Chapter 5

Truth-functional Logic: Proofs

What Will I Learn in This Chapter?

We turn now to the construction of proofs in truth-functional logic. Proofs can be challenging, but many students find their construction the most interesting part of logic. Beginning in Section 5.2, we will introduce eight basic rules that are adequate to construct a proof for every valid truth-functional argument. In Sections 5.5 and 5.6, we present and justify numerous rules that make proofs shorter and easier. Section 5.7 explains some useful strategies and tactics for constructing proofs.

5.1 Statement Forms and Their Instances

Truth tables are a systematic, though sometimes cumbersome, way of refuting invalid truth-functional arguments. By contrast, we will now approach truth-functional logic by constructing proofs for valid arguments. We will neither use truth tables nor base anything on them, except in the sense that none of the rules we adopt will ever allow a valid argument to have true premises and a false conclusion. Truth-functional proofs are sequences of statements. Each sequence begins with assumptions. To advance a proof, one attaches new statements to the sequence. The sequence ends when one attaches the statement one is trying to prove. For example, in proving an argument valid, one begins by assuming the premises, and the goal is to attach the conclusion.

In constructing a proof sequence, we follow certain rules carefully. Proofs establish validity only because, when these rules are followed, each statement attached to the sequence is true if all earlier statements are true. That is, after the premises, each statement is implied by earlier statements in the sequence.

Thus, if we begin a proof with the premises of an argument, and if we succeed in attaching the conclusion, the argument must be valid.

In Chapter 4, we represented simple statements by uppercase letters such as 'A', 'B', and 'C', and we used lowercase letters like 'p', 'q', and 'r' as statement variables. We observed that statement variables are placeholders, or blanks, that can be filled by any single statement, simple or compound, by any uppercase letter representing a simple statement, or by any well-formed truth-functional combination of uppercase letters representing a compound statement (p. 204). Like the uppercase letters that represent statements, statement variables can be combined by truth-functional connectors. We refer to such combinations as **statement forms;** thus, p, $\sim q$, $p \vee q$, and $p \rightarrow (q \vee \sim r)$ are statement forms. If a particular symbolized statement can be obtained by correctly filling the blanks in a statement form, we say that the statement has that form or that it is an **instance** of that form; for example, 'A ∨ B' and '∼C ∨ (B ∨ D)' are instances of $p \vee q$.

By using lowercase letters as blanks, we can state rules that apply to all instances of certain statement forms. Truth-functional proofs proceed by applying such rules. Thus, before we can construct proofs, we must learn to identify instances of statement forms.

Every conditional, however complex, has (or is an instance of) the form $p \rightarrow q$. Thus, 'A → B', '∼A → B', and ' (A ∨ ∼(B ∨ C)) → (D ∨ ∼A)' are all instances of $p \rightarrow q$. Indeed, even 'A → A' is an instance of $p \rightarrow q$, because there is no requirement that different statements must fill differently labeled blanks in a statement form. In contrast, only conditionals in which the same statement is both antecedent and consequent have the form $p \rightarrow p$. Thus, of the preceding instances, only 'A → A' is an instance of $p \rightarrow p$. So the convention is this: *Given some statement form, while the same statement must be substituted for each occurrence of a given variable in the form, one need not substitute different statements for different variables.*

Chapter 5

The preceding examples illustrate another point: 'A → A' is an instance both of $p \rightarrow q$ and of $p \rightarrow p$. Thus, just as there are many different instances of each statement form, each statement typically has (or is an instance of) more than one form.

Next, consider the statement form $p \rightarrow \sim q$; here are some of its instances: 'A → ∼B', 'A → ∼A', '∼A → ∼A', '(B → C) → ∼A', and '(B → ∼D) → ∼(A → B)'. By contrast, 'A → B' is *not* an instance of $p \rightarrow \sim q$, because its consequent is not preceded by '∼'. Here is the convention: *In forming instances of a statement form, one cannot fill the blanks marked by variables in such a way as to eliminate connectors or parentheses.* In other words, each connector in any statement form must be identifiable in every instance of that form. This stricture leads to the following result: '∼∼A' is an instance of p and of $\sim p$. Yet, although 'A' is equivalent to '$\sim\sim$A', 'A' is an instance of p, but *not* of $\sim p$. This is because the '$\sim$' of $\sim p$ is no longer identifiable in 'A'. The point is important enough to warrant another example put in a slightly different way: 'B $\vee$ C' is an instance of p, but not of $\sim p$. (Given $\sim p$, there is nothing you can put into the blank marked by p that will yield 'B $\vee$ C'). However, '$\sim\sim$(B $\vee$ C)' is an instance of p and also of $\sim p$. (Given $\sim p$, just fill the blank marked by p with '$\sim$(B $\vee$ C)', and you get '$\sim\sim$(B $\vee$ C)'). And this is true even though 'B $\vee$ C' is equivalent to '$\sim\sim$(B $\vee$ C)'.

Confused? Very possibly. But take heart: whether you are able to state or even fully to understand these conventions, with a little practice you will probably be able to recognize instances. The best way to learn is by trying some examples and by comparing your results with the solutions.

Exercises 5.1

Identify the statements in the right-hand column (if any) that are instances of each of the statement forms in the left-hand column:

★**L1.** $\sim p$	**R1.** $(A \rightarrow B) \vee C$
★**L2.** $p \vee q$	**R2.** $\sim(A \rightarrow B)$
★**L3.** $\sim(p \vee q)$	**R3.** $\sim(A \rightarrow B) \rightarrow \sim\sim A$
★**L4.** $\sim p \vee q$	**R4.** A
★**L5.** $\sim\sim p$	**R5.** $A \rightarrow \sim A$
★**L6.** $p \rightarrow q$	**R6.** $A \rightarrow (B \cdot \sim A)$
★**L7.** $p \rightarrow \sim q$	**R7.** $\sim\sim A \rightarrow B$
★**L8.** $\sim p \rightarrow \sim q$	**R8.** $\sim(A \vee B) \vee (A \vee B)$
★**L9.** $\sim(p \rightarrow q)$	**R9.** $\sim(\sim(A \vee B) \vee (A \vee B))$
★**L10.** $\sim p \rightarrow p$	**R10.** $\sim(A \cdot \sim B) \rightarrow \sim\sim(A \cdot \sim B)$
★**L11.** $p \rightarrow (q \cdot \sim p)$	**R11.** $\sim\sim(A \rightarrow B)$
★**L12.** $\sim\sim p \rightarrow p$	**R12.** $(A \rightarrow B) \rightarrow \sim(A \rightarrow B)$

5.2 Modus Ponens and Rules for Conjunctions, Disjunctions, and Biconditionals

Proofs of valid truth-functional arguments are sequences of truth-functional statements. In attaching statements to truth-functional proof sequences, we need eight basic rules. These rules are adequate for constructing a proof of any valid truth-functional argument. Of the eight rules, six are introduced in this section and two more are given in the next two sections. Later, we will adopt rules that can make proofs shorter and easier. These shortcut rules are not really necessary, because, as we will see, they can all be derived from the eight basic rules. Most of the rules (both basic and shortcut) resemble specific implications and equivalences from among those listed in Section 4.5. All the rules are presented as statement forms; as you will see, we apply the rules by forming instances of those forms.

Modus Ponens, Simplification, and Conjunction

The first rule is the familiar inference called **modus ponens.** In stating this and all other rules, the word 'given' is shorthand either for 'given a line in a proof sequence, the entire statement on which has the form' or for 'given lines in a proof sequence, the entire statements on which have the forms', and 'attach' is shorthand for 'attach a line, the entire statement on which has the form'. The preceding sentence is complicated, but it enables us to state the rules quite simply. Our first rule is

RULE **MODUS PONENS (MP):** Given $p \rightarrow q$ and p, one can attach q.

For example, if a proof sequence includes, as entire lines, both '∼A → (B ∨ C)' and '∼A', we can attach 'B ∨ C' to the sequence (because the three statements are, respectively, instances of the statement forms $p \rightarrow q$, p, and q, as specified in the rule).

The next two rules deal with conjunctions. The first is called **simplification:**

RULE **SIMPLIFICATION (SIMP):** Given $p \cdot q$, one can attach p, or q, or both (one at a time).

Simplification tells us that if a conjunction appears in a proof sequence, we can attach *either* (or both) conjunct(s). For example, given 'A · (B → C)', one could attach 'A', 'B → C', or (on different lines) 'A' and 'B → C'.

Our third rule is called **conjunction:**

RULE **CONJUNCTION (CONJ):** Given p and q, one can attach $p \cdot q$.

Given any statement or statements on one or more lines (not necessarily successive), conjunction enables us to attach their conjunction. For example, if one line is 'A ∨ B' and another line is 'B → C', one can attach '(A ∨ B) · (B → C)'. Notice that, as the rule is stated, it is acceptable to conjoin a line with itself: given 'A ∨ B', one can attach '(A ∨ B) · (A ∨ B)'. In a sense, conjunction and simplification reverse one another. Conjunction is sometimes called "'and' introduction," and simplification can be called "'and' elimination."

We can see how these first three rules function in a short proof of this simple argument:

A · B	
B → C	∴ C · A

As in every attempt to prove validity, the proof sequence for this argument begins with its premises. We then use appropriate substitution instances of the rules to attach statements to the sequence until we can attach the conclusion, 'C · A'. At that point, the proof is complete. The proof runs as follows:

Example 5.2.A

1. A · B	
2. B → C	∴ C · A
3. B	from line 1 by Simp
4. C	from lines 2 and 3 by MP
5. A	from line 1 by Simp
6. C · A	from lines 4 and 5 by Conj

Notice three things: (1) the column at the right, which is not really part of the proof, contains explanations of how each line has been attached. (2) Although 'C' is on line 4, we cannot simply attach 'C · A' on line 5 from lines 1 and 4. To follow conjunction exactly, we must first isolate 'A' on a line by itself (line 5) and only then conjoin it with 'C'. (3) It often happens that a given line is used several times in the course of a proof. Thus, in Example 5.2.A, line 1 is used in attaching both line 3 and line 5.

