s \leq t \leq 60 \, \text{s; } \quad v = (-2t + 120) \, \text{m/s}; \quad \int_{500}^s \, ds = \int_{10}^{t'} \, (-2t + 120) \, \text{dt}$$

$$s - 500 = -t^2 + 120t - \left[ (-10)^2 + 120(10) \right]$$

$$s = -t^2 + 120t - 600 \, \text{m}$$

When $t' = 60 \, \text{s}$, the position is

$$s = -(60)^2 + 120(60) - 600 = 3000 \, \text{m} \quad \text{Ans.}$$

The $s$–$t$ graph is shown in Fig. 12–14c.

NOTE: A direct solution for $s$ is possible when $t' = 60 \, \text{s}$, since the triangular area under the $v$–$t$ graph would yield the displacement $\Delta s = s - 0$ from $t = 0$ to $t' = 60 \, \text{s}$. Hence,

$$\Delta s = \frac{1}{2}(60 \, \text{s})(100 \, \text{m/s}) = 3000 \, \text{m} \quad \text{Ans.}$$
EXAMPLE 12.8

The $v$–$s$ graph describing the motion of a motorcycle is shown in Fig. 12–15a. Construct the $a$–$s$ graph of the motion and determine the time needed for the motorcycle to reach the position $s = 400$ ft.

**SOLUTION**

**a–s Graph.** Since the equations for segments of the $v$–$s$ graph are given, the $a$–$s$ graph can be determined using $a \, ds = v \, dv$.

For $0 \leq s < 200$ ft;

$v = (0.2s + 10) \text{ ft/s}$

$a = v \frac{dv}{ds} = (0.2s + 10) \frac{d}{ds} (0.2s + 10) = 0.04s + 2$

For $200$ ft $< s \leq 400$ ft;

$v = 50$ ft/s

$a = v \frac{dv}{ds} = (50) \frac{d}{ds} (50) = 0$

The results are plotted in Fig. 12–15b.

**Time.** The time can be obtained using the $v$–$s$ graph and $v = ds/dt$, because this equation relates $v$, $s$, and $t$. For the first segment of motion, $s = 0$ when $t = 0$, so

For $0 \leq s < 200$ ft;

$v = (0.2s + 10) \text{ ft/s}$

$dt = \frac{ds}{v} = \frac{ds}{0.2s + 10}$

$$
t = \int_0^t dt = \int_0^s \frac{ds}{0.2s + 10}
$$

At $s = 200$ ft, $t = 5 \ln(0.2(200) + 10) - 5 \ln 10 = 8.05$ s. Therefore, using these initial conditions for the second segment of motion,

For $200$ ft $< s \leq 400$ ft;

$v = 50$ ft/s

$dt = \frac{ds}{v} = \frac{ds}{50}$

$$
t = \int_{8.05}^t dt = \int_{200}^s \frac{ds}{50};
$$

At $s = 400$ ft,

Therefore, at $s = 400$ ft,

$$
t = \frac{400}{50} + 4.05 = 12.0 \text{ s} \quad \text{Ans.}
$$

**NOTE:** The graphical results can be checked in part by calculating slopes. For example, at $s = 0$, $a = v(dv/ds) = 10(50 - 10)/200 = 2$ m/s$^2$. Also, the results can be checked in part by inspection. The $v$–$s$ graph indicates the initial increase in velocity (acceleration) followed by constant velocity $(a = 0)$. 

12.3 Rectilinear Kinematics: Erratic Motion

FUNDAMENTAL PROBLEMS

F12–9. The particle travels along a straight track such that its position is described by the $s-t$ graph. Construct the $v-t$ graph for the same time interval.

F12–10. A van travels along a straight road with a velocity described by the graph. Construct the $s-t$ and $a-t$ graphs during the same period. Take $s = 0$ when $t = 0$.

F12–11. A bicycle travels along a straight road where its velocity is described by the $v-s$ graph. Construct the $a-s$ graph for the same time interval.

F12–12. The sports car travels along a straight road such that its position is described by the graph. Construct the $v-t$ and $a-t$ graphs for the time interval $0 \leq t \leq 10$.

F12–13. The dragster starts from rest and has an acceleration described by the graph. Construct the $v-t$ graph for the time interval $0 \leq t \leq t'$, where $t'$ is the time for the car to come to rest.

F12–14. The dragster starts from rest and has a velocity described by the graph. Construct the $s-t$ graph during the time interval $0 \leq t \leq 15$. Also, determine the total distance traveled during this time interval.
12–42. The speed of a train during the first minute has been recorded as follows:

<table>
<thead>
<tr>
<th>$t$ (s)</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v$ (m/s)</td>
<td>0</td>
<td>16</td>
<td>21</td>
<td>24</td>
</tr>
</tbody>
</table>

Plot the $v$–$t$ graph, approximating the curve as straight-line segments between the given points. Determine the total distance traveled.

12–43. A two-stage missile is fired vertically from rest with the acceleration shown. In 15 s the first stage $A$ burns out and the second stage $B$ ignites. Plot the $v$–$t$ and $s$–$t$ graphs which describe the two-stage motion of the missile for $0 \leq t \leq 20$ s.

12–46. A train starts from station $A$ and for the first kilometer, it travels with a uniform acceleration. Then, for the next two kilometers, it travels with a uniform speed. Finally, the train decelerates uniformly for another kilometer before coming to rest at station $B$. If the time for the whole journey is six minutes, draw the $v$–$t$ graph and determine the maximum speed of the train.

12–47. The particle travels along a straight line with the velocity described by the graph. Construct the $a$–$s$ graph.

12–44. A freight train starts from rest and travels with a constant acceleration of 0.5 ft/s$^2$. After a time $t'$ it maintains a constant speed so that when $t = 160$ s it has traveled 2000 ft. Determine the time $t'$ and draw the $v$–$t$ graph for the motion.

12–45. If the position of a particle is defined by $s = [2 \sin (\pi/5)t + 4]$ m, where $t$ is in seconds, construct the $s$–$t$, $v$–$t$, and $a$–$t$ graphs for $0 \leq t \leq 10$ s.
12-49. A particle travels along a curve defined by the equation \( s = (t^3 - 3t^2 + 2t) \) m, where \( t \) is in seconds. Draw the \( s - t \), \( v - t \), and \( a - t \) graphs for the particle for \( 0 \leq t \leq 3 \) s.

12-50. A truck is traveling along the straight line with a velocity described by the graph. Construct the \( a - s \) graph for \( 0 \leq s \leq 1500 \) ft.

12-51. A car starts from rest and travels along a straight road with a velocity described by the graph. Determine the total distance traveled until the car stops. Construct the \( s - t \) and \( a - t \) graphs.

12-52. A car travels up a hill with the speed shown. Determine the total distance the car travels until it stops (\( t = 60 \) s). Plot the \( a - t \) graph.

12-53. The snowmobile moves along a straight course according to the \( v - t \) graph. Construct the \( s - t \) and \( a - t \) graphs for the same 50-s time interval. When \( t = 0, s = 0 \).

12-54. A motorcyclist at \( A \) is traveling at 60 ft/s when he wishes to pass the truck \( T \) which is traveling at a constant speed of 60 ft/s. To do so the motorcyclist accelerates at 6 ft/s\(^2\) until reaching a maximum speed of 85 ft/s. If he then maintains this speed, determine the time needed for him to reach a point located 100 ft in front of the truck. Draw the \( v - t \) and \( s - t \) graphs for the motorcycle during this time.
12–55. An airplane traveling at 70 m/s lands on a straight runway and has a deceleration described by the graph. Determine the time $t'$ and the distance traveled for it to reach a speed of 5 m/s. Construct the $v$–$t$ and $s$–$t$ graphs for this time interval, $0 \leq t \leq t'$.

![Graph of an airplane's deceleration](image1)

**Prob. 12–55**

12–57. The dragster starts from rest and travels along a straight track with an acceleration-deceleration described by the graph. Construct the $v$–$s$ graph for $0 \leq s \leq s'$, and determine the distance $s'$ traveled before the dragster again comes to rest.

![Graph of a dragster's acceleration-deceleration](image2)

**Prob. 12–57**

*12–56. The position of a cyclist traveling along a straight road is described by the graph. Construct the $v$–$t$ and $a$–$t$ graphs.

![Graph of a cyclist's position](image3)

**Prob. 12–56**

12–58. A sports car travels along a straight road with an acceleration-deceleration described by the graph. If the car starts from rest, determine the distance $s'$ the car travels until it stops. Construct the $v$–$s$ graph for $0 \leq s \leq s'$.

![Graph of a sports car's acceleration-deceleration](image4)

**Prob. 12–58**

12–59. A missile starting from rest travels along a straight track and for 10 s has an acceleration as shown. Draw the $v-t$ graph that describes the motion and find the distance traveled in 10 s.

12–60. A motorcyclist starting from rest travels along a straight road and for 10 s has an acceleration as shown. Draw the $v-t$ graph that describes the motion and find the distance traveled in 10 s.

$12–61$. The $v-t$ graph of a car while traveling along a road is shown. Draw the $s-t$ and $a-t$ graphs for the motion.

$12–62$. The boat travels in a straight line with the acceleration described by the $a-s$ graph. If it starts from rest, construct the $v-s$ graph and determine the boat’s maximum speed. What distance $s'$ does it travel before it stops?
12–63. The rocket has an acceleration described by the graph. If it starts from rest, construct the \( v-t \) and \( s-t \) graphs for the motion for the time interval \( 0 \leq t \leq 14 \) s.

12–65. The acceleration of the speed boat starting from rest is described by the graph. Construct the \( v-s \) graph.

12–66. The boat travels along a straight line with the speed described by the graph. Construct the \( s-t \) and \( a-s \) graphs. Also, determine the time required for the boat to travel a distance \( s = 400 \) m if \( s = 0 \) when \( t = 0 \).
12-67. The \( s-t \) graph for a train has been determined experimentally. From the data, construct the \( v-t \) and \( a-t \) graphs for the motion.

\[ s = 24t - 360 \]
\[ s = 0.4t^2 \]

\( t \) (s) \hspace{2cm} 30 \hspace{1cm} 40

\( s \) (m) \hspace{2cm} 360 \hspace{1cm} 600

Pro. 12–67

12-69. The airplane travels along a straight runway with an acceleration described by the graph. If it starts from rest and requires a velocity of 90 m/s to take off, determine the minimum length of runway required and the time \( t' \) for take off. Construct the \( v-t \) and \( s-t \) graphs.

\[ a = 0.8t \]

\( a(m/s^2) \)

\( t(s) \)

10 \hspace{1cm} \text{to} \hspace{1cm} t'

Pro. 12–69

12-68. The airplane lands at 250 ft/s on a straight runway and has a deceleration described by the graph. Determine the distance \( s' \) traveled before its speed is decreased to 25 ft/s. Draw the \( s-t \) graph.

\[ a = \begin{cases} 8 & \text{for } 0 \leq t < 10 \\ 0.8t & \text{for } t \geq 10 \end{cases} \]

\( a(m/s^2) \)

\( t(s) \)

10 \hspace{1cm} \text{to} \hspace{1cm} t'

Pro. 12–68

12-70. The \( a-t \) graph of the bullet train is shown. If the train starts from rest, determine the elapsed time \( t' \) before it again comes to rest. What is the total distance traveled during this time interval? Construct the \( v-t \) and \( s-t \) graphs.

\[ a = \begin{cases} 0.1t & \text{for } 0 \leq t < 30 \\ -\frac{1}{15}t + 5 & \text{for } 30 \leq t \leq 75 \end{cases} \]

\( a(m/s^2) \)

\( t(s) \)

30 \hspace{1cm} 75 \hspace{1cm} t'

Pro. 12–70
12.4 General Curvilinear Motion

Curvilinear motion occurs when a particle moves along a curved path. Since this path is often described in three dimensions, vector analysis will be used to formulate the particle’s position, velocity, and acceleration. In this section the general aspects of curvilinear motion are discussed, and in subsequent sections we will consider three types of coordinate systems often used to analyze this motion.

**Position.** Consider a particle located at a point on a space curve defined by the path function \( s(t) \), Fig. 12–16a. The position of the particle, measured from a fixed point \( O \), will be designated by the position vector \( \mathbf{r} = \mathbf{r}(t) \). Notice that both the magnitude and direction of this vector will change as the particle moves along the curve.

**Displacement.** Suppose that during a small time interval \( \Delta t \) the particle moves a distance \( \Delta s \) along the curve to a new position, defined by \( \mathbf{r}' = \mathbf{r} + \Delta \mathbf{r} \), Fig. 12–16b. The displacement \( \Delta \mathbf{r} \) represents the change in the particle’s position and is determined by vector subtraction; i.e., \( \Delta \mathbf{r} = \mathbf{r}' - \mathbf{r} \).

**Velocity.** During the time \( \Delta t \), the average velocity of the particle is

\[
\mathbf{v}_{\text{avg}} = \frac{\Delta \mathbf{r}}{\Delta t}
\]

The instantaneous velocity is determined from this equation by letting \( \Delta t \to 0 \), and consequently the direction of \( \Delta \mathbf{r} \) approaches the tangent to the curve. Hence, \( \mathbf{v} = \lim_{\Delta t \to 0} (\Delta \mathbf{r}/\Delta t) \) or

\[
\mathbf{v} = \frac{d\mathbf{r}}{dt}
\]  

(12–7)

Since \( d\mathbf{r} \) will be tangent to the curve, the direction of \( \mathbf{v} \) is also tangent to the curve, Fig. 12–16c. The magnitude of \( \mathbf{v} \), which is called the speed, is obtained by realizing that the length of the straight line segment \( \Delta \mathbf{r} \) in Fig. 12–16b approaches the arc length \( \Delta s \) as \( \Delta t \to 0 \), we have

\[
v = \lim_{\Delta t \to 0} (\Delta r/\Delta t) = \lim_{\Delta t \to 0} (\Delta s/\Delta t), \text{ or}
\]

\[
v = \frac{d\mathbf{s}}{dt}
\]  

(12–8)

Thus, the speed can be obtained by differentiating the path function \( s \) with respect to time.

* A summary of some of the important concepts of vector analysis is given in Appendix B.
**Acceleration.** If the particle has a velocity \( v \) at time \( t \) and a velocity \( v' = v + \Delta v \) at \( t + \Delta t \), Fig. 12–16, then the *average acceleration* of the particle during the time interval \( \Delta t \) is

\[
a_{\text{avg}} = \frac{\Delta v}{\Delta t}
\]

where \( \Delta v = v' - v \). To study this time rate of change, the two velocity vectors in Fig. 12–16d are plotted in Fig. 12–16e such that their tails are located at the fixed point \( O' \) and their arrowheads touch points on a curve. This curve is called a *hodograph*, and when constructed, it describes the locus of points for the arrowhead of the velocity vector in the same manner as the *path* \( s \) describes the locus of points for the arrowhead of the position vector, Fig. 12–16a.

To obtain the *instantaneous acceleration*, let \( \Delta t \to 0 \) in the above equation. In the limit \( \Delta v \) will approach the *tangent to the hodograph*, and so \( \mathbf{a} = \lim_{\Delta t \to 0} (\Delta v/\Delta t) \), or

\[
\mathbf{a} = \frac{d\mathbf{v}}{dt}
\]

Substituting Eq. 12–7 into this result, we can also write

\[
\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}
\]

By definition of the derivative, \( \mathbf{a} \) acts *tangent to the hodograph*, Fig. 12–16f, and, *in general it is not tangent to the path of motion*, Fig. 12–16g. To clarify this point, realize that \( \Delta v \) and consequently \( \mathbf{a} \) must account for the change made in both the magnitude and direction of the velocity \( \mathbf{v} \) as the particle moves from one point to the next along the path, Fig. 12–16d. However, in order for the particle to follow any curved path, the directional change always “swings” the velocity vector toward the “inside” or “concave side” of the path, and therefore \( \mathbf{a} \) *cannot* remain tangent to the path. In summary, \( \mathbf{v} \) is always tangent to the *path* and \( \mathbf{a} \) is always tangent to the *hodograph.*
12.5 **Curvilinear Motion: Rectangular Components**

Occasionally the motion of a particle can best be described along a path that can be expressed in terms of its $x, y, z$ coordinates.

**Position.** If the particle is at point $(x, y, z)$ on the curved path $s$ shown in Fig. 12–17a, then its location is defined by the *position vector*

$$
\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}
$$

(12–10)

When the particle moves, the $x, y, z$ components of $\mathbf{r}$ will be functions of time; i.e., $x = x(t), y = y(t), z = z(t)$, so that $\mathbf{r} = \mathbf{r}(t)$.

At any instant the *magnitude* of $\mathbf{r}$ is defined from Eq. C–3 in Appendix C as

$$
r = \sqrt{x^2 + y^2 + z^2}
$$

And the *direction* of $\mathbf{r}$ is specified by the unit vector $\mathbf{u}_r = \mathbf{r}/r$.

**Velocity.** The first time derivative of $\mathbf{r}$ yields the velocity of the particle. Hence,

$$
\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}
$$

(12–11)

When taking this derivative, it is necessary to account for changes in both the magnitude and direction of each of the vector’s components. For example, the derivative of the $\mathbf{i}$ component of $\mathbf{r}$ is

$$
\frac{d}{dt}(x\mathbf{i}) = \frac{dx}{dt}\mathbf{i} + x\frac{d\mathbf{i}}{dt}
$$

The second term on the right side is zero, provided the $x, y, z$ reference frame is fixed, and therefore the *direction* (and the *magnitude*) of $\mathbf{i}$ does not change with time. Differentiation of the $\mathbf{j}$ and $\mathbf{k}$ components may be carried out in a similar manner, which yields the final result,

$$
\mathbf{v} = \frac{d\mathbf{r}}{dt} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}
$$

(12–11)

where

$$
v_x = \dot{x} \quad v_y = \dot{y} \quad v_z = \dot{z}
$$

(12–12)
The “dot” notation \( \dot{x}, \dot{y}, \dot{z} \) represents the first time derivatives of \( x = x(t), y = y(t), z = z(t) \), respectively.

The velocity has a magnitude that is found from

\[
v = \sqrt{v_x^2 + v_y^2 + v_z^2}
\]

and a direction that is specified by the unit vector \( \mathbf{u}_v = \mathbf{v}/v \). As discussed in Sec. 12–4, this direction is always tangent to the path, as shown in Fig. 12–17b.

**Acceleration.** The acceleration of the particle is obtained by taking the first time derivative of Eq. 12–11 (or the second time derivative of Eq. 12–10). We have

\[
\mathbf{a} = \frac{d\mathbf{v}}{dt} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}
\]

(12–13)

where

\[
\begin{align*}
a_x &= \dot{v}_x = \ddot{x} \\
a_y &= \dot{v}_y = \ddot{y} \\
a_z &= \dot{v}_z = \ddot{z}
\end{align*}
\]

(12–14)

Here \( a_x, a_y, a_z \) represent, respectively, the first time derivatives of \( v_x = v_x(t), v_y = v_y(t), v_z = v_z(t) \), or the second time derivatives of the functions \( x = x(t), y = y(t), z = z(t) \).

The acceleration has a magnitude

\[
a = \sqrt{a_x^2 + a_y^2 + a_z^2}
\]

and a direction specified by the unit vector \( \mathbf{u}_a = \mathbf{a}/a \). Since \( \mathbf{a} \) represents the time rate of change in both the magnitude and direction of the velocity, in general \( \mathbf{a} \) will not be tangent to the path, Fig. 12–17c.
## Important Points

- Curvilinear motion can cause changes in both the magnitude and direction of the position, velocity, and acceleration vectors.
- The velocity vector is always directed tangent to the path.
- In general, the acceleration vector is not tangent to the path, but rather, it is tangent to the hodograph.
- If the motion is described using rectangular coordinates, then the components along each of the axes do not change direction, only their magnitude and sense (algebraic sign) will change.
- By considering the component motions, the change in magnitude and direction of the particle’s position and velocity are automatically taken into account.

## Procedure for Analysis

### Coordinate System.

- A rectangular coordinate system can be used to solve problems for which the motion can conveniently be expressed in terms of its \( x, y, z \) components.

### Kinematic Quantities.

- Since rectilinear motion occurs along each coordinate axis, the motion along each axis is found using \( v = ds/dt \) and \( a = dv/dt \); or in cases where the motion is not expressed as a function of time, the equation \( a \, ds = v \, dv \) can be used.
- In two dimensions, the equation of the path \( y = f(x) \) can be used to relate the \( x \) and \( y \) components of velocity and acceleration by applying the chain rule of calculus. A review of this concept is given in Appendix C.
- Once the \( x, y, z \) components of \( \mathbf{v} \) and \( \mathbf{a} \) have been determined, the magnitudes of these vectors are found from the Pythagorean theorem, Eq. B–3, and their coordinate direction angles from the components of their unit vectors, Eqs. B–4 and B–5.
EXAMPLE 12.9

At any instant the horizontal position of the weather balloon in Fig. 12–18a is defined by where \( t \) is in seconds. If the equation of the path is determine the magnitude and direction of the velocity and the acceleration when \( t = 2 \text{ s} \).

SOLUTION

Velocity. The velocity component in the \( x \) direction is

\[
v_x = \frac{d}{dt} (8t) = 8 \text{ ft/s}
\]

To find the relationship between the velocity components we will use the chain rule of calculus. (See Appendix A for a full explanation.)

\[
v_y = \frac{d}{dt} \left( \frac{x^2}{10} \right) = 2x\dot{x}/10 = 2(16)(8)/10 = 25.6 \text{ ft/s}
\]

When \( t = 2 \text{ s} \), the magnitude of velocity is therefore

\[
v = \sqrt{(8 \text{ ft/s})^2 + (25.6 \text{ ft/s})^2} = 26.8 \text{ ft/s} \quad \text{Ans.}
\]

The direction is tangent to the path, Fig. 12–18b, where

\[
\theta_y = \tan^{-1} \frac{v_y}{v_x} = \tan^{-1} \frac{25.6}{8} = 72.6^\circ \quad \text{Ans.}
\]

Acceleration. The relationship between the acceleration components is determined using the chain rule. (See Appendix C.) We have

\[
a_x = \dot{v}_x = \frac{d}{dt} (8) = 0
\]

\[
a_y = \dot{v}_y = \frac{d}{dt} \left( \frac{2x\dot{x}}{10} \right) = 2(\dot{x})\ddot{x}/10 + 2x(\dddot{x})/10
\]

\[
= 2(8^2)/10 + 2(16)(0)/10 = 12.8 \text{ ft/s}^2
\]

Thus,

\[
a = \sqrt{(0)^2 + (12.8)^2} = 12.8 \text{ ft/s}^2 \quad \text{Ans.}
\]

The direction of \( \mathbf{a} \), as shown in Fig. 12–18c, is

\[
\theta_a = \tan^{-1} \frac{12.8}{0} = 90^\circ \quad \text{Ans.}
\]

NOTE: It is also possible to obtain \( v_y \) and \( a_y \) by first expressing \( y = f(t) = (8t)^2/10 = 6.4t^2 \) and then taking successive time derivatives.
EXAMPLE 12.10

For a short time, the path of the plane in Fig. 12–19a is described by \( y = (0.001x^2) \) m. If the plane is rising with a constant velocity of 10 m/s, determine the magnitudes of the velocity and acceleration of the plane when it is at \( y = 100 \) m.

**SOLUTION**

When \( y = 100 \) m, then \( 100 = 0.001x^2 \) or \( x = 316.2 \) m. Also, since \( v_y = 10 \) m/s, then

\[
y = v_y t; \quad 100 \text{ m} = (10 \text{ m/s}) t \quad t = 10 \text{ s}
\]

**Velocity.** Using the chain rule (see Appendix C) to find the relationship between the velocity components, we have

\[
v_y = \frac{dy}{dt} = \frac{d}{dt}(0.001x^2) = 0.002x \frac{dx}{dt} = 0.002xv_x
\]

Thus

\[
10 \text{ m/s} = 0.002(316.2 \text{ m})(v_x) \Rightarrow v_x = 15.81 \text{ m/s}
\]

The magnitude of the velocity is therefore

\[
v = \sqrt{v_x^2 + v_y^2} = \sqrt{(15.81 \text{ m/s})^2 + (10 \text{ m/s})^2} = 18.7 \text{ m/s} \quad \text{Ans.}
\]

**Acceleration.** Using the chain rule, the time derivative of Eq. (1) gives the relation between the acceleration components.

\[
a_y = \frac{dv_y}{dt} = 0.002xv_x + 0.002x\frac{dv_x}{dt} = 0.002(v_x^2 + xa_x)
\]

When \( x = 316.2 \) m, \( v_x = 15.81 \text{ m/s}, \frac{dv_x}{dt} = a_x = 0 \),

\[
0 = 0.002((15.81 \text{ m/s})^2 + 316.2 \text{ m}(a_x)) \Rightarrow a_x = -0.791 \text{ m/s}^2
\]

The magnitude of the plane’s acceleration is therefore

\[
a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-0.791 \text{ m/s}^2)^2 + (0 \text{ m/s}^2)^2} = 0.791 \text{ m/s}^2 \quad \text{Ans.}
\]

These results are shown in Fig. 12–19b.
12.6 Motion of a Projectile

The free-flight motion of a projectile is often studied in terms of its rectangular components. To illustrate the kinematic analysis, consider a projectile launched at point \((x_0, y_0)\), with an initial velocity of \(v_0\), having components \((v_0)_x\) and \((v_0)_y\), Fig. 12–20. When air resistance is neglected, the only force acting on the projectile is its weight, which causes the projectile to have a constant downward acceleration of approximately \(a_c = g = 9.81 \text{ m/s}^2\) or \(g = 32.2 \text{ ft/s}^2\).

### Horizontal Motion

Since \(a_x = 0\), application of the constant acceleration equations, 12–4 to 12–6, yields

1. \[ v = v_0 + a_x t; \quad v_x = (v_0)_x \]
2. \[ x = x_0 + v_0 t + \frac{1}{2} a_x t^2; \quad x = x_0 + (v_0)_x t \]
3. \[ v_x^2 = v_0^2 + 2a_x (x - x_0); \quad v_x = (v_0)_x \]

The first and last equations indicate that the horizontal component of velocity always remains constant during the motion.

### Vertical Motion

Since the positive \(y\) axis is directed upward, then \(a_y = -g\). Applying Eqs. 12–4 to 12–6, we get

1. \[ v = v_0 + a_y t; \quad v_y = (v_0)_y - gt \]
2. \[ y = y_0 + v_0 t + \frac{1}{2} a_y t^2; \quad y = y_0 + (v_0)_y t - \frac{1}{2} gt^2 \]
3. \[ v_y^2 = v_0^2 + 2a_y (y - y_0); \quad v_y = (v_0)_y - 2g(y - y_0) \]

Recall that the last equation can be formulated on the basis of eliminating the time \(t\) from the first two equations, and therefore only two of the above three equations are independent of one another.

* This assumes that the earth’s gravitational field does not vary with altitude.

Each picture in this sequence is taken after the same time interval. The red ball falls from rest, whereas the yellow ball is given a horizontal velocity when released. Both balls accelerate downward at the same rate, and so they remain at the same elevation at any instant. This acceleration causes the difference in elevation between the balls to increase between successive photos. Also, note the horizontal distance between successive photos of the yellow ball is constant since the velocity in the horizontal direction remains constant.
To summarize, problems involving the motion of a projectile can have at most three unknowns since only three independent equations can be written; that is, one equation in the horizontal direction and two in the vertical direction. Once \( v_x \) and \( v_y \) are obtained, the resultant velocity \( \mathbf{v} \), which is always tangent to the path, can be determined by the vector sum as shown in Fig. 12–20.

### Procedure for Analysis

**Coordinate System.**
- Establish the fixed \( x, y \) coordinate axes and sketch the trajectory of the particle. Between any two points on the path specify the given problem data and identify the three unknowns. In all cases the acceleration of gravity acts downward and equals 9.81 m/s\(^2\) or 32.2 ft/s\(^2\). The particle’s initial and final velocities should be represented in terms of their \( x \) and \( y \) components.
- Remember that positive and negative position, velocity, and acceleration components always act in accordance with their associated coordinate directions.

**Kinematic Equations.**
- Depending upon the known data and what is to be determined, a choice should be made as to which three of the following four equations should be applied between the two points on the path to obtain the most direct solution to the problem.

**Horizontal Motion.**
- The velocity in the horizontal or \( x \) direction is constant, i.e.,
  \[ v_x = (v_0)_x \]
  \[ x = x_0 + (v_0)_x t \]

**Vertical Motion.**
- In the vertical or \( y \) direction only two of the following three equations can be used for solution.
  \[ v_y = (v_0)_y + at \]
  \[ y = y_0 + (v_0)_y t + \frac{1}{2}at^2 \]
  \[ v_y^2 = (v_0)_y^2 + 2a(y - y_0) \]

For example, if the particle’s final velocity \( v_y \) is not needed, then the first and third of these equations will not be useful.

Gravel falling off the end of this conveyor belt follows a path that can be predicted using the equations of constant acceleration. In this way the location of the accumulated pile can be determined. Rectangular coordinates are used for the analysis since the acceleration is only in the vertical direction.
EXAMPLE 12.11

A sack slides off the ramp, shown in Fig. 12–21, with a horizontal velocity of 12 m/s. If the height of the ramp is 6 m from the floor, determine the time needed for the sack to strike the floor and the range $R$ where sacks begin to pile up.

![Fig. 12–21](image)

**SOLUTION**

**Coordinate System.** The origin of coordinates is established at the beginning of the path, point $A$, Fig. 12–21. The initial velocity of a sack has components $(v_A)_x = 12$ m/s and $(v_A)_y = 0$. Also, between points $A$ and $B$ the acceleration is $a_y = -9.81$ m/s$^2$. Since $(v_B)_x = (v_A)_x = 12$ m/s, the three unknowns are $(v_B)_y$, $R$, and the time of flight $t_{AB}$. Here we do not need to determine $(v_B)_y$.

**Vertical Motion.** The vertical distance from $A$ to $B$ is known, and therefore we can obtain a direct solution for $t_{AB}$ by using the equation

$$y_B = y_A + (v_A)_y t_{AB} + \frac{1}{2}a_y t_{AB}^2$$

$$-6 = 0 + 0 + \frac{1}{2}(-9.81 \text{ m/s}^2) t_{AB}^2$$

$$t_{AB} = 1.11 \text{ s} \quad \text{Ans.}$$

**Horizontal Motion.** Since $t_{AB}$ has been calculated, $R$ is determined as follows:

$$x_B = x_A + (v_A)_x t_{AB}$$

$$R = 0 + 12 \text{ m/s} \times (1.11 \text{ s})$$

$$R = 13.3 \text{ m} \quad \text{Ans.}$$

**NOTE:** The calculation for $t_{AB}$ also indicates that if a sack were released from rest at $A$, it would take the same amount of time to strike the floor at $C$, Fig. 12–21.
EXAMPLE 12.12

The chipping machine is designed to eject wood chips at \( v_O = 25 \text{ ft/s} \) as shown in Fig. 12–22. If the tube is oriented at \( 30^\circ \) from the horizontal, determine how high, \( h \), the chips strike the pile if at this instant they land on the pile 20 ft from the tube.

\[ v_O = 25 \text{ ft/s} \]

**SOLUTION**

**Coordinate System.** When the motion is analyzed between points \( O \) and \( A \), the three unknowns are the height \( h \), time of flight \( t_{OA} \), and vertical component of velocity \( (v_A)_y \). [Note that \( (v_A)_x = (v_O)_x \).] With the origin of coordinates at \( O \), Fig. 12–22, the initial velocity of a chip has components of

\[
(v_O)_x = (25 \cos 30^\circ) \text{ ft/s} = 21.65 \text{ ft/s}
\]

\[
(v_O)_y = (25 \sin 30^\circ) \text{ ft/s} = 12.5 \text{ ft/s}
\]

Also, \((v_A)_x = (v_O)_x = 21.65 \text{ ft/s}\) and \( a_y = -32.2 \text{ ft/s}^2\). Since we do not need to determine \((v_A)_y\), we have

**Horizontal Motion.**

\[
(-) \quad x_A = x_O + (v_O)_x t_{OA}
\]

\[
20 \text{ ft} = 0 + (21.65 \text{ ft/s}) t_{OA}
\]

\[ t_{OA} = 0.9238 \text{ s} \]

**Vertical Motion.** Relating \( t_{OA} \) to the initial and final elevations of a chip, we have

\[
(+) \quad y_A = y_O + (v_O)_y t_{OA} + \frac{1}{2} a_y t_{OA}^2
\]

\[
h - 4 \text{ ft} = 0 + (12.5 \text{ ft/s})(0.9238 \text{ s}) + \frac{1}{2}(-32.2 \text{ ft/s}^2)(0.9238 \text{ s})^2
\]

\[ h = 1.81 \text{ ft} \quad \text{Ans.} \]

**NOTE:** We can determine \((v_A)_y\) by using \((v_A)_y = (v_O)_y + a_y t_{OA}\).
EXAMPLE 12.13

The track for this racing event was designed so that riders jump off the slope at 30°, from a height of 1 m. During a race it was observed that the rider shown in Fig. 12–23 remained in mid air for 1.5 s. Determine the speed at which he was traveling off the ramp, the horizontal distance he travels before striking the ground, and the maximum height he attains. Neglect the size of the bike and rider.

SOLUTION

Coordinate System. As shown in Fig. 12–23, the origin of the coordinates is established at A. Between the end points of the path AB the three unknowns are the initial speed \( v_A \), range \( R \), and the vertical component of velocity \( (v_B)_y \).

Vertical Motion. Since the time of flight and the vertical distance between the ends of the path are known, we can determine \( v_A \).

\[
(+ \uparrow) \quad y_B = y_A + (v_A)t_{AB} + \frac{1}{2}a_y t_{AB}^2 \\
-1 \text{ m} = 0 + v_A \sin 30° (1.5 \text{ s}) + \frac{1}{2}(-9.81 \text{ m/s}^2) (1.5 \text{ s})^2 \\
v_A = 13.38 \text{ m/s} = 13.4 \text{ m/s} \quad \text{Ans.}
\]

Horizontal Motion. The range \( R \) can now be determined.

\[
(\Rightarrow) \quad x_B = x_A + (v_A)t_{AB} \\
R = 0 + 13.38 \cos 30° \text{ m/s} (1.5 \text{ s}) \\
= 17.4 \text{ m} \quad \text{Ans.}
\]

In order to find the maximum height \( h \) we will consider the path AC, Fig. 12–23b. Here the three unknowns are the time of flight \( t_{AC} \), the horizontal distance from A to C, and the height \( h \). At the maximum height \((v_C)_y = 0\), and since \( v_A \) is known, we can determine \( h \) directly without considering \( t_{AC} \) using the following equation.

\[
(v_C)^2_y = (v_A)^2 + 2a_y[y_C - y_A] \\
0^2 = (13.38 \sin 30° \text{ m/s})^2 + 2(-9.81 \text{ m/s}^2)[(h - 1 \text{ m}) - 0] \\
h = 3.28 \text{ m} \quad \text{Ans.}
\]

NOTE: Show that the bike will strike the ground at B with a velocity having components of \( (v_B)_x = 11.6 \text{ m/s} \), \( (v_B)_y = 8.02 \text{ m/s} \).
F12–15. If the $x$ and $y$ components of a particle’s velocity are $v_x = (32t)$ m/s and $v_y = 8$ m/s, determine the equation of the path $y = f(x)$. $x = 0$ and $y = 0$ when $t = 0$.

F12–16. A particle is traveling along the straight path. If its position along the $x$ axis is $x = (8t)$ m, where $t$ is in seconds, determine its speed when $t = 2$ s.

F12–18. A particle travels along a straight-line path $y = 0.5x$. If the $x$ component of the particle’s velocity is $v_x = (2t^2)$ m/s, where $t$ is in seconds, determine the magnitude of the particle’s velocity and acceleration when $t = 4$ s.

F12–19. A particle is traveling along the parabolic path $y = 0.25x^2$. If $x = (2t^2)$ m, where $t$ is in seconds, determine the magnitude of the particle’s velocity and acceleration when $t = 2$ s.

F12–20. The position of a box sliding down the spiral can be described by $r = [2 \sin (2t)i + 2 \cos (2t)j - 2t^2k]$ ft, where $t$ is in seconds and the arguments for the sine and cosine are in radians. Determine the velocity and acceleration of the box when $t = 2$ s.

F12–17. A particle is constrained to travel along the path. If $x = (4t^4)$ m, where $t$ is in seconds, determine the magnitude of the particle’s velocity and acceleration when $t = 0.5$ s.
12.6 MOTION OF A PROJECTILE

F12–21. The ball is kicked from point A with the initial velocity \( v_A = 10 \text{ m/s} \). Determine the maximum height \( h \) it reaches.

F12–22. The ball is kicked from point A with the initial velocity \( v_A = 10 \text{ m/s} \). Determine the range \( R \), and the speed when the ball strikes the ground.

\[ \begin{align*}
&v_A = 10 \text{ m/s} \\
&x_B \\
&h \\
&30^\circ \\
&y
\end{align*} \]

F12–21/22

F12–23. Determine the speed at which the basketball at A must be thrown at the angle of 30° so that it makes it to the basket at B.

\[ \begin{align*}
&v_A = 10 \text{ m/s} \\
&30^\circ \\
&x_B \\
&h \\
&3 \text{ m} \\
&1.5 \text{ m} \\
&10 \text{ m} \\
&y
\end{align*} \]

F12–23

F12–24. Water is sprayed at an angle of 90° from the slope at 20 m/s. Determine the range \( R \).

\[ \begin{align*}
&v_B = 20 \text{ m/s} \\
&R \\
&y
\end{align*} \]

F12–24

F12–25. A ball is thrown from A. If it is required to clear the wall at B, determine the minimum magnitude of its initial velocity \( v_A \).

\[ \begin{align*}
&v_A = 150 \text{ m/s} \\
&x_B \\
&8 \text{ ft} \\
&3 \text{ ft} \\
&12 \text{ ft} \\
&y
\end{align*} \]

F12–25

F12–26. A projectile is fired with an initial velocity of \( v_A = 150 \text{ m/s} \) off the roof of the building. Determine the range \( R \) where it strikes the ground at B.

\[ \begin{align*}
&v_A = 150 \text{ m/s} \\
&x_B \\
&B \\
&R \\
&150 \text{ m} \\
&y
\end{align*} \]

F12–26
12–71. The position of a particle is \( \mathbf{r} = \left\{ (3t^3 - 2t)i - (4t^{1/2} + t)j + (3t^2 - 2)k \right\} \text{ m, where } t \text{ is in seconds. Determine the magnitude of the particle’s velocity and acceleration when } t = 2 \text{ s.}

*12–72. The velocity of a particle is \( \mathbf{v} = \{3i + (6 - 2t)j\} \text{ m/s, where } t \text{ is in seconds. If } \mathbf{r} = 0 \text{ when } t = 0, \text{ determine the displacement of the particle during the time interval } t = 1 \text{ s to } t = 3 \text{ s.}

• 12–73. A particle travels along the parabolic path \( y = bx^2. \) If its component of velocity along the \( y \) axis is \( v_y = ct^2, \) determine the \( x \) and \( y \) components of the particle’s acceleration. Here \( b \) and \( c \) are constants.

12–74. The velocity of a particle is given by \( \mathbf{v} = \{16t^2i + 4t^3j + (5t + 2)k\} \text{ m/s, where } t \text{ is in seconds. If the particle is at the origin when } t = 0, \text{ determine the magnitude of the particle’s acceleration when } t = 2 \text{ s. Also, what is the } x, y, z \text{ coordinate position of the particle at this instant?}

12–75. A particle travels along the circular path \( x^2 + y^2 = r^2. \) If the \( y \) component of the particle’s velocity is \( v_y = 2r \cos 2t, \) determine the \( x \) and \( y \) components of its acceleration at any instant.

*12–76. The box slides down the slope described by the equation \( y = (0.05x^2) \text{ m, where } x \text{ is in meters. If the box has } x \text{ components of velocity and acceleration of } v_x = -3 \text{ m/s and } a_x = -1.5 \text{ m/s}^2 \text{ at } x = 5 \text{ m, determine the } y \text{ components of the velocity and the acceleration of the box at this instant.}

12–77. The position of a particle is defined by \( \mathbf{r} = \{5 \cos 2t i + 4 \sin 2t j\} \text{ m, where } t \text{ is in seconds and the arguments for the sine and cosine are given in radians. Determine the magnitudes of the velocity and acceleration of the particle when } t = 1 \text{ s. Also, prove that the path of the particle is elliptical.}

12–78. Pegs \( A \) and \( B \) are restricted to move in the elliptical slots due to the motion of the slotted link. If the link moves with a constant speed of 10 m/s, determine the magnitude of the velocity and acceleration of peg \( A \) when \( x = 1 \text{ m.}

12–79. A particle travels along the path \( y^2 = 4x \) with a constant speed of \( v = 4 \text{ m/s. Determine the } x \text{ and } y \text{ components of the particle’s velocity and acceleration when the particle is at } x = 4 \text{ m.}

*12–80. The van travels over the hill described by \( y = (-1.5 \times 10^{-3}) x^2 + 15 \) ft. If it has a constant speed of 75 ft/s, determine the \( x \) and \( y \) components of the van’s velocity and acceleration when \( x = 50 \text{ ft.} \)
12.81. A particle travels along the circular path from $A$ to $B$ in 1 s. If it takes 3 s for it to go from $A$ to $C$, determine its average velocity when it goes from $B$ to $C$.

![Diagram](Prob.12-81)

12.82. A car travels east 2 km for 5 minutes, then north 3 km for 8 minutes, and then west 4 km for 10 minutes. Determine the total distance traveled and the magnitude of displacement of the car. Also, what is the magnitude of the average velocity and the average speed?

12.83. The roller coaster car travels down the helical path at constant speed such that the parametric equations that define its position are $x = c \sin kt$, $y = c \cos kt$, $z = h - bt$, where $c$, $h$, and $b$ are constants. Determine the magnitudes of its velocity and acceleration.

![Diagram](Prob.12-83)

12.84. The path of a particle is defined by $y^2 = 4kx$, and the component of velocity along the $y$ axis is $v_y = ct$, where both $k$ and $c$ are constants. Determine the $x$ and $y$ components of acceleration when $y = y_0$.

12.85. A particle moves along the curve $y = x - (x^2/400)$, where $x$ and $y$ are in ft. If the velocity component in the $x$ direction is $v_x = 2$ ft/s and remains constant, determine the magnitudes of the velocity and acceleration when $x = 20$ ft.

12.86. The motorcycle travels with constant speed $v_0$ along the path that, for a short distance, takes the form of a sine curve. Determine the $x$ and $y$ components of its velocity at any instant on the curve.

![Diagram](Prob.12-86)

12.87. The skateboard rider leaves the ramp at $A$ with an initial velocity $v_A$ at a 30° angle. If he strikes the ground at $B$, determine $v_A$ and the time of flight.

![Diagram](Prob.12-87)

12.88. The pitcher throws the baseball horizontally with a speed of 140 ft/s from a height of 5 ft. If the batter is 60 ft away, determine the time for the ball to arrive at the batter and the height $h$ at which it passes the batter.

![Diagram](Prob.12-88)
12–89. The ball is thrown off the top of the building. If it strikes the ground at $B$ in 3 s, determine the initial velocity $v_A$ and the inclination angle $\theta_A$ at which it was thrown. Also, find the magnitude of the ball’s velocity when it strikes the ground.

12–90. A projectile is fired with a speed of $v = 60 \text{ m/s}$ at an angle of $60^\circ$. A second projectile is then fired with the same speed 0.5 s later. Determine the angle $\theta$ of the second projectile so that the two projectiles collide. At what position $(x, y)$ will this happen?

12–91. The fireman holds the hose at an angle $\theta = 30^\circ$ with horizontal, and the water is discharged from the hose at $A$ with a speed of $v_A = 40 \text{ ft/s}$. If the water stream strikes the building at $B$, determine his two possible distances $s$ from the building.

12–92. Water is discharged from the hose with a speed of $40 \text{ ft/s}$. Determine the two possible angles $\theta$ the fireman can hold the hose so that the water strikes the building at $B$. Take $s = 20 \text{ ft}$.
12.93. The pitching machine is adjusted so that the baseball is launched with a speed of \( v_A = 30 \text{ m/s} \). If the ball strikes the ground at \( B \), determine the two possible angles \( \theta_A \) at which it was launched.

\[ \text{Prob. 12–93} \]

12.94. It is observed that the time for the ball to strike the ground at \( B \) is 2.5 s. Determine the speed \( v_A \) and angle \( \theta_A \) at which the ball was thrown.

\[ \text{Prob. 12–94} \]

12.95. If the motorcycle leaves the ramp traveling at 110 ft/s, determine the height \( h \) ramp \( B \) must have so that the motorcycle lands safely.

\[ \text{Prob. 12–95} \]

12.96. The baseball player \( A \) hits the baseball with \( v_A = 40 \text{ ft/s} \) and \( \theta_A = 60^\circ \). When the ball is directly above of player \( B \) he begins to run under it. Determine the constant speed \( v_B \) and the distance \( d \) at which \( B \) must run in order to make the catch at the same elevation at which the ball was hit.

\[ \text{Prob. 12–96} \]

12.97. A boy throws a ball at \( O \) in the air with a speed \( v_0 \) at an angle \( \theta_1 \). If he then throws another ball with the same speed \( v_0 \) at an angle \( \theta_2 < \theta_1 \), determine the time between the throws so that the balls collide in mid air at \( B \).

\[ \text{Prob. 12–97} \]
12–98. The golf ball is hit at A with a speed of $v_A = 40 \text{ m/s}$ and directed at an angle of $30^\circ$ with the horizontal as shown. Determine the distance $d$ where the ball strikes the slope at B.

12–99. If the football is kicked at the $45^\circ$ angle, determine its minimum initial speed $v_A$ so that it passes over the goal post at C. At what distance $s$ from the goal post will the football strike the ground at B?

*12–100. The velocity of the water jet discharging from the orifice can be obtained from $v = \sqrt{2gh}$, where $h = 2 \text{ m}$ is the depth of the orifice from the free water surface. Determine the time for a particle of water leaving the orifice to reach point B and the horizontal distance $x$ where it hits the surface.

12–101. A projectile is fired from the platform at B. The shooter fires his gun from point A at an angle of $30^\circ$. Determine the muzzle speed of the bullet if it hits the projectile at C.
12–102. A golf ball is struck with a velocity of 80 ft/s as shown. Determine the distance \( d \) to where it will land.

![Diagram of a golf ball being struck](image)

**Prob. 12–102**

12–103. The football is to be kicked over the goalpost, which is 15 ft high. If its initial speed is \( v_A = 80 \) ft/s, determine if it makes it over the goalpost, and if so, by how much, \( h \).

*12–104. The football is kicked over the goalpost with an initial velocity of \( v_A = 80 \) ft/s as shown. Determine the point \( B (x, y) \) where it strikes the bleachers.

![Diagram of a football being kicked](image)

**Probs. 12–103/104**

12–105. The boy at \( A \) attempts to throw a ball over the roof of a barn with an initial speed of \( v_A = 15 \) m/s. Determine the angle \( \theta_A \) at which the ball must be thrown so that it reaches its maximum height at \( C \). Also, find the distance \( d \) where the boy should stand to make the throw.

![Diagram of a boy throwing a ball](image)

**Prob. 12–105**

12–106. The boy at \( A \) attempts to throw a ball over the roof of a barn such that it is launched at an angle \( \theta_A = 40^\circ \). Determine the minimum speed \( v_A \) at which he must throw the ball so that it reaches its maximum height at \( C \). Also, find the distance \( d \) where the boy must stand so that he can make the throw.

![Diagram of a boy throwing a ball at an angle](image)

**Prob. 12–106**
12–107. The fireman wishes to direct the flow of water from his hose to the fire at B. Determine two possible angles \( \theta_1 \) and \( \theta_2 \) at which this can be done. Water flows from the hose at \( v_A = 80 \text{ ft/s} \).

![Diagram of Prob. 12–107]

\( \text{Prob. 12–107} \)

12–108. Small packages traveling on the conveyor belt fall off into a 1-m-long loading car. If the conveyor is running at a constant speed of \( v_C = 2 \text{ m/s} \), determine the smallest and largest distance \( R \) at which the end \( A \) of the car may be placed from the conveyor so that the packages enter the car.

![Diagram of Prob. 12–108]

\( \text{Prob. 12–108} \)

12–109. Determine the horizontal velocity \( v_A \) of a tennis ball at \( A \) so that it just clears the net at \( B \). Also, find the distance \( s \) where the ball strikes the ground.

![Diagram of Prob. 12–109]

\( \text{Prob. 12–109} \)

12–110. It is observed that the skier leaves the ramp \( A \) at an angle \( \theta_A = 25^\circ \) with the horizontal. If he strikes the ground at \( B \), determine his initial speed \( v_A \) and the time of flight \( t_{AB} \).

![Diagram of Prob. 12–110]

\( \text{Prob. 12–110} \)
12.7 Curvilinear Motion: Normal and Tangential Components

When the path along which a particle travels is known, then it is often convenient to describe the motion using \( n \) and \( t \) coordinate axes which act normal and tangent to the path, respectively, and at the instant considered have their origin located at the particle.

**Planar Motion.** Consider the particle shown in Fig. 12–24a, which moves in a plane along a fixed curve, such that at a given instant it is at position \( s \), measured from point \( O \). We will now consider a coordinate system that has its origin at a fixed point on the curve, and at the instant considered this origin happens to coincide with the location of the particle. The \( t \) axis is tangent to the curve at the point and is positive in the direction of increasing \( s \). We will designate this positive direction with the unit vector \( \mathbf{u}_t \). A unique choice for the normal axis can be made by noting that geometrically the curve is constructed from a series of differential arc segments \( ds \), Fig. 12–24b. Each segment \( ds \) is formed from the arc of an associated circle having a radius of curvature \( \rho \) (rho) and center of curvature \( O' \). The normal axis \( n \) is perpendicular to the \( t \) axis with its positive sense directed toward the center of curvature \( O' \), Fig. 12–24a. This positive direction, which is always on the concave side of the curve, will be designated by the unit vector \( \mathbf{u}_n \). The plane which contains the \( n \) and \( t \) axes is referred to as the embracing or osculating plane, and in this case it is fixed in the plane of motion.*

**Velocity.** Since the particle moves, \( s \) is a function of time. As indicated in Sec. 12.4, the particle’s velocity \( \mathbf{v} \) has a direction that is always tangent to the path, Fig. 12–24c, and a magnitude that is determined by taking the time derivative of the path function \( s = s(t) \), i.e., \( v = ds/dt \) (Eq. 12–8). Hence

\[
\mathbf{v} = v \mathbf{u}_t
\]

where

\[
v = \dot{s} \tag{12–16}
\]

---

*The osculating plane may also be defined as the plane which has the greatest contact with the curve at a point. It is the limiting position of a plane contacting both the point and the arc segment \( ds \). As noted above, the osculating plane is always coincident with a plane curve; however, each point on a three-dimensional curve has a unique osculating plane.
Acceleration. The acceleration of the particle is the time rate of change of the velocity. Thus,

$$\mathbf{a} = \mathbf{v} = \dot{\mathbf{r}} + \mathbf{v} \mathbf{u}_n$$  \hspace{1cm} (12–17)

In order to determine the time derivative \(\dot{\mathbf{u}}_i\), note that as the particle moves along the arc \(ds\) in time \(dt\), \(\mathbf{u}_i\) preserves its magnitude of unity; however, its direction changes, and becomes \(\mathbf{u}'_i\), Fig. 12–24. As shown in Fig. 12–24e, we require \(\mathbf{u}'_i = \mathbf{u}_i + d\mathbf{u}_i\). Here \(d\mathbf{u}_i\) stretches between the arrowheads of \(\mathbf{u}_i\) and \(\mathbf{u}'_i\), which lie on an infinitesimal arc of radius \(\rho = 1\). Hence, \(d\mathbf{u}_i\) has a magnitude of \(du_i = (1) \, d\theta\), and its direction is defined by \(\mathbf{u}_n\). Consequently, \(d\mathbf{u}_i = d\theta \mathbf{u}_n\), and therefore the time derivative becomes \(\dot{\mathbf{u}}_i = \dot{\theta} \mathbf{u}_n\). Since \(ds = \rho d\theta\), Fig. 12–24d, then \(\dot{\theta} = \dot{s}/\rho\), and therefore

$$\dot{\mathbf{u}}_i = \dot{\theta} \mathbf{u}_n = \frac{\dot{s}}{\rho} \mathbf{u}_n = \frac{v}{\rho} \mathbf{u}_n$$

Substituting into Eq. 12–17, \(\mathbf{a}\) can be written as the sum of its two components,

$$\mathbf{a} = a_r \mathbf{u}_r + a_n \mathbf{u}_n$$  \hspace{1cm} (12–18)

where

$$a_r = \dot{v}$$  \hspace{1cm} or  \hspace{1cm} $$a_r ds = v \, dv$$  \hspace{1cm} (12–19)

and

$$a_n = \frac{v^2}{\rho}$$  \hspace{1cm} (12–20)

These two mutually perpendicular components are shown in Fig. 12–24f. Therefore, the magnitude of acceleration is the positive value of

$$a = \sqrt{a_r^2 + a_n^2}$$  \hspace{1cm} (12–21)
To better understand these results, consider the following two special cases of motion.

1. If the particle moves along a straight line, then $\rho \to \infty$ and from Eq. 12–20, $a_n = 0$. Thus $a = a_t = \dot{v}$, and we can conclude that the **tangential component of acceleration** represents the time rate of change in the magnitude of the velocity.

2. If the particle moves along a curve with a constant speed, then $a_t = \dot{v} = 0$ and $a = a_n = v^2/\rho$. Therefore, the **normal component of acceleration** represents the time rate of change in the direction of the velocity. Since $a_n$ always acts towards the center of curvature, this component is sometimes referred to as the *centripetal* (or center seeking) acceleration.

As a result of these interpretations, a particle moving along the curved path in Fig. 12–25 will have accelerations directed as shown.

**Fig. 12–25**

**Three-Dimensional Motion.** If the particle moves along a space curve, Fig. 12–26, then at a given instant the $t$ axis is uniquely specified; however, an infinite number of straight lines can be constructed normal to the tangent axis. As in the case of planar motion, we will choose the positive $n$ axis directed toward the path’s center of curvature $O'$. This axis is referred to as the *principal normal* to the curve. With the $n$ and $t$ axes so defined, Eqs. 12–15 through 12–21 can be used to determine $\mathbf{v}$ and $\mathbf{a}$. Since $\mathbf{u}_t$ and $\mathbf{u}_n$ are always perpendicular to one another and lie in the osculating plane, for spatial motion a third unit vector, $\mathbf{u}_b$, defines the *binormal axis* $b$ which is perpendicular to $\mathbf{u}_t$ and $\mathbf{u}_n$, Fig. 12–26.

Since the three unit vectors are related to one another by the vector cross product, e.g., $\mathbf{u}_b = \mathbf{u}_t \times \mathbf{u}_n$, Fig. 12–26, it may be possible to use this relation to establish the direction of one of the axes, if the directions of the other two are known. For example, if no motion occurs in the $\mathbf{u}_b$ direction, and this direction and $\mathbf{u}_t$ are known, then $\mathbf{u}_n$ can be determined, where in this case $\mathbf{u}_n = \mathbf{u}_b \times \mathbf{u}_t$, Fig. 12–26. Remember, though, that $\mathbf{u}_n$ is always on the concave side of the curve.

**Fig. 12–26**
CHAPTER 12 KINEMATICS OF A PARTICLE

Procedure for Analysis

Coordinate System.
- Provided the path of the particle is known, we can establish a set of n and t coordinates having a fixed origin, which is coincident with the particle at the instant considered.
- The positive tangent axis acts in the direction of motion and the positive normal axis is directed toward the path’s center of curvature.

Velocity.
- The particle’s velocity is always tangent to the path.
- The magnitude of velocity is found from the time derivative of the path function.
  \[ v = \dot{s} \]

Tangential Acceleration.
- The tangential component of acceleration is the result of the time rate of change in the magnitude of velocity. This component acts in the positive s direction if the particle’s speed is increasing or in the opposite direction if the speed is decreasing.
- The relations between \( a_t \), \( v \), \( t \) and \( s \) are the same as for rectilinear motion, namely,
  \[ a_t = \dot{v} \quad a_t \, ds = v \, dv \]
- If \( a_t \) is constant, \( a_t = (a_t)_c \), the above equations, when integrated, yield
  \[ s = s_0 + v_0t + \frac{1}{2}(a_t)_ct^2 \]
  \[ v = v_0 + (a_t)_ct \]
  \[ v^2 = v_0^2 + 2(a_t)_c(s - s_0) \]

Normal Acceleration.
- The normal component of acceleration is the result of the time rate of change in the direction of the velocity. This component is always directed toward the center of curvature of the path, i.e., along the positive n axis.
- The magnitude of this component is determined from
  \[ a_n = \frac{v^2}{\rho} \]
- If the path is expressed as \( y = f(x) \), the radius of curvature \( \rho \) at any point on the path is determined from the equation
  \[ \rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|} \]
- The derivation of this result is given in any standard calculus text.
EXAMPLE 12.14

When the skier reaches point A along the parabolic path in Fig. 12–27a, he has a speed of 6 m/s which is increasing at 2 m/s². Determine the direction of his velocity and the direction and magnitude of his acceleration at this instant. Neglect the size of the skier in the calculation.

SOLUTION

Coordinate System. Although the path has been expressed in terms of its x and y coordinates, we can still establish the origin of the n, t axes at the fixed point A on the path and determine the components of $v$ and $a$ along these axes, Fig. 12–27a.

Velocity. By definition, the velocity is always directed tangent to the path. Since $y = \frac{1}{20}x²$, $dy/dx = \frac{1}{10}x$, then at $x = 10$ m, $dy/dx = 1$. Hence, at A, $v$ makes an angle of $\theta = \tan^{-1}1 = 45°$ with the x axis, Fig. 12–27a. Therefore,

$$v_A = 6 \text{ m/s} \quad 45° \hat{\theta}$$

Ans.

The acceleration is determined from $a = \dot{v}u_r + (v^2/\rho)u_n$. However, it is first necessary to determine the radius of curvature of the path at A (10 m, 5 m). Since $d²y/dx² = \frac{1}{10}$, then

$$\rho = \left[1 + (dy/dx)^2\right]^{3/2} \left[1 + \left(\frac{1}{10}x\right)^2\right]^{3/2} \bigg|_{x=10 \text{ m}} = 28.28 \text{ m}$$

The acceleration becomes

$$a_A = \dot{v}u_r + \frac{v^2}{\rho}u_n$$
$$= 2u_r + \frac{(6 \text{ m/s})²}{28.28 \text{ m}}u_n$$
$$= \{2u_r + 1.273u_n\} \text{ m/s}²$$

As shown in Fig. 12–27b,

$$a = \sqrt{(2 \text{ m/s}²)^² + (1.273 \text{ m/s}²)^²} = 2.37 \text{ m/s}²$$

$$\phi = \tan^{-1} \frac{2}{1.273} = 57.5°$$

Thus, $45° + 90° + 57.5° - 180° = 12.5°$ so that,

$$a = 2.37 \text{ m/s}² \quad 12.5° \hat{\theta}$$

Ans.

NOTE: By using n, t coordinates, we were able to readily solve this problem through the use of Eq. 12–18, since it accounts for the separate changes in the magnitude and direction of $v$. Fig. 12–27
EXAMPLE 12.15

A race car $C$ travels around the horizontal circular track that has a radius of 300 ft, Fig. 12–28. If the car increases its speed at a constant rate of $7 \text{ ft/s}^2$, starting from rest, determine the time needed for it to reach an acceleration of $8 \text{ ft/s}^2$. What is its speed at this instant?

![Fig. 12–28](image)

**SOLUTION**

**Coordinate System.** The origin of the $n$ and $t$ axes is coincident with the car at the instant considered. The $t$ axis is in the direction of motion, and the positive $n$ axis is directed toward the center of the circle. This coordinate system is selected since the path is known.

**Acceleration.** The magnitude of acceleration can be related to its components using $a = \sqrt{a_t^2 + a_n^2}$. Here $a_t = 7 \text{ ft/s}^2$. Since $a_n = v^2/r$, the velocity as a function of time must be determined first.

$$v = v_0 + (a_t)t$$

$$v = 0 + 7t$$

Thus

$$a_n = \frac{v^2}{r} = \frac{(7t)^2}{300} = 0.163t^2 \text{ ft/s}^2$$

The time needed for the acceleration to reach $8 \text{ ft/s}^2$ is therefore

$$a = \sqrt{a_t^2 + a_n^2}$$

$$8 \text{ ft/s}^2 = \sqrt{(7 \text{ ft/s}^2)^2 + (0.163t^2)^2}$$

Solving for the positive value of $t$ yields

$$0.163t^2 = \sqrt{(8 \text{ ft/s}^2)^2 - (7 \text{ ft/s}^2)^2}$$

$$t = 4.87 \text{ s} \quad \text{Ans.}$$

**Velocity.** The speed at time $t = 4.87 \text{ s}$ is

$$v = 7t = 7(4.87) = 34.1 \text{ ft/s} \quad \text{Ans.}$$

**NOTE:** Remember the velocity will always be tangent to the path, whereas the acceleration will be directed within the curvature of the path.
EXAMPLE 12.16

The boxes in Fig. 12–29a travel along the industrial conveyor. If a box as in Fig. 12–29b starts from rest at A and increases its speed such that $a_t = (0.2t)$ m/s$^2$, where $t$ is in seconds, determine the magnitude of its acceleration when it arrives at point B.

SOLUTION

Coordinate System. The position of the box at any instant is defined from the fixed point A using the position or path coordinate $s$, Fig. 12–29b. The acceleration is to be determined at B, so the origin of the $n$, $t$ axes is at this point.

Acceleration. To determine the acceleration components $a_t = \dot{v}$ and $a_n = v^2/\rho$, it is first necessary to formulate $v$ and $\dot{v}$ so that they may be evaluated at B. Since $v_A = 0$ when $t = 0$, then

$$a_t = \dot{v} = 0.2t \quad (1)$$

$$\int_0^v \dot{v} \, dv = \int_0^t 0.2t \, dt$$

$$v = 0.1t^2 \quad (2)$$

The time needed for the box to reach point B can be determined by realizing that the position of B is $s_B = 3 + 2\pi(2)/4 = 6.142$ m, Fig. 12–29b, and since $s_A = 0$ when $t = 0$ we have

$$v = \frac{ds}{dt} = 0.1t^2$$

$$\int_0^{6.142} ds = \int_0^{t_B} 0.1t^2 \, dt$$

$$6.142 \, m = 0.0333 t_B^3$$

$$t_B = 5.690 \, s$$

Substituting into Eqs. 1 and 2 yields

$$(a_B)_t = \ddot{v}_B = 0.2(5.690) = 1.138 \, m/s^2$$

$$v_B = 0.1(5.69)^2 = 3.238 \, m/s$$

At $B$, $\rho_B = 2$ m, so that

$$(a_B)_n = \frac{v_B^2}{\rho_B} = \frac{(3.238 \, m/s)^2}{2 \, m} = 5.242 \, m/s^2$$

The magnitude of $a_B$, Fig. 12–29c, is therefore

$$a_B = \sqrt{(1.138 \, m/s^2)^2 + (5.242 \, m/s^2)^2} = 5.36 \, m/s^2 \quad \text{Ans.}$$
12

FUNDAMENTAL PROBLEMS

F12–27. The boat is traveling along the circular path with a speed of \( v = (0.0625t^2) \) m/s, where \( t \) is in seconds. Determine the magnitude of its acceleration when \( t = 10 \) s.

\[ v = 0.0625t^2 \]

**F12–27**

F12–28. The car is traveling along the road with a speed of \( v = (300/s) \) m/s, where \( s \) is in meters. Determine the magnitude of its acceleration when \( t = 3 \) s if \( t = 0 \) at \( s = 0 \).

\[ v = \frac{300}{s} \text{ m/s} \]

**F12–28**

F12–29. If the car decelerates uniformly along the curved road from \( 25 \) m/s at \( A \) to \( 15 \) m/s at \( C \), determine the acceleration of the car at \( B \).

\[ \rho_B = 300 \text{ m} \]

**F12–29**

F12–30. When \( x = 10 \) ft, the crate has a speed of \( 20 \) ft/s which is increasing at \( 6 \) ft/s\(^2\). Determine the direction of the crate’s velocity and the magnitude of the crate’s acceleration at this instant.

\[ x = 20 \text{ ft/s} \]

**F12–30**

F12–31. If the motorcycle has a deceleration of \( a_t = -(0.001s) \) m/s\(^2\) and its speed at position \( A \) is \( 25 \) m/s, determine the magnitude of its acceleration when it passes point \( B \).

\[ a_t = -0.001s \text{ m/s}^2 \]

**F12–31**

F12–32. The car travels up the hill with a speed of \( v = (0.2s) \) m/s, where \( s \) is in meters, measured from \( A \). Determine the magnitude of its acceleration when it is at point \( s = 50 \) m, where \( \rho = 500 \) m.

\[ v = 0.2s \text{ m/s} \]

**F12–32**
12–111. When designing a highway curve it is required that cars traveling at a constant speed of 25 m/s must not have an acceleration that exceeds 3 m/s\(^2\). Determine the minimum radius of curvature of the curve.