Here is a second example:

Example 5.2.B

A	
(A · ~B) → (~C · D)	
~B	∴ D
1. A	
2. (A · ~B) → (~C · D)	
3. ~B	∴ D
4. A · ~B	from lines 1 and 3 by Conj
5. ~C · D	from lines 2 and 4 by MP
6. D	from line 5 by Simp

Beginning with the next example, we will shorten proofs in two ways: we will number premises as they are given rather than rewriting them to begin the proof, and we will shorten the explanations in the right-hand column to line numbers and rules.

Example 5.2.C

1.	A	
2.	B → E	
3.	A → D	
4.	B · ~C	
5.	(D · E) → ~G	∴ A · ~G
6.	D	1, 3 MP
7.	B	4 Simp
8.	E	2, 7 MP
9.	D · E	6, 8 Conj
10.	~G	5, 9 MP
11.	A · ~G	1, 10 Conj

Here are two important facts about using the rules: (1) The rules can be applied only to entire lines. For example, the following inference is *incorrect:*

Example 5.2.D

1.	~(A · B)	
2.	~A	*misapplying* Simp to 1

(2) In applying rules, like modus ponens, that require two (or more) earlier lines, the order of those earlier lines in the sequence is immaterial. For instance, in Example 5.2.C, in attaching line 6, the conditional to which MP was applied (line 3) came *after* the line affirming its antecedent (line 1); however, in attaching line 8, the conditional to which MP was applied (line 2) came *before* its antecedent (line 7).

Remember, we construct proofs by taking *instances* of the rules—instances in the sense in which they were explained in Section 5.1. The statements we use and the statement we attach must have *exactly* the forms specified in one of the rules—they must be exact instances of a rule. This idea warrants illustration. Here are lines 5 and 9 from Example 5.2.C:

5. (D · E) → ~G
9. D · E

From these lines, by modus ponens, we attach line 10:

10. ~G 5, 9 MP

Attaching line 10 is justified because, together, lines 5 and 9 meet the conditions for MP, namely, they are instances of $p \rightarrow q$ and p, so we can attach '~G', which, in this case, is the instance of q. If line 9 had been just 'D' or, say, 'E · D', lines 5 and 9 would not have allowed us to use MP. Be careful to apply rules only to exact instances.

Now for a word about strategy. The most difficult thing about proofs is knowing, at any given point, what statement to attach. The number and nature of the rules you will ultimately have available are such that, at any given point in a proof, literally infinitely many statements can be attached; the trick is figuring out one or two choices that are likely to bring you closer to the conclusion you seek. We will learn more about strategy later. For now, just remember this: in approaching a new proof, be sure to examine the *conclusion*. The conclusion often suggests the best approach for a proof. Look again at Example 5.2.C; the conclusion is the conjunction ‘A · ~G’. So, before beginning the proof, ask yourself, How can I get a conjunction in a proof sequence? Often, as in Example 5.2.C, a conjunction comes by conjoining two statements already in the sequence. So, to get ‘A · ~G’, we seek ‘A’ and ‘~G’, and when we have them, we will conjoin them. But ‘A’ is the first premise—we have that already. So how can we get ‘~G’? Since it is the consequent in line 5, we can try to get the antecedent in 5, namely, ‘D · E’, and then apply modus ponens. These thoughts take us a long way toward finding a proof. Notice that the steps go *backwards* from the conclusion: we use a “retro” approach based on how the proof may end. Of course, we could carry this example further by trying to see how to get ‘D · E’ (another conjunction). But what we have is already sufficient to illustrate the point: often, the best way to find a proof is by working backwards from the conclusion.

Before trying some exercises, here is something else to keep in mind: Even for the simple proofs in this section, it is usually possible to reach a conclusion in more than one way. For instance, instead of starting Example 5.2.C by using modus ponens to get ‘D’ (line 6), we could have begun by simplifying ‘B’ from line 4 (which we did in line 7). Next, we could have applied modus ponens to get ‘E’ and only then used MP on lines 1 and 3 to get ‘D’. The resulting proof would be just slightly different from Example 5.2.C, but we see this important point: if you construct a proof different from one given in the text, your proof may still be correct. As more rules become available, there will be many different, yet equally correct, proofs of the same valid argument.

Exercises 5.2

Construct a proof for each of the following arguments:

★**1.** (A · D) → (E · G)
E → ~A
A · B
C · D ∴ ~A

★**2.** A → (B → C)
B · D
D → A ∴ C

★**3.** B
(B → C) · (C → D)
(B · D) → (~(E ∨ D) → H)
~(E ∨ D) ∴ (B · D) · H

4. A · ~B
C · ~D
(~B · ~D) → (E · (G ∨ H)) ∴ C · (G ∨ H)

5. A → (C → (D · ~E))
D → (A · (C ∨ ~B))
G · D
C ∴ D · ~E

6. E → ~(G ∨ H)
A · ~D
~(B ∨ C) → (A → E)
~D → ~(B ∨ C) ∴ ~(G ∨ H)

Disjunctive Syllogism and Addition

Simplification and conjunction enable us to work with conjunctions. The next two rules, disjunctive syllogism and addition, concern disjunctions. These rules form a pair much like simplification and conjunction. Disjunctive syllogism can be called "'Or' elimination," and addition can be called "'Or' introduction." As these rules are presented, see if you understand why this is so.

There are two parts to **disjunctive syllogism:**

RULE

Disjunctive Syllogism (DS): Given *p* ∨ *q* and ~*p*, one can attach *q*. Given *p* ∨ *q* and ~*q*, one can attach *p*.

For example, given '(A → B) ∨ ~C' and '~(A → B)', one can attach '~C' or given '(A ∨ ~B) ∨ (C → D)' and '~(C → D)', one can attach 'A ∨ ~B'.

If you have read Chapter 3 on syllogisms, the name of this rule may seem odd because, as we are using the term, the rule has nothing to do with syllogisms. In days of yore, when syllogisms were thought to encompass all arguments, this nonsyllogistic form of reasoning was given its name to distinguish it from ordinary syllogisms (which, at the time, were called "categorical syllogisms"); the name stuck. Later we will encounter another name, "hypothetical syllogism," that originated in a similar way and that also applies to a nonsyllogistic form of reasoning. Our use of these quaint names is an essentially harmless concession to history and common usage.

Now we must look again at forming instances, because our new rule often causes students grief in a certain respect. To see how, suppose you encounter these lines in some proof:

Example 5.2.E

~A ∨ B
A

You may want to attach 'B' by disjunctive syllogism, but doing so would be *incorrect* because, as DS is stated, it can be applied only to lines of the forms $p \vee q$ and $\sim p$, and, as we saw earlier (p. 272–273), 'A' is not an instance of $\sim p$. Thus, although '~A' (in the first line) and 'A' (in the second) deny one another, the statements do not warrant the use of disjunctive syllogism. Obviously, we need some means of inferring 'B' from these lines—don't worry, that will come soon enough. For now, what matters is being clear about applying the rules we have. Here are a few exercises intended to ensure that you see what you can and cannot do with disjunctive syllogism (and, by analogy, with the other rules as well).

Exercises 5.2

For each of the following pairs of lines, decide whether one can apply disjunctive syllogism; if the rule can be applied, state the result of applying it:

★**7.** ~(A → B)
(A → B) ∨ C

★**8.** ~A ∨ B
~~A

★**9.** A ∨ (B ∨ C)
~(B ∨ C)

★**10.** A → (B ∨ C)
~B

★**11.** (A → (B ∨ C)) ∨ (D → E)
~(A → (B → C))

★**12.** ~(A ∨ (B ∨ C))
(A ∨ (B ∨ C)) ∨ (D → E)

★**13.** (A → (B ∨ C)) ∨ (D → E)
~(A → (B ∨ C))

★**14.** ~~(A → (B ∨ C)) ∨ (D → E)
~(A → (B ∨ C))

★**15.** $\sim\sim(A \rightarrow (B \vee C)) \vee (D \rightarrow E)$
$\sim\sim\sim(A \rightarrow (B \vee C))$

Our second rule for dealing with disjunctions is called **addition:**

RULE **ADDITION (ADD):** Given *p*, one can attach $p \vee q$ or $q \vee p$.

Given *any* statement in a proof, addition enables us to attach a disjunction in which the given statement is one disjunct and *any statement whatsoever* is added as the other disjunct. For example, given '$A \rightarrow B$', one could attach '$(A \rightarrow B) \vee E$' or '$(C \cdot D) \vee (A \rightarrow B)$'. This rule requires some explanation.

You may find addition surprising because it seems to give you something for nothing. Students ask, "Must the disjunct you add appear earlier in the proof?" or "Can you really add any disjunct you like?" or "Where does the disjunct you add come from?" The answer to the first question is "No" and to the second question is "Yes." (We will get to the third question later in the paragraph.) This is because using addition always yields a statement *weaker* than the statement it came from. In using the rule, you *aren't* getting something for nothing; instead, you are getting (attaching) a statement that tells you less than the statement you started with. Think about it: Suppose you know you are having pizza for dinner, and you tell your gluttonous brother, "We are having either pizza or liver for dinner, but I won't tell you which." Clearly, your brother knows *less* than you do, and, to be safe, he may grab some burgers on the way home. Replacing a known fact (we are having pizza) by a disjunction (we are having pizza or liver) in which the known fact is one disjunct yields a statement that, while true, is less informative than the fact you started with. So your next question may be "Then why use this rule at all?" In fact, one seldom encounters addition in real-life arguments, but, as we will soon see, for technical reasons, it is occasionally useful in constructing proofs. Now we can answer the third of the three questions posed at the beginning of this paragraph: the added disjunct in the statement you attach doesn't really *come from* anywhere—you simply make it up as required by the proof at hand.

It's time for some examples:

Example 5.2.F

1. $(B \vee C) \rightarrow \sim(A \rightarrow D)$
2. $(A \rightarrow D) \vee \sim E$
3. B
4. $B \rightarrow \sim G$ $\therefore \sim G \cdot \sim E$

Look at the conclusion; it's a conjunction. This means that we will probably get it by conjoining '~G' and '~E'. How can we get these? Attaching '~G' is easy:

5.	~G	3,4 MP

We can get '~E' only from line 2. (It doesn't appear elsewhere.) But since line 2 is a disjunction, we might be able to get '~E' by disjunctive syllogism. To use DS in this way, we must first attach the consequent in line 1. We can do that by modus ponens if we can attach 'B ∨ C'. But one of these disjuncts, 'B', already appears on line 3. Thus, we use addition to attach what we need. So the proof goes like this:

6.	B ∨ C	3 Add
7.	~(A → D)	1,6 MP
8.	~E	2,7 DS
9.	~G · ~E	5,8 Conj

Here is a second example:

Example 5.2.G

1.	(A ∨ ~H) → (B ∨ E)	
2.	~D ∨ B	
3.	A	
4.	(E · ~D) → (B ∨ ~I)	
5.	~(G ∨ ~B) ∨ I	
6.	~B	∴ ~(G ∨ ~B)

In this argument, examining the conclusion alone doesn't help much. But, in looking at the premises, you may notice that the conclusion is the first disjunct in premise 5. If we could get '~I' and use disjunctive syllogism, we would have the conclusion. '~I' can come from line 4 by modus ponens and disjunctive syllogism. So we need to get the antecedent of line 4; that is, we need to attach 'E' and '~D' and then conjoin them.

7.	~D	2, 6 DS
8.	A ∨ ~H	3 Add
9.	B ∨ E	1, 8 MP
10.	E	6, 9 DS
11.	E · ~D	7, 10 Conj
12.	B ∨ ~I	4, 11 MP
13.	~I	6, 12 DS
14.	~(G ∨ ~B)	5, 13 DS

EXERCISES 5.2

Construct a proof for each of the following arguments:

16. $\sim B \vee A$
$\sim\sim B$
$A \rightarrow (C \vee D)$
$\sim C$ $\quad\therefore D$

17. $(B \vee D) \rightarrow (D \vee E)$
$((D \vee E) \vee J) \rightarrow (G \vee H)$
B
$(G \vee H) \rightarrow \sim D$
$\sim D \rightarrow \sim G$ $\quad\therefore H$

18. $(B \vee \sim(C \cdot D)) \vee (\sim A \cdot \sim B)$
$(D \vee G) \rightarrow (E \rightarrow H)$
$\sim A \rightarrow D$
$\sim(B \vee \sim(C \cdot D))$ $\quad\therefore E \rightarrow H$

19. $\sim A \vee B$
$(G \vee C) \rightarrow \sim D$
$A \vee G$
$\sim B \cdot \sim C$ $\quad\therefore \sim D \cdot \sim A$

Biconditional Exchange

You may have noticed that, so far, none of our rules involve biconditionals. The rule of **biconditional exchange** compensates for this deficiency; it has two parts:

RULE

BICONDITIONAL EXCHANGE (BE): Given $p \leftrightarrow q$, one can attach $p \rightarrow q$, or $q \rightarrow p$, or both (one at a time).
Given $p \rightarrow q$ and $q \rightarrow p$, one can attach $p \leftrightarrow q$.

Together, the two parts of biconditional exchange form a pair that resembles the two earlier pairs simplification and conjunction, and disjunctive syllogism and addition. Like the first of each of these pairs, the first part of BE enables one to "eliminate" a connector (in this case '$\leftrightarrow$'); like the second of each pair, the second part of BE enables one to "introduce" that same connector.

Since biconditional exchange is our only rule for dealing with biconditionals, if a biconditional appears in a premise, one often (but not always) applies the first part of BE to "eliminate" the '$\leftrightarrow$' by attaching one or both conditionals, which can then be used with other rules. Here is a simple example:

Example 5.2.H

1.	$A \leftrightarrow B$	
2.	A	$\therefore B$
3.	$A \rightarrow B$	1 BE
4.	B	2,3 MP

Similarly, if a biconditional appears in a conclusion, one often (but not always) uses biconditional exchange to "introduce" the '$\leftrightarrow$' by attaching a biconditional based on a pair of conditionals included earlier in the sequence. This step may be the last line in the proof. Here is a simple example:

Example 5.2.I

1.	$C \rightarrow (A \rightarrow B)$	
2.	C	
3.	$B \rightarrow A$	$\therefore A \leftrightarrow B$
4.	$A \rightarrow B$	1,2 MP
5.	$A \leftrightarrow B$	3,4 BE

EXERCISES 5.2

For each line in the following proofs, identify the line (or lines) and the rule that warrant its having been attached:

★**20.**

1.	$(A \vee B) \rightarrow ((\sim C \cdot \sim D) \rightarrow E)$	
2.	$D \vee \sim C$	
3.	$A \leftrightarrow \sim D$	
4.	$(B \rightarrow \sim C) \cdot A$	$\therefore \sim C \cdot E$
5.	A	
6.	$A \rightarrow \sim D$	
7.	$\sim D$	
8.	$\sim C$	
9.	$A \vee B$	
10.	$(\sim C \cdot \sim D) \rightarrow E$	
11.	$\sim C \cdot \sim D$	
12.	E	
13.	$\sim C \cdot E$	

★**21.**
1. $\sim A \rightarrow \sim E$
2. $(B \leftrightarrow C) \rightarrow \sim A$
3. $D \rightarrow (B \rightarrow C)$
4. $D \cdot (C \rightarrow B)$
5. $E \vee (\sim B \vee \sim C)$
6. $\sim\sim C$ $\therefore \sim B \vee E$
7. D
8. $B \rightarrow C$
9. $C \rightarrow B$
10. $B \leftrightarrow C$
11. $\sim A$
12. $\sim E$
13. $\sim B \vee \sim C$
14. $\sim B$
15. $\sim B \vee E$

Construct a proof for each of the following arguments (remember that proofs may be correct even if they differ from those given as solutions):

★**22.** $A \rightarrow D$
$\sim\sim B$
$E \rightarrow (C \vee (D \rightarrow A))$
$\sim C$
$\sim B \vee E$ $\therefore D \leftrightarrow A$

★**23.** $\sim(A \cdot D) \vee E$
$(C \vee B) \leftrightarrow \sim E$
$(G \vee (A \cdot D)) \cdot C$ $\therefore G$

24. $A \leftrightarrow B$
$B \rightarrow C$
A $\therefore C$

25. $\sim D$
$\sim A \vee D$
$\sim C \vee A$
$E \rightarrow G$
$(A \vee C) \vee E$
$\sim C \rightarrow \sim(A \vee C)$ $\therefore G \cdot \sim C$

26. $B \rightarrow (E \rightarrow G)$
$A \rightarrow \sim C$
$A \cdot B$
$C \vee (G \rightarrow E)$ $\therefore E \leftrightarrow G$

27. $(B \vee \sim(C \cdot D)) \rightarrow (\sim A \cdot \sim B)$
$D \rightarrow (E \rightarrow G)$
$\sim A \rightarrow D$
$B \vee \sim(C \cdot D)$
$G \rightarrow E$ $\quad \therefore E \leftrightarrow G$

28. $(B \vee D) \rightarrow A$
$(A \vee J) \rightarrow (G \vee H)$
$\sim D \leftrightarrow (G \vee H)$
$(\sim D \rightarrow \sim G) \cdot B$ $\quad \therefore H$

29. $(A \cdot C) \rightarrow (D \rightarrow E)$
$C \rightarrow (A \cdot (H \vee \sim B))$
$G \cdot C$
$G \rightarrow (E \rightarrow D)$ $\quad \therefore D \leftrightarrow E$

5.3 Conditional Proof

We will now learn the two remaining basic rules in our system of proofs. These rules are different from any we have learned so far, but are similar to each other. In this section, we will discuss the first (conditional proof); the second (indirect proof) is discussed in the next section.

We begin with an argument that illustrates the form of conditional proof. A former student of mine—call him Homer—tried to win my sympathy with this pathetic argument:

Example 5.3.A

Suppose I buy a new Porsche. That would impress Hattie, but it would take all my money, and I'd have to get a second job. That would mean that I couldn't spend time with Hattie, and she would go out with Biff. If Hattie went out with Biff, I'd be miserable. So if I bought a new Porsche, I'd only be miserable. See 5.7.58.

Homer begins with a supposition: that he buys a new Porsche. In tracing out the consequences of this supposed purchase, he sees that it would make him miserable. He concludes that *if he were to buy the Porsche,* he would be miserable. In such reasoning, a supposition is temporarily adopted just to see what follows, but one knows all along that it is merely a supposition—it is never asserted as true. Indeed, as you saw if you read Zeno's argument (Example 1.2.A) in Section 1.2 (and as we will also see in the next section, on indirect proof), the whole point of making a supposition may be to prove that the supposition is *false.* Thus, in the end, Homer does not conclude that he *is* miserable (as, given his reasoning, he would be if he bought the Porsche); instead, he concludes only that *if he bought the Porsche,* he would be miserable.