*12–112. At a given instant, a car travels along a circular curved road with a speed of 20 m/s while decreasing its speed at the rate of 3 m/s\(^2\). If the magnitude of the car’s acceleration is 5 m/s\(^2\), determine the radius of curvature of the road.

•12–113. Determine the maximum constant speed a race car can have if the acceleration of the car cannot exceed 7.5 m/s\(^2\) while rounding a track having a radius of curvature of 200 m.

12–114. An automobile is traveling on a horizontal circular curve having a radius of 800 ft. If the acceleration of the automobile is 5 ft/s\(^2\), determine the constant speed at which the automobile is traveling.

12–115. A car travels along a horizontal circular curved road that has a radius of 600 m. If the speed is uniformly increased at a rate of 2000 km/h\(^2\), determine the magnitude of the acceleration at the instant the speed of the car is 60 km/h.

*12–116. The automobile has a speed of 80 ft/s at point A and an acceleration \(\mathbf{a}\) having a magnitude of 10 ft/s\(^2\), acting in the direction shown. Determine the radius of curvature of the path at point A and the tangential component of acceleration.

•12–117. Starting from rest the motorboat travels around the circular path, \(\rho = 50\) m, at a speed \(v = (0.8t)\) m/s, where \(t\) is in seconds. Determine the magnitudes of the boat’s velocity and acceleration when it has traveled 20 m.

12–118. Starting from rest, the motorboat travels around the circular path, \(\rho = 50\) m, at a speed \(v = (0.2t^2)\) m/s, where \(t\) is in seconds. Determine the magnitudes of the boat’s velocity and acceleration at the instant \(t = 3\) s.

Probs. 12–117/118

12–119. A car moves along a circular track of radius 250 ft, and its speed for a short period of time \(0 \leq t \leq 2\) s is \(v = 3(t + t^2)\) ft/s, where \(t\) is in seconds. Determine the magnitude of the car’s acceleration when \(t = 2\) s. How far has it traveled in \(t = 2\) s?

*12–120. The car travels along the circular path such that its speed is increased by \(a_t = (0.5t^2)\) m/s\(^2\), where \(t\) is in seconds. Determine the magnitudes of its velocity and acceleration after the car has traveled \(s = 18\) m starting from rest. Neglect the size of the car.
•12–121. The train passes point B with a speed of 20 m/s which is decreasing at \( a_t = -0.5 \text{ m/s}^2 \). Determine the magnitude of acceleration of the train at this point.

12–122. The train passes point A with a speed of 30 m/s and begins to decrease its speed at a constant rate of \( a_t = -0.25 \text{ m/s}^2 \). Determine the magnitude of the acceleration of the train when it reaches point B, where \( s_{AB} = 412 \text{ m} \).

•12–125. When the car reaches point A it has a speed of 25 m/s. If the brakes are applied, its speed is reduced by \( a_t = \left(-\frac{1}{4} t^{0.5}\right) \text{ m/s}^2 \). Determine the magnitude of acceleration of the car just before it reaches point C.

12–126. When the car reaches point A, it has a speed of 25 m/s. If the brakes are applied, its speed is reduced by \( a_t = (0.001s - 1) \text{ m/s}^2 \). Determine the magnitude of acceleration of the car just before it reaches point C.

12–127. Determine the magnitude of acceleration of the airplane during the turn. It flies along the horizontal circular path AB in 40 s, while maintaining a constant speed of 300 ft/s.

*12–128. The airplane flies along the horizontal circular path AB in 60 s. If its speed at point A is 400 ft/s, which decreases at a rate of \( a_t = (-0.1t) \text{ ft/s}^2 \), determine the magnitude of the plane’s acceleration when it reaches point B.
12–129. When the roller coaster is at B, it has a speed of 25 m/s, which is increasing at \( a_i = 3 \text{ m/s}^2 \). Determine the magnitude of the acceleration of the roller coaster at this instant and the direction angle it makes with the x axis.

12–130. If the roller coaster starts from rest at A and its speed increases at \( a_i = (6 - 0.06x) \text{ m/s}^2 \), determine the magnitude of its acceleration when it reaches B where \( s_B = 40 \text{ m} \).

12–131. The car is traveling at a constant speed of 30 m/s. The driver then applies the brakes at A and thereby reduces the car’s speed at the rate of \( a_i = (-0.08v) \text{ m/s}^2 \), where \( v \) is in m/s. Determine the acceleration of the car just before it reaches point C on the circular curve. It takes 15 s for the car to travel from A to C.

*12–132. The car is traveling at a speed of 30 m/s. The driver applies the brakes at A and thereby reduces the speed at the rate of \( a_i = \left( -\frac{1}{t} \right) \text{ m/s}^2 \), where \( t \) is in seconds. Determine the acceleration of the car just before it reaches point C on the circular curve. It takes 15 s for the car to travel from A to C.

12–133. A particle is traveling along a circular curve having a radius of 20 m. If it has an initial speed of 20 m/s and then begins to decrease its speed at the rate of \( a_i = (-0.25s) \text{ m/s}^2 \), determine the magnitude of the acceleration of the particle two seconds later.

12–134. A racing car travels with a constant speed of 240 km/h around the elliptical race track. Determine the acceleration experienced by the driver at A.

12–135. The racing car travels with a constant speed of 240 km/h around the elliptical race track. Determine the acceleration experienced by the driver at B.

*12–136. The position of a particle is defined by \( \mathbf{r} = \left\{ 2 \sin \left( \frac{\pi}{4} \right) \mathbf{i} + 2 \cos \left( \frac{\pi}{4} \right) \mathbf{j} + 3 \right\} \text{ m} \), where \( t \) is in seconds. Determine the magnitudes of the velocity and acceleration at any instant.

*12–137. The position of a particle is defined by \( \mathbf{r} = \left\{ t^2 \mathbf{i} + 3t^2 \mathbf{j} + 8t \mathbf{k} \right\} \text{ m} \), where \( t \) is in seconds. Determine the magnitude of the velocity and acceleration and the radius of curvature of the path when \( t = 2 \text{ s} \).
12–138. Car $B$ turns such that its speed is increased by $(a_B) = (0.5e^t)$ m/s$^2$, where $t$ is in seconds. If the car starts from rest when $\theta = 0^\circ$, determine the magnitudes of its velocity and acceleration when the arm $AB$ rotates $\theta = 30^\circ$. Neglect the size of the car.

12–139. Car $B$ turns such that its speed is increased by $(a_B) = (0.5e^t)$ m/s$^2$, where $t$ is in seconds. If the car starts from rest when $\theta = 0^\circ$, determine the magnitudes of its velocity and acceleration when $t = 2$ s. Neglect the size of the car.

*12–140. The truck travels at a speed of 4 m/s along a circular road that has a radius of 50 m. For a short distance from $s = 0$, its speed is then increased by $a = (0.05s)$ m/s$^2$, where $s$ is in meters. Determine its speed and the magnitude of its acceleration when it has moved $s = 10$ m.

•12–141. The truck travels along a circular road that has a radius of 50 m at a speed of 4 m/s. For a short distance when $t = 0$, its speed is then increased by $a = (0.4t)$ m/s$^2$, where $t$ is in seconds. Determine the speed and the magnitude of the truck’s acceleration when $t = 4$ s.

12–142. Two cyclists, $A$ and $B$, are traveling counterclockwise around a circular track at a constant speed of 8 ft/s at the instant shown. If the speed of $A$ is increased at $t = 1$ s, determine the distance measured counterclockwise along the track from $B$ to $A$ between the cyclists when $t = 1$ s. What is the magnitude of the acceleration of each cyclist at this instant?

12–143. A toboggan is traveling down along a curve which can be approximated by the parabola $y = 0.01x^2$. Determine the magnitude of its acceleration when it reaches point $A$, where its speed is $v_A = 10$ m/s, and it is increasing at the rate of $(a_A) = 3$ m/s$^2$. 

12–144. The jet plane is traveling with a speed of 120 m/s which is decreasing at 40 m/s² when it reaches point A. Determine the magnitude of its acceleration when it is at this point. Also, specify the direction of flight, measured from the x axis.

\[ v_B = 8 \text{ m/s}, \quad \frac{dv_B}{dt} = 4 \text{ m/s}^2 \]

\[ x_A = 2 \text{ m}, \quad y_A = 1.6 \text{ m} \]

\[ 12–145. \] The jet plane is traveling with a constant speed of 110 m/s along the curved path. Determine the magnitude of the acceleration of the plane at the instant it reaches point A (y = 0).

\[ y = 0.4x^2 \]

12–146. The motorcyclist travels along the curve at a constant speed of 30 ft/s. Determine his acceleration when he is located at point A. Neglect the size of the motorcycle and rider for the calculation.

\[ v = 30 \text{ ft/s} \]

12–147. The box of negligible size is sliding down along a curved path defined by the parabola \( y = 0.4x^2 \). When it is at A (\( x_A = 2 \text{ m}, \ y_A = 1.6 \text{ m} \)), the speed is \( v_B = 8 \text{ m/s} \) and the increase in speed is \( \frac{dv_B}{dt} = 4 \text{ m/s}^2 \). Determine the magnitude of the acceleration of the box at this instant.
**12–148.** A spiral transition curve is used on railroads to connect a straight portion of the track with a curved portion. If the spiral is defined by the equation 

\[ y = (10^{-6})x^3 \]

where \( x \) and \( y \) are in feet, determine the magnitude of the acceleration of a train engine moving with a constant speed of 40 ft/s when it is at point \( x = 600 \) ft.

**12–149.** Particles \( A \) and \( B \) are traveling counter-clockwise around a circular track at a constant speed of 8 m/s. If at the instant shown the speed of \( A \) begins to increase by \( (a_t)_A = (0.4s_A) \) m/s\(^2\), where \( s_A \) is in meters, determine the distance measured counterclockwise along the track from \( B \) to \( A \) when \( t = 1 \) s. What is the magnitude of the acceleration of each particle at this instant?

**12–150.** Particles \( A \) and \( B \) are traveling around a circular track at a speed of 8 m/s at the instant shown. If the speed of \( B \) is increasing by \( (a_t)_B = 4 \) m/s\(^2\), and at the same instant \( A \) has an increase in speed of \( (a_t)_A = 0.8t \) m/s\(^2\), determine how long it takes for a collision to occur. What is the magnitude of the acceleration of each particle just before the collision occurs?

**12–151.** The race car travels around the circular track with a speed of 16 m/s. When it reaches point \( A \) it increases its speed at \( a_t = (\frac{v}{4})^{1/3} \) m/s\(^2\), where \( v \) is in m/s. Determine the magnitudes of the velocity and acceleration of the car when it reaches point \( B \). Also, how much time is required for it to travel from \( A \) to \( B \)?
12–152. A particle travels along the path \( y = a + bx + cx^2 \), where \( a, b, c \) are constants. If the speed of the particle is constant, \( v = v_0 \), determine the \( x \) and \( y \) components of velocity and the normal component of acceleration when \( x = 0 \).

12–153. The ball is kicked with an initial speed \( v_A = 8 \text{ m/s} \) at an angle \( \theta_A = 40^\circ \) with the horizontal. Find the equation of the path, \( y = f(x) \), and then determine the normal and tangential components of its acceleration when \( t = 0.25 \text{ s} \).

12–154. The motion of a particle is defined by the equations \( x = (2t + t^3) \) m and \( y = (t^3) \) m, where \( t \) is in seconds. Determine the normal and tangential components of the particle’s velocity and acceleration when \( t = 2 \text{ s} \).

12–155. The motorcycle travels along the elliptical track at a constant speed \( v \). Determine the greatest magnitude of the acceleration if \( a > b \).

12.8 Curvilinear Motion: Cylindrical Components

Sometimes the motion of the particle is constrained on a path that is best described using cylindrical coordinates. If motion is restricted to the plane, then polar coordinates are used.

**Polar Coordinates.** We can specify the location of the particle shown in Fig. 12–30a using a radial coordinate \( r \), which extends outward from the fixed origin \( O \) to the particle, and a transverse coordinate \( \theta \), which is the counterclockwise angle between a fixed reference line and the \( r \) axis. The angle is generally measured in degrees or radians, where \( 1 \text{ rad} = 180^\circ / \pi \). The positive directions of the \( r \) and \( \theta \) coordinates are defined by the unit vectors \( \mathbf{u}_r \) and \( \mathbf{u}_\theta \), respectively. Here \( \mathbf{u}_r \) is in the direction of increasing \( r \) when \( \theta \) is held fixed, and \( \mathbf{u}_\theta \) is in a direction of increasing \( \theta \) when \( r \) is held fixed. Note that these directions are perpendicular to one another.

![Fig. 12–30](image-url)
Position. At any instant the position of the particle, Fig. 12–30a, is defined by the position vector

\[ \mathbf{r} = r \mathbf{u}_r \]  

(12–22)

Velocity. The instantaneous velocity \( \mathbf{v} \) is obtained by taking the time derivative of \( \mathbf{r} \). Using a dot to represent the time derivative, we have

\[ \mathbf{v} = \dot{\mathbf{r}} = \dot{r} \mathbf{u}_r + r \dot{\mathbf{u}}_r \]

To evaluate \( \dot{\mathbf{u}}_r \), notice that \( \mathbf{u}_r \) only changes its direction with respect to time, since by definition the magnitude of this vector is always one unit. Hence, during the time \( \Delta t \), a change \( \Delta r \) will not cause a change in the direction of \( \mathbf{u}_r \); however, a change \( \Delta \theta \) will cause \( \mathbf{u}_r \) to become \( \mathbf{u}_r' \), where \( \mathbf{u}_r' = \mathbf{u}_r + \Delta \mathbf{u}_r \), Fig. 12–30b. The time change in \( \mathbf{u}_r \) is then \( \Delta \mathbf{u}_r \). For small angles \( \Delta \theta \) this vector has a magnitude \( \Delta r_r \approx \Delta (\Delta \theta) \) and acts in the \( \mathbf{u}_\theta \) direction. Therefore, \( \Delta \mathbf{u}_r = \Delta \theta \mathbf{u}_\theta \), and so

\[ \dot{\mathbf{u}}_r = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{u}_r}{\Delta t} = \left( \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} \right) \mathbf{u}_\theta \]

(12–23)

Substituting into the above equation, the velocity can be written in component form as

\[ \mathbf{v} = v_r \mathbf{u}_r + v_\theta \mathbf{u}_\theta \]  

(12–24)

where

\[ v_r = \dot{r} \]

\[ v_\theta = r \dot{\theta} \]  

(12–25)

These components are shown graphically in Fig. 12–30c. The radial component \( v_r \) is a measure of the rate of increase or decrease in the length of the radial coordinate, i.e., \( \dot{r} \); whereas the transverse component \( v_\theta \) can be interpreted as the rate of motion along the circumference of a circle having a radius \( r \). In particular, the term \( \dot{\theta} = \frac{d\theta}{dt} \) is called the angular velocity, since it indicates the time rate of change of the angle \( \theta \). Common units used for this measurement are rad/s.

Since \( v_r \) and \( v_\theta \) are mutually perpendicular, the magnitude of velocity or speed is simply the positive value of

\[ v = \sqrt{(\dot{r})^2 + (r \dot{\theta})^2} \]  

(12–26)

and the direction of \( \mathbf{v} \) is, of course, tangent to the path, Fig. 12–30c.
12.8 CURVILINEAR MOTION: CYLINDRICAL COMPONENTS

**Acceleration.** Taking the time derivatives of Eq. 12–24, using Eqs. 12–25, we obtain the particle’s instantaneous acceleration,

\[ \mathbf{a} = \dot{\mathbf{v}} = r\dot{\mathbf{u}}_r + r\dot{\mathbf{u}}_r + r\dot{\mathbf{u}}_\theta + r\dot{\mathbf{u}}_\theta + r\dot{\mathbf{u}}_\phi \]

To evaluate \( \dot{\mathbf{u}}_\theta \), it is necessary only to find the change in the direction of \( \mathbf{u}_\theta \) since its magnitude is always unity. During the time \( \Delta t \), a change \( \Delta r \) will not change the direction of \( \mathbf{u}_\theta \), however, a change \( \Delta \theta \) will cause \( \mathbf{u}_\theta \) to become \( \Delta \mathbf{u}_\theta \), where \( \mathbf{u}_\theta = \mathbf{u}_\theta + \Delta \mathbf{u}_\theta \), Fig. 12–30d. The time change in \( \mathbf{u}_\theta \) is thus \( \Delta \mathbf{u}_\theta \). For small angles this vector has a magnitude \( \Delta \mathbf{u}_\theta \approx 1(\Delta \theta) \) and acts in the \( -\mathbf{u}_r \) direction; i.e., \( \Delta \mathbf{u}_\theta = -\Delta \mathbf{u}_r \). Thus,

\[ \dot{\mathbf{u}}_\theta = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{u}_\theta}{\Delta t} = -\left( \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} \right) \mathbf{u}_r \]

\[ \dot{\mathbf{u}}_\theta = -\dot{\theta} \mathbf{u}_r \quad (12–27) \]

Substituting this result and Eq. 12–23 into the above equation for \( \mathbf{a} \), we can write the acceleration in component form as

\[ \mathbf{a} = a_r \mathbf{u}_r + a_\theta \mathbf{u}_\theta \quad (12–28) \]

where

\[ a_r = \ddot{r} - r \dot{\theta}^2 \]
\[ a_\theta = \dot{r} \dot{\theta} + 2r \ddot{\theta} \quad (12–29) \]

The term \( \dot{\theta} = d^2\theta/dt^2 = d/dt(d\theta/dt) \) is called the angular acceleration since it measures the change made in the angular velocity during an instant of time. Units for this measurement are rad/s².

Since \( a_r \), and \( a_\theta \) are always perpendicular, the magnitude of acceleration is simply the positive value of

\[ a = \sqrt{\left(\ddot{r} - r \dot{\theta}^2\right)^2 + \left(\dot{r} \dot{\theta} + 2r \ddot{\theta}\right)^2} \quad (12–30) \]

The direction is determined from the vector addition of its two components. In general, \( \mathbf{a} \) will not be tangent to the path, Fig. 12–30e.
Cylindrical Coordinates. If the particle moves along a space curve as shown in Fig. 12–31, then its location may be specified by the three cylindrical coordinates, \( r, \theta, z \). The \( z \) coordinate is identical to that used for rectangular coordinates. Since the unit vector defining its direction, \( \mathbf{u}_z \), is constant, the time derivatives of this vector are zero, and therefore the position, velocity, and acceleration of the particle can be written in terms of its cylindrical coordinates as follows:

\[
\mathbf{r}_P = r \mathbf{u}_r + z \mathbf{u}_z \\
\mathbf{v} = \dot{r} \mathbf{u}_r + r \dot{\theta} \mathbf{u}_\theta + \dot{z} \mathbf{u}_z \\
\mathbf{a} = (\ddot{r} - r \ddot{\theta}) \mathbf{u}_r + (r \dddot{\theta} + 2 \dot{r} \dot{\theta}) \mathbf{u}_\theta + \ddot{z} \mathbf{u}_z
\]

\( \text{(12–31)} \)

\( \text{(12–32)} \)

Time Derivatives. The above equations require that we obtain the time derivatives \( \dot{r}, \ddot{r}, \dot{\theta}, \text{ and } \dddot{\theta} \) in order to evaluate the \( r \) \( \text{ and } \theta \) components of \( \mathbf{v} \) \( \text{ and } \mathbf{a} \). Two types of problems generally occur:

1. If the polar coordinates are specified as time parametric equations, \( r = r(t) \) \( \text{ and } \theta = \theta(t) \), then the time derivatives can be found directly.

2. If the time-parametric equations are not given, then the path \( r = f(\theta) \) must be known. Using the chain rule of calculus we can then find the relation between \( \dot{r} \) \( \text{ and } \dot{\theta} \), and between \( \ddot{r} \) \( \text{ and } \dddot{\theta} \). Application of the chain rule, along with some examples, is explained in Appendix C.

Procedure for Analysis

Coordinate System.
- Polar coordinates are a suitable choice for solving problems when data regarding the angular motion of the radial coordinate \( r \) is given to describe the particle’s motion. Also, some paths of motion can conveniently be described in terms of these coordinates.
- To use polar coordinates, the origin is established at a fixed point, and the radial line \( r \) is directed to the particle.
- The transverse coordinate \( \theta \) is measured from a fixed reference line to the radial line.

Velocity and Acceleration.
- Once \( r \) and the four time derivatives \( \dot{r}, \ddot{r}, \dot{\theta}, \text{ and } \dddot{\theta} \) have been evaluated at the instant considered, their values can be substituted into Eqs. 12–25 and 12–29 to obtain the radial and transverse components of \( \mathbf{v} \) \( \text{ and } \mathbf{a} \).
- If it is necessary to take the time derivatives of \( r = f(\theta) \), then the chain rule of calculus must be used. See Appendix C.
- Motion in three dimensions requires a simple extension of the above procedure to include \( z \) \( \text{ and } \dot{z} \).
EXAMPLE 12.17

The amusement park ride shown in Fig. 12–32a consists of a chair that is rotating in a horizontal circular path of radius \( r \) such that the arm \( OB \) has an angular velocity \( \dot{\theta} \) and angular acceleration \( \ddot{\theta} \). Determine the radial and transverse components of velocity and acceleration of the passenger. Neglect his size in the calculation.

![Diagram](image)

**Fig. 12–32**

**SOLUTION**

**Coordinate System.** Since the angular motion of the arm is reported, polar coordinates are chosen for the solution, Fig. 12–32a. Here \( \theta \) is not related to \( r \), since the radius is constant for all \( \theta \).

**Velocity and Acceleration.** It is first necessary to specify the first and second time derivatives of \( r \) and \( \theta \). Since \( r \) is constant, we have

\[
\begin{align*}
  r &= r \\
  \dot{r} &= 0 \\
  \ddot{r} &= 0
\end{align*}
\]

Thus,

\[
\begin{align*}
  v_r &= \dot{r} = 0 & \text{Ans.} \\
  v_\theta &= r\dot{\theta} & \text{Ans.} \\
  a_r &= \ddot{r} - r\ddot{\theta} = -r\ddot{\theta} & \text{Ans.} \\
  a_\theta &= r\ddot{\theta} + 2r\dot{\theta} = r\ddot{\theta} & \text{Ans.}
\end{align*}
\]

These results are shown in Fig. 12–32b.

**NOTE:** The \( n, t \) axes are also shown in Fig. 12–32b, which in this special case of circular motion happen to be collinear with the \( r \) and \( \theta \) axes, respectively. Since \( v = v_\theta = v_r = r\theta \), then by comparison,

\[
\begin{align*}
  -a_r &= a_n = \frac{v^2}{\rho} = \frac{(r\dot{\theta})^2}{r} = r\ddot{\theta} \\
  a_\theta &= a_t = \frac{dv}{dt} = \frac{d}{dt}(r\dot{\theta}) = \frac{dr}{dt}\dot{\theta} + r\frac{d\dot{\theta}}{dt} = 0 + \ddot{\theta}
\end{align*}
\]
The rod $OA$ in Fig. 12–33a rotates in the horizontal plane such that $\theta = (t^3)$ rad. At the same time, the collar $B$ is sliding outward along $OA$ so that $r = (100t^2)$ mm. If in both cases $t$ is in seconds, determine the velocity and acceleration of the collar when $t = 1$ s.

**SOLUTION**

**Coordinate System.** Since time-parametric equations of the path are given, it is not necessary to relate $r$ to $\theta$.

**Velocity and Acceleration.** Determining the time derivatives and evaluating them when $t = 1$ s, we have

$$
\begin{align*}
    r &= 100t^2 \\&= 100\text{mm} \\
    \theta &= t^3 \\&= 1\text{ rad} = 57.3^\circ \\
    \dot{r} &= 200t \\&= 200\text{ mm/s} \\
    \dot{\theta} &= 3t^2 \\&= 3\text{ rad/s} \\
    \ddot{r} &= 200 \\&= 200\text{ mm/s}^2 \\
    \ddot{\theta} &= 6t \\&= 6\text{ rad/s}^2.
\end{align*}
$$

As shown in Fig. 12–33b,

$$
\mathbf{v} = \dot{r}\mathbf{u}_r + \dot{\theta}\mathbf{u}_\theta
$$

$$
= 200\mathbf{u}_r + 100(3)\mathbf{u}_\theta = \{200\mathbf{u}_r + 300\mathbf{u}_\theta\} \text{ mm/s}
$$

The magnitude of $\mathbf{v}$ is

$$
\begin{align*}
    v &= \sqrt{(200)^2 + (300)^2} = 361\text{ mm/s} \\
    \delta &= \tan^{-1}\left(\frac{300}{200}\right) = 56.3^\circ \\
    \delta + 57.3^\circ &= 114^\circ
\end{align*}
$$

As shown in Fig. 12–33c,

$$
\mathbf{a} = (\dot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta
$$

$$
= [200 - 100(3)^2]\mathbf{u}_r + [100(6) + 2(200)3]\mathbf{u}_\theta
$$

$$
= \{-700\mathbf{u}_r + 1800\mathbf{u}_\theta\} \text{ mm/s}^2
$$

The magnitude of $\mathbf{a}$ is

$$
\begin{align*}
    a &= \sqrt{(700)^2 + (1800)^2} = 1930\text{ mm/s}^2 \\
    \phi &= \tan^{-1}\left(\frac{1800}{700}\right) = 68.7^\circ \quad (180^\circ - \phi) + 57.3^\circ = 169^\circ
\end{align*}
$$

**NOTE:** The velocity is tangent to the path; however, the acceleration is directed within the curvature of the path, as expected.
EXAMPLE 12.19

The searchlight in Fig. 12–34a casts a spot of light along the face of a wall that is located 100 m from the searchlight. Determine the magnitudes of the velocity and acceleration at which the spot appears to travel across the wall at the instant \( \theta = 45^\circ \). The searchlight rotates at a constant rate of \( \dot{\theta} = 4 \text{ rad/s} \).

**SOLUTION**

**Coordinate System.** Polar coordinates will be used to solve this problem since the angular rate of the searchlight is given. To find the necessary time derivatives it is first necessary to relate \( r \) to \( \theta \) from Fig. 12–34a,

\[
r = 100 \cos \theta = 100 \sec \theta
\]

**Velocity and Acceleration.** Using the chain rule of calculus, noting that \( d(\sec \theta) = \sec \theta \tan \theta \, d\theta \), and \( d(\tan \theta) = \sec^2 \theta \, d\theta \), we have

\[
\begin{align*}
\dot{r} &= 100(\sec \theta \tan \theta) \dot{\theta} \\
\ddot{r} &= 100(\sec \theta \tan \theta)(\tan \theta) \ddot{\theta} + 100 \sec \theta (\sec^2 \theta) \dot{\theta}^2 \\
&= 100 \sec \theta \tan^2 \theta (\dot{\theta})^2 + 100 \sec^3 \theta (\dot{\theta})^2 + 100(\sec \theta \tan \theta) \dddot{\theta}
\end{align*}
\]

Since \( \dot{\theta} = 4 \text{ rad/s} \) = constant, then \( \dddot{\theta} = 0 \), and the above equations, when \( \theta = 45^\circ \), become

\[
\begin{align*}
r &= 100 \sec 45^\circ = 141.4 \\
\dot{r} &= 400 \sec 45^\circ \tan 45^\circ = 565.7 \\
\ddot{r} &= 1600 \left( \sec 45^\circ \tan^2 45^\circ + \sec^3 45^\circ \right) = 6788.2
\end{align*}
\]

As shown in Fig. 12–34b,

\[
\begin{align*}
v &= \dot{r} \mathbf{u}_r + r \dot{\theta} \mathbf{u}_\theta \\
&= 565.7 \mathbf{u}_r + 141.4(4) \mathbf{u}_\theta \\
&= \{565.7 \mathbf{u}_r + 565.7 \mathbf{u}_\theta \} \text{ m/s}
\end{align*}
\]

\[
v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{(565.7)^2 + (565.7)^2} = 800 \text{ m/s}
\]

**Ans.**

As shown in Fig. 12–34c,

\[
\begin{align*}
a &= \left( \ddot{r} - r \dddot{\theta}^2 \right) \mathbf{u}_r + (r \dddot{\theta} + 2 \dot{r} \dot{\theta}) \mathbf{u}_\theta \\
&= \left( 6788.2 - 141.4(4)^2 \right) \mathbf{u}_r + \left[ 141.4(0) + 2(565.7)(4) \right] \mathbf{u}_\theta \\
&= \{4525.5 \mathbf{u}_r + 4525.5 \mathbf{u}_\theta \} \text{ m/s}^2
\end{align*}
\]

\[
a = \sqrt{a_r^2 + a_\theta^2} = \sqrt{(4525.5)^2 + (4525.5)^2} = 6400 \text{ m/s}^2
\]

**Ans.**

**NOTE:** It is also possible to find \( a \) without having to calculate \( \dddot{r} \) (or \( a_r \)). As shown in Fig. 12–34d, since \( a_\theta = 4525.5 \text{ m/s}^2 \), then by vector resolution, \( a = 4525.5/\cos 45^\circ = 6400 \text{ m/s}^2 \).
Due to the rotation of the forked rod, the ball in Fig. 12–35a travels around the slotted path, a portion of which is in the shape of a cardioid, \( r = 0.5(1 - \cos \theta) \) ft, where \( \theta \) is in radians. If the ball’s velocity is \( v = 4 \text{ ft/s} \) and its acceleration is \( a = 30 \text{ ft/s}^2 \) at the instant \( \theta = 180^\circ \), determine the angular velocity \( \dot{\theta} \) and angular acceleration \( \ddot{\theta} \) of the fork.

**SOLUTION**

**Coordinate System.** This path is most unusual, and mathematically it is best expressed using polar coordinates, as done here, rather than rectangular coordinates. Also, since \( \dot{\theta} \) and \( \ddot{\theta} \) must be determined, then \( r, \theta \) coordinates are an obvious choice.

**Velocity and Acceleration.** The time derivatives of \( r \) and \( \theta \) can be determined using the chain rule.

\[
\begin{align*}
\dot{r} &= 0.5(1 - \cos \theta) \\
\ddot{r} &= 0.5(\sin \theta)\dot{\theta} + 0.5(\cos \theta)\ddot{\theta} \\
\end{align*}
\]

Evaluating these results at \( \theta = 180^\circ \), we have

\[
\begin{align*}
\dot{r} &= 0 \quad \ddot{r} = -0.5\dot{\theta}^2 \\
\end{align*}
\]

Since \( v = 4 \text{ ft/s} \), using Eq. 12–26 to determine \( \dot{\theta} \) yields

\[
\begin{align*}
\dot{\theta} &= 4 \text{ rad/s} \quad \text{Ans.}
\end{align*}
\]

In a similar manner, \( \ddot{\theta} \) can be found using Eq. 12–30.

\[
\begin{align*}
\ddot{\theta} &= \frac{(-24)^2 + \ddot{\theta}^2}{(30)^2} \\
\ddot{\theta} &= 18 \text{ rad/s}^2 \quad \text{Ans.}
\end{align*}
\]

Vectors \( \mathbf{a} \) and \( \mathbf{v} \) are shown in Fig. 12–35b.

**NOTE:** At this location, the \( \theta \) and \( t \) (tangential) axes will coincide. The \( +n \) (normal) axis is directed to the right, opposite to \( +r \).
F12–33. The car has a speed of 55 ft/s. Determine the angular velocity $\dot{\theta}$ of the radial line $OA$ at this instant.

F12–34. The platform is rotating about the vertical axis such that at any instant its angular position is $\theta = (4t^{3/2})$ rad, where $t$ is in seconds. A ball rolls outward along the radial groove so that its position is $r = (0.1t^3)$ m, where $t$ is in seconds. Determine the magnitudes of the velocity and acceleration of the ball when $t = 1.5$ s.

F12–35. Peg $P$ is driven by the forked link $OA$ along the curved path described by $r = (2\theta)$ ft. At the instant $\theta = \pi/4$ rad, the angular velocity and angular acceleration of the link are $\dot{\theta} = 3$ rad/s and $\ddot{\theta} = 1$ rad/s$^2$. Determine the magnitude of the peg's acceleration at this instant.

F12–36. Peg $P$ is driven by the forked link $OA$ along the path described by $r = e^\theta$. When $\theta = \frac{\pi}{4}$ rad, the link has an angular velocity and angular acceleration of $\dot{\theta} = 2$ rad/s and $\ddot{\theta} = 4$ rad/s$^2$. Determine the radial and transverse components of the peg's acceleration at this instant.

F12–37. The collars are pin-connected at $B$ and are free to move along rod $OA$ and the curved guide $OC$ having the shape of a cardioid, $r = [0.2(1 + \cos \theta)]$ m. At $\theta = 30^\circ$, the angular velocity of $OA$ is $\dot{\theta} = 3$ rad/s. Determine the magnitudes of the velocity of the collars at this point.

F12–38. At the instant $\theta = 45^\circ$, the athlete is running with a constant speed of 2 m/s. Determine the angular velocity at which the camera must turn in order to follow the motion.
12–156. A particle moves along a circular path of radius 300 mm. If its angular velocity is \( \dot{\theta} = (2t^2) \text{ rad/s} \), where \( t \) is in seconds, determine the magnitude of the particle’s acceleration when \( t = 2 \text{ s} \).

12–157. A particle moves along a circular path of radius 300 mm. If its angular velocity is \( \theta = (3t^2) \text{ rad/s} \), where \( t \) is in seconds, determine the magnitudes of the particle’s velocity and acceleration when \( \theta = 45^\circ \). The particle starts from rest when \( \theta = 0^\circ \).

12–158. A particle moves along a circular path of radius 5 ft. If its position is \( \theta = (e^{0.5t}) \text{ rad} \), where \( t \) is in seconds, determine the magnitude of the particle’s acceleration when \( \theta = 90^\circ \).

12–159. The position of a particle is described by \( r = (t^3 + 4t - 4) \text{ m} \) and \( \theta = (t^{3/2}) \text{ rad} \), where \( t \) is in seconds. Determine the magnitudes of the particle’s velocity and acceleration at the instant \( t = 2 \text{ s} \).

12–160. The position of a particle is described by \( r = (300e^{-0.5t}) \text{ mm} \) and \( \theta = (0.3t^2) \text{ rad} \), where \( t \) is in seconds. Determine the magnitudes of the particle’s velocity and acceleration at the instant \( t = 1.5 \text{ s} \).

12–161. An airplane is flying in a straight line with a velocity of 200 m/s and an acceleration of 3 m/s\(^2\). If the propeller has a diameter of 6 ft and is rotating at an angular rate of 120 rad/s, determine the magnitudes of velocity and acceleration of a particle located on the tip of the propeller.

12–162. A particle moves along a circular path having a radius of 4 in. such that its position as a function of time is given by \( \theta = (\cos 2t) \text{ rad} \), where \( t \) is in seconds. Determine the magnitude of the acceleration of the particle when \( \theta = 30^\circ \).

12–163. A particle travels around a limaçon, defined by the equation \( r = b - a \cos \theta \), where \( a \) and \( b \) are constants. Determine the particle’s radial and transverse components of velocity and acceleration as a function of \( \theta \) and its time derivatives.

12–164. A particle travels around a lituus, defined by the equation \( r^2 \theta = a^2 \), where \( a \) is a constant. Determine the particle’s radial and transverse components of velocity and acceleration as a function of \( \theta \) and its time derivatives.

12–165. A car travels along the circular curve of radius \( r = 300 \text{ ft} \). At the instant shown, its angular rate of rotation is \( \dot{\theta} = 0.4 \text{ rad/s} \), which is increasing at the rate of \( \ddot{\theta} = 0.2 \text{ rad/s}^2 \). Determine the magnitudes of the car’s velocity and acceleration at this instant.

12–166. The slotted arm \( OA \) rotates counterclockwise about \( O \) with a constant angular velocity of \( \dot{\theta} \). The motion of pin \( B \) is constrained such that it moves on the fixed circular surface and along the slot in \( OA \). Determine the magnitudes of the velocity and acceleration of pin \( B \) as a function of \( \theta \).

12–167. The slotted arm \( OA \) rotates counterclockwise about \( O \) such that when \( \theta = \pi/4 \), arm \( OA \) is rotating with an angular velocity of \( \dot{\theta} \) and an angular acceleration of \( \ddot{\theta} \). Determine the magnitudes of the velocity and acceleration of pin \( B \) at this instant. The motion of pin \( B \) is constrained such that it moves on the fixed circular surface and along the slot in \( OA \).
12–168. The car travels along the circular curve having a radius \( r = 400 \text{ ft} \). At the instant shown, its angular rate of rotation is \( \dot{\theta} = 0.025 \text{ rad/s} \), which is decreasing at the rate \( \ddot{\theta} = -0.008 \text{ rad/s}^2 \). Determine the radial and transverse components of the car's velocity and acceleration at this instant and sketch these components on the curve.

12–169. The car travels along the circular curve of radius \( r = 400 \text{ ft} \) with a constant speed of \( v = 30 \text{ ft/s} \). Determine the angular rate of rotation \( \dot{\theta} \) of the radial line \( r \) and the magnitude of the car's acceleration.

12–170. Starting from rest, the boy runs outward in the radial direction from the center of the platform with a constant acceleration of \( 0.5 \text{ m/s}^2 \). If the platform is rotating at a constant rate \( \dot{\theta} = 0.2 \text{ rad/s} \), determine the radial and transverse components of the velocity and acceleration of the boy when \( t = 3 \text{ s} \). Neglect his size.

12–171. The small washer slides down the cord \( OA \). When it is at the midpoint, its speed is \( 200 \text{ mm/s} \) and its acceleration is \( 10 \text{ mm/s}^2 \). Express the velocity and acceleration of the washer at this point in terms of its cylindrical components.

*12–172. If arm \( OA \) rotates counterclockwise with a constant angular velocity of \( \dot{\theta} = 2 \text{ rad/s} \), determine the magnitudes of the velocity and acceleration of peg \( P \) at \( \theta = 30^\circ \). The peg moves in the fixed groove defined by the lemniscate, and along the slot in the arm.

12–173. The peg moves in the curved slot defined by the lemniscate, and through the slot in the arm. At \( \theta = 30^\circ \), the angular velocity is \( \dot{\theta} = 2 \text{ rad/s} \), and the angular acceleration is \( \ddot{\theta} = 1.5 \text{ rad/s}^2 \). Determine the magnitudes of the velocity and acceleration of peg \( P \) at this instant.
12–174. The airplane on the amusement park ride moves along a path defined by the equations \( r = 4 \text{ m}, \theta = (0.2t) \text{ rad}, \) and \( z = (0.5 \cos \theta) \text{ m}, \) where \( t \) is in seconds. Determine the cylindrical components of the velocity and acceleration of the airplane when \( t = 6 \text{ s}. \)

\[ r = 4 \text{ m}, \theta = 0.2t \text{ rad}, z = 0.5 \cos(0.2t) \text{ m} \]

12–175. The motion of peg \( P \) is constrained by the lemniscate curved slot in \( OB \) and by the slotted arm \( OA. \) If \( OA \) rotates counterclockwise with a constant angular velocity of \( \dot{\theta} = 3 \text{ rad/s}, \) determine the magnitudes of the velocity and acceleration of peg \( P \) at \( \theta = 30^\circ. \)

12–176. The motion of peg \( P \) is constrained by the lemniscate curved slot in \( OB \) and by the slotted arm \( OA. \) If \( OA \) rotates counterclockwise with an angular velocity of \( \dot{\theta} = (3t^{3/2}) \text{ rad/s}, \) where \( t \) is in seconds, determine the magnitudes of the velocity and acceleration of peg \( P \) at \( \theta = 30^\circ. \) When \( t = 0, \dot{\theta} = 0^\circ. \)

\[ \dot{\theta} = 3 \text{ rad/s}, \theta = 30^\circ \]

12–177. The driver of the car maintains a constant speed of 40 m/s. Determine the angular velocity of the camera tracking the car when \( \theta = 15^\circ. \)

12–178. When \( \theta = 15^\circ, \) the car has a speed of 50 m/s which is increasing at 6 m/s\(^2\). Determine the angular velocity of the camera tracking the car at this instant.

12–179. If the cam rotates clockwise with a constant angular velocity of \( \dot{\theta} = 5 \text{ rad/s}, \) determine the magnitudes of the velocity and acceleration of the follower rod \( AB \) at the instant \( \theta = 30^\circ. \) The surface of the cam has a shape of limaçon defined by \( r = (200 + 100 \cos \theta) \text{ mm}. \)

12–180. At the instant \( \theta = 30^\circ, \) the cam rotates with a clockwise angular velocity of \( \dot{\theta} = 5 \text{ rad/s} \) and an angular acceleration of \( \ddot{\theta} = 6 \text{ rad/s}^2. \) Determine the magnitudes of the velocity and acceleration of the follower rod \( AB \) at this instant. The surface of the cam has a shape of a limaçon defined by \( r = (200 + 100 \cos \theta) \text{ mm}. \)
12–181. The automobile travels from a parking deck down along a cylindrical spiral ramp at a constant speed of \( v = 1.5 \text{ m/s} \). If the ramp descends a distance of 12 m for every full revolution, \( \theta = 2\pi \text{ rad} \), determine the magnitude of the car’s acceleration as it moves along the ramp, \( r = 10 \text{ m} \). \textit{Hint:} For part of the solution, note that the tangent to the ramp at any point is at an angle of \( \phi = \tan^{-1} \left( \frac{12}{(2\pi(10))} \right) = 10.81^\circ \) from the horizontal. Use this to determine the velocity components \( v_\theta \) and \( v_z \), which in turn are used to determine \( \theta \) and \( z \).

12–182. The box slides down the helical ramp with a constant speed of \( v = 2 \text{ m/s} \). Determine the magnitude of its acceleration. The ramp descends a vertical distance of 1 m for every full revolution. The mean radius of the ramp is \( r = 0.5 \text{ m} \).

12–183. The box slides down the helical ramp which is defined by \( r = 0.5 \text{ m}, \theta = (0.5t^3) \text{ rad}, \text{ and } z = (2 - 0.2t^2) \text{ m} \), where \( t \) is in seconds. Determine the magnitudes of the velocity and acceleration of the box at the instant \( \theta = 2\pi \text{ rad} \).

*12–184. Rod OA rotates counterclockwise with a constant angular velocity of \( \dot{\theta} = 6 \text{ rad/s} \). Through mechanical means collar \( B \) moves along the rod with a speed of \( \dot{r} = (4t^2) \text{ m/s} \), where \( t \) is in seconds. If \( r = 0 \) when \( t = 0 \), determine the magnitudes of velocity and acceleration of the collar when \( t = 0.75 \text{ s} \).

*12–185. Rod OA is rotating counterclockwise with an angular velocity of \( \dot{\theta} = 2t^3 \text{ rad/s} \). Through mechanical means collar \( B \) moves along the rod with a speed of \( \dot{r} = (4t^2) \text{ m/s} \). If \( \theta = 0 \) and \( r = 0 \) when \( t = 0 \), determine the magnitudes of velocity and acceleration of the collar at \( \theta = 60^\circ \).

12–186. The slotted arm \( AB \) drives pin \( C \) through the spiral groove described by the equation \( r = a\theta \). If the angular velocity is constant at \( \dot{\theta} \), determine the radial and transverse components of velocity and acceleration of the pin.

12–187. The slotted arm \( AB \) drives pin \( C \) through the spiral groove described by the equation \( r = (1.5 \theta) \text{ ft} \), where \( \theta \) is in radians. If the arm starts from rest when \( \theta = 60^\circ \) and is driven at an angular velocity of \( \dot{\theta} = 4t \text{ rad/s} \), where \( t \) is in seconds, determine the radial and transverse components of velocity and acceleration of the pin \( C \) when \( t = 1 \text{ s} \).
12–188. The partial surface of the cam is that of a logarithmic spiral $r = (40e^{0.05\theta})$ mm, where $\theta$ is in radians. If the cam rotates at a constant angular velocity of $\dot{\theta} = 4$ rad/s, determine the magnitudes of the velocity and acceleration of the point on the cam that contacts the follower rod at the instant $\theta = 30^\circ$.

12–189. Solve Prob. 12–188, if the cam has an angular acceleration of $\ddot{\theta} = 2$ rad/s$^2$ when its angular velocity is $\dot{\theta} = 4$ rad/s at $\theta = 30^\circ$.

12–190. A particle moves along an Archimedean spiral $r = (8\theta)$ ft, where $\theta$ is given in radians. If $\dot{\theta} = 4$ rad/s (constant), determine the radial and transverse components of the particle's velocity and acceleration at the instant $\theta = \pi/2$ rad. Sketch the curve and show the components on the curve.

12–191. Solve Prob. 12–190 if the particle has an angular acceleration $\ddot{\theta} = 5$ rad/s$^2$ when $\dot{\theta} = 4$ rad/s at $\theta = \pi/2$ rad.

12–192. The boat moves along a path defined by $r^2 = [10(10^3) \cos 2\theta]$ ft$^2$, where $\theta$ is in radians. If $\theta = (0.4t^2)$ rad, where $t$ is in seconds, determine the radial and transverse components of the boat's velocity and acceleration at the instant $t = 1$ s.

12–193. A car travels along a road, which for a short distance is defined by $r = (200/\theta)$ ft, where $\theta$ is in radians. If it maintains a constant speed of $v = 35$ ft/s, determine the radial and transverse components of its velocity when $\theta = \pi/3$ rad.

12–194. For a short time the jet plane moves along a path in the shape of a lemniscate, $r^2 = (2500 \cos 2\theta)$ km$^2$. At the instant $\theta = 30^\circ$, the radar tracking device is rotating at $\dot{\theta} = 5(10^{-3})$ rad/s with $\ddot{\theta} = 2(10^{-3})$ rad/s$^2$. Determine the radial and transverse components of velocity and acceleration of the plane at this instant.
12.9 Absolute Dependent Motion Analysis of Two Particles

In some types of problems the motion of one particle will depend on the corresponding motion of another particle. This dependency commonly occurs if the particles, here represented by blocks, are interconnected by inextensible cords which are wrapped around pulleys. For example, the movement of block $A$ downward along the inclined plane in Fig. 12–36 will cause a corresponding movement of block $B$ up the other incline. We can show this mathematically by first specifying the location of the blocks using position coordinates $s_A$ and $s_B$. Note that each of the coordinate axes is (1) measured from a fixed point $(O)$ or fixed datum line, (2) measured along each inclined plane in the direction of motion of each block, and (3) has a positive sense from $C$ to $A$ and $D$ to $B$. If the total cord length is $l_T$, the two position coordinates are related by the equation

$$s_A + l_{CD} + s_B = l_T$$

Here $l_{CD}$ is the length of the cord passing over arc $CD$. Taking the time derivative of this expression, realizing that $l_{CD}$ and $l_T$ remain constant, while $s_A$ and $s_B$ measure the segments of the cord that change in length. We have

$$\frac{ds_A}{dt} + \frac{ds_B}{dt} = 0 \quad \text{or} \quad v_B = -v_A$$

The negative sign indicates that when block $A$ has a velocity downward, i.e., in the direction of positive $s_A$, it causes a corresponding upward velocity of block $B$; i.e., $B$ moves in the negative $s_B$ direction.

In a similar manner, time differentiation of the velocities yields the relation between the accelerations, i.e.,

$$a_B = -a_A$$

A more complicated example is shown in Fig. 12–37a. In this case, the position of block $A$ is specified by $s_A$, and the position of the end of the cord from which block $B$ is suspended is defined by $s_B$. As above, we have chosen position coordinates which (1) have their origin at fixed points or datums, (2) are measured in the direction of motion of each block, and (3) are positive to the right for $s_A$ and positive downward for $s_B$. During the motion, the length of the red colored segments of the cord in Fig. 12–37a remains constant. If $l$ represents the total length of cord minus these segments, then the position coordinates can be related by the equation

$$2s_B + h + s_A = l$$

Since $l$ and $h$ are constant during the motion, the two time derivatives yield

$$2v_B = -v_A \quad 2a_B = -a_A$$

Hence, when $B$ moves downward ($+s_B$), $A$ moves to the left ($-s_A$) with twice the motion.
This example can also be worked by defining the position of block B from the center of the bottom pulley (a fixed point), Fig. 12–37b. In this case

\[ 2(h - s_B) + h + s_A = l \]

Time differentiation yields

\[ 2v_B = v_A \quad 2a_B = a_A \]

Here the signs are the same. Why?

---

**Procedure for Analysis**

The above method of relating the dependent motion of one particle to that of another can be performed using algebraic scalars or position coordinates provided each particle moves along a rectilinear path. When this is the case, only the magnitudes of the velocity and acceleration of the particles will change, not their line of direction.

**Position-Coordinate Equation.**

- Establish each position coordinate with an origin located at a **fixed** point or datum.
- It is **not necessary** that the origin be the same for each of the coordinates; however, it is **important** that each coordinate axis selected be directed along the **path of motion** of the particle.
- Using geometry or trigonometry, relate the position coordinates to the total length of the cord, \( l_T \), or to that portion of cord, \( l \), which **excludes** the segments that do not change length as the particles move—such as arc segments wrapped over pulleys.
- If a problem involves a **system** of two or more cords wrapped around pulleys, then the position of a point on one cord must be related to the position of a point on another cord using the above procedure. Separate equations are written for a fixed length of each cord of the system and the positions of the two particles are then related by these equations (see Examples 12.22 and 12.23).

**Time Derivatives.**

- Two successive time derivatives of the position-coordinate equations yield the required velocity and acceleration equations which relate the motions of the particles.
- The signs of the terms in these equations will be consistent with those that specify the positive and negative sense of the position coordinates.
EXAMPLE 12.21

Determine the speed of block $A$ in Fig. 12–38 if block $B$ has an upward speed of $6 \text{ ft/s}$.

![Diagram of two blocks connected by a cord, with one block moving upward at 6 ft/s.]

**Fig. 12–38**

**SOLUTION**

**Position-Coordinate Equation.** There is *one cord* in this system having segments which change length. Position coordinates $s_A$ and $s_B$ will be used since each is measured from a fixed point ($C$ or $D$) and extends along each block’s *path of motion*. In particular, $s_B$ is directed to point $E$ since motion of $B$ and $E$ is the *same*.

The red colored segments of the cord in Fig. 12–38 remain at a constant length and do not have to be considered as the blocks move. The remaining length of cord, $l$, is also constant and is related to the changing position coordinates $s_A$ and $s_B$ by the equation

$$s_A + 3s_B = l$$

**Time Derivative.** Taking the time derivative yields

$$v_A + 3v_B = 0$$

so that when $v_B = -6 \text{ ft/s}$ (upward),

$$v_A = 18 \text{ ft/s} \downarrow \quad \text{Ans.}$$
EXAMPLE 12.22

Determine the speed of \( A \) in Fig. 12–39 if \( B \) has an upward speed of 6 ft/s.

\[ \text{Ans. } v_A = 24 \text{ ft/s} \]

**SOLUTION**

**Position-Coordinate Equation.** As shown, the positions of blocks \( A \) and \( B \) are defined using coordinates \( s_A \) and \( s_B \). Since the system has two cords with segments that change length, it will be necessary to use a third coordinate, \( s_C \), in order to relate \( s_A \) to \( s_B \). In other words, the length of one of the cords can be expressed in terms of \( s_A \) and \( s_C \), and the length of the other cord can be expressed in terms of \( s_B \) and \( s_C \).

The red colored segments of the cords in Fig. 12–39 do not have to be considered in the analysis. Why? For the remaining cord lengths, say \( l_1 \) and \( l_2 \), we have

\[ s_A + 2s_C = l_1 \quad s_B + (s_B - s_C) = l_2 \]

**Time Derivative.** Taking the time derivative of these equations yields

\[ v_A + 2v_C = 0 \quad 2v_B - v_C = 0 \]

Eliminating \( v_C \) produces the relationship between the motions of each cylinder.

\[ v_A + 4v_B = 0 \]

so that when \( v_B = -6 \text{ ft/s} \) (upward),

\[ v_A = +24 \text{ ft/s} = 24 \text{ ft/s} \]

\[ \text{Ans. } v_A = 24 \text{ ft/s} \]
EXAMPLE 12.23

Determine the speed of block $B$ in Fig. 12–40 if the end of the cord at $A$ is pulled down with a speed of 2 m/s.

**SOLUTION**

**Position-Coordinate Equation.** The position of point $A$ is defined by $s_A$, and the position of block $B$ is specified by $s_B$ since point $E$ on the pulley will have the same motion as the block. Both coordinates are measured from a horizontal datum passing through the fixed pin at pulley $D$. Since the system consists of two cords, the coordinates $s_A$ and $s_B$ cannot be related directly. Instead, by establishing a third position coordinate, $s_C$, we can now express the length of one of the cords in terms of $s_B$ and $s_C$, and the length of the other cord in terms of $s_A$, $s_B$, and $s_C$.

Excluding the red colored segments of the cords in Fig. 12–40, the remaining constant cord lengths $l_1$ and $l_2$ (along with the hook and link dimensions) can be expressed as

$$ s_C + s_B = l_1 $$

$$ (s_A - s_C) + (s_B - s_C) + s_B = l_2 $$

**Time Derivative.** The time derivative of each equation gives

$$ v_C + v_B = 0 $$

$$ v_A - 2v_C + 2v_B = 0 $$

Eliminating $v_C$, we obtain

$$ v_A + 4v_B = 0 $$

so that when $v_A = 2$ m/s (downward),

$$ v_B = -0.5 \text{ m/s} = 0.5 \text{ m/s} \uparrow \ \text{Ans.} $$
A man at $A$ is hoisting a safe $S$ as shown in Fig. 12–41 by walking to the right with a constant velocity $v_A = 0.5 \text{ m/s}$. Determine the velocity and acceleration of the safe when it reaches the elevation of 10 m. The rope is 30 m long and passes over a small pulley at $D$.

**SOLUTION**

**Position-Coordinate Equation.** This problem is unlike the previous examples since rope segment $DA$ changes both direction and magnitude. However, the ends of the rope, which define the positions of $S$ and $A$, are specified by means of the $x$ and $y$ coordinates since they must be measured from a fixed point and directed along the paths of motion of the ends of the rope.

The $x$ and $y$ coordinates may be related since the rope has a fixed length which at all times is equal to the length of segment $DA$ plus $CD$. Using the Pythagorean theorem to determine we have

$$l = l_{DA} + l_{CD}$$

$$30 = \sqrt{(15)^2 + x^2} + (15 - y)$$

$$y = \sqrt{225 + x^2 - 15} \quad (1)$$

**Time Derivatives.** Taking the time derivative, using the chain rule (see Appendix C), where $v_S = dy/dt$ and $v_A = dx/dt$, yields

$$v_S = \frac{dy}{dt} = \frac{1}{2} \frac{2x}{\sqrt{225 + x^2}} \frac{dx}{dt} = \frac{x}{\sqrt{225 + x^2}} v_A \quad (2)$$

At $y = 10 \text{ m}$, $x$ is determined from Eq. 1, i.e., $x = 20 \text{ m}$. Hence, from Eq. 2 with $v_A = 0.5 \text{ m/s}$,

$$v_S = \frac{20}{\sqrt{225 + (20)^2}} (0.5) = 0.4 \text{ m/s} = 400 \text{ mm/s} \uparrow \text{ Ans.}$$

The acceleration is determined by taking the time derivative of Eq. 2. Since $v_A$ is constant, then $a_A = dv_A/dt = 0$, and we have

$$a_S = \frac{d^2y}{dt^2} = \left[ \frac{-x(dx/dt)}{(225 + x^2)^{3/2}} \right] v_A + \left[ \frac{1}{(225 + x^2)^{3/2}} \right] \frac{dx}{dt} v_A + \left[ \frac{1}{(225 + x^2)^{3/2}} \right] \frac{dv_A}{dt} = \frac{225v_A^2}{(225 + x^2)^{3/2}}$$

At $x = 20 \text{ m}$, with $v_A = 0.5 \text{ m/s}$, the acceleration becomes

$$a_S = \frac{225(0.5 \text{ m/s})^2}{[225 + (20 \text{ m})^2]^{3/2}} = 0.00360 \text{ m/s}^2 = 3.60 \text{ mm/s}^2 \uparrow \text{ Ans.}$$

**NOTE:** The constant velocity at $A$ causes the other end $C$ of the rope to have an acceleration since $v_A$ causes segment $DA$ to change its direction as well as its length.
12.10 Relative-Motion of Two Particles Using Translating Axes

Throughout this chapter the absolute motion of a particle has been determined using a single fixed reference frame. There are many cases, however, where the path of motion for a particle is complicated, so that it may be easier to analyze the motion in parts by using two or more frames of reference. For example, the motion of a particle located at the tip of an airplane propeller, while the plane is in flight, is more easily described if one observes first the motion of the airplane from a fixed reference and then superimposes (vectorially) the circular motion of the particle measured from a reference attached to the airplane.

In this section translating frames of reference will be considered for the analysis. Relative-motion analysis of particles using rotating frames of reference will be treated in Secs. 16.8 and 20.4, since such an analysis depends on prior knowledge of the kinematics of line segments.

**Position.** Consider particles $A$ and $B$, which move along the arbitrary paths shown in Fig. 12–42. The absolute position of each particle, $\mathbf{r}_A$ and $\mathbf{r}_B$, is measured from the common origin $O$ of the fixed $x$, $y$, $z$ reference frame. The origin of a second frame of reference $x'$, $y'$, $z'$ is attached to and moves with particle $A$. The axes of this frame are only permitted to translate relative to the fixed frame. The position of $B$ measured relative to $A$ is denoted by the relative-position vector $\mathbf{r}_{B/A}$. Using vector addition, the three vectors shown in Fig. 12–42 can be related by the equation

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \quad (12–33)$$

**Velocity.** An equation that relates the velocities of the particles is determined by taking the time derivative of the above equation; i.e.,

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \quad (12–34)$$

Here $\mathbf{v}_B = d\mathbf{r}_B/dt$ and $\mathbf{v}_A = d\mathbf{r}_A/dt$ refer to absolute velocities, since they are observed from the fixed frame; whereas the relative velocity $\mathbf{v}_{B/A} = d\mathbf{r}_{B/A}/dt$ is observed from the translating frame. It is important to note that since the $x'$, $y'$, $z'$ axes translate, the components of $\mathbf{r}_{B/A}$ will not change direction and therefore the time derivative of these components will only have to account for the change in their magnitudes. Equation 12–34 therefore states that the velocity of $B$ is equal to the velocity of $A$ plus (vectorially) the velocity of “$B$ with respect to $A$,” as measured by the translating observer fixed in the $x'$, $y'$, $z'$ reference frame.
Acceleration. The time derivative of Eq. 12–34 yields a similar vector relation between the absolute and relative accelerations of particles A and B.

\[ \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \]  

(12–35)

Here \( \mathbf{a}_{B/A} \) is the acceleration of \( B \) as seen by the observer located at \( A \) and translating with the \( x', y', z' \) reference frame.*

### Procedure For Analysis

- When applying the relative velocity and acceleration equations, it is first necessary to specify the particle \( A \) that is the origin for the translating \( x', y', z' \) axes. Usually this point has a known velocity or acceleration.

- Since vector addition forms a triangle, there can be at most two unknowns, represented by the magnitudes and/or directions of the vector quantities.

- These unknowns can be solved for either graphically, using trigonometry (law of sines, law of cosines), or by resolving each of the three vectors into rectangular or Cartesian components, thereby generating a set of scalar equations.

The pilots of these jet planes flying close to one another must be aware of their relative positions and velocities at all times in order to avoid a collision.

* An easy way to remember the setup of these equations, is to note the “cancellation” of the subscript \( A \) between the two terms, e.g., \( \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \).
EXAMPLE 12.25

A train travels at a constant speed of 60 mi/h, crosses over a road as shown in Fig. 12–43a. If the automobile $A$ is traveling at 45 mi/h along the road, determine the magnitude and direction of the velocity of the train relative to the automobile.

**SOLUTION I**

**Vector Analysis.** The relative velocity $v_{T/A}$ is measured from the translating $x', y'$ axes attached to the automobile, Fig. 12–43a. It is determined from $v_T = v_A + v_{T/A}$. Since $v_T$ and $v_A$ are known in both magnitude and direction, the unknowns become the $x$ and $y$ components of $v_{T/A}$. Using the $x, y$ axes in Fig. 12–43a, we have

$$v_T = v_A + v_{T/A}$$

$$60i = (45 \cos 45°i + 45 \sin 45°j) + v_{T/A}$$

$$v_{T/A} = \{28.2i - 31.8j\} \text{ mi/h} \quad \text{Ans.}$$

The magnitude of $v_{T/A}$ is thus

$$v_{T/A} = \sqrt{(28.2)^2 + (-31.8)^2} = 42.5 \text{ mi/h} \quad \text{Ans.}$$

From the direction of each component, Fig. 12–43b, the direction of $v_{T/A}$ is

$$\tan \theta = \frac{(v_{T/A})_y}{(v_{T/A})_x} = \frac{31.8}{28.2}$$

$$\theta = 48.5° \quad \text{Ans.}$$

Note that the vector addition shown in Fig. 12–43b indicates the correct sense for $v_{T/A}$. This figure anticipates the answer and can be used to check it.

**SOLUTION II**

**Scalar Analysis.** The unknown components of $v_{T/A}$ can also be determined by applying a scalar analysis. We will assume these components act in the positive $x$ and $y$ directions. Thus,

$$v_T = v_A + v_{T/A}$$

$$\begin{bmatrix} 60 \text{ mi/h} \end{bmatrix} = \begin{bmatrix} 45 \text{ mi/h} \end{bmatrix} \angle 45° + \begin{bmatrix} (v_{T/A})_x \end{bmatrix} + \begin{bmatrix} (v_{T/A})_y \end{bmatrix}$$

Resolving each vector into its $x$ and $y$ components yields

$$(\rightarrow) \quad 60 = 45 \cos 45° + (v_{T/A})_x + 0$$

$$(\uparrow) \quad 0 = 45 \sin 45° + 0 + (v_{T/A})_y$$

Solving, we obtain the previous results,

$$(v_{T/A})_x = 28.2 \text{ mi/h} = 28.2 \text{ mi/h} \rightarrow$$

$$(v_{T/A})_y = -31.8 \text{ mi/h} = 31.8 \text{ mi/h} \downarrow$$

**Fig. 12–43**
CHAPTER 12  
KINEMATICS OF A PARTICLE

EXAMPLE 12.26

Plane A in Fig. 12–44a is flying along a straight-line path, whereas plane B is flying along a circular path having a radius of curvature of $\rho_B = 400 \text{ km}$. Determine the velocity and acceleration of B as measured by the pilot of A.

SOLUTION

Velocity. The origin of the $x$ and $y$ axes are located at an arbitrary fixed point. Since the motion relative to plane A is to be determined, the translating frame of reference $x'$, $y'$ is attached to it, Fig. 12–44a. Applying the relative-velocity equation in scalar form since the velocity vectors of both planes are parallel at the instant shown, we have

$$v_B = v_A + v_{B/A}$$

$$600 \text{ km/h} = 700 \text{ km/h} + v_{B/A}$$

$$v_{B/A} = -100 \text{ km/h} = 100 \text{ km/h} \downarrow$$

Ans.

The vector addition is shown in Fig. 12–44b.

Acceleration. Plane B has both tangential and normal components of acceleration since it is flying along a curved path. From Eq. 12–20, the magnitude of the normal component is

$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{(600 \text{ km/h})^2}{400 \text{ km}} = 900 \text{ km/h}^2$$

Applying the relative-acceleration equation gives

$$a_B = a_A + a_{B/A}$$

$$900\mathbf{i} - 100\mathbf{j} = 50\mathbf{j} + a_{B/A}$$

Thus,

$$a_{B/A} = \{900\mathbf{i} - 150\mathbf{j}\} \text{ km/h}^2$$

From Fig. 12–44c, the magnitude and direction of $a_{B/A}$ are therefore

$$a_{B/A} = 912 \text{ km/h}^2 \quad \theta = \tan^{-1}\frac{150}{900} = 9.46^\circ$$

Ans.