The rule of conditional proof makes precise this informal way of reasoning. In a proof, one can suppose anything one likes (e.g., *I buy a Porsche*), as long as one remembers throughout the proof that the supposition is merely temporary and not something known to be true. To remind us of the status of suppositions, we use a system of arrows as a bookkeeping device. The arrows indicate that whatever follows from a supposition is not known to follow from the original premises alone, as is otherwise true of the statements we attach in proofs, but depends also on one or more temporary suppositions. A proof can conclude only after all suppositions have been "closed." For now, this means writing them as antecedents in conditionals (e.g., *if I buy a Porsche,* I'll be miserable). That is why this rule is called *conditional proof.*

As an example of conditional proof, consider this simple and obviously valid argument.

$$A \to B \qquad \therefore A \to (A \cdot B)$$

Using only the rules in Section 5.2, it is impossible to construct a proof of the argument. Here is how we use our new rule of conditional proof to do so:

Example 5.3.B

	1. $A \to B$	$\therefore A \to (A \cdot B)$
┌	2. A	
│	3. B	1,2 MP
└→	4. $A \cdot B$	2,3 Conj
	5. $A \to (A \cdot B)$	2–4 CP

As Example 5.3.B illustrates, a supposition is marked by beginning an arrow; the arrow bends down to continue along the sequence, showing which lines are attached while the supposition is still available for use (unclosed). The arrow then points to the last line in which the supposition is available. The statement immediately below the arrow (line 5) *closes the supposition.* Thereafter, neither the supposition *nor any line enclosed by the arrow* can be used again in the proof, a condition that is sometimes expressed by saying that these lines are "no longer in force."

Before any proof can end, every supposition *must* be closed. For now, a supposition *must* be closed with a conditional (in Section 5.4 we will learn another way of closing suppositions) whose antecedent *must* be the most recent unclosed supposition (line 2) and whose consequent *must* be the immediately preceding statement (the one to which the arrow points—line 4). The rule of **conditional proof** is subject to these restrictions and *cannot be used otherwise.*

In stating the rule, the word ‘supposing’ is shorthand for ‘attaching a line in a proof sequence, the entire statement on which has the form’, and, as before, ‘attach’ is shorthand for ‘attach a line, the entire statement on which has the form’.

RULE

CONDITIONAL PROOF (CP): If supposing p enables one to attach q, one can attach $p \to q$ (which closes the supposition).

Here are two simple examples that use CP:

Example 5.3.C

	1. $A \to B$	
	2. $B \to C$	$\therefore A \to C$
┌	3. A	
│	4. B	1,3 MP
└→	5. C	2,4 MP
	6. $A \to C$	3–5 CP

Example 5.3.D

	1. $A \to C$	$\therefore (A \cdot B) \to C$
┌	2. $A \cdot B$	
│	3. A	2 Simp
└→	4. C	1,3 MP
	5. $(A \cdot B) \to C$	2–4 CP

“But,” students often ask, “can you suppose *anything* you like?” Yes, anything! At first, you may feel uneasy about a rule that enables you to suppose anything at all. But a supposition, like a loan, is only temporary and must always be “paid back.” We “pay back” a supposition by closing it through conditionalizing—that is, by writing it as the antecedent of a conditional—and, as previously noted, no proof can end until all suppositions are closed. The conditional that closes a supposition says, in effect, that *if* we knew the antecedent (our supposition), we could infer the consequent (the final statement derived from the supposition). Thus, the conditional merely expresses *horizontally* what the enclosed sequence has demonstrated *vertically.*

In using conditional proof correctly, it is impossible to prove anything unwarranted. As an extreme example, let’s see what happens if we simply suppose, early in a proof, the very statement we are trying to prove. Let’s consider again the same argument for which we constructed Example 5.3.B:

Example 5.3.E

1. $A \to B$	$\therefore A \to (A \cdot B)$

What will happen if, instead of supposing 'A', as we did in Example 5.3.B, we suppose the conclusion itself?

2. $A \rightarrow (A \cdot B)$

This step is entirely correct. But where do we go from here? We cannot just stop, since the supposition has not been closed. Using our rules, we can attach a line or two to extend the proof a bit:

3. $(A \rightarrow (A \cdot B)) \vee C$	2 Add
4. $(A \rightarrow (A \cdot B)) \cdot ((A \rightarrow (A \cdot B)) \vee C)$	2,3 Conj

But we are still no closer to closing the supposition on line 2. If we simplify and close the supposition, we get this:

5. $A \rightarrow (A \cdot B)$	4 Simp
6. $(A \rightarrow (A \cdot B)) \rightarrow (A \rightarrow (A \cdot B))$	2-5 CP

Whatever the merits of this proof, it does *not* establish what we set out to prove. As the discussion illustrates, suppositions (like bank loans) are advantageous only if suited to genuine needs: they are never a free gift.

Once a supposition is closed by a conditional, neither that supposition nor any line enclosed by its arrow can be used again. A "proof" of an obviously invalid argument shows what can happen if we disregard this restriction.

Example 5.3.F

If I'd been born in 1503, I'd have died a long time ago. Therefore, I died a long time ago.

Here is the erroneous "proof":

1. $B \rightarrow D$	∴ D
2. B	
3. D	1,2 MP
4. $B \rightarrow D$	2–3 CP
5. D	2,4 *misapplying* MP

But 'B' in line 2 is only a supposition; once it is closed in line 4, we cannot use it again to attach line 5.

Here is a special application of conditional proof:

Example 5.3.G

1. A	
2. $A \rightarrow A$	

This proof may seem odd because we conditionalize immediately after making a supposition. However, as long as the most recent supposition (namely, 'A') is the antecedent and the preceeding line (namely, 'A' (again)) is the consequent, there is nothing wrong with the proof. Moreover, a proof like 5.3.G is occasionally useful in incorporating some statement of the form $p \rightarrow p$ into a proof.

Some proofs require more than one supposition. This can happen in two ways. The following proof exemplifies the first way:

Example 5.3.H

1. $(A \vee B) \rightarrow C$	
2. $C \rightarrow (D \cdot E)$	
3. $D \rightarrow A$	$\therefore (A \vee B) \leftrightarrow D$
4. $A \vee B$	
5. C	1,4 MP
6. $D \cdot E$	2,5 MP
7. D	6 Simp
8. $(A \vee B) \rightarrow D$	4–7 CP
9. D	
10. A	3,9 MP
11. $A \vee B$	10 Add
12. $D \rightarrow (A \vee B)$	9–11 CP
13. $(A \vee B) \leftrightarrow D$	8,12 BE

In this example, two suppositions entered the proof successively.

Here is a second way multiple suppositions can enter proofs:

Example 5.3.I

1. $A \rightarrow D$	
2. $D \rightarrow (C \vee E)$	
3. $B \rightarrow \sim E$	$\therefore A \rightarrow (B \rightarrow C)$
4. A	
5. D	1,4 MP
6. $C \vee E$	2,5 MP
7. B	
8. $\sim E$	3,7 MP
9. C	6,8 DS
10. $B \rightarrow C$	7–9 CP
11. $A \rightarrow (B \rightarrow C)$	4–10 CP

When, as we do here, we make a supposition before an earlier one is closed, the suppositions must be closed in reverse order: the most recent supposition is always closed first.

Since we often use suppositions, we will make explicit a terminological distinction that has been with us from the beginning: we use the term ‘**assumption**’ to include both premises (which normally begin proofs for the validity of arguments) and suppositions (which can be adopted for technical purposes at any point in the course of any proof).

We can now make an important addition to our backwards strategy: conditional proof is especially useful in proving conditionals. To prove a conditional, suppose its antecedent; the proof continues until you attach the desired consequent. Closing the supposition then yields the result. This strategy is used twice in each of Examples 5.3.H and 5.3.I.

In harmony with our earlier observations about “elimination” and “introduction” rules (pp. 274, 278, and 282), modus ponens can be thought of as a way of *eliminating* conditionals, and our new rule, conditional proof, can be thought of as a way of *introducing* them. Thus, the seven rules we have so far learned can be grouped into four pairs: simplification and conjunction, disjunctive syllogism and addition, the two parts of biconditional exchange, and modus ponens and conditional proof. The first of each pair enables us to “eliminate” ‘ $\cdot$ ’, ‘ $\vee$ ’, ‘ $\leftrightarrow$ ’, and ‘ $\to$ ’, respectively; the second of each pair enables us to “introduce” these same connectors. You may find this way of thinking about the rules useful in learning and applying them.

EXERCISES 5.3

Construct a proof for each of the following arguments:

★**1.**	$A \cdot B$	$\therefore\ (B \to C) \to (A \cdot C)$
★**2.**	$((A \vee B) \cdot C) \to D$	
	$D \to (B \to E)$	
	$B \cdot (E \vee G)$	$\therefore\ C \to E$
★**3.**	$A \vee B$	
	$\sim C \vee D$	
	$\sim A \vee C$	
	$B \to \sim D$	$\therefore\ B \leftrightarrow \sim C$
4.	$A \to B$	$\therefore\ A \to (B \vee C)$
5.	$A \to (B \to C)$	$\therefore\ (A \cdot B) \to C$
6.	$(A \cdot B) \to C$	$\therefore\ A \to (B \to C)$
7.	$A \to (B \to C)$	$\therefore\ B \to (A \to C)$
8.	$A \cdot \sim A$	$\therefore\ C$
9.	$A \leftrightarrow B$	
	$B \leftrightarrow C$	$\therefore\ A \leftrightarrow C$

10. $\sim A \vee \sim B$
$C \rightarrow \sim\sim A$
$\sim D \vee B$ $\therefore C \rightarrow \sim D$

11. $C \rightarrow \sim G$
$A \rightarrow B$
$H \rightarrow D$
$G \vee H$
$B \leftrightarrow C$ $\therefore A \rightarrow D$

12. $A \rightarrow (B \rightarrow C)$
$B \rightarrow (A \rightarrow D)$
$C \rightarrow (D \rightarrow E)$ $\therefore (A \cdot B) \rightarrow E$

Symbolize each of the following arguments, and construct a proof of each:

★**13.** Either wealth is an evil, or wealth is a good; but wealth is not an evil; therefore, wealth is a good. (Sextus Empiricus, *Against the Dogmatists*) See 1.1.3, 1.3.2, 2.7.II.