NOTE: The solution to this problem was possible using a translating frame of reference, since the pilot in plane A is “translating.” Observation of the motion of plane A with respect to the pilot of plane B, however, must be obtained using a rotating set of axes attached to plane B. (This assumes, of course, that the pilot of B is fixed in the rotating frame, so he does not turn his eyes to follow the motion of A.) The analysis for this case is given in Example 16.21.
At the instant shown in Fig. 12–45a, cars A and B are traveling with speeds of 18 m/s and 12 m/s, respectively. Also at this instant, A has a decrease in speed of 2 m/s², and B has an increase in speed of 3 m/s². Determine the velocity and acceleration of B with respect to A.

**SOLUTION**

**Velocity.** The fixed x, y axes are established at an arbitrary point on the ground and the translating x', y' axes are attached to car A, Fig. 12–45a. Why? The relative velocity is determined from \( \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \). What are the two unknowns? Using a Cartesian vector analysis, we have

\[
\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} = (18 \cos 60^\circ \mathbf{i} - 18 \sin 60^\circ \mathbf{j}) + \{9 \mathbf{i} + 3.588 \mathbf{j}\} \text{ m/s}
\]

Thus,

\[
v_{B/A} = \sqrt{(9)^2 + (3.588)^2} = 9.69 \text{ m/s} \quad \text{Ans.}
\]

Noting that \( \mathbf{v}_{B/A} \) has +i and +j components, Fig. 12–45b, its direction is

\[
\tan \theta = \frac{(v_{B/A})_y}{(v_{B/A})_x} = \frac{3.588}{9}
\]

\[
\theta = 21.7^\circ \quad \text{Ans.}
\]

**Acceleration.** Car B has both tangential and normal components of acceleration. Why? The magnitude of the normal component is

\[
(a_B)_n = \frac{v_{B/A}^2}{\rho} = \frac{(12 \text{ m/s})^2}{100 \text{ m}} = 1.440 \text{ m/s}^2
\]

Applying the equation for relative acceleration yields

\[
\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}
\]

\[
(-1.440\mathbf{i} - 3\mathbf{j}) = (2 \cos 60^\circ \mathbf{i} + 2 \sin 60^\circ \mathbf{j}) + \mathbf{a}_{B/A}
\]

\[
\mathbf{a}_{B/A} = \{-2.440 \mathbf{i} - 4.732 \mathbf{j}\} \text{ m/s}^2
\]

Here \( \mathbf{a}_{B/A} \) has −i and −j components. Thus, from Fig. 12–45c,

\[
a_{B/A} = \sqrt{(2.440)^2 + (4.732)^2} = 5.32 \text{ m/s}^2 \quad \text{Ans.}
\]

\[
\tan \phi = \frac{(a_{B/A})_y}{(a_{B/A})_x} = \frac{4.732}{2.440}
\]

\[
\phi = 62.7^\circ \quad \text{Ans.}
\]

**NOTE:** Is it possible to obtain the relative acceleration of \( \mathbf{a}_{A/B} \) using this method? Refer to the comment made at the end of Example 12.26.
**FUNDAMENTAL PROBLEMS**

**F12–39.** Determine the speed of block $D$ if end $A$ of the rope is pulled down with a speed of $v_A = 3 \text{ m/s}$. 

**F12–40.** Determine the speed of block $A$ if end $B$ of the rope is pulled down with a speed of $6 \text{ m/s}$. 

**F12–39**

**F12–40**

**F12–41.** Determine the speed of block $A$ if end $B$ of the rope is pulled down with a speed of $1.5 \text{ m/s}$. 

**F12–41**

**F12–42.** Determine the speed of block $A$ if end $F$ of the rope is pulled down with a speed of $v_F = 3 \text{ m/s}$. 

**F12–42**

**F12–43.** Determine the speed of car $A$ if point $P$ on the cable has a speed of $4 \text{ m/s}$ when the motor $M$ winds the cable in. 

**F12–43**

**F12–44.** Determine the speed of cylinder $B$ if cylinder $A$ moves downward with a speed of $v_A = 4 \text{ ft/s}$. 

**F12–44**
**F12–45.** Car $A$ is traveling with a constant speed of 80 km/h due north, while car $B$ is traveling with a constant speed of 100 km/h due east. Determine the velocity of car $B$ relative to car $A$.

**F12–46.** Two planes $A$ and $B$ are traveling with the constant velocities shown. Determine the magnitude and direction of the velocity of plane $B$ relative to plane $A$.

**F12–47.** The boats $A$ and $B$ travel with constant speeds of $v_A = 15$ m/s and $v_B = 10$ m/s when they leave the pier at $O$ at the same time. Determine the distance between them when $t = 4$ s.

**F12–48.** At the instant shown, cars $A$ and $B$ are traveling at the speeds shown. If $B$ is accelerating at 1200 km/h$^2$ while $A$ maintains a constant speed, determine the velocity and acceleration of $A$ with respect to $B$. 
PROBLEMS

12–195. The mine car $C$ is being pulled up the incline using the motor $M$ and the rope-and-pulley arrangement shown. Determine the speed $v_p$ at which a point $P$ on the cable must be traveling toward the motor to move the car up the plane with a constant speed of $v = 2 \, \text{m/s}$.

12–196. Determine the displacement of the log if the truck at $C$ pulls the cable 4 ft to the right.

12–197. If the hydraulic cylinder $H$ draws in rod $BC$ at 2 ft/s, determine the speed of slider $A$.

12–198. If end $A$ of the rope moves downward with a speed of $5 \, \text{m/s}$, determine the speed of cylinder $B$.

12–199. Determine the speed of the elevator if each motor draws in the cable with a constant speed of $5 \, \text{m/s}$.
12–200. Determine the speed of cylinder $A$, if the rope is drawn towards the motor $M$ at a constant rate of 10 m/s.

12–201. If the rope is drawn towards the motor $M$ at a speed of $v_M = (5t^{1/2})$ m/s, where $t$ is in seconds, determine the speed of cylinder $A$ when $t = 1$ s.

12–202. If the end of the cable at $A$ is pulled down with a speed of 2 m/s, determine the speed at which block $B$ rises.

12–203. Determine the speed of $B$ if $A$ is moving downwards with a speed of $v_A = 4$ m/s at the instant shown.

12–204. The crane is used to hoist the load. If the motors at $A$ and $B$ are drawing in the cable at a speed of 2 ft/s and 4 ft/s, respectively, determine the speed of the load.
•12–205. The cable at B is pulled downwards at 4 ft/s, and the speed is decreasing at 2 ft/s². Determine the velocity and acceleration of block A at this instant.

12–206. If block A is moving downward with a speed of 4 ft/s while C is moving up at 2 ft/s, determine the speed of block B.

12–207. If block A is moving downward at 6 ft/s while block C is moving down at 18 ft/s, determine the speed of block B.

•12–208. If the end of the cable at A is pulled down with a speed of 2 m/s, determine the speed at which block E rises.

•12–209. If motors at A and B draw in their attached cables with an acceleration of \( a = (0.2t) \text{ m/s}^2 \), where \( t \) is in seconds, determine the speed of the block when it reaches a height of \( h = 4 \text{ m} \), starting from rest at \( h = 0 \). Also, how much time does it take to reach this height?
12–210. The motor at C pulls in the cable with an acceleration \( a_C = (3t^2) \text{ m/s}^2 \), where \( t \) is in seconds. The motor at D draws in its cable at \( a_D = 5 \text{ m/s}^2 \). If both motors start at the same instant from rest when \( d = 3 \text{ m} \), determine (a) the time needed for \( d = 0 \), and (b) the velocities of blocks A and B when this occurs.

12–211. The motion of the collar at A is controlled by a motor at B such that when the collar is at \( s_A = 3 \text{ ft} \) it is moving upwards at 2 ft/s and decreasing at 1 ft/s\(^2\). Determine the velocity and acceleration of a point on the cable as it is drawn into the motor B at this instant.

12–212. The man pulls the boy up to the tree limb C by walking backward at a constant speed of 1.5 m/s. Determine the speed at which the boy is being lifted at the instant \( x_A = 4 \text{ m} \). Neglect the size of the limb. When \( x_A = 0 \), \( y_B = 8 \text{ m} \), so that A and B are coincident, i.e., the rope is 16 m long.

12–213. The man pulls the boy up to the tree limb C by walking backward. If he starts from rest when \( x_A = 0 \) and moves backward with a constant acceleration \( a_A = 0.2 \text{ m/s}^2 \), determine the speed of the boy at the instant \( y_B = 4 \text{ m} \). Neglect the size of the limb. When \( x_A = 0 \), \( y_B = 8 \text{ m} \), so that A and B are coincident, i.e., the rope is 16 m long.

12–214. If the truck travels at a constant speed of \( v_T = 6 \text{ ft/s} \), determine the speed of the crate for any angle \( \theta \) of the rope. The rope has a length of 100 ft and passes over a pulley of negligible size at A. Hint: Relate the coordinates \( x_T \) and \( x_C \) to the length of the rope and take the time derivative. Then substitute the trigonometric relation between \( x_C \) and \( \theta \).
12–215. At the instant shown, car $A$ travels along the straight portion of the road with a speed of 25 m/s. At this same instant car $B$ travels along the circular portion of the road with a speed of 15 m/s. Determine the velocity of car $B$ relative to car $A$.

12–218. The ship travels at a constant speed of $v_s = 20$ m/s and the wind is blowing at a speed of $v_w = 10$ m/s, as shown. Determine the magnitude and direction of the horizontal component of velocity of the smoke coming from the smoke stack as it appears to a passenger on the ship.

12–216. Car $A$ travels along a straight road at a speed of 25 m/s while accelerating at 1.5 m/s$^2$. At this same instant car $C$ is traveling along the straight road with a speed of 30 m/s while decelerating at 3 m/s$^2$. Determine the velocity and acceleration of car $A$ relative to car $C$.

12–217. Car $B$ is traveling along the curved road with a speed of 15 m/s while decreasing its speed at 2 m/s$^2$. At this same instant car $C$ is traveling along the straight road with a speed of 30 m/s while decelerating at 3 m/s$^2$. Determine the velocity and acceleration of car $B$ relative to car $C$.

12–219. The car is traveling at a constant speed of 100 km/h. If the rain is falling at 6 m/s in the direction shown, determine the velocity of the rain as seen by the driver.

12–220. The man can row the boat in still water with a speed of 5 m/s. If the river is flowing at 2 m/s, determine the speed of the boat and the angle $\theta$ he must direct the boat so that it travels from $A$ to $B$. 
12–221. At the instant shown, cars A and B travel at speeds of 30 mi/h and 20 mi/h, respectively. If B is increasing its speed by 1200 mi/h^2, while A maintains a constant speed, determine the velocity and acceleration of B with respect to A.

12–222. At the instant shown, cars A and B travel at speeds of 30 m/h and 20 m/h, respectively. If A is increasing its speed at 400 m/s^2 whereas the speed of B is decreasing at 800 m/s^2, determine the velocity and acceleration of B with respect to A.

12–223. Two boats leave the shore at the same time and travel in the directions shown. If \( v_A = 20 \text{ ft/s} \) and \( v_B = 15 \text{ ft/s} \), determine the velocity of boat A with respect to boat B. How long after leaving the shore will the boats be 800 ft apart?

12–224. At the instant shown, cars A and B travel at speeds of 70 mi/h and 50 mi/h, respectively. If B is increasing its speed by 1100 mi/h^2, while A maintains a constant speed, determine the velocity and acceleration of B with respect to A. Car B moves along a curve having a radius of curvature of 0.7 mi.

12–225. At the instant shown, cars A and B travel at speeds of 70 mi/h and 50 mi/h, respectively. If A is increasing its speed at whereas the speed of B is decreasing at \( 800 \text{ mi/h}^2 \), determine the acceleration of B with respect to A. Car B moves along a curve having a radius of curvature of 0.7 mi.

12–226. An aircraft carrier is traveling forward with a velocity of 50 km/h. At the instant shown, the plane at A has just taken off and has attained a forward horizontal air speed of 200 km/h, measured from still water. If the plane at B is traveling along the runway of the carrier at 175 km/h in the direction shown, determine the velocity of A with respect to B.

12–227. A car is traveling north along a straight road at 50 km/h. An instrument in the car indicates that the wind is directed towards the east. If the car’s speed is 80 km/h, the instrument indicates that the wind is directed towards the north-east. Determine the speed and direction of the wind.
12–228. At the instant shown car \( A \) is traveling with a velocity of 30 m/s and has an acceleration of 2 m/s\(^2\) along the highway. At the same instant \( B \) is traveling on the trumpet interchange curve with a speed of 15 m/s, which is decreasing at 0.8 m/s\(^2\). Determine the relative velocity and relative acceleration of \( B \) with respect to \( A \) at this instant.

12–230. A man walks at 5 km/h in the direction of a 20-km/h wind. If raindrops fall vertically at 7 km/h in still air, determine the direction in which the drops appear to fall with respect to the man. Assume the horizontal speed of the raindrops is equal to that of the wind.

12–229. Two cyclists \( A \) and \( B \) travel at the same constant speed \( v \). Determine the velocity of \( A \) with respect to \( B \) if \( A \) travels along the circular track, while \( B \) travels along the diameter of the circle.

12–231. A man can row a boat at 5 m/s in still water. He wishes to cross a 50-m-wide river to point \( B \), 50 m downstream. If the river flows with a velocity of 2 m/s, determine the speed of the boat and the time needed to make the crossing.
CONCEPTUAL PROBLEMS

P12-1. If you measured the time it takes for the construction elevator to go from A to B, then B to C, and then C to D, and you also know the distance between each of the points, how could you determine the average velocity and average acceleration of the elevator as it ascends from A to D? Use numerical values to explain how this can be done.

P12-2. If the sprinkler at A is 1 m from the ground, then scale the necessary measurements from the photo to determine the approximate velocity of the water jet as it flows from the nozzle of the sprinkler.

P12-3. The basketball was thrown at an angle measured from the horizontal to the man’s outstretched arms. If the basket is 10 ft from the ground, make appropriate measurements in the photo and determine if the ball located as shown will pass through the basket.

P12-4. The pilot tells you the wingspan of her plane and her constant airspeed. How would you determine the acceleration of the plane at the moment shown? Use numerical values and take any necessary measurements from the photo.
### CHAPTER REVIEW

**Rectilinear Kinematics**
Rectilinear kinematics refers to motion along a straight line. A position coordinate \( s \) specifies the location of the particle on the line, and the displacement \( \Delta s \) is the change in this position.

The average velocity is a vector quantity, defined as the displacement divided by the time interval.

\[
v_{\text{avg}} = \frac{-\Delta s}{\Delta t}
\]

The average speed is a scalar, and is the total distance traveled divided by the time of travel.

\[
(v_{\text{sp}})_{\text{avg}} = \frac{s_f}{\Delta t}
\]

The time, position, velocity, and acceleration are related by three differential equations.

If the acceleration is known to be constant, then the differential equations relating time, position, velocity, and acceleration can be integrated.

\[
a = \frac{dv}{dt}, \quad v = \frac{ds}{dt}, \quad a \; ds = v \; dv
\]

\[
v = v_0 + a_c t
\]

\[
s = s_0 + v_0 t + \frac{1}{2} a_c t^2
\]

\[
v^2 = v_0^2 + 2a_c(s - s_0)
\]

**Graphical Solutions**
If the motion is erratic, then it can be described by a graph. If one of these graphs is given, then the others can be established using the differential relations between \( a, v, s, \) and \( t \).

\[
a = \frac{dv}{dt}
\]

\[
v = \frac{ds}{dt}
\]

\[
a \; ds = v \; dv
\]
Curvilinear Motion, $x, y, z$
Curvilinear motion along the path can be resolved into rectilinear motion along the $x, y, z$ axes. The equation of the path is used to relate the motion along each axis.

\[
\begin{align*}
\dot{v}_x &= \dot{x} & a_x &= \ddot{x} \\
\dot{v}_y &= \dot{y} & a_y &= \ddot{y} \\
\dot{v}_z &= \dot{z} & a_z &= \ddot{z}
\end{align*}
\]

Projectile Motion
Free-flight motion of a projectile follows a parabolic path. It has a constant velocity in the horizontal direction, and a constant downward acceleration of $g = 9.81 \, \text{m/s}^2$ or $32.2 \, \text{ft/s}^2$ in the vertical direction. Any two of the three equations for constant acceleration apply in the vertical direction, and in the horizontal direction only one equation applies.

\[
\begin{align*}
(+\uparrow) \quad v_y &= (v_0)_y + a_y t \\
(+\uparrow) \quad y &= y_0 + (v_0)_y t + \frac{1}{2} a_y t^2 \\
(+\uparrow) \quad v_y^2 &= (v_0)_y^2 + 2a_y(y - y_0) \\
(\downarrow) \quad x &= x_0 + (v_0)_x t
\end{align*}
\]
### Curvilinear Motion \( n, t \)

If normal and tangential axes are used for the analysis, then \( \mathbf{v} \) is always in the positive \( t \) direction.

The acceleration has two components. The tangential component, \( \mathbf{a}_t \), accounts for the change in the magnitude of the velocity; a slowing down is in the negative \( t \) direction, and a speeding up is in the positive \( t \) direction. The normal component \( \mathbf{a}_n \), accounts for the change in the direction of the velocity. This component is always in the positive \( n \) direction.

\[
\mathbf{a}_t = \dot{v} \quad \text{or} \quad a_t ds = v \, dv
\]
\[
a_n = \frac{v^2}{\rho}
\]

### Curvilinear Motion \( r, \theta \)

If the path of motion is expressed in polar coordinates, then the velocity and acceleration components can be related to the time derivatives of \( r \) and \( \theta \).

To apply the time-derivative equations, it is necessary to determine \( r, \dot{r}, \ddot{r}, \dot{\theta}, \ddot{\theta} \) at the instant considered. If the path \( r = f(\theta) \) is given, then the chain rule of calculus must be used to obtain time derivatives. (See Appendix C.)

Once the data are substituted into the equations, then the algebraic sign of the results will indicate the direction of the components of \( \mathbf{v} \) or \( \mathbf{a} \) along each axis.
Absolute Dependent Motion of Two Particles
The dependent motion of blocks that are suspended from pulleys and cables can be related by the geometry of the system. This is done by first establishing position coordinates, measured from a fixed origin to each block. Each coordinate must be directed along the line of motion of a block.

Using geometry and/or trigonometry, the coordinates are then related to the cable length in order to formulate a position coordinate equation.

The first time derivative of this equation gives a relationship between the velocities of the blocks, and a second time derivative gives the relation between their accelerations.

Relative-Motion Analysis Using Translating Axes
If two particles A and B undergo independent motions, then these motions can be related to their relative motion using a translating set of axes attached to one of the particles (A).

For planar motion, each vector equation produces two scalar equations, one in the x, and the other in the y direction. For solution, the vectors can be expressed in Cartesian form, or the x and y scalar components can be written directly.
The design of conveyors for a bottling plant requires knowledge of the forces that act on them and the ability to predict the motion of the bottles they transport.
CHAPTER OBJECTIVES

• To state Newton’s Second Law of Motion and to define mass and weight.

• To analyze the accelerated motion of a particle using the equation of motion with different coordinate systems.

• To investigate central-force motion and apply it to problems in space mechanics.

13.1 Newton’s Second Law of Motion

Kinetics is a branch of dynamics that deals with the relationship between the change in motion of a body and the forces that cause this change. The basis for kinetics is Newton’s second law, which states that when an unbalanced force acts on a particle, the particle will accelerate in the direction of the force with a magnitude that is proportional to the force.

This law can be verified experimentally by applying a known unbalanced force \( \mathbf{F} \) to a particle, and then measuring the acceleration \( \mathbf{a} \). Since the force and acceleration are directly proportional, the constant of proportionality, \( m \), may be determined from the ratio \( m = F/a \). This positive scalar \( m \) is called the mass of the particle. Being constant during any acceleration, \( m \) provides a quantitative measure of the resistance of the particle to a change in its velocity, that is its inertia.
If the mass of the particle is \( m \), Newton’s second law of motion may be written in mathematical form as

\[
F = ma
\]

The above equation, which is referred to as the *equation of motion*, is one of the most important formulations in mechanics.* As previously stated, its validity is based solely on *experimental evidence*. In 1905, however, Albert Einstein developed the theory of relativity and placed limitations on the use of Newton’s second law for describing general particle motion. Through experiments it was proven that *time* is not an absolute quantity as assumed by Newton; and as a result, the equation of motion fails to predict the exact behavior of a particle, especially when the particle’s speed approaches the speed of light \((0.3 \text{ Gm/s})\). Developments of the theory of quantum mechanics by Erwin Schrödinger and others indicate further that conclusions drawn from using this equation are also invalid when particles are the size of an atom and move close to one another. For the most part, however, these requirements regarding particle speed and size are not encountered in engineering problems, so their effects will not be considered in this book.

**Newton’s Law of Gravitational Attraction.** Shortly after formulating his three laws of motion, Newton postulated a law governing the mutual attraction between any two particles. In mathematical form this law can be expressed as

\[
F = G \frac{m_1 m_2}{r^2}
\]

(13–1)

where

- \( F \) = force of attraction between the two particles
- \( G \) = universal constant of gravitation; according to experimental evidence \( G = 66.73(10^{-12}) \text{ m}^3/(\text{kg} \cdot \text{s}^2) \)
- \( m_1, m_2 \) = mass of each of the two particles
- \( r \) = distance between the centers of the two particles

*Since \( m \) is constant, we can also write \( F = d(mv)/dt \), where \( mv \) is the particle’s linear momentum. Here the unbalanced force acting on the particle is proportional to the time rate of change of the particle’s linear momentum.
13.1 NEWTON'S SECOND LAW OF MOTION

In the case of a particle located at or near the surface of the earth, the only gravitational force having any sizable magnitude is that between the earth and the particle. This force is termed the "weight" and, for our purpose, it will be the only gravitational force considered.

From Eq. 13–1, we can develop a general expression for finding the weight \( W \) of a particle having a mass \( m_1 = m \). Let \( m_2 = M_e \) be the mass of the earth and \( r \) the distance between the earth's center and the particle. Then, if \( g = \frac{GM_e}{r^2} \), we have

\[
W = mg
\]

By comparison with \( F = ma \), we term \( g \) the acceleration due to gravity. For most engineering calculations \( g \) is a point on the surface of the earth at sea level, and at a latitude of 45°, which is considered the "standard location." Here the values \( g = 9.81 \text{ m/s}^2 = 32.2 \text{ ft/s}^2 \) will be used for calculations.

In the SI system the mass of the body is specified in kilograms, and the weight must be calculated using the above equation, Fig. 13–1a. Thus,

\[
W = mg \quad (g = 9.81 \text{ m/s}^2)
\]

(13–2)

As a result, a body of mass 1 kg has a weight of 9.81 N; a 2-kg body weighs 19.62 N; and so on.

In the FPS system the weight of the body is specified in pounds. The mass is measured in slugs, a term derived from "sluggish" which refers to the body's inertia. It must be calculated, Fig. 13–1b, using

\[
m = \frac{W}{g} \quad (g = 32.2 \text{ ft/s}^2)
\]

(13–3)

Therefore, a body weighing 32.2 lb has a mass of 1 slug; a 64.4-lb body has a mass of 2 slugs; and so on.
13.2 The Equation of Motion

When more than one force acts on a particle, the resultant force is determined by a vector summation of all the forces; i.e., \( \mathbf{F}_R = \Sigma \mathbf{F} \). For this more general case, the equation of motion may be written as

\[
\Sigma \mathbf{F} = m \mathbf{a}
\]  
(13–4)

To illustrate application of this equation, consider the particle shown in Fig. 13–2a, which has a mass \( m \) and is subjected to the action of two forces, \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \). We can graphically account for the magnitude and direction of each force acting on the particle by drawing the particle’s free-body diagram, Fig. 13–2b. Since the resultant of these forces produces the vector \( m \mathbf{a} \), its magnitude and direction can be represented graphically on the kinetic diagram, shown in Fig. 13–2c.* The equal sign written between the diagrams symbolizes the graphical equivalency between the free-body diagram and the kinetic diagram; i.e., \( \Sigma \mathbf{F} = m \mathbf{a} \).† In particular, note that if \( \mathbf{F}_R = \Sigma \mathbf{F} = \mathbf{0} \), then the acceleration is also zero, so that the particle will either remain at rest or move along a straight-line path with constant velocity. Such are the conditions of static equilibrium, Newton’s first law of motion.

**Inertial Reference Frame.** When applying the equation of motion, it is important that the acceleration of the particle be measured with respect to a reference frame that is either fixed or translates with a constant velocity. In this way, the observer will not accelerate and measurements of the particle’s acceleration will be the same from any reference of this type. Such a frame of reference is commonly known as a Newtonian or inertial reference frame, Fig. 13–3.

When studying the motions of rockets and satellites, it is justifiable to consider the inertial reference frame as fixed to the stars, whereas dynamics problems concerned with motions on or near the surface of the earth may be solved by using an inertial frame which is assumed fixed to the earth. Even though the earth both rotates about its own axis and revolves about the sun, the accelerations created by these rotations are relatively small and so they can be neglected for most applications.

---

*Recall the free-body diagram considers the particle to be free of its surrounding supports and shows all the forces acting on the particle. The kinetic diagram pertains to the particle’s motion as caused by the forces.

†The equation of motion can also be rewritten in the form \( \Sigma \mathbf{F} - m \mathbf{a} = \mathbf{0} \). The vector \(-m \mathbf{a}\) is referred to as the inertial force vector. If it is treated in the same way as a “force vector,” then the state of “equilibrium” created is referred to as dynamic equilibrium. This method of application is often referred to as the D’Alembert principle, named after the French mathematician Jean le Rond d’Alembert.
We are all familiar with the sensation one feels when sitting in a car that is subjected to a forward acceleration. Often people think this is caused by a “force” which acts on them and tends to push them back in their seats; however, this is not the case. Instead, this sensation occurs due to their inertia or the resistance of their mass to a change in velocity.

Consider the passenger who is strapped to the seat of a rocket sled. Provided the sled is at rest or is moving with constant velocity, then no force is exerted on his back as shown on his free-body diagram.

When the thrust of the rocket engine causes the sled to accelerate, then the seat upon which he is sitting exerts a force \( F \) on him which pushes him forward with the sled. In the photo, notice that the inertia of his head resists this change in motion (acceleration), and so his head moves back against the seat and his face, which is nonrigid, tends to distort backward.

Upon deceleration the force of the seatbelt \( F' \) tends to pull his body to a stop, but his head leaves contact with the back of the seat and his face distorts forward, again due to his inertia or tendency to continue to move forward. No force is pulling him forward, although this is the sensation he receives.
13.3 Equation of Motion for a System of Particles

The equation of motion will now be extended to include a system of particles isolated within an enclosed region in space, as shown in Fig. 13–4a. In particular, there is no restriction in the way the particles are connected, so the following analysis applies equally well to the motion of a solid, liquid, or gas system.

At the instant considered, the arbitrary \(i\)-th particle, having a mass \(m_i\), is subjected to a system of internal forces and a resultant external force. The \textit{internal force}, represented symbolically as \(\mathbf{f}_i\), is the resultant of all the forces the other particles exert on the \(i\)th particle. The \textit{resultant external force} \(\mathbf{F}_i\) represents, for example, the effect of gravitational, electrical, magnetic, or contact forces between the \(i\)th particle and adjacent bodies or particles \textit{not} included within the system.

The free-body and kinetic diagrams for the \(i\)th particle are shown in Fig. 13–4b. Applying the equation of motion,

\[ \Sigma \mathbf{F} = m \mathbf{a}_i; \quad \mathbf{F}_i + \mathbf{f}_i = m_i \mathbf{a}_i \]

When the equation of motion is applied to each of the other particles of the system, similar equations will result. And, if all these equations are added together \textit{vectorially}, we obtain

\[ \Sigma \mathbf{F}_i + \Sigma \mathbf{f}_i = \Sigma m_i \mathbf{a}_i \]

\[ \text{Inertial coordinate system} \quad \text{Free-body diagram} \quad \text{Kinetic diagram} \]

\[ \text{Fig. 13–4} \]
The summation of the internal forces, if carried out, will equal zero, since internal forces between any two particles occur in equal but opposite collinear pairs. Consequently, only the sum of the external forces will remain, and therefore the equation of motion, written for the system of particles, becomes

$$\sum F_i = \sum m_i a_i$$  \hspace{1cm} (13–5)

If \( \mathbf{r}_G \) is a position vector which locates the *center of mass* \( G \) of the particles, Fig. 13–4a, then by definition of the center of mass, \( m\mathbf{r}_G = \sum m_i \mathbf{r}_i \), where \( m = \sum m_i \) is the total mass of all the particles. Differentiating this equation twice with respect to time, assuming that no mass is entering or leaving the system, yields

$$ma_G = \sum m_i a_i$$

Substituting this result into Eq. 13–5, we obtain

$$\sum F = ma_G$$  \hspace{1cm} (13–6)

Hence, the sum of the external forces acting on the system of particles is equal to the total mass of the particles times the acceleration of its center of mass \( G \). Since in reality all particles must have a finite size to possess mass, Eq. 13–6 justifies application of the equation of motion to a *body* that is represented as a single particle.

**Important Points**

- The equation of motion is based on experimental evidence and is valid only when applied within an inertial frame of reference.
- The equation of motion states that the *unbalanced force* on a particle causes it to *accelerate*.
- An inertial frame of reference does not rotate, rather its axes either translate with constant velocity or are at rest.
- Mass is a property of matter that provides a quantitative measure of its resistance to a change in velocity. It is an absolute quantity and so it does not change from one location to another.
- Weight is a force that is caused by the earth’s gravitation. It is not absolute; rather it depends on the altitude of the mass from the earth’s surface.
13.4 Equations of Motion: Rectangular Coordinates

When a particle moves relative to an inertial $x, y, z$ frame of reference, the forces acting on the particle, as well as its acceleration, can be expressed in terms of their $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components, Fig. 13–5. Applying the equation of motion, we have

$$\sum \mathbf{F} = m \mathbf{a}; \quad \sum F_x \mathbf{i} + \sum F_y \mathbf{j} + \sum F_z \mathbf{k} = m(a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k})$$

For this equation to be satisfied, the respective $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components on the left side must equal the corresponding components on the right side. Consequently, we may write the following three scalar equations:

$$\begin{align*}
\sum F_x &= ma_x \\
\sum F_y &= ma_y \\
\sum F_z &= ma_z
\end{align*} \quad (13–7)$$

In particular, if the particle is constrained to move only in the $x$–$y$ plane, then the first two of these equations are used to specify the motion.

### Procedure for Analysis

The equations of motion are used to solve problems which require a relationship between the forces acting on a particle and the accelerated motion they cause.

**Free-Body Diagram.**

- Select the inertial coordinate system. Most often, rectangular or $x, y, z$ coordinates are chosen to analyze problems for which the particle has rectilinear motion.
- Once the coordinates are established, draw the particle’s free-body diagram. Drawing this diagram is very important since it provides a graphical representation that accounts for all the forces (\(\Sigma \mathbf{F}\)) which act on the particle, and thereby makes it possible to resolve these forces into their $x, y, z$ components.
- The direction and sense of the particle’s acceleration $\mathbf{a}$ should also be established. If the sense is unknown, for mathematical convenience assume that the sense of each acceleration component acts in the same direction as its positive inertial coordinate axis.
- The acceleration may be represented as the $ma$ vector on the kinetic diagram.*
- Identify the unknowns in the problem.

---

*It is a convention in this text always to use the kinetic diagram as a graphical aid when developing the proofs and theory. The particle’s acceleration or its components will be shown as blue colored vectors near the free-body diagram in the examples.
13.4 EQUATIONS OF MOTION: RECTANGULAR COORDINATES

**Equations of Motion.**
- If the forces can be resolved directly from the free-body diagram, apply the equations of motion in their scalar component form.
- If the geometry of the problem appears complicated, which often occurs in three dimensions, Cartesian vector analysis can be used for the solution.

**Friction.** If a moving particle contacts a rough surface, it may be necessary to use the frictional equation, which relates the frictional and normal forces \( F_f \) and \( N \) acting at the surface of contact by using the coefficient of kinetic friction, i.e., \( F_f = \mu_k N \). Remember that \( F_f \) always acts on the free-body diagram such that it opposes the motion of the particle relative to the surface it contacts. If the particle is on the verge of relative motion, then the coefficient of static friction should be used.

**Spring.** If the particle is connected to an elastic spring having negligible mass, the spring force \( F_s \) can be related to the deformation of the spring by the equation \( F_s = ks \). Here \( k \) is the spring’s stiffness measured as a force per unit length, and \( s \) is the stretch or compression defined as the difference between the deformed length \( l \) and the undeformed length \( l_0 \), i.e., \( s = l - l_0 \).

**Kinematics.**
- If the velocity or position of the particle is to be found, it will be necessary to apply the necessary kinematic equations once the particle’s acceleration is determined from \( \Sigma F = ma \).
- If acceleration is a function of time, use \( a = dv/dt \) and \( v = ds/dt \) which, when integrated, yield the particle’s velocity and position, respectively.
- If acceleration is a function of displacement, integrate \( a \, ds = v \, dv \) to obtain the velocity as a function of position.
- If acceleration is constant, use \( v = v_0 + a_t t \), \( s = s_0 + v_0 t + \frac{1}{2} a_t t^2 \), \( v^2 = v_0^2 + 2a_t (s - s_0) \) to determine the velocity or position of the particle.
- If the problem involves the dependent motion of several particles, use the method outlined in Sec. 12.9 to relate their accelerations. In all cases, make sure the positive inertial coordinate directions used for writing the kinematic equations are the same as those used for writing the equations of motion; otherwise, simultaneous solution of the equations will result in errors.
- If the solution for an unknown vector component yields a negative scalar, it indicates that the component acts in the direction opposite to that which was assumed.
The 50-kg crate shown in Fig. 13–6a rests on a horizontal surface for which the coefficient of kinetic friction is \( \mu_k = 0.3 \). If the crate is subjected to a 400-N towing force as shown, determine the velocity of the crate in 3 s starting from rest.

**SOLUTION**

Using the equations of motion, we can relate the crate’s acceleration to the force causing the motion. The crate’s velocity can then be determined using kinematics.

**Free-Body Diagram.** The weight of the crate is \( W = mg = 50 \text{ kg} \times (9.81 \text{ m/s}^2) = 490.5 \text{ N} \). As shown in Fig. 13–6b, the frictional force has a magnitude \( F = \mu_k N_C \) and acts to the left, since it opposes the motion of the crate. The acceleration \( a \) is assumed to act horizontally, in the positive \( x \) direction. There are two unknowns, namely \( N_C \) and \( a \).

**Equations of Motion.** Using the data shown on the free-body diagram, we have

\[
\begin{align*}
\downarrow \sum F_x &= ma_x; \quad 400 \cos 30^\circ - 0.3 N_C = 50a \\
\uparrow \sum F_y &= ma_y; \quad N_C - 490.5 + 400 \sin 30^\circ = 0
\end{align*}
\]

Solving Eq. 2 for \( N_C \), substituting the result into Eq. 1, and solving for \( a \) yields

\[
N_C = 290.5 \text{ N} \\
a = 5.185 \text{ m/s}^2
\]

**Kinematics.** Notice that the acceleration is *constant*, since the applied force \( P \) is constant. Since the initial velocity is zero, the velocity of the crate in 3 s is

\[
(\downarrow) \quad v = v_0 + at = 0 + 5.185(3) \\
= 15.6 \text{ m/s} \rightarrow \quad \text{Ans.}
\]

**NOTE:** We can also use the alternative procedure of drawing the crate’s free-body and kinetic diagrams, Fig. 13–6c, prior to applying the equations of motion.
EXAMPLE 13.2

A 10-kg projectile is fired vertically upward from the ground, with an initial velocity of 50 m/s, Fig. 13–7a. Determine the maximum height to which it will travel if (a) atmospheric resistance is neglected; and (b) atmospheric resistance is measured as where is the speed of the projectile at any instant, measured in m/s.

SOLUTION

In both cases the known force on the projectile can be related to its acceleration using the equation of motion. Kinematics can then be used to relate the projectile’s acceleration to its position.

Part (a) Free-Body Diagram. As shown in Fig. 13–7b, the projectile’s weight is \( W = mg = 10(9.81) = 98.1 \) N. We will assume the unknown acceleration \( a \) acts upward in the positive \( z \) direction.

Equation of Motion.

\[
+ \sum F_z = ma_z: \quad -98.1 = 10a, \quad a = -9.81 \text{ m/s}^2
\]

The result indicates that the projectile, like every object having free-flight motion near the earth’s surface, is subjected to a constant downward acceleration of 9.81 m/s².

Kinematics. Initially, \( z_0 = 0 \) and \( v_0 = 50 \) m/s, and at the maximum height \( z = h, v = 0 \). Since the acceleration is constant, then

\[
\begin{align*}
(+ \uparrow) & \quad v^2 = v_0^2 + 2a(z - z_0) \\
& \quad 0 = (50)^2 + 2(-9.81)(h - 0) \\
& \quad h = 127 \text{ m} \quad \text{Ans.}
\end{align*}
\]

Part (b) Free-Body Diagram. Since the force \( F_D = (0.01v^2) \) N tends to retard the upward motion of the projectile, it acts downward as shown on the free-body diagram, Fig. 13–7c.

Equation of Motion.

\[
+ \sum F_z = ma_z: \quad -0.01v^2 - 9.81 = 10a, \quad a = -(0.001v^2 + 9.81)
\]

Kinematics. Here the acceleration is not constant since \( F_D \) depends on the velocity. Since \( a = f(v) \), we can relate \( a \) to position using

\[
(+ \uparrow) \ a \, dz = v \, dv; \quad -(0.001v^2 + 9.81) \, dz = v \, dv
\]

Separating the variables and integrating, realizing that initially \( z_0 = 0, v_0 = 50 \) m/s (positive upward), and at \( z = h, v = 0 \), we have

\[
\begin{align*}
\int_0^h dz &= - \int_{50}^0 \frac{v \, dv}{0.001v^2 + 9.81} \\
&= -500 \ln(\frac{v^2 + 9810}{50^2\text{m/s}}) \\
&= 114 \text{ m} \quad \text{Ans.}
\end{align*}
\]

NOTE: The answer indicates a lower elevation than that obtained in part (a) due to atmospheric resistance or drag.
EXAMPLE 13.3

The baggage truck $A$ shown in the photo has a weight of 900 lb and tows a 550-lb cart $B$ and a 325-lb cart $C$. For a short time the driving frictional force developed at the wheels of the truck is $F_A = (40t)$ lb, where $t$ is in seconds. If the truck starts from rest, determine its speed in 2 seconds. Also, what is the horizontal force acting on the coupling between the truck and cart $B$ at this instant? Neglect the size of the truck and carts.

\[
FA = 140t^2 \text{ lb},
\]

\[\text{EXAMPLE 13.3}\]

\[\text{SOLUTION}\]

\text{Free-Body Diagram.} As shown in Fig. 13–8a, it is the frictional driving force that gives both the truck and carts an acceleration. Here we have considered all three vehicles as a single system.

\text{Equation of Motion.} Only motion in the horizontal direction has to be considered.

\[
\sum F_x = ma_x; \quad 40t = \left(\frac{900 + 550 + 325}{32.2}\right)a
\]

\[a = 0.7256t
\]

\text{Kinematics.} Since the acceleration is a function of time, the velocity of the truck is obtained using $a = dv/dt$ with the initial condition that $v_0 = 0$ at $t = 0$. We have

\[
\int_0^2 dv = \int_0^{2s} 0.7256t \, dt; \quad v = 0.3628t^2 \bigg|_0^{2s} = 1.45 \text{ ft/s} \quad \text{Ans.}
\]

\text{Free-Body Diagram.} In order to determine the force between the truck and cart $B$, we will consider a free-body diagram of the truck so that we can “expose” the coupling force $T$ as external to the free-body diagram, Fig. 13–8b.

\text{Equation of Motion.} When $t = 2$ s, then

\[
\sum F_x = ma_x; \quad 40(2) - T = \left(\frac{900}{32.2}\right)[0.7256(2)]
\]

\[T = 39.4 \text{ lb} \quad \text{Ans.}
\]

\text{NOTE:} Try and obtain this same result by considering a free-body diagram of carts $B$ and $C$ as a single system.
EXAMPLE 13.4

A smooth 2-kg collar $C$, shown in Fig. 13–9a, is attached to a spring having a stiffness $k = 3 \text{ N/m}$ and an unstretched length of 0.75 m. If the collar is released from rest at $A$, determine its acceleration and the normal force of the rod on the collar at the instant $y = 1 \text{ m}$.

SOLUTION

Free-Body Diagram. The free-body diagram of the collar when it is located at the arbitrary position $y$ is shown in Fig. 13–9b. Furthermore, the collar is assumed to be accelerating so that “$a$” acts downward in the positive $y$ direction. There are four unknowns, namely, $N_C$, $F_s$, $a$, and $\theta$.

Equations of Motion.

$$\Sigma F_x = ma_x; \quad -N_C + F_s \cos \theta = 0 \quad (1)$$

$$\Sigma F_y = ma_y; \quad 19.62 - F_s \sin \theta = 2a \quad (2)$$

From Eq. 2 it is seen that the acceleration depends on the magnitude and direction of the spring force. Solution for $N_C$ and $a$ is possible once $F_s$ and $\theta$ are known.

The magnitude of the spring force is a function of the stretch $s$ of the spring; i.e., $F_s = ks$. Here the unstretched length is $AB = 0.75 \text{ m}$, Fig. 13–9a; therefore, $s = CB - AB = \sqrt{y^2 + (0.75)^2} - 0.75$. Since $k = 3 \text{ N/m}$, then

$$F_s = ks = 3\left(\sqrt{y^2 + (0.75)^2} - 0.75\right) \quad (3)$$

From Fig. 13–9a, the angle $\theta$ is related to $y$ by trigonometry.

$$\tan \theta = \frac{y}{0.75} \quad (4)$$

Substituting $y = 1 \text{ m}$ into Eqs. 3 and 4 yields $F_s = 1.50 \text{ N}$ and $\theta = 53.1^\circ$. Substituting these results into Eqs. 1 and 2, we obtain

$$N_C = 0.900 \text{ N} \quad \text{Ans.}$$

$$a = 9.21 \text{ m/s}^2 \downarrow \quad \text{Ans.}$$

NOTE: This is not a case of constant acceleration, since the spring force changes both its magnitude and direction as the collar moves downward.
The 100-kg block shown in Fig. 13–10a is released from rest. If the masses of the pulleys and the cord are neglected, determine the speed of the 20-kg block B in 2 s.

**SOLUTION**

**Free-Body Diagrams.** Since the mass of the pulleys is neglected, then for pulley C, \( ma = 0 \) and we can apply \( \Sigma F_y = 0 \) as shown in Fig. 13–10b. The free-body diagrams for blocks A and B are shown in Fig. 13–10c and d, respectively. Notice that for A to remain stationary \( T = 490.5 \) N, whereas for B to remain static \( T = 196.2 \) N. Hence A will move down while B moves up. Although this is the case, we will assume both blocks accelerate downward, in the direction of \(+s_A\) and \(+s_B\). The three unknowns are \( T, a_A, \) and \( a_B \).

**Equations of Motion.** Block A,
\[ +\Sigma F_y = ma_y; \quad 981 - 2T = 100a_A \quad (1) \]
Block B,
\[ +\Sigma F_y = ma_y; \quad 196.2 - T = 20a_B \quad (2) \]

**Kinematics.** The necessary third equation is obtained by relating \( a_A \) to \( a_B \) using a dependent motion analysis, discussed in Sect. 12.9. The coordinates \( s_A \) and \( s_B \) in Fig. 13–10a measure the positions of A and B from the fixed datum. It is seen that
\[ 2s_A + s_B = l \]
where \( l \) is constant and represents the total vertical length of cord. Differentiating this expression twice with respect to time yields
\[ 2a_A = -a_B \quad (3) \]
Notice that when writing Eqs. 1 to 3, the positive direction was always assumed downward. It is very important to be consistent in this assumption since we are seeking a simultaneous solution of equations. The results are
\[ T = 327.0 \text{ N} \]
\[ a_A = 3.27 \text{ m/s}^2 \]
\[ a_B = -6.54 \text{ m/s}^2 \]
Hence when block A accelerates downward, block B accelerates upward as expected. Since \( a_B \) is constant, the velocity of block B in 2 s is thus
\[
\left(\begin{array}{l}
+ \downarrow \\
\end{array}
\right)
\]
\[ v = v_0 + a_BT \\
= 0 + (-6.54)(2) \\
= -13.1 \text{ m/s} \quad \text{Ans.} \]
The negative sign indicates that block B is moving upward.
FUNDAMENTAL PROBLEMS

F13–1. The motor winds in the cable with a constant acceleration, such that the 20-kg crate moves a distance $s = 6 \text{ m}$ in 3 s, starting from rest. Determine the tension developed in the cable. The coefficient of kinetic friction between the crate and the plane is $\mu_k = 0.3$.

F13–2. If motor $M$ exerts a force of $F = (10t^2 + 10) \text{ N}$ on the cable, where $t$ is in seconds, determine the velocity of the 25-kg crate when $t = 4 \text{ s}$. The coefficients of static and kinetic friction between the crate and the plane are $\mu_s = 0.3$ and $\mu_k = 0.25$, respectively. The crate is initially at rest.

F13–3. A spring of stiffness $k = 500 \text{ N/m}$ is mounted against the 10-kg block. If the block is subjected to the force of $F = 500 \text{ N}$, determine its velocity at $s = 0.5 \text{ m}$. When $s = 0$, the block is at rest and the spring is uncompressed. The contact surface is smooth.

F13–4. The 2-Mg car is being towed by a winch. If the winch exerts a force of $T = (100s) \text{ N}$ on the cable, where $s$ is the displacement of the car in meters, determine the speed of the car when $s = 10 \text{ m}$, starting from rest. Neglect rolling resistance of the car.

F13–5. The spring has a stiffness $k = 200 \text{ N/m}$ and is unstretched when the 25-kg block is at $A$. Determine the acceleration of the block when $s = 0.4 \text{ m}$. The contact surface between the block and the plane is smooth.

F13–6. Block $B$ rests upon a smooth surface. If the coefficients of static and kinetic friction between $A$ and $B$ are $\mu_s = 0.4$ and $\mu_k = 0.3$, respectively, determine the acceleration of each block if $P = 6 \text{ lb}$.
PROBLEMS

13–1. The casting has a mass of 3 Mg. Suspended in a vertical position and initially at rest, it is given an upward speed of 200 mm/s in 0.3 s using a crane hook H. Determine the tension in cables AC and AB during this time interval if the acceleration is constant.

13–2. The 160-Mg train travels with a speed of 80 km/h when it starts to climb the slope. If the engine exerts a traction force F of 1/20 of the weight of the train and the rolling resistance \( F_D \) is equal to 1/500 of the weight of the train, determine the deceleration of the train.

13–3. The 160-Mg train starts from rest and begins to climb the slope as shown. If the engine exerts a traction force F of 1/8 of the weight of the train, determine the speed of the train when it has traveled up the slope a distance of 1 km. Neglect rolling resistance.

13–4. The 2-Mg truck is traveling at 15 m/s when the brakes on all its wheels are applied, causing it to skid for a distance of 10 m before coming to rest. Determine the constant horizontal force developed in the coupling C, and the frictional force developed between the tires of the truck and the road during this time. The total mass of the boat and trailer is 1 Mg.

13–5. If blocks A and B of mass 10 kg and 6 kg, respectively, are placed on the inclined plane and released, determine the force developed in the link. The coefficients of kinetic friction between the blocks and the inclined plane are \( \mu_A = 0.1 \) and \( \mu_B = 0.3 \). Neglect the mass of the link.

13–6. Motors A and B draw in the cable with the accelerations shown. Determine the acceleration of the 300-lb crate C and the tension developed in the cable. Neglect the mass of all the pulleys.
13–7. The van is traveling at 20 km/h when the coupling of the trailer at A fails. If the trailer has a mass of 250 kg and coasts 45 m before coming to rest, determine the constant horizontal force $F$ created by rolling friction which causes the trailer to stop.

13–10. The crate has a mass of 80 kg and is being towed by a chain which is always directed at $20^\circ$ from the horizontal as shown. If the magnitude of $P$ is increased until the crate begins to slide, determine the crate’s initial acceleration if the coefficient of static friction is $\mu_s = 0.5$ and the coefficient of kinetic friction is $\mu_k = 0.3$.

13–11. The crate has a mass of 80 kg and is being towed by a chain which is always directed at $20^\circ$ from the horizontal as shown. Determine the crate’s acceleration if the coefficient of static friction is $\mu_s = 0.4$, the coefficient of kinetic friction is $\mu_k = 0.3$, and the towing force is $P = (90t^2)$ N, where $t$ is in seconds.

13–8. If the 10-lb block $A$ slides down the plane with a constant velocity when $\theta = 30^\circ$, determine the acceleration of the block when $\theta = 45^\circ$.

13–9. Each of the three barges has a mass of 30 Mg, whereas the tugboat has a mass of 12 Mg. As the barges are being pulled forward with a constant velocity of 4 m/s, the tugboat must overcome the frictional resistance of the water, which is 2 kN for each barge and 1.5 kN for the tugboat. If the cable between $A$ and $B$ breaks, determine the acceleration of the tugboat.

13–12. Determine the acceleration of the system and the tension in each cable. The inclined plane is smooth, and the coefficient of kinetic friction between the horizontal surface and block $C$ is $(\mu_k)_C = 0.2$. 
•13–13. The two boxcars \( A \) and \( B \) have a weight of 20,000 lb and 30,000 lb, respectively. If they coast freely down the incline when the brakes are applied to all the wheels of car \( A \) causing it to skid, determine the force in the coupling \( C \) between the two cars. The coefficient of kinetic friction between the wheels of \( A \) and the tracks is \( \mu_k = 0.5 \). The wheels of car \( B \) are free to roll. Neglect their mass in the calculation. Suggestion: Solve the problem by representing single resultant normal forces acting on \( A \) and \( B \), respectively.

![Prob. 13–13](image1)

•13–16. The man pushes on the 60-lb crate with a force \( F \). The force is always directed down at 30° from the horizontal as shown, and its magnitude is increased until the crate begins to slide. Determine the crate’s initial acceleration if the coefficient of static friction is \( \mu_s = 0.6 \) and the coefficient of kinetic friction is \( \mu_k = 0.3 \).

![Prob. 13–16](image2)

13–14. The 3.5-Mg engine is suspended from a spreader beam \( AB \) having a negligible mass and is hoisted by a crane which gives it an acceleration of 4 m/s² when it has a velocity of 2 m/s. Determine the force in chains \( CA \) and \( CB \) during the lift.

13–15. The 3.5-Mg engine is suspended from a spreader beam having a negligible mass and is hoisted by a crane which exerts a force of 40 kN on the hoisting cable. Determine the distance the engine is hoisted in 4 s, starting from rest.

13–17. A force of \( F = 15 \text{ lb} \) is applied to the cord. Determine how high the 30-lb block \( A \) rises in 2 s starting from rest. Neglect the weight of the pulleys and cord.

13–18. Determine the constant force \( F \) which must be applied to the cord in order to cause the 30-lb block \( A \) to have a speed of 12 ft/s when it has been displaced 3 ft upward starting from rest. Neglect the weight of the pulleys and cord.
13–19. The 800-kg car at \( B \) is connected to the 350-kg car at \( A \) by a spring coupling. Determine the stretch in the spring if (a) the wheels of both cars are free to roll and (b) the brakes are applied to all four wheels of car \( B \), causing the wheels to skid. Take \((\mu_k)_B = 0.4\). Neglect the mass of the wheels.

13–21. Block \( B \) has a mass \( m \) and is released from rest when it is on top of cart \( A \), which has a mass of \( 3m \). Determine the tension in cord \( CD \) needed to hold the cart from moving while \( B \) slides down \( A \). Neglect friction.

13–22. Block \( B \) has a mass \( m \) and is released from rest when it is on top of cart \( A \), which has a mass of \( 3m \). Determine the tension in cord \( CD \) needed to hold the cart from moving while \( B \) slides down \( A \). The coefficient of kinetic friction between \( A \) and \( B \) is \( \mu_k \).

13–20. The 10-lb block \( A \) travels to the right at \( v_A = 2 \text{ ft/s} \) at the instant shown. If the coefficient of kinetic friction is \( \mu_k = 0.2 \) between the surface and \( A \), determine the velocity of \( A \) when it has moved 4 ft. Block \( B \) has a weight of 20 lb.

13–23. The 2-kg shaft \( CA \) passes through a smooth journal bearing at \( B \). Initially, the springs, which are coiled loosely around the shaft, are unstretched when no force is applied to the shaft. In this position \( s = s' = 250 \text{ mm} \) and the shaft is at rest. If a horizontal force of \( F = 5 \text{ kN} \) is applied, determine the speed of the shaft at the instant \( s = 50 \text{ mm} \), \( s' = 450 \text{ mm} \). The ends of the springs are attached to the bearing at \( B \) and the caps at \( C \) and \( A \).
13–24. If the force of the motor $M$ on the cable is shown in the graph, determine the velocity of the cart when $t = 3$ s. The load and cart have a mass of 200 kg and the cart starts from rest.

![Graph showing force vs. time](image)

**Prob. 13–24**

13–26. A freight elevator, including its load, has a mass of 500 kg. It is prevented from rotating by the track and wheels mounted along its sides. When $t = 2$ s, the motor $M$ draws in the cable with a speed of 6 m/s, measured relative to the elevator. If it starts from rest, determine the constant acceleration of the elevator and the tension in the cable. Neglect the mass of the pulleys, motor, and cables.

![Elevator diagram](image)

**Prob. 13–26**

13–25. If the motor draws in the cable with an acceleration of $3 \text{ m/s}^2$, determine the reactions at the supports $A$ and $B$. The beam has a uniform mass of 30 kg/m, and the crate has a mass of 200 kg. Neglect the mass of the motor and pulleys.

![Beam and crate diagram](image)

**Prob. 13–25**

13–27. Determine the required mass of block $A$ so that when it is released from rest it moves the 5-kg block $B$ a distance of 0.75 m up along the smooth inclined plane in $t = 2$ s. Neglect the mass of the pulleys and cords.

![Inclined plane diagram](image)

**Prob. 13–27**
13.4 EQUATIONS OF MOTION: RECTANGULAR COORDINATES

13–28. Blocks $A$ and $B$ have a mass of $m_A$ and $m_B$, where $m_A > m_B$. If pulley $C$ is given an acceleration of $a_0$, determine the acceleration of the blocks. Neglect the mass of the pulley.

13–29. The tractor is used to lift the 150-kg load $B$ with the 24-m-long rope, boom, and pulley system. If the tractor travels to the right at a constant speed of 4 m/s, determine the tension in the rope when $s_A = 5$ m. When $s_A = 0$, $s_B = 0$.

13–30. The tractor is used to lift the 150-kg load $B$ with the 24-m-long rope, boom, and pulley system. If the tractor travels to the right with an acceleration of 3 m/s$^2$ and has a velocity of 4 m/s at the instant $s_A = 5$ m, determine the tension in the rope at this instant. When $s_A = 0$, $s_B = 0$.

13–31. The 75-kg man climbs up the rope with an acceleration of 0.25 m/s$^2$, measured relative to the rope. Determine the tension in the rope and the acceleration of the 80-kg block.

13–32. Motor $M$ draws in the cable with an acceleration of 4ft/s$^2$, measured relative to the 200-lb mine car. Determine the acceleration of the car and the tension in the cable. Neglect the mass of the pulleys.
13–33. The 2-lb collar \( C \) fits loosely on the smooth shaft. If the spring is unstretched when \( s = 0 \) and the collar is given a velocity of 15 ft/s, determine the velocity of the collar when \( s = 1 \) ft.

![Prob. 13–33](image)

13–34. In the cathode-ray tube, electrons having a mass \( m \) are emitted from a source point \( S \) and begin to travel horizontally with an initial velocity \( v_0 \). While passing between the grid plates a distance \( l \), they are subjected to a vertical force having a magnitude \( eV/w \), where \( e \) is the charge of an electron, \( V \) the applied voltage acting across the plates, and \( w \) the distance between the plates. After passing clear of the plates, the electrons then travel in straight lines and strike the screen at \( A \). Determine the deflection \( d \) of the electrons in terms of the dimensions of the voltage plate and tube. Neglect gravity which causes a slight vertical deflection when the electron travels from \( S \) to the screen, and the slight deflection between the plates.

![Prob. 13–34](image)

13–35. The 2-kg collar \( C \) is free to slide along the smooth shaft \( AB \). Determine the acceleration of collar \( C \) if (a) the shaft is fixed from moving, (b) collar \( A \), which is fixed to shaft \( AB \), moves to the left at constant velocity along the horizontal guide, and (c) collar \( A \) is subjected to an acceleration of 2 m/s\(^2\) to the left. In all cases, the motion occurs in the vertical plane.

![Prob. 13–35](image)

13–36. Blocks \( A \) and \( B \) each have a mass \( m \). Determine the largest horizontal force \( P \) which can be applied to \( B \) so that \( A \) will not move relative to \( B \). All surfaces are smooth.

13–37. Blocks \( A \) and \( B \) each have a mass \( m \). Determine the largest horizontal force \( P \) which can be applied to \( B \) so that \( A \) will not slip on \( B \). The coefficient of static friction between \( A \) and \( B \) is \( \mu_s \). Neglect any friction between \( B \) and \( C \).
13–38. If a force $F = 200 \text{ N}$ is applied to the 30-kg cart, show that the 20-kg block $A$ will slide on the cart. Also determine the time for block $A$ to move on the cart 1.5 m. The coefficients of static and kinetic friction between the block and the cart are $\mu_s = 0.3$ and $\mu_k = 0.25$. Both the cart and the block start from rest.

![Prob. 13–38](image)

13–39. Suppose it is possible to dig a smooth tunnel through the earth from a city at $A$ to a city at $B$ as shown. By the theory of gravitation, any vehicle $C$ of mass $m$ placed within the tunnel would be subjected to a gravitational force which is always directed toward the center of the earth $D$. This force $F$ has a magnitude that is directly proportional to its distance $r$ from the earth’s center. Hence, if the vehicle has a weight of $W = mg$ when it is located on the earth’s surface, then at an arbitrary location $r$ the magnitude of force $F$ is $F = (mg/R)r$, where $R = 6328 \text{ km}$, the radius of the earth. If the vehicle is released from rest when it is at $B$, $x = s = 2 \text{ Mm}$, determine the time needed for it to reach $A$, and the maximum velocity it attains. Neglect the effect of the earth’s rotation in the calculation and assume the earth has a constant density. Hint: Write the equation of motion in the $x$ direction, noting that $r \cos \theta = x$. Integrate, using the kinematic relation $v \, dv = a \, dx$, then integrate the result using $v = dx/dt$.

![Prob. 13–39](image)

13–40. The 30-lb crate is being hoisted upward with a constant acceleration of $6 \text{ ft/s}^2$. If the uniform beam $AB$ has a weight of 200 lb, determine the components of reaction at the fixed support $A$. Neglect the size and mass of the pulley at $B$. Hint: First find the tension in the cable, then analyze the forces in the beam using statics.

![Prob. 13–40](image)

13–41. If a horizontal force of $P = 10 \text{ lb}$ is applied to block $A$, determine the acceleration of block $B$. Neglect friction. Hint: Show that $a_B = a_A \tan 15^\circ$.

![Prob. 13–41](image)
13–42. Block A has a mass \( m_A \) and is attached to a spring having a stiffness \( k \) and unstretched length \( l_0 \). If another block \( B \), having a mass \( m_B \), is pressed against \( A \) so that the spring deforms a distance \( d \), determine the distance both blocks slide on the smooth surface before they begin to separate. What is their velocity at this instant?

13–43. Block A has a mass \( m_A \) and is attached to a spring having a stiffness \( k \) and unstretched length \( l_0 \). If another block \( B \), having a mass \( m_B \), is pressed against \( A \) so that the spring deforms a distance \( d \), show that for separation to occur it is necessary that \( d > 2 \mu_k g (m_A + m_B)/k \), where \( \mu_k \) is the coefficient of kinetic friction between the blocks and the ground. Also, what is the distance the blocks slide on the surface before they separate?

\[ F_D = (100v)N \]

13–44. The 600-kg dragster is traveling with a velocity of 125 m/s when the engine is shut off and the braking parachute is deployed. If air resistance imposed on the dragster due to the parachute is \( F_D = (6000 + 0.9v^2) \) N, where \( v \) is in m/s, determine the time required for the dragster to come to rest.

\[ F_D = C v^2 \]

13–45. The buoyancy force on the 500-kg balloon is \( F = 6 \) kN, and the air resistance is \( F_D = (100v) \) N, where \( v \) is in m/s. Determine the terminal or maximum velocity of the balloon if it starts from rest.

13–46. The parachutist of mass \( m \) is falling with a velocity of \( v_0 \) at the instant he opens the parachute. If air resistance is \( F_D = C v^2 \), determine her maximum velocity (terminal velocity) during the descent.

13–47. The weight of a particle varies with altitude such that \( W = m (gr_0^2)/r^2 \), where \( r_0 \) is the radius of the earth and \( r \) is the distance from the particle to the earth’s center. If the particle is fired vertically with a velocity \( v_0 \) from the earth’s surface, determine its velocity as a function of position \( r \). What is the smallest velocity \( v_0 \) required to escape the earth’s gravitational field, what is \( r_{\text{max}} \), and what is the time required to reach this altitude?
13.5 **Equations of Motion: Normal and Tangential Coordinates**

When a particle moves along a curved path which is known, the equation of motion for the particle may be written in the tangential, normal, and binormal directions, Fig. 13–11. Note that there is no motion of the particle in the binormal direction, since the particle is constrained to move along the path. We have

\[ \sum F = ma \]

\[ \sum F_t \hat{u}_t + \sum F_n \hat{u}_n + \sum F_b \hat{u}_b = ma_t + ma_n \]

This equation is satisfied provided

\[ \begin{align*}
\sum F_t &= ma_t \\
\sum F_n &= ma_n \\
\sum F_b &= 0
\end{align*} \quad (13–8) \]

Recall that \( a_t = \frac{dv}{dt} \) represents the time rate of change in the magnitude of velocity. So if \( \sum F_t \) acts in the direction of motion, the particle’s speed will increase, whereas if it acts in the opposite direction, the particle will slow down. Likewise, \( a_n = \frac{v^2}{\rho} \) represents the time rate of change in the velocity’s direction. It is caused by \( \sum F_n \), which always acts in the positive \( n \) direction, i.e., toward the path’s center of curvature. From this reason it is often referred to as the *centripetal force*.

The centrifuge is used to subject a passenger to a very large normal acceleration caused by rapid rotation. Realize that this acceleration is *caused by* the unbalanced normal force exerted on the passenger by the seat of the centrifuge.
CHAPTER 13  KINETICS OF A PARTICLE: FORCE AND ACCELERATION

**Procedure for Analysis**

When a problem involves the motion of a particle along a *known curved path*, normal and tangential coordinates should be considered for the analysis since the acceleration components can be readily formulated. The method for applying the equations of motion, which relate the forces to the acceleration, has been outlined in the procedure given in Sec. 13.4. Specifically, for \( t, n, b \) coordinates it may be stated as follows:

**Free-Body Diagram.**

- Establish the inertial \( t, n, b \) coordinate system at the particle and draw the particle's free-body diagram.

- The particle's normal acceleration \( a_n \) *always* acts in the positive \( n \) direction.

- If the tangential acceleration \( a_t \) is unknown, assume it acts in the positive \( t \) direction.

- There is no acceleration in the \( b \) direction.

- Identify the unknowns in the problem.

**Equations of Motion.**

- Apply the equations of motion, Eqs. 13–8.

**Kinematics.**

- Formulate the tangential and normal components of acceleration; i.e., \( a_t = \frac{dv}{dt} \) or \( a_t = v \frac{dv}{ds} \) and \( a_n = \frac{v^2}{\rho} \).

- If the path is defined as \( y = f(x) \), the radius of curvature at the point where the particle is located can be obtained from \( \rho = \left[1 + (dy/dx)^2\right]^{3/2}/|d^2y/dx^2| \).
EXAMPLE 13.6

Determine the banking angle $\theta$ for the race track so that the wheels of the racing cars shown in Fig. 13–12a will not have to depend upon friction to prevent any car from sliding up or down the track. Assume the cars have negligible size, a mass $m$, and travel around the curve of radius $\rho$ with a constant speed $v$.

SOLUTION

Before looking at the following solution, give some thought as to why it should be solved using $t, n, b$ coordinates.

Free-Body Diagram. As shown in Fig. 13–12b, and as stated in the problem, no frictional force acts on the car. Here $N_C$ represents the resultant of the ground on all four wheels. Since $a_n$ can be calculated, the unknowns are $N_C$ and $\theta$.