★**14.** The number five is neither larger than six nor smaller than four. If five is not larger than six, it is not larger than seven. If five is not smaller than four, it is not smaller than three. Therefore, five is not larger than seven, and it is not smaller than three. See 4.6.73.

★**15.** If the consequent of a conditional is contingent, the consequent is sometimes true. If the consequent is sometimes true, the conditional is sometimes true, that is, it is noncontradictory. So if a conditional has a contingent consequent, the conditional is noncontradictory. (K. Codell Carter, *A First Course in Logic*) See 4.6.92.

5.4 Indirect Proof

To introduce our final basic rule, recall this argument from Chapter 1:

Example 5.4.A
If you know you are dead, you are dead. If you know you are dead, you are not dead. So if you know you are dead, you are dead and not dead. Therefore, you do not know you are dead. (Benson Mates, *Elementary Logic*) See 1.2.C, 1.3.12, 2.7.29, 4.6.82.

Hopefully, you can see that this argument is valid. If you are in doubt, you can persuade yourself by constructing a truth table. We can think of the argument as beginning with a supposition (you know you are dead); the supposition leads to a contradiction (you are dead and not dead). We then conclude that the supposition is false (so that you do not know you are dead). Many of Zeno's famous arguments, some of which we saw in Chapter 1, also have this form, and you may have encountered similar arguments if you have studied mathematics.

The kind of reasoning illustrated in Example 5.4.A is sometimes called proof by contradiction or *reductio ad absurdum;* we will call it **indirect proof.** In stating the rule for indirect proof, the word 'supposing' is shorthand for 'supposing a line in a proof sequence, the entire statement on which has the form', and, as usual, 'attach' is shorthand for 'attach a line, the entire statement on which has the form'. The rule has two parts:

RULE

INDIRECT PROOF (IP): If supposing *p* enables one to attach *q* · ~*q,* one can attach ~*p* (which closes the supposition).
If supposing ~*p* enables one to attach *q* · ~*q,* one can attach *p* (which closes the supposition).

The dual insight behind Indirect Proof is this: If supposing that some statement is true leads to a contradiction, that statement must be false; if supposing that some statement is false leads to a contradiction, that statement must be true.

Indirect proof resembles conditional proof in that both employ suppositions. In using IP, as in using CP, we invoke our system of arrows to help us remember the status of suppositions and of the lines derived from them. As before, proofs cannot end with unclosed suppositions. The difference is this: in IP, having reached a contradiction, we close the supposition, *not* by attaching a conditional (as in CP), but by attaching the *opposite* of the supposition that led to the contradiction. (That is, if the supposition was *p*, we attach ~*p;* if it was ~*p,* we attach *p*.)

Here are three examples of indirect proof:

Chapter 5

Example 5.4.B

1.	A → (B → C)	
2.	A → ~(B → C)	∴ ~A
┌3.	A	
│4.	B → C	1,3 MP
│5.	~(B → C)	2,3 MP
└→6.	(B → C) · ~(B → C)	4,5 Conj
7.	~A	3–6 IP

Example 5.4.C

1.	$A \cdot B$	
2.	$\sim B \vee D$	$\therefore D$
┌3.	$\sim D$	
│4.	$\sim B$	2,3 DS
│5.	B	1 Simp
└→6.	$B \cdot \sim B$	4,5 Conj
7.	D	3–6 IP

Example 5.4.D

1.	$\sim C$	
2.	$(A \vee D) \vee (B \cdot C)$	$\therefore A \vee D$
┌3.	$\sim(A \vee D)$	
│4.	$B \cdot C$	2,3 DS
│5.	C	4 Simp
└→6.	$C \cdot \sim C$	1,5 Conj
7.	$A \vee D$	3–6 IP

Here is a more complex example that includes two uses of IP as parts of the main line of reasoning:

Example 5.4.E

1.	$E \rightarrow A$	
2.	$((B \rightarrow C) \vee (E \rightarrow D)) \rightarrow \sim A$	
3.	$D \rightarrow (B \rightarrow C)$	
4.	$D \cdot (C \rightarrow B)$	
5.	$E \vee (B \rightarrow \sim C)$	
6.	$\sim\sim C$	$\therefore \sim B$
7.	D	4 Simp
8.	$B \rightarrow C$	3,7 MP
9.	$(B \rightarrow C) \vee (E \rightarrow D)$	8 Add
10.	$\sim A$	2,9 MP
┌11.	E	
│12.	A	1,11 MP
└→13.	$A \cdot \sim A$	10,12 Conj
14.	$\sim E$	11–13 IP
15.	$B \rightarrow \sim C$	5,14 DS
┌16.	B	
│17.	$\sim C$	15,16 MP
└→18.	$\sim C \cdot \sim\sim C$	6,17 Conj
19.	$\sim B$	16–18 IP

It is also possible to combine conditional proof and indirect proof. For example, in proving a conditional, one may suppose its antecedent (as in a typical use of CP) and then suppose the *denial* of the consequent. If one can reach a contradiction, IP enables one to attach the desired consequent; CP then yields the conditional. Here is a proof that illustrates this tactic; the proof also includes a preliminary use of IP that is subordinate to the main line of proof:

Example 5.4.F

1.	$A \rightarrow B$	
2.	$\sim A \rightarrow \sim C$	$\therefore C \rightarrow B$
3.	C	
4.	$\sim B$	
5.	A	
6.	B	1,5 MP
7.	$B \cdot \sim B$	4,6 Conj
8.	$\sim A$	5–7 IP
9.	$\sim C$	2,8 MP
10.	$C \cdot \sim C$	3,9 Conj
11.	B	4–10 IP
12.	$C \rightarrow B$	3–11 CP

EXERCISES 5.4

Construct a proof for each of the following arguments:

★**1.** $(A \vee B) \rightarrow ((\sim C \cdot \sim D) \rightarrow E)$
$C \rightarrow D$
$A \rightarrow \sim D$
$A \cdot (B \rightarrow \sim C)$ $\therefore \sim C \cdot E$

2. $A \rightarrow B$
$\sim B$ $\therefore \sim A$

3. $\sim\sim A$ $\therefore A$

4. $A \rightarrow B$ $\therefore \sim B \rightarrow \sim A$

5. $B \leftrightarrow (A \cdot C)$
$A \rightarrow \sim C$ $\therefore \sim B$

6. $(A \cdot D) \rightarrow E$
$(C \vee B) \rightarrow \sim E$
$G \rightarrow (A \cdot D)$
C $\therefore \sim G$

7. $\sim(A \cdot \sim B)$
$A \cdot (B \rightarrow C)$ $\therefore C$

8. $\sim A \vee B$
$D \vee A$
$D \rightarrow C$ $\therefore\ \sim C \rightarrow B$

9. $\sim(A \vee B)$
$\sim A \rightarrow \sim E$ $\therefore\ E \leftrightarrow A$

10. $\sim A \cdot \sim B$ $\therefore\ \sim(A \vee B)$

11. $\sim(A \vee B)$ $\therefore \sim A \cdot \sim B$

Symbolize and construct a proof for each of the following arguments:

★**12.** If wages are treated as the only cost, in accord with the labor theory of value, the same price cannot be charged for the two different outputs of wheat, even though the kernels of wheat are identical. The good-land wheat involves lower labor costs and will sell for less than the poor-land wheat. This, of course, is absurd. The only correct procedure is to put an accounting price tag on each land, with the good-land having the higher price. (Paul A. Samuelson, *Economics*) See 4.2.59, 4.6.85.

★**13.** If commerce does not include navigation, the government of the Union has no direct power over the subject, and can make no law prescribing what shall constitute American vessels, or requiring that they shall be navigated by American seamen. Yet this power has been exercised from the commencement of the government, has been exercised with the consent of all, and has been understood by all to be a commercial regulation. The word used in the constitution, then, comprehends, and has been always understood to comprehend, navigation within its meaning; and a power to regulate navigation is as expressly granted, as if that term had been added to the word 'commerce'. (Supreme Court Decision, *Gibbons* v. *Ogden,* 1824) See 4.6.70.

★**14.** God, or substance consisting of infinite attributes each of which expresses eternal and infinite essence, necessarily exists. If this be denied, it follows that His essence does not involve existence. But this is absurd. (Baruch Spinoza, *Ethics*) See 4.2.57, 4.6.69.

★**15.** There can be no way of proving the existence of god is even probable. For if the existence of such a god could be proven to be probable, the proposition that he existed would be an empirical hypothesis. And in that case, it would be possible to deduce from it and other empirical hypotheses certain experiential propositions which were not deducible from those other hypotheses alone. But in fact this is not possible. (A. J. Ayer, *Language, Truth, and Logic*) See 4.6.100.

★**16.** If there is a decision procedure for quantificational logic, the halting problem is solvable. But if Church's thesis is correct, the halting problem is unsolvable. Therefore, given Church's thesis, there is no decision procedure for quantificational logic. (George S. Boolos and Richard C. Jeffrey, *Computability and Logic*) See 4.6.96.

We have now learned eight basic rules that are adequate to construct a proof for any valid truth-functional argument. These rules are summarized in the following box, in which a line means that, given a statement of the form (or statements of each of the forms) above the line, one can attach a statement of the form (or of any one or more of the forms separated by commas) below the line. Of course, the rules are all subject to the explanations given in the text.