Equations of Motion. Using the $n, b$ axes shown,

$$\sum F_n = ma_n; \quad N_C \sin \theta = \frac{mv^2}{\rho} \quad (1)$$
$$\sum F_b = 0; \quad N_C \cos \theta - mg = 0 \quad (2)$$

Eliminating $N_C$ and $m$ from these equations by dividing Eq. 1 by Eq. 2, we obtain

$$\tan \theta = \frac{v^2}{g \rho}$$

$$\theta = \tan^{-1} \left( \frac{v^2}{g \rho} \right) \quad \text{Ans.}$$

NOTE: The result is independent of the mass of the car. Also, a force summation in the tangential direction is of no consequence to the solution. If it were considered, then $a_t = \frac{dv}{dt} = 0$, since the car moves with constant speed. A further analysis of this problem is discussed in Prob. 21–47.

Fig. 13–12
The 3-kg disk $D$ is attached to the end of a cord as shown in Fig. 13–13a. The other end of the cord is attached to a ball-and-socket joint located at the center of a platform. If the platform rotates rapidly, and the disk is placed on it and released from rest as shown, determine the time it takes for the disk to reach a speed great enough to break the cord. The maximum tension the cord can sustain is 100 N, and the coefficient of kinetic friction between the disk and the platform is $\mu_k = 0.1$.

**EXAMPLE 13.7**

Free-Body Diagram. The frictional force has a magnitude $F = \mu_k N_D = 0.1 N_D$ and a sense of direction that opposes the *relative motion* of the disk with respect to the platform. It is this force that gives the disk a tangential component of acceleration causing $v$ to increase, thereby causing $T$ to increase until it reaches 100 N. The weight of the disk is $W = 3(9.81) = 29.43$ N. Since $a_n$ can be related to $v$, the unknowns are $N_D$, $a_t$, and $v$.

**Equations of Motion.**

$$\Sigma F_n = ma_n; \quad T = 3 \left( \frac{v^2}{T} \right) \quad (1)$$
$$\Sigma F_t = ma_t; \quad 0.1 N_D = 3a_t \quad (2)$$
$$\Sigma F_b = 0; \quad N_D - 29.43 = 0 \quad (3)$$

Setting $T = 100$ N, Eq. 1 can be solved for the critical speed $v_{cr}$ of the disk needed to break the cord. Solving all the equations, we obtain

$$N_D = 29.43 \text{ N}$$
$$a_t = 0.981 \text{ m/s}^2$$
$$v_{cr} = 5.77 \text{ m/s}$$

**Kinematics.** Since $a_t$ is *constant*, the time needed to break the cord is

$$v_{cr} = v_0 + a_t t$$
$$5.77 = 0 + (0.981)t$$
$$t = 5.89 \text{ s} \quad \text{Ans.}$$
EXAMPLE 13.8

Design of the ski jump shown in the photo requires knowing the type of forces that will be exerted on the skier and her approximate trajectory. If in this case the jump can be approximated by the parabola shown in Fig. 13–14a, determine the normal force on the 150-lb skier the instant she arrives at the end of the jump, point A, where her velocity is 65 ft/s. Also, what is her acceleration at this point?

SOLUTION

Why consider using \( n \), \( t \) coordinates to solve this problem?

**Free-Body Diagram.** Since \( dy/dx = x/100 \mid_{x=0} = 0 \), the slope at \( A \) is horizontal. The free-body diagram of the skier when she is at \( A \) is shown in Fig. 13–14b. Since the path is curved, there are two components of acceleration, \( a_n \) and \( a_t \). Since \( a_n \) can be calculated, the unknowns are \( a_t \) and \( N_A \).

**Equations of Motion.**

\[
+ \sum F_{n} = ma_n; \quad N_A - 150 = \frac{150}{32.2} \left( \frac{(65)^2}{\rho} \right) \tag{1}
\]

\[
\sum F_{t} = ma_t; \quad 0 = \frac{150}{32.2} a_t \tag{2}
\]

The radius of curvature \( \rho \) for the path must be determined at point \( A(0, -200 \text{ ft}) \). Here \( y = \frac{1}{200} x^2 - 200 \), \( dy/dx = \frac{1}{100} x \), \( d^2y/dx^2 = \frac{1}{100} \), so that at \( x = 0 \),

\[
\rho = \left[ \frac{1 + (dy/dx)^2}{d^2y/dx^2} \right]^{3/2} \bigg|_{x=0} = \frac{1 + (0)^2}{\frac{1}{100}} = 100 \text{ ft}
\]

Substituting this into Eq. 1 and solving for \( N_A \), we obtain

\[
N_A = 347 \text{ lb} \quad \text{Ans.}
\]

**Kinematics.** From Eq. 2,

\[
a_t = 0
\]

Thus,

\[
a_n = \frac{v^2}{\rho} = \frac{(65)^2}{100} = 42.2 \text{ ft/s}^2
\]

\[
a_A = a_n = 42.2 \text{ ft/s}^2 \quad \text{Ans.}
\]

**NOTE:** Apply the equation of motion in the \( y \) direction and show that when the skier is in midair her acceleration is 32.2 ft/s\(^2\).
The 60-kg skateboarder in Fig. 13–15a coasts down the circular track. If he starts from rest when $\theta = 0^\circ$, determine the magnitude of the normal reaction the track exerts on him when $\theta = 60^\circ$. Neglect his size for the calculation.

**SOLUTION**

**Free-Body Diagram.** The free-body diagram of the skateboarder when he is at an arbitrary position $\theta$ is shown in Fig. 13–15b. At $\theta = 60^\circ$ there are three unknowns, $N_s$, $a_t$, and $a_n$ (or $v$).

**Equations of Motion.**

1. $\downarrow \Sigma F_n = ma_n: N_s - [60(9.81)N] \sin \theta = (60 \text{ kg}) \left( \frac{v^2}{4\text{ m}} \right)$  

2. $\downarrow \Sigma F_t = ma_t: [60(9.81)N] \cos \theta = (60 \text{ kg}) a_t$

   $$a_t = 9.81 \cos \theta$$

**Kinematics.** Since $a_t$ is expressed in terms of $\theta$, the equation $v \, dv = a_t \, ds$ must be used to determine the speed of the skateboarder when $\theta = 60^\circ$. Using the geometric relation $s = \theta r$, where $ds = r \, d\theta = (4 \text{ m}) \, d\theta$, Fig. 13–15c, and the initial condition $v = 0$ at $\theta = 0^\circ$, we have,

$$v \, dv = a_t \, ds$$

$$\int_0^v v \, dv = \int_0^{60^\circ} 9.81 \cos \theta (4 \, d\theta)$$

$$\left. \frac{v^2}{2} \right|_0^v = 39.24 \sin \theta \bigg|_0^{60^\circ}$$

$$\frac{v^2}{2} - 0 = 39.24(\sin 60^\circ - 0)$$

$$v^2 = 67.97 \text{ m}^2/\text{s}^2$$

Substituting this result and $\theta = 60^\circ$ into Eq. (1), yields

$$N_s = 1529.23 \text{ N} = 1.53 \text{ kN}$$

*Ans.*
FUNDAMENTAL PROBLEMS

F13–7. The block rests at a distance of 2 m from the center of the platform. If the coefficient of static friction between the block and the platform is $\mu_s = 0.3$, determine the maximum speed which the block can attain before it begins to slip. Assume the angular motion of the disk is slowly increasing.

F13–8. Determine the maximum speed that the jeep can travel over the crest of the hill and not lose contact with the road.

F13–9. A pilot weighs 150 lb and is traveling at a constant speed of 120 ft/s. Determine the normal force he exerts on the seat of the plane when he is upside down at $A$. The loop has a radius of curvature of 400 ft.

F13–10. The sports car is traveling along a $30^\circ$ banked road having a radius of curvature of $\rho = 500$ ft. If the coefficient of static friction between the tires and the road is $\mu_s = 0.2$, determine the maximum safe speed so no slipping occurs. Neglect the size of the car.

F13–11. If the 10-kg ball has a velocity of 3 m/s when it is at the position $A$, along the vertical path, determine the tension in the cord and the increase in the speed of the ball at this position.

F13–12. The motorcycle has a mass of 0.5 Mg and a negligible size. It passes point $A$ traveling with a speed of 15 m/s, which is increasing at a constant rate of 1.5 m/s$^2$. Determine the resultant frictional force exerted by the road on the tires at this instant.
*13–48. The 2-kg block $B$ and 15-kg cylinder $A$ are connected to a light cord that passes through a hole in the center of the smooth table. If the block is given a speed of $v = 10 \text{ m/s}$, determine the radius $r$ of the circular path along which it travels.

*13–49. The 2-kg block $B$ and 15-kg cylinder $A$ are connected to a light cord that passes through a hole in the center of the smooth table. If the block travels along a circular path of radius $r = 1.5 \text{ m}$, determine the speed of the block.

13–50. At the instant shown, the 50-kg projectile travels in the vertical plane with a speed of $v = 40 \text{ m/s}$. Determine the tangential component of its acceleration and the radius of curvature $\rho$ of its trajectory at this instant.

13–51. At the instant shown, the radius of curvature of the vertical trajectory of the 50-kg projectile is $\rho = 200 \text{ m}$. Determine the speed of the projectile at this instant.

*13–52. Determine the mass of the sun, knowing that the distance from the earth to the sun is $149.6(10^6) \text{ km}$. Hint: Use Eq. 13–1 to represent the force of gravity acting on the earth.

*13–53. The sports car, having a mass of 1700 kg, travels horizontally along a $20^\circ$ banked track which is circular and has a radius of curvature of $\rho = 100 \text{ m}$. If the coefficient of static friction between the tires and the road is $\mu_s = 0.2$, determine the maximum constant speed at which the car can travel without sliding up the slope. Neglect the size of the car.

13–54. Using the data in Prob. 13–53, determine the minimum speed at which the car can travel around the track without sliding down the slope.

13–55. The device shown is used to produce the experience of weightlessness in a passenger when he reaches point $A$, $\theta = 90^\circ$, along the path. If the passenger has a mass of 75 kg, determine the minimum speed he should have when he reaches $A$ so that he does not exert a normal reaction on the seat. The chair is pin-connected to the frame $BC$ so that he is always seated in an upright position. During the motion his speed remains constant.

*13–56. A man having the mass of 75 kg sits in the chair which is pin-connected to the frame $BC$. If the man is always seated in an upright position, determine the horizontal and vertical reactions of the chair on the man at the instant $\theta = 45^\circ$. At this instant he has a speed of $6 \text{ m/s}$, which is increasing at $0.5 \text{ m/s}^2$. 

13–57. Determine the tension in wire CD just after wire AB is cut. The small bob has a mass \( m \).

13–58. Determine the time for the satellite to complete its orbit around the earth. The orbit has a radius \( r \) measured from the center of the earth. The masses of the satellite and the earth are \( m_s \) and \( M_e \), respectively.

13–59. An acrobat has a weight of 150 lb and is sitting on a chair which is perched on top of a pole as shown. If by a mechanical drive the pole rotates downward at a constant rate from \( \theta = 0^\circ \), such that the acrobat’s center of mass \( G \) maintains a constant speed of \( v_u = 10 \text{ ft/s} \), determine the angle \( \theta \) at which he begins to “fly” out of the chair. Neglect friction and assume that the distance from the pivot \( O \) to \( G \) is \( \rho = 15 \text{ ft} \).

13–60. A spring, having an unstretched length of 2 ft, has one end attached to the 10-lb ball. Determine the angle \( \theta \) of the spring if the ball has a speed of 6 ft/s tangent to the horizontal circular path.
13–61. If the ball has a mass of 30 kg and a speed \( v = 4 \text{ m/s} \) at the instant it is at its lowest point, \( \theta = 0^\circ \), determine the tension in the cord at this instant. Also, determine the angle \( \theta \) to which the ball swings and momentarily stops. Neglect the size of the ball.

13–62. The ball has a mass of 30 kg and a speed \( v = 4 \text{ m/s} \) at the instant it is at its lowest point, \( \theta = 0^\circ \). Determine the tension in the cord and the rate at which the ball’s speed is decreasing at the instant \( \theta = 20^\circ \). Neglect the size of the ball.

13–63. The vehicle is designed to combine the feel of a motorcycle with the comfort and safety of an automobile. If the vehicle is traveling at a constant speed of 80 km/h along a circular curved road of radius 100 m, determine the tilt angle \( \theta \) of the vehicle so that only a normal force from the seat acts on the driver. Neglect the size of the driver.

13–64. The ball has a mass \( m \) and is attached to the cord of length \( l \). The cord is tied at the top to a swivel and the ball is given a velocity \( \mathbf{v}_0 \). Show that the angle \( \theta \) which the cord makes with the vertical as the ball travels around the circular path must satisfy the equation \( \tan \theta \sin \theta = \frac{\mathbf{v}_0^2}{gl} \). Neglect air resistance and the size of the ball.

13–65. The smooth block \( B \), having a mass of 0.2 kg, is attached to the vertex \( A \) of the right circular cone using a light cord. If the block has a speed of 0.5 m/s around the cone, determine the tension in the cord and the reaction which the cone exerts on the block. Neglect the size of the block.
13–66. Determine the minimum coefficient of static friction between the tires and the road surface so that the 1.5-Mg car does not slide as it travels at 80 km/h on the curved road. Neglect the size of the car.

13–67. If the coefficient of static friction between the tires and the road surface is \( \mu_s = 0.25 \), determine the maximum speed of the 1.5-Mg car without causing it to slide when it travels on the curve. Neglect the size of the car.

**Probs. 13–66/67**

*13–68. At the instant shown, the 3000-lb car is traveling with a speed of 75 ft/s, which is increasing at a rate of 6 ft/s\(^2\). Determine the magnitude of the resultant frictional force the road exerts on the tires of the car. Neglect the size of the car.

**Probs. 13–70/71**

*13–72. The 0.8-Mg car travels over the hill having the shape of a parabola. If the driver maintains a constant speed of 9 m/s, determine both the resultant normal force and the resultant frictional force that all the wheels of the car exert on the road at the instant it reaches point \( A \). Neglect the size of the car.

*13–73. The 0.8-Mg car travels over the hill having the shape of a parabola. When the car is at point \( A \), it is traveling at 9 m/s and increasing its speed at 3 m/s\(^2\). Determine both the resultant normal force and the resultant frictional force that all the wheels of the car exert on the road at this instant. Neglect the size of the car.
**13–74.** The 6-kg block is confined to move along the smooth parabolic path. The attached spring restricts the motion and, due to the roller guide, always remains horizontal as the block descends. If the spring has a stiffness of \( k = 10 \text{ N/m} \), and unstretched length of 0.5 m, determine the normal force of the path on the block at the instant \( x = 1 \text{ m} \) when the block has a speed of 4 m/s. Also, what is the rate of increase in speed of the block at this point? Neglect the mass of the roller and the spring.

![Diagram of a block on a parabolic path with a spring](image1)

**Prob. 13–74**

**13–75.** Prove that if the block is released from rest at point \( B \) of a smooth path of arbitrary shape, the speed it attains when it reaches point \( A \) is equal to the speed it attains when it falls freely through a distance \( h \); i.e., \( v = \sqrt{2gh} \).

![Diagram of a block on a smooth path](image2)

**Prob. 13–75**

**13–76.** A toboggan and rider of total mass 90 kg travel down along the (smooth) slope defined by the equation \( y = 0.08x^2 \). At the instant \( x = 10 \text{ m} \), the toboggan’s speed is 5 m/s. At this point, determine the rate of increase in speed and the normal force which the slope exerts on the toboggan. Neglect the size of the toboggan and rider for the calculation.

![Diagram of a toboggan and rider on a slope](image3)

**Prob. 13–76**

**13–77.** The skier starts from rest at \( A(10 \text{ m}, 0) \) and descends the smooth slope, which may be approximated by a parabola. If she has a mass of 52 kg, determine the normal force the ground exerts on the skier at the instant she arrives at point \( B \). Neglect the size of the skier. *Hint: Use the result of Prob. 13–75.*

![Diagram of a skier on a smooth slope](image4)

**Prob. 13–77**
13–78. The 5-lb box is projected with a speed of 20 ft/s at A up the vertical circular smooth track. Determine the angle \( \theta \) when the box leaves the track.

13–79. Determine the minimum speed that must be given to the 5-lb box at A in order for it to remain in contact with the circular path. Also, determine the speed of the box when it reaches point B.

13–82. Determine the maximum speed the 1.5-Mg car can have and still remain in contact with the road when it passes point A. If the car maintains this speed, what is the normal reaction of the road on it when it passes point B? Neglect the size of the car.

*13–80. The 800-kg motorbike travels with a constant speed of 80 km/h up the hill. Determine the normal force the surface exerts on its wheels when it reaches point A. Neglect its size.

13–83. The 5-lb collar slides on the smooth rod, so that when it is at A it has a speed of 10 ft/s. If the spring to which it is attached has an unstretched length of 3 ft and a stiffness of \( k = 10 \text{ lb/ft} \), determine the normal force on the collar and the acceleration of the collar at this instant.
13.6 Equations of Motion: Cylindrical Coordinates

When all the forces acting on a particle are resolved into cylindrical components, i.e., along the unit-vector directions $\mathbf{u}_r$, $\mathbf{u}_\theta$, $\mathbf{u}_z$, Fig. 13–16, the equation of motion can be expressed as

$$\sum \mathbf{F} = m \mathbf{a}$$

$$\sum F_r \mathbf{u}_r + \sum F_\theta \mathbf{u}_\theta + \sum F_z \mathbf{u}_z = ma_r \mathbf{u}_r + ma_\theta \mathbf{u}_\theta + ma_z \mathbf{u}_z$$

To satisfy this equation, we require

$$\begin{align*}
\sum F_r &= ma_r \\
\sum F_\theta &= ma_\theta \\
\sum F_z &= ma_z
\end{align*}$$  \hspace{1cm} (13–9)

If the particle is constrained to move only in the $r-\theta$ plane, then only the first two of Eqs. 13–9 are used to specify the motion.

**Tangential and Normal Forces.** The most straightforward type of problem involving cylindrical coordinates requires the determination of the resultant force components $\sum F_r$, $\sum F_\theta$, $\sum F_z$ which cause a particle to move with a known acceleration. If, however, the particle’s accelerated motion is not completely specified at the given instant, then some information regarding the directions or magnitudes of the forces acting on the particle must be known or computed in order to solve Eqs. 13–9.

For example, the force $\mathbf{P}$ causes the particle in Fig. 13–17a to move along a path $r = f(\theta)$. The normal force $\mathbf{N}$ which the path exerts on the particle is always perpendicular to the tangent of the path, whereas the frictional force $\mathbf{F}$ always acts along the tangent in the opposite direction of motion. The directions of $\mathbf{N}$ and $\mathbf{F}$ can be specified relative to the radial coordinate by using the angle $\psi$ (psi), Fig. 13–17b, which is defined between the extended radial line and the tangent to the curve.

As the car of weight $W$ descends the spiral track, the resultant normal force which the track exerts on the car can be represented by its three cylindrical components. $-\mathbf{N}_r$ creates a radial acceleration, $-\mathbf{a}_r$, $\mathbf{N}_\theta$ creates a transverse acceleration $\mathbf{a}_\theta$, and the difference $W - \mathbf{N}_z$ creates an azimuthal acceleration $-\mathbf{a}_z$. 

This angle can be obtained by noting that when the particle is displaced a distance $ds$ along the path, Fig. 13–17c, the component of displacement in the radial direction is $dr$ and the component of displacement in the transverse direction is $r\,d\theta$. Since these two components are mutually perpendicular, the angle $\psi$ can be determined from $\tan \psi = r\,d\theta/dr$, or

$$\tan \psi = \frac{r}{dr/d\theta} \quad (13-10)$$

If $\psi$ is calculated as a positive quantity, it is measured from the extended radial line to the tangent in a counterclockwise sense or in the positive direction of $\theta$. If it is negative, it is measured in the opposite direction to positive $\theta$. For example, consider the cardioid $r = a(1 + \cos \theta)$, shown in Fig. 13–18. Because $dr/d\theta = -a \sin \theta$, then when $\theta = 30^\circ$, $\tan \psi = a(1 + \cos 30^\circ)/(-a \sin 30^\circ) = -3.732$, or $\psi = -75^\circ$, measured clockwise, opposite to $+\theta$ as shown in the figure.

### Procedure for Analysis

Cylindrical or polar coordinates are a suitable choice for the analysis of a problem for which data regarding the angular motion of the radial line $r$ are given, or in cases where the path can be conveniently expressed in terms of these coordinates. Once these coordinates have been established, the equations of motion can then be applied in order to relate the forces acting on the particle to its acceleration components. The method for doing this has been outlined in the procedure for analysis given in Sec. 13.4. The following is a summary of this procedure.

**Free-Body Diagram.**
- Establish the $r$, $\theta$, $z$ inertial coordinate system and draw the particle’s free-body diagram.
- Assume that $a_r$, $a_\theta$, $a_z$ act in the positive directions of $r$, $\theta$, $z$ if they are unknown.
- Identify all the unknowns in the problem.

**Equations of Motion.**
- Apply the equations of motion, Eqs. 13–9.

**Kinematics.**
- Use the methods of Sec. 12.8 to determine $r$ and the time derivatives $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, $\ddot{\theta}$, $\dot{z}$, and then evaluate the acceleration components $a_r = \ddot{r} - r\dot{\theta}^2$, $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$, $a_z = \ddot{z}$.
- If any of the acceleration components is computed as a negative quantity, it indicates that it acts in its negative coordinate direction.
- When taking the time derivatives of $r = f(\theta)$, it is very important to use the chain rule of calculus, which is discussed at the end of Appendix C.
The smooth 0.5-kg double-collar in Fig. 13–19a can freely slide on arm AB and the circular guide rod. If the arm rotates with a constant angular velocity of \( \theta = 3 \text{ rad/s} \), determine the force the arm exerts on the collar at the instant \( \theta = 45^\circ \). Motion is in the horizontal plane.

**SOLUTION**

**Free-Body Diagram.** The normal reaction \( N_C \) of the circular guide rod and the force \( F \) of arm AB act on the collar in the plane of motion, Fig. 13–19b. Note that \( F \) acts perpendicular to the axis of arm AB, that is, in the direction of the \( \theta \) axis, while \( N_C \) acts perpendicular to the tangent of the circular path at \( \theta = 45^\circ \). The four unknowns are \( N_C, F, a_r, a_\theta \).

**Equations of Motion.**

\[
\begin{align*}
\sum F_r &= ma_r; \quad -N_C \cos 45^\circ = (0.5 \text{ kg}) a_r \quad (1) \\
\sum F_\theta &= ma_\theta; \quad F - N_C \sin 45^\circ = (0.5 \text{ kg}) a_\theta \quad (2)
\end{align*}
\]

**Kinematics.** Using the chain rule (see Appendix C), the first and second time derivatives of \( r \) when \( \theta = 45^\circ, \theta = 3 \text{ rad/s}, \dot{\theta} = 0 \), are

\[
\begin{align*}
r &= 0.8 \cos \theta = 0.8 \cos 45^\circ = 0.5657 \text{ m} \\
\dot{r} &= -0.8 \sin \theta \dot{\theta} = -0.8 \sin 45^\circ (3) = -1.6971 \text{ m/s} \\
\ddot{r} &= -0.8[\sin \theta \ddot{\theta} + \cos \theta \dot{\theta}^2] \\
&= -0.8[\sin 45^\circ (0) + \cos 45^\circ (3^2)] = -5.091 \text{ m/s}^2
\end{align*}
\]

We have

\[
\begin{align*}
a_r &= \ddot{r} - r\dot{\theta}^2 = -5.091 \text{ m/s}^2 - (0.5657 \text{ m})(3 \text{ rad/s})^2 = -10.18 \text{ m/s}^2 \\
a_\theta &= r\ddot{\theta} + 2r\dot{\theta} = (0.5657 \text{ m})(0) + 2(-1.6971 \text{ m/s})(3 \text{ rad/s}) \\
&= -10.18 \text{ m/s}^2
\end{align*}
\]

Substituting these results into Eqs. (1) and (2) and solving, we get

\[
\begin{align*}
N_C &= 7.20 \text{ N} \\
F &= 0 \quad \text{Ans.}
\end{align*}
\]
EXAMPLE 13.11

The smooth 2-kg cylinder C in Fig. 13–20a has a pin P through its center which passes through the slot in arm OA. If the arm is forced to rotate in the vertical plane at a constant rate \( \theta = 0.5 \text{ rad/s} \), determine the force that the arm exerts on the peg at the instant \( \theta = 60^\circ \).

**SOLUTION**

Why is it a good idea to use polar coordinates to solve this problem?

**Free-Body Diagram.** The free-body diagram for the cylinder is shown in Fig. 13–20b. The force on the peg, \( F_P \), acts perpendicular to the slot in the arm. As usual, \( a_r \) and \( a_\theta \) are assumed to act in the directions of positive \( r \) and \( \theta \), respectively. Identify the four unknowns.

**Equations of Motion.** Using the data in Fig. 13–20b, we have

\[
\sum F_r = ma_r; \quad 19.62 \sin \theta - N_C \sin \theta = 2a_r \tag{1}
\]

\[
\sum F_\theta = ma_\theta; \quad 19.62 \cos \theta + F_P - N_C \cos \theta = 2a_\theta \tag{2}
\]

**Kinematics.** From Fig. 13–20a, \( r \) can be related to \( \theta \) by the equation

\[
r = \frac{0.4}{\sin \theta} = 0.4 \csc \theta
\]

Since \( d(\csc \theta) = -(\csc \theta \cot \theta) d\theta \) and \( d(\cot \theta) = -(\csc^2 \theta) d\theta \), then \( r \) and the necessary time derivatives become

\[
\dot{\theta} = 0.5 \quad r = 0.4 \csc \theta
\]

\[
\ddot{\theta} = 0 \quad \dot{r} = -0.4(\csc \theta \cot \theta) \dot{\theta}
\]

\[
= -0.2 \csc \theta \cot \theta
\]

\[
r = -0.2(\csc \theta \cot \theta)(\dot{\theta}) \cot \theta - 0.2 \csc \theta(-\csc^2 \theta) \dot{\theta}
\]

\[
= 0.1 \csc \theta(\cot^2 \theta + \csc^2 \theta)
\]

Evaluating these formulas at \( \theta = 60^\circ \), we get

\[
\dot{\theta} = 0.5 \quad r = 0.462
\]

\[
\ddot{\theta} = 0 \quad \dot{r} = -0.133
\]

\[
\dddot{r} = 0.192
\]

\[
a_r = \dddot{r} - r\ddot{\theta}^2 = 0.192 - 0.462(0.5)^2 = 0.0770
\]

\[
a_\theta = r\dddot{\theta} + 2r\dot{\theta} = 0 + 2(-0.133)(0.5) = -0.133
\]

Substituting these results into Eqs. 1 and 2 with \( \theta = 60^\circ \) and solving yields

\[
N_C = 19.5 \text{ N} \quad F_P = -0.356 \text{ N} \quad \text{Ans.}
\]

The negative sign indicates that \( F_P \) acts opposite to the direction shown in Fig. 13–20b.
A can $C$, having a mass of 0.5 kg, moves along a grooved horizontal slot shown in Fig. 13–21a. The slot is in the form of a spiral, which is defined by the equation $r = (0.1\theta)$ m, where $\theta$ is in radians. If the arm $OA$ rotates with a constant rate $\dot{\theta} = 4$ rad/s in the horizontal plane, determine the force it exerts on the can at the instant $\theta = \pi$ rad. Neglect friction and the size of the can.

**SOLUTION**

**Free-Body Diagram.** The driving force $F_C$ acts perpendicular to the arm $OA$, whereas the normal force of the wall of the slot on the can, $N_C$, acts perpendicular to the tangent to the curve at $\theta = \pi$ rad, Fig. 13–21b. As usual, $\mathbf{a}_r$ and $\mathbf{a}_\theta$ are assumed to act in the positive directions of $r$ and $\theta$, respectively. Since the path is specified, the angle $\psi$ which the extended radial line $r$ makes with the tangent, Fig. 13–21c, can be determined from Eq. 13–10. We have $r = 0.1\theta$, so that $dr/d\theta = 0.1$, and therefore

$$\tan \psi = \frac{r}{dr/d\theta} = \frac{0.1\theta}{0.1} = \theta$$

When $\theta = \pi$, $\psi = \tan^{-1}\pi = 72.3^\circ$, so that $\phi = 90^\circ - \psi = 17.7^\circ$, as shown in Fig. 13–21c. Identify the four unknowns in Fig. 13–21b.

**Equations of Motion.** Using $\phi = 17.7^\circ$ and the data shown in Fig. 13–21b, we have

$$\sum F_r = ma_r; \quad N_C \cos 17.7^\circ = 0.5a_r \quad (1)$$

$$\sum F_\theta = ma_\theta; \quad F_C - N_C \sin 17.7^\circ = 0.5a_\theta \quad (2)$$

**Kinematics.** The time derivatives of $r$ and $\theta$ are

$$\dot{r} = 4 \text{ rad/s} \quad r = 0.1\theta$$

$$\dot{\theta} = 0 \quad \ddot{r} = 0.1\dot{\theta} = 0.1(4) = 0.4 \text{ m/s}$$

$$\dddot{\theta} = 0$$

At the instant $\theta = \pi$ rad,

$$a_r = \dddot{r} - r\ddot{\theta}^2 = 0 - 0.1(\pi)(4)^2 = -5.03 \text{ m/s}^2$$

$$a_\theta = \frac{1}{2}\dddot{\theta} + 2\dddot{\theta} = 0 + 2(0.4)(4) = 3.20 \text{ m/s}^2$$

Substituting these results into Eqs. 1 and 2 and solving yields

$$N_C = -2.64 \text{ N}$$

$$F_C = 0.800 \text{ N} \quad \text{Ans.}$$

What does the negative sign for $N_C$ indicate?
**FUNDAMENTAL PROBLEMS**

**F13–13.** Determine the constant angular velocity $\dot{\theta}$ of the vertical shaft of the amusement ride if $\phi = 45^\circ$. Neglect the mass of the cables and the size of the passengers.

**F13–15.** The 2-Mg car is traveling along the curved road described by $r = (50e^{2\theta})$ m, where $\theta$ is in radians. If a camera is located at $A$ and it rotates with an angular velocity of $\dot{\theta} = 0.05$ rad/s and an angular acceleration of $\ddot{\theta} = 0.01$ rad/s$^2$ at the instant $\theta = \frac{\pi}{6}$ rad, determine the resultant friction force developed between the tires and the road at this instant.

**F13–14.** The 0.2-kg ball is blown through the smooth vertical circular tube whose shape is defined by $r = (0.6 \sin \theta)$ m, where $\theta$ is in radians. If $\theta = (\pi \dot{t})$ rad, where $t$ is in seconds, determine the magnitude of force $F$ exerted by the blower on the ball when $t = 0.5$ s.

**F13–16.** The 0.2-kg pin $P$ is constrained to move in the smooth curved slot, which is defined by the lemniscate $r = (0.6 \cos 2\theta)$ m. Its motion is controlled by the rotation of the slotted arm $OA$, which has a constant clockwise angular velocity of $\dot{\theta} = -3$ rad/s. Determine the force arm $OA$ exerts on the pin $P$ when $\theta = 0^\circ$. Motion is in the vertical plane.
**PROBLEMS**

13–84. The path of motion of a 5-lb particle in the horizontal plane is described in terms of polar coordinates as \( r = (2t + 1) \) ft and \( \theta = (0.5t^2 - t) \) rad, where \( t \) is in seconds. Determine the magnitude of the resultant force acting on the particle when \( t = 2 \) s.

13–85. Determine the magnitude of the resultant force acting on a 5-kg particle at the instant \( t = 2 \) s, if the particle is moving along a horizontal path defined by the equations \( r = (2t + 10) \) m and \( \theta = (1.5t^2 - 6t) \) rad, where \( t \) is in seconds.

13–86. A 2-kg particle travels along a horizontal smooth path defined by

\[
r = \left(\frac{1}{4}t^3 + 2\right) \text{ m}, \quad \theta = \left(\frac{t^2}{4}\right) \text{ rad},
\]

where \( t \) is in seconds. Determine the radial and transverse components of force exerted on the particle when \( t = 2 \) s.

13–87. A 2-kg particle travels along a path defined by

\[
r = (3 + 2t^2) \text{ m}, \quad \theta = \left(\frac{1}{3}t^3 + 2\right) \text{ rad}
\]

and \( z = (5 - 2t^2) \) m, where \( t \) is in seconds. Determine the \( r, \theta, z \) components of force that the path exerts on the particle at the instant \( t = 1 \) s.

13–88. If the coefficient of static friction between the block of mass \( m \) and the turntable is \( \mu_s \), determine the maximum constant angular velocity of the platform without causing the block to slip.

13–89. The 0.5-kg collar \( C \) can slide freely along the smooth rod \( AB \). At a given instant, rod \( AB \) is rotating with an angular velocity of \( \dot{\theta} = 2 \text{ rad/s} \) and has an angular acceleration of \( \ddot{\theta} = 2 \text{ rad/s}^2 \). Determine the normal force of rod \( AB \) and the radial reaction of the end plate \( B \) on the collar at this instant. Neglect the mass of the rod and the size of the collar.

13–90. The 2-kg rod \( AB \) moves up and down as its end slides on the smooth contoured surface of the cam, where \( r = 0.1 \) m and \( z = (0.02 \sin \theta) \) m. If the cam is rotating with a constant angular velocity of \( 5 \text{ rad/s} \), determine the force on the roller \( A \) when \( \theta = 90^\circ \). Neglect friction at the bearing \( C \) and the mass of the roller.

13–91. The 2-kg rod \( AB \) moves up and down as its end slides on the smooth contoured surface of the cam, where \( r = 0.1 \) m and \( z = (0.02 \sin \theta) \) m. If the cam is rotating at a constant angular velocity of \( 5 \text{ rad/s} \), determine the maximum and minimum force the cam exerts on the roller at \( A \). Neglect friction at the bearing \( C \) and the mass of the roller.
**13–92.** If the coefficient of static friction between the conical surface and the block of mass \( m \) is \( \mu_s = 0.2 \), determine the minimum constant angular velocity \( \dot{\theta} \) so that the block does not slide downwards.

**13–93.** If the coefficient of static friction between the conical surface and the block is \( \mu_s = 0.2 \), determine the maximum constant angular velocity \( \dot{\theta} \) without causing the block to slide upwards.

**13–94.** If the position of the 3-kg collar \( C \) on the smooth rod \( AB \) is held at \( r = 720 \text{ mm} \), determine the constant angular velocity \( \dot{\theta} \) at which the mechanism is rotating about the vertical axis. The spring has an unstretched length of 400 mm. Neglect the mass of the rod and the size of the collar.

**13–95.** The mechanism is rotating about the vertical axis with a constant angular velocity of \( \dot{\theta} = 6 \text{ rad/s} \). If rod \( AB \) is smooth, determine the constant position \( r \) of the 3-kg collar \( C \). The spring has an unstretched length of 400 mm. Neglect the mass of the rod and the size of the collar.

**13–96.** Due to the constraint, the 0.5-kg cylinder \( C \) travels along the path described by \( r = (0.6 \cos \theta) \text{ m} \). If arm \( OA \) rotates counterclockwise with an angular velocity of \( \dot{\theta} = 2 \text{ rad/s} \) and an angular acceleration of \( \ddot{\theta} = 0.8 \text{ rad/s}^2 \) at the instant \( \theta = 30^\circ \), determine the force exerted by the arm on the cylinder at this instant. The cylinder is in contact with only one edge of the smooth slot, and the motion occurs in the horizontal plane.
13–97. The 0.75-lb smooth can is guided along the circular path using the arm guide. If the arm has an angular velocity \( \dot{\theta} = 2 \text{ rad/s} \) and an angular acceleration \( \ddot{\theta} = 0.4 \text{ rad/s}^2 \) at the instant \( \theta = 30^\circ \), determine the force of the guide on the can. Motion occurs in the horizontal plane.

13–98. Solve Prob. 13–97 if motion occurs in the vertical plane.

13–99. The forked rod is used to move the smooth 2-lb particle around the horizontal path in the shape of a limaçon, \( r = (2 + \cos \theta) \text{ ft} \). If at all times \( \dot{\theta} = 0.5 \text{ rad/s} \), determine the force which the rod exerts on the particle at the instant \( \theta = 90^\circ \). The fork and path contact the particle on only one side.

*13–100. Solve Prob. 13–99 at the instant \( \theta = 60^\circ \).

*13–101. The forked rod is used to move the smooth 2-lb particle around the horizontal path in the shape of a limaçon, \( r = (2 + \cos \theta) \text{ ft} \). If \( \theta = (0.5t^2) \text{ rad} \), where \( t \) is in seconds, determine the force which the rod exerts on the particle at the instant \( t = 1 \text{ s} \). The fork and path contact the particle on only one side.

13–102. The amusement park ride rotates with a constant angular velocity of \( \dot{\theta} = 0.8 \text{ rad/s} \). If the path of the ride is defined by \( r = (3\sin \theta + 5) \text{ m} \) and \( z = (3\cos \theta) \text{ m} \), determine the \( r \), \( \theta \), and \( z \) components of force exerted by the seat on the 20-kg boy when \( \theta = 120^\circ \).

13–98. Solve Prob. 13–97 if motion occurs in the vertical plane.

13–103. The airplane executes the vertical loop defined by \( r^2 = [810(10^3) \cos 2\theta] \text{ m}^2 \). If the pilot maintains a constant speed \( v = 120 \text{ m/s} \) along the path, determine the normal force the seat exerts on him at the instant \( \theta = 0^\circ \). The pilot has a mass of 75 kg.
13–104. A boy standing firmly spins the girl sitting on a circular “dish” or sled in a circular path of radius \( r_0 = 3 \) m such that her angular velocity is \( \theta_0 = 0.1 \) rad/s. If the attached cable \( OC \) is drawn inward such that the radial coordinate \( r \) changes with a constant speed of \( \dot{r} = -0.5 \) m/s, determine the tension it exerts on the sled at the instant \( r = 2 \) m. The sled and girl have a total mass of 50 kg. Neglect the size of the girl and sled and the effects of friction between the sled and ice. *Hint*: First show that the equation of motion in the \( \theta \) direction yields \( a_\theta = r \ddot{\theta} + 2 \dot{r} \dot{\theta} = (1/r) d/dt(r^2 \dot{\theta}) = 0 \). When integrated, \( r^2 \dot{\theta} = C \), where the constant \( C \) is determined from the problem data.

Probs. 13–107/108

13–105. The smooth particle has a mass of 80 g. It is attached to an elastic cord extending from \( O \) to \( P \) and due to the slotted arm guide moves along the horizontal circular path \( r = (0.8 \sin \theta) \) m. If the cord has a stiffness \( k = 30 \) N/m and an unstretched length of 0.25 m, determine the force of the guide on the particle when \( \theta = 60^\circ \). The guide has a constant angular velocity \( \dot{\theta} = 5 \) rad/s.

13–106. Solve Prob. 13–105 if \( \dot{\theta} = 2 \) rad/s² when \( \dot{\theta} = 5 \) rad/s and \( \theta = 60^\circ \).

13–107. The 1.5-kg cylinder \( C \) travels along the path described by \( r = (0.6 \sin \theta) \) m. If arm \( OA \) rotates counterclockwise with a constant angular velocity of \( \dot{\theta} = 3 \) rad/s, determine the force exerted by the smooth slot in arm \( OA \) on the cylinder at the instant \( \theta = 60^\circ \). The spring has a stiffness of 100 N/m and is unstretched when \( \theta = 30^\circ \). The cylinder is in contact with only one edge of the slotted arm. Neglect the size of the cylinder. Motion occurs in the horizontal plane.

13–108. The 1.5-kg cylinder \( C \) travels along the path described by \( r = (0.6 \sin \theta) \) m. If arm \( OA \) is rotating counterclockwise with an angular velocity of \( \dot{\theta} = 3 \) rad/s, determine the force exerted by the smooth slot in arm \( OA \) on the cylinder at the instant \( \theta = 60^\circ \). The spring has a stiffness of 100 N/m and is unstretched when \( \theta = 30^\circ \). The cylinder is in contact with only one edge of the slotted arm. Neglect the size of the cylinder. Motion occurs in the vertical plane.

13–109. Using air pressure, the 0.5-kg ball is forced to move through the tube lying in the horizontal plane and having the shape of a logarithmic spiral. If the tangential force exerted on the ball due to air pressure is 6 N, determine the rate of increase in the ball’s speed at the instant \( \theta = \pi/2 \). Also, what is the angle \( \psi \) from the extended radial coordinate \( r \) to the line of action of the 6-N force?
13–110. The tube rotates in the horizontal plane at a constant rate of $\dot{\theta} = 4 \text{ rad/s}$. If a 0.2-kg ball $B$ starts at the origin $O$ with an initial radial velocity of $\dot{r} = 1.5 \text{ m/s}$ and moves outward through the tube, determine the radial and transverse components of the ball’s velocity at the instant it leaves the outer end at $C$, $r = 0.5 \text{ m}$. Hint: Show that the equation of motion in the $r$ direction is $\ddot{r} = 16r = 0$. The solution is of the form $r = Ae^{-4t} + Be^{4t}$. Evaluate the integration constants $A$ and $B$, and determine the time $t$ when $r = 0.5 \text{ m}$. Proceed to obtain $v_r$ and $v_\theta$.

13–111. The pilot of an airplane executes a vertical loop which in part follows the path of a cardioid, $r = 600(1 + \cos \theta) \text{ ft}$. If his speed at $A$ ($\theta = 0^\circ$) is a constant $v_P = 80 \text{ ft/s}$, determine the vertical force the seat belt must exert on him to hold him to his seat when the plane is upside down at $A$. He weighs 150 lb.

13–112. The 0.5-lb ball is guided along the vertical circular path $r = 2r_c \cos \theta$ using the arm $OA$. If the arm has an angular velocity $\dot{\theta} = 0.4 \text{ rad/s}$ and an angular acceleration $\ddot{\theta} = 0.8 \text{ rad/s}^2$ at the instant $\theta = 30^\circ$, determine the force of the arm on the ball. Neglect friction and the size of the ball. Set $r_c = 0.4 \text{ ft}$.

13–113. The ball of mass $m$ is guided along the vertical circular path $r = 2r_c \cos \theta$ using the arm $OA$. If the arm has a constant angular velocity $\dot{\theta}_0$, determine the angle $\theta = 45^\circ$ at which the ball starts to leave the surface of the semicylinder. Neglect friction and the size of the ball.

13–114. The ball has a mass of 1 kg and is confined to move along the smooth vertical slot due to the rotation of the smooth arm $OA$. Determine the force of the rod on the ball and the normal force of the slot on the ball when $\theta = 30^\circ$. The rod is rotating with a constant angular velocity $\dot{\theta} = 3 \text{ rad/s}$. Assume the ball contacts only one side of the slot at any instant.

13–115. Solve Prob. 13–114 if the arm has an angular acceleration of $\ddot{\theta} = 2 \text{ rad/s}^2$ when $\dot{\theta} = 3 \text{ rad/s}$ at $\theta = 30^\circ$. 
If a particle is moving only under the influence of a force having a line of action which is always directed toward a fixed point, the motion is called *central-force motion*. This type of motion is commonly caused by electrostatic and gravitational forces.

In order to analyze the motion, we will consider the particle $P$ shown in Fig. 13–22a, which has a mass $m$ and is acted upon only by the central force $F$. The free-body diagram for the particle is shown in Fig. 13–22b. Using polar coordinates $(r, \theta)$, the equations of motion, Eqs. 13–9, become

$$\Sigma F_r = ma_r; \quad -F = m \left[ \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right]$$

$$\Sigma F_\theta = ma_\theta; \quad 0 = m \left( r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right)$$

(13–11)

The second of these equations may be written in the form

$$\frac{1}{r} \frac{d}{dt} \left( r^2 \frac{d\theta}{dt} \right) = 0$$

so that integrating yields

$$\frac{r^2}{dt} \frac{d\theta}{dt} = h$$

(13–12)

Here $h$ is the constant of integration.

From Fig. 13–22a notice that the shaded area described by the radius $r$, as $r$ moves through an angle $d\theta$, is $dA = \frac{1}{2} r^2 d\theta$. If the *areal velocity* is defined as

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{h}{2}$$

(13–13)

then it is seen that the areal velocity for a particle subjected to central-force motion is constant. In other words, the particle will sweep out equal segments of area per unit of time as it travels along the path. To obtain the *path of motion*, $r = f(\theta)$, the independent variable $t$ must be eliminated from Eqs. 13–11. Using the chain rule of calculus and Eq. 13–12, the time derivatives of Eqs. 13–11 may be replaced by

$$\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} = \frac{h}{r^2} \frac{dr}{d\theta}$$

$$\frac{d^2r}{dt^2} = \frac{d}{dt} \left( \frac{h}{r^2} \frac{dr}{d\theta} \right) = \frac{d}{d\theta} \left( \frac{h}{r^2} \frac{dr}{d\theta} \right) \frac{d\theta}{dt} = \left[ \frac{d}{d\theta} \left( \frac{h}{r^2} \frac{dr}{d\theta} \right) \right] \frac{h}{r^2}$$
Substituting a new dependent variable \((\xi)\) \(\xi = 1/r\) into the second equation, we have
\[
\frac{d^2r}{dt^2} = -h^2 \xi^2 \frac{d^2 \xi}{d\theta^2}
\]
Also, the square of Eq. 13–12 becomes
\[
\left(\frac{d\theta}{dt}\right)^2 = h^2 \xi^4
\]
Substituting these two equations into the first of Eqs. 13–11 yields
\[
-h^2 \xi^2 \frac{d^2 \xi}{d\theta^2} - h^2 \xi^3 = -\frac{F}{m}
\]
or
\[
\frac{d^2 \xi}{d\theta^2} + \xi = \frac{F}{mh^2 \xi^2}
\]
Equation (13–14)

This differential equation defines the path over which the particle travels when it is subjected to the central force \(F\).* For application, the force of gravitational attraction will be considered. Some common examples of central-force systems which depend on gravitation include the motion of the moon and artificial satellites about the earth, and the motion of the planets about the sun. As a typical problem in space mechanics, consider the trajectory of a space satellite or space vehicle launched into free-flight orbit with an initial velocity \(v_0\), Fig. 13–23. It will be assumed that this velocity is initially parallel to the tangent at the surface of the earth, as shown in the figure.† Just after the satellite is released into free flight, the only force acting on it is the gravitational force of the earth. (Gravitational attractions involving other bodies such as the moon or sun will be neglected, since for orbits close to the earth their effect is small in comparison with the earth’s gravitation.) According to Newton’s law of gravitation, force \(F\) will always act between the mass centers of the earth and the satellite, Fig. 13–23. From Eq. 13–1, this force of attraction has a magnitude of
\[
F = G \frac{M_e m}{r^2}
\]
where \(M_e\) and \(m\) represent the mass of the earth and the satellite, respectively, \(G\) is the gravitational constant, and \(r\) is the distance between

*In the derivation, \(F\) is considered positive when it is directed toward point \(O\). If \(F\) is oppositely directed, the right side of Eq. 13–14 should be negative.
†The case where \(v_0\) acts at some initial angle \(\theta\) to the tangent is best described using the conservation of angular momentum (see Prob. 15–100).
the mass centers. To obtain the orbital path, we set $\xi = 1/r$ in the foregoing equation and substitute the result into Eq. 13–14. We obtain

$$\frac{d^2\xi}{d\theta^2} + \xi = \frac{GM_e}{h^2}$$

(13–15)

This second-order differential equation has constant coefficients and is nonhomogeneous. The solution is the sum of the complementary and particular solutions given by

$$\xi = \frac{1}{r} = C \cos(\theta - \phi) + \frac{GM_e}{h^2}$$

(13–16)

This equation represents the free-flight trajectory of the satellite. It is the equation of a conic section expressed in terms of polar coordinates.

A geometric interpretation of Eq. 13–16 requires knowledge of the equation for a conic section. As shown in Fig. 13–24, a conic section is defined as the locus of a point $P$ that moves in such a way that the ratio of its distance to a focus, or fixed point $F$, to its perpendicular distance to a fixed line $DD$ called the directrix, is constant. This constant ratio will be denoted as $e$ and is called the eccentricity. By definition

$$e = \frac{FP}{PA}$$

From Fig. 13–24,

$$FP = r = e(PA) = e[p - r \cos(\theta - \phi)]$$

or

$$\frac{1}{r} = \frac{1}{p} \cos(\theta - \phi) + \frac{1}{ep}$$

Comparing this equation with Eq. 13–16, it is seen that the fixed distance from the focus to the directrix is

$$p = \frac{1}{C}$$

(13–17)

And the eccentricity of the conic section for the trajectory is

$$e = \frac{Ch^2}{GM_e}$$

(13–18)
Provided the polar angle $\theta$ is measured from the $x$ axis (an axis of symmetry since it is perpendicular to the directrix), the angle $\phi$ is zero, Fig. 13–24, and therefore Eq. 13–16 reduces to

$$\frac{1}{r} = C \cos \theta + \frac{GM_e}{h^2}$$  \hspace{1cm} (13–19)

The constants $h$ and $C$ are determined from the data obtained for the position and velocity of the satellite at the end of the power-flight trajectory. For example, if the initial height or distance to the space vehicle is $r_0$, measured from the center of the earth, and its initial speed is $v_0$ at the beginning of its free flight, Fig. 13–25, then the constant $h$ may be obtained from Eq. 13–12. When $\theta = \phi = 0^\circ$, the velocity $v_0$ has no radial component; therefore, from Eq. 12–25, $v_0 = r_0(d\theta/dt)$, so that

$$h = r_0^2 \frac{d\theta}{dt}$$

or

$$h = r_0v_0$$  \hspace{1cm} (13–20)

To determine $C$, use Eq. 13–19 with $\theta = 0^\circ$, $r = r_0$, and substitute Eq. 13–20 for $h$:

$$C = \frac{1}{r_0} \left( 1 - \frac{GM_e}{r_0v_0^2} \right)$$  \hspace{1cm} (13–21)

The equation for the free-flight trajectory therefore becomes

$$\frac{1}{r} = \frac{1}{r_0} \left( 1 - \frac{GM_e}{r_0v_0^2} \right) \cos \theta + \frac{GM_e}{r_0^2v_0^2}$$  \hspace{1cm} (13–22)

The type of path traveled by the satellite is determined from the value of the eccentricity of the conic section as given by Eq. 13–18. If

$$e = 0 \quad \text{free-flight trajectory is a circle}$$

$$e = 1 \quad \text{free-flight trajectory is a parabola}$$

$$e < 1 \quad \text{free-flight trajectory is an ellipse}$$

$$e > 1 \quad \text{free-flight trajectory is a hyperbola}$$  \hspace{1cm} (13–23)
Parabolic Path. Each of these trajectories is shown in Fig. 13–25. From the curves it is seen that when the satellite follows a parabolic path, it is “on the border” of never returning to its initial starting point. The initial launch velocity, \( v_0 \), required for the satellite to follow a parabolic path is called the escape velocity. The speed, \( v_e \), can be determined by using the second of Eqs. 13–23, \( e = 1 \), with Eqs. 13–18, 13–20, and 13–21. It is left as an exercise to show that

\[
v_e = \sqrt{\frac{2GM_e}{r_0}}
\]

Circular Orbit. The speed \( v_c \) required to launch a satellite into a circular orbit can be found using the first of Eqs. 13–23, \( e = 0 \). Since \( e \) is related to \( h \) and \( C \), Eq. 13–18, \( C \) must be zero to satisfy this equation (from Eq. 13–20, \( h \) cannot be zero); and therefore, using Eq. 13–21, we have

\[
v_c = \sqrt{\frac{GM_e}{r_0}}
\]

Provided \( r_0 \) represents a minimum height for launching, in which frictional resistance from the atmosphere is neglected, speeds at launch which are less than \( v_c \) will cause the satellite to reenter the earth’s atmosphere and either burn up or crash, Fig. 13–25.
Elliptical Orbit. All the trajectories attained by planets and most satellites are elliptical, Fig. 13–26. For a satellite’s orbit about the earth, the minimum distance from the orbit to the center of the earth $O$ (which is located at one of the foci of the ellipse) is $r_p$ and can be found using Eq. 13–22 with $\theta = 0^\circ$. Therefore;

$$r_p = r_0$$  \hspace{1cm} (13–26)

This minimum distance is called the *perigee* of the orbit. The *apogee* or maximum distance $r_a$ can be found using Eq. 13–22 with $\theta = 180^\circ$. Thus,

$$r_a = \frac{r_0}{(2GM_e/r_0v_0^2) - 1}$$  \hspace{1cm} (13–27)

With reference to Fig. 13–26, the half length of the major axis of the ellipse is

$$a = \frac{r_p + r_a}{2}$$  \hspace{1cm} (13–28)

Using analytical geometry, it can be shown that the half length of the minor axis is determined from the equation

$$b = \sqrt{r_pr_a}$$  \hspace{1cm} (13–29)

*Actually, the terminology perigee and apogee pertains only to orbits about the earth. If any other heavenly body is located at the focus of an elliptical orbit, the minimum and maximum distances are referred to respectively as the *periapsis* and *apoapsis* of the orbit.*
Furthermore, by direct integration, the area of an ellipse is

\[ A = \pi ab = \frac{\pi}{2} (r_p + r_a) \sqrt{r_pr_a} \]  

(13–30)

The areal velocity has been defined by Eq. 13–13, \( \frac{dA}{dt} = h/2 \). Integrating yields \( A = hT/2 \), where \( T \) is the period of time required to make one orbital revolution. From Eq. 13–30, the period is

\[ T = \frac{\pi}{h} (r_p + r_a) \sqrt{r_pr_a} \]  

(13–31)

In addition to predicting the orbital trajectory of earth satellites, the theory developed in this section is valid, to a surprisingly close approximation, at predicting the actual motion of the planets traveling around the sun. In this case the mass of the sun, \( M_s \), should be substituted for \( M_e \) when the appropriate formulas are used.

The fact that the planets do indeed follow elliptic orbits about the sun was discovered by the German astronomer Johannes Kepler in the early seventeenth century. His discovery was made before Newton had developed the laws of motion and the law of gravitation, and so at the time it provided important proof as to the validity of these laws. Kepler’s laws, developed after 20 years of planetary observation, are summarized as follows:

1. Every planet travels in its orbit such that the line joining it to the center of the sun sweeps over equal areas in equal intervals of time, whatever the line’s length.

2. The orbit of every planet is an ellipse with the sun placed at one of its foci.

3. The square of the period of any planet is directly proportional to the cube of the major axis of its orbit.

A mathematical statement of the first and second laws is given by Eqs. 13–13 and 13–22, respectively. The third law can be shown from Eq. 13–31 using Eqs. 13–19, 13–28, and 13–29. (See Prob. 13–116.)
EXAMPLE 13.13

A satellite is launched 600 km from the surface of the earth, with an initial velocity of 30 Mm/h acting parallel to the tangent at the surface of the earth, Fig. 13–27. Assuming that the radius of the earth is 6378 km and that its mass is 5.976(10^{24}) kg, determine (a) the eccentricity of the orbital path, and (b) the velocity of the satellite at apogee.

**SOLUTION**

**Part (a).** The eccentricity of the orbit is obtained using Eq. 13–18. The constants $h$ and $C$ are first determined from Eqs. 13–20 and 13–21. Since

$$h = r_p v_0 = 6.978(10^6)(8333.3) = 58.15(10^9) \text{ m}^2/\text{s}$$

$$C = \frac{1}{r_p} \left(1 - \frac{GM_e}{r_p v_0^2}\right)$$

$$= \frac{1}{6.978(10^6)} \left\{1 - \frac{66.73(10^{-12})(5.976(10^{24})]}{6.978(10^6)(8333.3)^2}\right\} = 25.4(10^{-9}) \text{ m}^{-1}$$

Hence,

$$e = \frac{Ch^2}{GM_e} = \frac{2.54(10^{-9})(58.15(10^9))^2}{66.73(10^{-12})(5.976(10^{24})]} = 0.215 < 1 \quad \text{Ans.}$$

From Eq. 13–23, observe that the orbit is an ellipse.

**Part (b).** If the satellite were launched at the apogee $A$ shown in Fig. 13–27, with a velocity $v_A$, the same orbit would be maintained provided

$$h = r_p v_0 = r_a v_A = 58.15(10^9) \text{ m}^2/\text{s}$$

Using Eq. 13–27, we have

$$r_a = \frac{r_p}{2GM_e} - 1 = \frac{6.978(10^6)}{2[66.73(10^{-12})(5.976(10^{24})]} - 1 = 10.804(10^6)$$

Thus,

$$v_A = \frac{58.15(10^9)}{10.804(10^6)} = 5382.2 \text{ m/s} = 19.4 \text{ Mm/h} \quad \text{Ans.}$$

**NOTE:** The farther the satellite is from the earth, the slower it moves, which is to be expected since $h$ is constant.
PROBLEMS

In the following problems, except where otherwise indicated, assume that the radius of the earth is 6378 km, the earth’s mass is \(5.976(10^{24})\) kg, the mass of the sun is \(1.99(10^{30})\) kg, and the gravitational constant is \(G = 6.673(10^{-11}) \text{ m}^3/(\text{kg} \cdot \text{s}^2)\).


13–117. The Viking explorer approaches the planet Mars on a parabolic trajectory as shown. When it reaches point \(A\) its velocity is 10 Mm/h. Determine \(r_0\) and the required velocity at \(A\) so that it can then maintain a circular orbit as shown. The mass of Mars is 0.1074 times the mass of the earth.

13–119. The satellite is moving in an elliptical orbit with an eccentricity \(e = 0.25\). Determine its speed when it is at its maximum distance \(A\) and minimum distance \(B\) from the earth.

13–118. The satellite is in an elliptical orbit around the earth as shown. Determine its velocity at perigee \(P\) and apogee \(A\), and the period of the satellite.

13–120. The space shuttle is launched with a velocity of 17 500 mi/h parallel to the tangent of the earth’s surface at point \(P\) and then travels around the elliptical orbit. When it reaches point \(A\), its engines are turned on and its velocity is suddenly increased. Determine the required increase in velocity so that it enters the second elliptical orbit. Take \(G = 34.4(10^{-9}) \text{ ft}^3/\text{lb} \cdot \text{s}^2\), \(M_e = 409(10^{21})\) slug, and \(r_e = 3960\) mi, where 5280 ft = mi.

13–121. Determine the increase in velocity of the space shuttle at point \(P\) so that it travels from a circular orbit to an elliptical orbit that passes through point \(A\). Also, compute the speed of the shuttle at \(A\).
13–122. The rocket is in free flight along an elliptical trajectory \( A' \). The planet has no atmosphere, and its mass is 0.60 times that of the earth. If the orbit has the apoapsis and periapsis shown, determine the rocket’s velocity when it is at point \( A \). Take \( G = 34.4(10^{-9})(\text{lb} \cdot \text{ft}^2)/\text{slug}^2 \), \( M_e = 409(10^{21}) \text{ slug} \), 1 mi = 5280 ft.

13–123. If the rocket is to land on the surface of the planet, determine the required free-flight speed it must have at \( A' \) so that the landing occurs at \( B \). How long does it take for the rocket to land, in going from \( A' \) to \( B \)? The planet has no atmosphere, and its mass is 0.6 times that of the earth. Take \( G = 34.4(10^{-9})(\text{lb} \cdot \text{ft}^2)/\text{slug}^2 \), \( M_e = 409(10^{21}) \text{ slug} \), 1 mi = 5280 ft.

13–124. A communications satellite is to be placed into an equatorial circular orbit around the earth so that it always remains directly over a point on the earth’s surface. If this requires the period to be 24 hours (approximately), determine the radius of the orbit and the satellite’s velocity.

13–125. The speed of a satellite launched into a circular orbit about the earth is given by Eq. 13–25. Determine the speed of a satellite launched parallel to the surface of the earth so that it travels in a circular orbit 800 km from the earth’s surface.

13–126. The earth has an orbit with eccentricity \( e = 0.0821 \) around the sun. Knowing that the earth’s minimum distance from the sun is \( 151.3(10^9) \text{ km} \), find the speed at which a rocket travels when it is at this distance. Determine the equation in polar coordinates which describes the earth’s orbit about the sun.
13–131. The satellite is launched parallel to the tangent of the earth’s surface with a velocity of \( v_0 = 30 \text{ Mm/h} \) from an altitude of 2 Mm above the earth as shown. Show that the orbit is elliptical, and determine the satellite’s velocity when it reaches point \( A \).

13–132. The satellite is in an elliptical orbit having an eccentricity of \( e = 0.15 \). If its velocity at perigee is \( v_p = 15 \text{ Mm/h} \), determine its velocity at apogee \( A \) and the period of the satellite.

13–133. The satellite is in an elliptical orbit. When it is at perigee \( P \), its velocity is \( v_p = 25 \text{ Mm/h} \), and when it reaches point \( A \), its velocity is \( v_A = 15 \text{ Mm/h} \) and its altitude above the earth’s surface is 18 Mm. Determine the period of the satellite.

13–134. A satellite is launched with an initial velocity \( v_0 = 4000 \text{ km/h} \) parallel to the surface of the earth. Determine the required altitude (or range of altitudes) above the earth’s surface for launching if the free-flight trajectory is to be (a) circular, (b) parabolic, (c) elliptical, and (d) hyperbolic.

13–135. The rocket is in a free-flight elliptical orbit about the earth such that as shown. Determine its speed when it is at point \( A \). Also determine the sudden change in speed the rocket must experience at \( B \) in order to travel in free flight along the orbit indicated by the dashed path.

13–136. A communications satellite is in a circular orbit above the earth such that it always remains directly over a point on the earth’s surface. As a result, the period of the satellite must equal the rotation of the earth, which is approximately 24 hours. Determine the satellite’s altitude \( h \) above the earth’s surface and its orbital speed.

13–137. Determine the constant speed of satellite \( S \) so that it circles the earth with an orbit of radius \( r = 15 \text{ Mm} \). Hint: Use Eq. 13–1.
CONCEPTUAL PROBLEMS

P13-1. If the box is released from rest at A, use numerical values to show how you would estimate the time for it to arrive at B. Also, list the assumptions for your analysis.

P13-2. The tugboat has a known mass and its propeller provides a known maximum thrust. When the tug is fully powered you observe the time it takes for the tug to reach a speed of known value starting from rest. Show how you could determine the mass of the barge. Neglect the drag force of the water on the tug. Use numerical values to explain your answer.

P13-3. Determine the smallest speed of each car A and B so that the passengers do not lose contact with the seat while the arms turn at a constant rate. What is the largest normal force of the seat on each passenger? Use numerical values to explain your answer.

P13-4. Each car is pin connected at its ends to the rim of the wheel which turns at a constant speed. Using numerical values, show how to determine the resultant force the seat exerts on the passenger located in the top car A. The passengers are seated towards the center of the wheel. Also, list the assumptions for your analysis.
### Kinetics

Kinetics is the study of the relation between forces and the acceleration they cause. This relation is based on Newton’s second law of motion, expressed mathematically as \( \sum F = ma \).

Before applying the equation of motion, it is important to first draw the particle’s free-body diagram in order to account for all of the forces that act on the particle. Graphically, this diagram is equal to the kinetic diagram, which shows the result of the forces, that is, the \( ma \) vector.

### Inertial Coordinate Systems

When applying the equation of motion, it is important to measure the acceleration from an inertial coordinate system. This system has axes that do not rotate but are either fixed or translate with a constant velocity. Various types of inertial coordinate systems can be used to apply \( \sum F = ma \) in component form.

- Rectangular \( x, y, z \) axes are used to describe rectilinear motion along each of the axes.
  \[ \sum F_x = ma_x, \sum F_y = ma_y, \sum F_z = ma_z \]

- Normal and tangential \( n, t \) axes are often used when the path is known. Recall that \( a_n \) is always directed in the +\( n \) direction. It indicates the change in the velocity direction. Also recall that \( a_t \) is tangent to the path. It indicates the change in the velocity magnitude.
  \[ \sum F_i = ma_i, \sum F_n = ma_n, \sum F_b = 0 \]
  \[ a_i = \frac{dv}{dt} \quad \text{or} \quad a_t = v \frac{dv}{ds} \]
  \[ a_n = \frac{v^2}{\rho} \quad \text{where} \quad \rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|} \]

- Cylindrical coordinates are useful when angular motion of the radial line \( r \) is specified or when the path can conveniently be described with these coordinates.
  \[ \sum F_r = m(\dot{r} - r\dot{\theta}^2) \]
  \[ \sum F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \]
  \[ \sum F_z = m\ddot{z} \]

### Central-Force Motion

When a single force acts upon a particle, such as during the free-flight trajectory of a satellite in a gravitational field, then the motion is referred to as central-force motion. The orbit depends upon the eccentricity \( e' \); and as a result, the trajectory can either be circular, parabolic, elliptical, or hyperbolic.