Basic Rules

Simplification (Simp)

$$\begin{array}{l} p \cdot q \\ \hline p,\ q \end{array}$$

Conjunction (Conj)

$$\begin{array}{l} p \\ q \\ \hline p \cdot q \end{array}$$

Disjunctive Syllogism (DS)

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline q \end{array} \qquad \begin{array}{l} p \vee q \\ \sim q \\ \hline p \end{array}$$

Addition (Add)

$$\begin{array}{l} p \\ \hline p \vee q,\ q \vee p \end{array}$$

Biconditional Exchange (BE)

$$\begin{array}{l} p \leftrightarrow q \\ \hline p \rightarrow q,\ q \rightarrow p \end{array} \qquad \begin{array}{l} p \rightarrow q \\ q \rightarrow p \\ \hline p \leftrightarrow q \end{array}$$

Modus Ponens (MP)

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline q \end{array}$$

Conditional Proof (CP)

$$\begin{array}{l} \ulcorner p \\ \llcorner q \\ \hline p \rightarrow q \end{array}$$

Indirect Proof (IP)

$$\begin{array}{l} \ulcorner p \\ \llcorner q \cdot \sim q \\ \hline \sim p \end{array} \qquad \begin{array}{l} \ulcorner \sim p \\ \llcorner q \cdot \sim q \\ \hline p \end{array}$$

5.5 Shortcut Rules

In Sections 5.2, 5.3, and 5.4, we learned eight rules by means of which one can construct a proof for every valid argument. However, an important issue from an earlier discussion remains unresolved. In Example 5.2.E, we considered the following lines that one might encounter in a proof:

Example 5.5.A

$\sim A \vee B$
A

We observed that we cannot apply disjunctive syllogism directly to these lines because, although '∼A' and 'A' deny one another, the lines do not have the forms $p \vee q$ and $\sim p$ as is required by DS. At that point, we noted, "We need some means of inferring 'B' from these lines." Now, with indirect proof, we can easily make the inference. Do you see how? Here is one way:

Example 5.5.B

1.	$\sim A \vee B$	
2.	A	
→3.	$\sim B$	
4.	$\sim A$	1,3 DS
→5.	$A \cdot \sim A$	2,4 Conj
6.	B	3–5 IP

However, while IP solves the immediate problem of Example 5.5.A, the solution is not entirely adequate. One often confronts pairs of lines similar in form to those in Example 5.5.A, and it would be tedious to require, in each case, a "subroutine" of the sort illustrated by Example 5.5.B. We need a rule enabling us to move more efficiently from such pairs of lines to the desired result.

Let us examine a similar case. Imbedded in potential proofs for several earlier exercises are lines of which the following are typical:

$B \rightarrow C$
$\sim C$

Given such lines, we may want to attach '∼B'; again, we can do so by using indirect proof:

Example 5.5.C

1.	$B \rightarrow C$	
2.	$\sim C$	
→3.	B	
4.	C	1,3 MP
→5.	$C \cdot \sim C$	2,4 Conj
6.	$\sim B$	3–5 IP

However, once again, it would be nice to have a rule enabling us to bypass lines 3 through 5, which are probably a digression in the overall proof, and move directly from lines 1 and 2 to line 6. So should we adopt further rules that warrant such shortcuts?

Choosing rules for a system of proofs is a balancing act. Most texts adopt 18 or more basic rules (in place of our 8). On the one hand, such extensive sets of rules enable one to avoid many redundant subroutines, like Examples 5.5.B and 5.5.C, but beginning students are often bewildered by such an unordered plethora. On the other hand, given only a minimal set of rules (such as ours), proofs are frequently interrupted by tedious subroutines, and one can get lost in these digressions. The best solution is a compromise: having learned a minimal set of rules adequate to construct a proof for every valid argument, we will now make available, as shortcuts, all the other rules that appear in other logic texts—and more, to boot.

Five of these shortcut rules are explained in this section: double negation, De Morgan's theorems, conditional exchange, modus tollens, and constructive dilemma. In Section 5.6, we will introduce several additional shortcut rules.

Double Negation

Double negation (like most shortcut rules) has two parts, each enabling us to go in one direction between a pair of statements that are truth-functionally equivalent. Of course, we do not use the truth-functional equivalence of the statements in proofs; rather, the truth-functional equivalence of the statements is a heuristic that guides us in adopting double negation as a rule. As in Section 5.2, in stating the shortcut rules, the word 'given' is shorthand either for 'given a line in a proof sequence, the entire statement on which has the form' or for 'given lines in a proof sequence, the entire statements on which have the forms', and 'attach' is shorthand for 'attach a line, the entire statement on which has the form'. Here is the two-part rule:

RULE

DOUBLE NEGATION (DN): Given $\sim\sim p$, one can attach p.
Given p, one can attach $\sim\sim p$.

Since we want double negation (and subsequent shortcut rules) to apply to all statements with the appropriate forms (in DN, the forms are p and $\sim\sim p$), rather than merely to a particular pair of statements (e.g., to 'A' and '∼∼A'), the rule must be *justified* in terms of statement forms. Here are the two parts of a justification:

Example 5.5.D

1.	$\sim\sim p$	$\therefore p$
┌2.	$\sim p$	
└→3.	$\sim p \cdot \sim\sim p$	1,2 Conj
4.	p	2,3 IP

Example 5.5.E

1.	p	$\therefore \sim\sim p$
┌2.	$\sim\sim\sim p$	
│┌3.	$\sim\sim p$	
│└→4.	$\sim\sim p \cdot \sim\sim\sim p$	2,3 Conj
│5.	$\sim p$	3–4 IP
└→6.	$p \cdot \sim p$	1,5 Conj
7.	$\sim\sim p$	2–6 IP

Example 5.5.D shows that if any line has the form $\sim\sim p$, one can attach a line of the form p; Example 5.5.E shows that if any line has the form p, one can attach a line of the form $\sim\sim p$.

Remember, having justified a shortcut rule by proving it by means of statement forms (as in Examples 5.5.D and 5.5.E), we are free to use that rule in subsequent proofs, just as we would use any other rule—after all, that's the whole point in having shortcuts. The following proof uses both parts of double negation:

Example 5.5.F

1.	$\sim\sim C \rightarrow \sim\sim D$	
2.	$\sim A \cdot \sim B$	
3.	$\sim A \rightarrow C$	$\therefore D \cdot \sim B$
4.	$\sim A$	2 Simp
5.	C	3,4 MP
6.	$\sim\sim C$	5 DN
7.	$\sim\sim D$	1,6 MP
8.	D	7 DN
9.	$\sim B$	2 Simp
10.	$D \cdot \sim B$	8,9 Conj

Before going on to our next shortcut rule, I want to emphasize that double negation, like all the shortcut rules that are to follow, is really *inessential* to any proof in which it is used. For example, we could rewrite Example 5.5.F to avoid both uses of DN by inserting subroutines of the forms of Examples 5.5.D and 5.5.E at the points where Example 5.5.F uses DN. Here is one such proof:

Example 5.5.G

1.	$\sim\sim C \rightarrow \sim\sim D$	
2.	$\sim A \cdot \sim B$	
3.	$\sim A \rightarrow C$	$\therefore D \cdot \sim B$
4.	$\sim A$	2 Simp
5.	C	3,4 MP
6.	$\sim\sim\sim C$	
7.	$\sim\sim C$	
8.	$\sim\sim C \cdot \sim\sim\sim C$	6,7 Conj
9.	$\sim C$	7–8 IP
10.	$C \cdot \sim C$	5,9 Conj
11.	$\sim\sim C$	6–10 IP
12.	$\sim\sim D$	1,11 MP
13.	$\sim D$	
14.	$\sim D \cdot \sim\sim D$	12,13 Conj
15.	D	13–14 IP
16.	$\sim B$	2 Simp
17.	$D \cdot \sim B$	15,16 Conj

This proof is exactly like Example 5.5.F, except that it does not use DN. In Example 5.5.G, lines 5 through 11 (which exactly follow Example 5.5.E) arrive at line 6 in Example 5.5.F, and lines 12 through 15 (which exactly follow Example 5.5.D) arrive at line 8 in Example 5.5.F.

De Morgan's Theorems

De Morgan's theorems are familiar from Section 4.2. The rule that enables us to use these theorems has two *2-part* parts corresponding to De Morgan's two theorems:

RULE

De Morgan's Theorems (DeM): Given $\sim(p \cdot q)$, one can attach $\sim p \vee \sim q$.
Given $\sim p \vee \sim q$, one can attach $\sim(p \cdot q)$.
Given $\sim(p \vee q)$, one can attach $\sim p \cdot \sim q$.
Given $\sim p \cdot \sim q$, one can attach $\sim(p \vee q)$.

Since this rule has two *2-part* parts, its justification requires four *1-part* proofs. I will prove the first and third parts listed and leave the remaining two parts as exercises for you to complete in the next exercise set.

Example 5.5.H

1.	$\sim(p \cdot q)$	$\therefore \sim p \vee \sim q$
2.	$\sim(\sim p \vee \sim q)$	
3.	$\sim p$	
4.	$\sim p \vee \sim q$	3 Add
5.	$(\sim p \vee \sim q) \cdot \sim(\sim p \vee \sim q)$	2,4 Conj
6.	p	3–5 IP
7.	$\sim q$	
8.	$\sim p \vee \sim q$	7 Add
9.	$(\sim p \vee \sim q) \cdot \sim(\sim p \vee \sim q)$	2,8 Conj
10.	q	7–9 IP
11.	$p \cdot q$	6,10 Conj
12.	$(p \cdot q) \cdot \sim(p \cdot q)$	1,11 Conj
13.	$\sim p \vee \sim q$	2–12 IP

Example 5.5.I

1.	$\sim(p \vee q)$	$\therefore \sim p \cdot \sim q$
2.	p	
3.	$p \vee q$	2 Add
4.	$(p \vee q) \cdot \sim(p \vee q)$	1,3 Conj
5.	$\sim p$	2–4 IP
6.	q	
7.	$p \vee q$	6 Add
8.	$(p \vee q) \cdot \sim(p \vee q)$	1,7 Conj
9.	$\sim q$	6–8 IP
10.	$\sim p \cdot \sim q$	5,9 Conj

Here is a simple proof that uses one-half of each of the two parts of De Morgan's theorems:

Example 5.5.J

1.	$\sim(\mathrm{A} \vee \mathrm{B})$	$\therefore \sim(\mathrm{A} \cdot \mathrm{B})$
2.	$\sim \mathrm{A} \cdot \sim \mathrm{B}$	1 DeM
3.	$\sim \mathrm{A}$	2 Simp
4.	$\sim \mathrm{A} \vee \sim \mathrm{B}$	3 Add
5.	$\sim(\mathrm{A} \cdot \mathrm{B})$	4 DeM

Conditional Exchange

Like De Morgan's theorems, conditional exchange has two 2-part parts:

RULE

CONDITIONAL EXCHANGE (CE): Given $p \rightarrow q$, one can attach $\sim p \vee q$.
Given $\sim p \vee q$, one can attach $p \rightarrow q$.
Given $p \rightarrow q$, one can attach $q \vee \sim p$.
Given $q \vee \sim p$, one can attach $p \rightarrow q$.

Here is an example that uses the first two parts:

Example 5.5.K

1.	$A \rightarrow B$	
2.	$\sim\sim A$	$\therefore C \rightarrow B$
3.	$\sim A \vee B$	1 CE
4.	B	2,3 DS
5.	$\sim C \vee B$	4 Add
6.	$C \rightarrow B$	5 CE

We will now prove the first part of conditional exchange. The remaining three parts are left for you to complete as an exercise in the next exercise set.

Example 5.5.L

1.	$p \rightarrow q$	$\therefore \sim p \vee q$
┌2.	$\sim(\sim p \vee q)$	
│3.	$\sim\sim p \cdot \sim q$	2 DeM
│4.	$\sim\sim p$	3 Simp
│5.	p	4 DN
│6.	q	1,5 MP
│7.	$\sim q$	3 Simp
└→8.	$q \cdot \sim q$	6,7 Conj
9.	$\sim p \vee q$	2–8 IP

You should now be cautioned against one common mistake: given '$A \vee B$', you *cannot* immediately attach '$\sim A \rightarrow B$' (even though these statements are truth-functionally equivalent). As conditional exchange is stated, it can be applied only to disjunctions in which one disjunct is preceded by '$\sim$'. For now, the only means of going from '$A \vee B$' to '$\sim A \rightarrow B$' is by an indirect proof in which you suppose '$\sim(\sim A \rightarrow B)$' and reach a contradiction.

Here are two examples that use the first three shortcut rules:

Example 5.5.M

1.	$\sim(A \cdot B)$	
2.	$\sim B \rightarrow \sim\sim C$	$\therefore\ A \rightarrow C$
3.	$\sim A \vee \sim B$	1 DeM
4.	$A \rightarrow \sim B$	3 CE
┌5.	A	
│6.	$\sim B$	4,5 MP
│7.	$\sim\sim C$	2,6 MP
└→8.	C	7 DN
9.	$A \rightarrow C$	5–8 CP

Example 5.5.N

1.	$\sim(A \cdot B) \rightarrow C$	
2.	$\sim A$	
3.	$C \rightarrow \sim(D \vee B)$	$\therefore\ \sim(C \rightarrow D)$
┌4.	$C \rightarrow D$	
│5.	$\sim A \vee \sim B$	2 Add
│6.	$\sim(A \cdot B)$	5 DeM
│7.	C	1,6 MP
│8.	D	4,7 MP
│9.	$\sim(D \vee B)$	3,7 MP
│10.	$\sim D \cdot \sim B$	9 DeM
│11.	$\sim D$	10 Simp
└→12.	$D \cdot \sim D$	8,11 Conj
13.	$\sim(C \rightarrow D)$	4–12 IP

EXERCISES 5.5

Construct proofs to justify the second and fourth parts of De Morgan's theorems and for the second, third, and fourth parts of conditional exchange. As in justifying the other shortcut rules, the proofs must be stated in terms of statement forms.

Construct a proof for each of the following arguments:

★**1.** C
$D \rightarrow G$
$\sim(\sim A \vee \sim B)$
$B \rightarrow (C \rightarrow \sim\sim(D \cdot E))$ $\quad\therefore\ G \cdot \sim\sim A$

★**2.** $(E \cdot \sim B) \rightarrow (G \leftrightarrow H)$
$\sim A \cdot \sim B$
$\sim(C \cdot D)$
$(\sim(A \vee B) \cdot (\sim C \vee \sim D)) \rightarrow E$ $\quad\therefore\ G \leftrightarrow H$

3. ~A	∴ A → B
4. B	∴ A → B
5. ~(~A · ~B)	
(A ∨ B) → ~~C	∴ C
6. A → ~B	
~(A · B) → (C ∨ D)	∴ ~C → D
7. ~(A · B)	
~(C ∨ ~B)	∴ ~A · ~C
8. ~(A → C) ∨ D	
B → C	
A → B	∴ D
9. A · (B → ~(C ∨ D))	
E	
D · B	
(~C · ~D) → H	∴ H · ~(~D ∨ ~E)
10. A · ~~B	
(~D → E) · (~D → G)	
B → ~(C ∨ D)	∴ E · G

Modus Tollens

Our next shortcut rule is **modus tollens;** we have encountered arguments of this kind earlier. The rule goes like this:

RULE **MODUS TOLLENS (MT):** Given $p \rightarrow q$ and $\sim q$, one can attach $\sim p$.

For example, given 'A → (B · C)' and '~(B · C)', one can attach '~A'. The justification is left to be completed in the next exercise set.

Constructive Dilemma

The final shortcut rule considered in this section is **constructive dilemma:**

RULE **CONSTRUCTIVE DILEMMA (CD):** Given $p \vee q$, $p \rightarrow r$, and $q \rightarrow s$, one can attach $r \vee s$.

For example, here is an argument with the form of constructive dilemma: "Gertie will invite either Doug or Walt. If Gertie invites Doug, Clyde will be jealous. If Gertie invites Walt, Clyde will be angry. So Clyde will be jealous or angry." The justification of constructive dilemma is left to be completed in the next exercise set.

One special case of constructive dilemma is frequently useful: Suppose you have statements such as '$A \rightarrow B$' and '$\sim A \rightarrow C$' and you want to infer '$B \vee C$'. Clearly, 'A' must be either true or false, and if we had '$A \vee \sim A$', we could use constructive dilemma to attach '$B \vee C$'; so how can we get '$A \vee \sim A$' into the proof? Here is one way: Suppose 'A' and immediately conditionalize to get '$A \rightarrow A$' (compare 5.3.G). Then use CE to get '$\sim A \vee A$'.

As you might guess, constructive dilemma is especially useful in dealing with disjunctions. It can often be used to prove the validity of arguments that have a disjunction as a premise or as the conclusion. Here is an example:

Example 5.5.O

1.	$A \rightarrow (B \rightarrow C)$	
2.	$(C \cdot D) \rightarrow \sim E$	
3.	$(B \rightarrow C) \rightarrow J$	
4.	$\sim E \rightarrow (G \vee \sim H)$	
5.	$(A \vee (C \cdot D)) \cdot (\sim E \rightarrow \sim C)$	$\therefore J \vee (G \vee \sim H)$
6.	A	
7.	$B \rightarrow C$	1,6 MP
8.	J	3,7 MP
9.	$A \rightarrow J$	6–8 CP
10.	$C \cdot D$	
11.	$\sim E$	2,10 MP
12.	$G \vee \sim H$	4,11 MP
13.	$(C \cdot D) \rightarrow (G \vee \sim H)$	10–12 CP
14.	$A \vee (C \cdot D)$	5 Simp
15.	$J \vee (G \vee \sim H)$	9,13,14 CD

Chapter 5

EXERCISES 5.5

Construct proofs to justify modus tollens and constructive dilemma. As in justifying the other shortcut rules, the proofs must be stated in terms of statement forms.

Identify the rule that has been applied in each of the following one-step proofs:

★**11.** **1.** $(A \vee B) \rightarrow \sim C$
2. $\sim A$
3. $((A \vee B) \rightarrow \sim C) \cdot \sim A$

★**12.** **1.** $\sim(A \vee B) \vee \sim C$
2. $(A \vee B) \rightarrow \sim C$

★13. 1. (A → B) · (C ∨ D)
2. ((A → B) · (C ∨ D)) ∨ (A → B)

★14. 1. ~(A ∨ B) ∨ ~C
2. ~((A ∨ B) · C)

★15. 1. ~A ∨ B
2. (~A ∨ B) · (~A ∨ B)

★16. 1. A ∨ ((B → ~C) · ~D)
2. ~A
3. (B → ~C) · ~D

★17. 1. (A ∨ B) → ~C
2. ~~C
3. ~(A ∨ B)

★18. 1. ~(A → B) ∨ (B → C)
2. (A → B) → (B → C)

★19. 1. ~(A ∨ B) ∨ ~(C ∨ D)
2. ~(A ∨ B) → D
3. ~(C ∨ D) → E
4. D ∨ E

★20. 1. ~(A ∨ B) → ~(C ∨ D)
2. ~~(C ∨ D)
3. ~~(A ∨ B)

★21. 1. (A · C) ↔ ~(D · G)
2. ~(D · G) → (A · C)

★22. 1. ~(A ∨ B)
2. ~~~(A ∨ B)

★23. 1. ((A → B) → (B → A)) · (A ↔ B)
2. (A → B) → (B → A)

★24. 1. (A ∨ ~B) ∨ ~(A ∨ B)
2. (A ∨ B) → (A ∨ ~B)

★25. 1. ~A · ~(B → C)
2. ~(A ∨ (B → C))

Extending DN, DeM, and CE to Apply within Lines

So far, all of our rules have been applied only to *entire* lines. Indeed, our rules have been stated to block any application within lines or to parts of lines, and you were expressly cautioned against attempting such applications (p. 276). Why was this necessary?

To see why we cannot ordinarily apply rules to parts of lines, recall an example we first encountered as 5.2.D:

Example 5.5.P

1. ~(A · B)
2. ~A *misapplying* Simp to 1

Why exactly is this inference invalid? Rules for proofs must ensure that each attached statement is true if all earlier statements are true. In Example 5.5.P, a mistaken application of simplification appears to yield '~A'. But if 'A' is true and 'B' is false, '~A' is false even though '~(A · B)' is true. Thus, if '~(A · B)' appears in a proof sequence, and if, by an incorrect use of simplification, we attach '~A', the truth of the attached statement is not ensured by the truth of the statement from whence it came.

Contrast Example 5.5.P with this proof (which is also incorrect, at least as DN was stated on page 298):

Example 5.5.Q

1. ~(A · B)
2. ~(A · ~~B) *misapplying* DN to 1

Now, we know that 'B' and '~~B' are equivalent—they must always have the same truth-value. Thus, any two statements that differ only in that one contains 'B' where the other contains '~~B' will also be equivalent. In particular, '~(A · B)' is equivalent to '~(A · ~~B)'. Thus, in a proof, given either '~(A · B)' or '~(A · ~~B)', we *could* safely attach the other without any risk of advancing from a true statement to a false one—that is, we *could,* except that, as the rules are stated, such a move is not allowed. So while our rule of double negation blocks Example 5.5.Q, in principle, a rule could be formulated that would allow such inferences, and that rule would not lead to mistakes. In other words, we could state an inference rule that uses double negation on a segment *within* a line. Similar reasoning applies to the use of De Morgan's theorems and conditional exchange.

Obviously, the preceding paragraph does not *prove* that the application of DN, DeM, and CE can be extended as we are here suggesting. A complete justification for such an extension depends on what is sometimes called the principle of extensionality,[1] and it goes beyond the bounds of this text. However, our example should at least make the proposed extension plausible.

So now the problem is restating double negation, De Morgan's theorems, and conditional exchange to allow their application to segments of lines (rather than to complete lines only). In the following rule, 'Segments with these forms are interchangeable' is shorthand for 'If a statement on any line in a proof sequence includes a segment of either of the following forms, one can attach a new line containing the same statement, except that that segment is replaced by a segment with the other of the two forms':

RULE

> **DOUBLE NEGATION (DN):** Segments with these forms are interchangeable: p and $\sim\sim p$

Here are two simple proofs that apply the extended DN rule within lines:

Example 5.5.R

1.	$C \lor \sim(\sim\sim A \lor B)$	$\therefore C \lor \sim(A \lor B)$
2.	$C \lor \sim(A \lor B)$	1 DN

To attach line 2, we use the extended version of DN to replace the segment '$\sim\sim A$' with 'A'.

Example 5.5.S

1.	$C \lor (A \lor B)$	$\therefore C \lor \sim\sim(A \lor B)$
2.	$C \lor \sim\sim(A \lor B)$	1 DN

To attach line 2, we use the extended version of DN to replace the segment '$A \lor B$' with '$\sim\sim(A \lor B)$'.

Want to see a proof like Example 5.5.R that does *not* use the extended DN rule? Here is one such proof:

Example 5.5.T

1.	$C \lor \sim(\sim\sim A \lor B)$	$\therefore C \lor \sim(A \lor B)$
2.	$\sim(C \lor \sim(A \lor B))$	
3.	$\sim C \cdot \sim\sim(A \lor B)$	2 DeM
4.	$\sim C$	3 Simp
5.	$\sim(\sim\sim A \lor B)$	1,4 DS
6.	$\sim\sim(A \lor B)$	3 Simp
7.	$A \lor B$	6 DN
8.	A	
9.	$\sim\sim A$	8 DN
10.	$\sim\sim A \lor B$	9 Add
11.	$(\sim\sim A \lor B) \cdot \sim(\sim\sim A \lor B)$	5,10 Conj
12.	$\sim A$	8–11 IP
13.	B	7,12 DS
14.	$\sim\sim A \lor B$	13 Add
15.	$(\sim\sim A \lor B) \cdot \sim(\sim\sim A \lor B)$	5,14 Conj
16.	$C \lor \sim(A \lor B)$	2–15 IP

As you see, extending DN to apply within lines is a great help in shortening some proofs.

What about the extended version of De Morgan's theorems? In the following rule, 'Segments with these forms are interchangeable' is shorthand for 'If a statement on any line in a proof sequence includes a segment with either of the forms of any one of the following pairs, one can attach a new line containing the same statement, except that that segment is replaced by a segment with the other of the two forms of that pair':

RULE

De Morgan's theorems (DeM): Segments with these forms are interchangeable: ~(*p* · *q*) and ~*p* ∨ ~*q*
~(*p* ∨ *q*) and ~*p* · ~*q*.

Here is a proof using the second part of this rule:

Example 5.5.U

1.	C ∨ ~(A ∨ B)	∴ C ∨ (~A · ~B)
2.	C ∨ (~A · ~B)	1 DeM

To attach line 2, we use the extended version of DeM to replace the segment '~(A ∨ B)' by '~A · ~B'.

Want to see a proof like Example 5.5.U that does not use the extended DeM and DN rules? Here is one such proof:

Example 5.5.V

1.	C ∨ ~(A ∨ B)	∴ C ∨ (~A · ~B)
2.	~(C ∨ (~A · ~B))	
3.	~C · ~(~A · ~B)	2 DeM
4.	~C	3 Simp
5.	~(A ∨ B)	1,4 DS
6.	~(~A · ~B)	3 Simp
7.	~~A ∨ ~~B	6 DeM
8.	~~A	
9.	A	8 DN
10.	A ∨ B	9 Add
→11.	(A ∨ B) · ~(A ∨ B)	5,10 Conj
12.	~~~A	8–11 IP
13.	~~B	7,12 DS
14.	B	13 DN
15.	A ∨ B	14 Add
→16.	(A ∨ B) · ~(A ∨ B)	5,15 Conj
17.	C ∨ (~A · ~B)	2–16 IP

Finally, here is the extended version of conditional exchange:

RULE

Conditional Exchange (CE): Segments with these forms are interchangeable: $p \rightarrow q$ and $\sim p \vee q$
$p \rightarrow q$ and $q \vee \sim p$.

Here is a simple example that uses the first part of this rule:

Example 5.5.W

1.	$C \vee \sim(\sim A \rightarrow B)$	$\therefore C \vee \sim(\sim\sim A \vee B)$
2.	$C \vee \sim(\sim\sim A \vee B)$	1 CE

To attach line 2, we use the extended version of CE to replace the segment '$\sim A \rightarrow B$' with '$\sim\sim A \vee B$'.

Want to see a proof like Example 5.5.W that does not use the extended CE, DeM, and DN rules? Probably not; you get the idea. However, just in case you still need to be persuaded of the value of these extended shortcuts, consider this proof, which uses all three rules (and which has been designed to combine Examples 5.5.R, 5.5.U, and 5.5.W):

Example 5.5.X

1.	$C \vee \sim(\sim A \rightarrow B)$	$\therefore C \vee (\sim A \cdot \sim B)$
2.	$C \vee \sim(\sim\sim A \vee B)$	1 CE
3.	$C \vee \sim(A \vee B)$	2 DN
4.	$C \vee (\sim A \cdot \sim B)$	3 DeM

A proof of this argument that does not use the extended shortcuts requires a combination of Examples 5.5.T and 5.5.V, plus one other comparable proof—a total of about fifty steps.

Before we see some further examples, two points need to be emphasized: (1) The extended shortcut rules are stated in terms of *segments* of lines, but an entire line can be thought of as a *segment* of itself; thus, these three rules really include (and therefore replace) the original versions of double negation, De Morgan's theorems, and conditional exchange. From a practical point of view, this means that in a proof, there is no need to indicate whether you are using the original version or the extended version—the citation is the same. (2) The extended versions of DN, DeM, and CE can be thought of as replacing the original shortcuts. However, these extensions (like the original shortcuts themselves) are *not* strictly required for any proofs. If you are a minimalist, you can ignore both the shortcuts and the extended shortcuts and do all your proofs using only the basic eight rules. Enjoy!

As another example, the following proof uses all three extended shortcut rules:

Example 5.5.Y

1.	$(A \rightarrow \sim B) \rightarrow C$	
2.	$(\sim(\sim A \vee \sim B) \vee C) \rightarrow (D \vee \sim\sim E)$	$\therefore D \vee E$
3.	$(\sim\sim(A \cdot B) \vee C) \rightarrow (D \vee \sim\sim E)$	2 DeM
4.	$(\sim(A \cdot B) \rightarrow C) \rightarrow (D \vee \sim\sim E)$	3 CE
5.	$(\sim(A \cdot B) \rightarrow C) \rightarrow (D \vee E)$	4 DN
6.	$(\sim A \vee \sim B) \rightarrow C$	1 CE
7.	$\sim(A \cdot B) \rightarrow C$	6 DeM
8.	$D \vee E$	5,7 MP

The extended shortcuts can frequently be used together in the following way: Suppose '$\sim$' precedes a compound statement; it often helps clarify the meaning of the statement if the '$\sim$' is "driven in," so that it applies to the shortest possible segment or segments of the statement. For example, suppose you seek a proof for an argument with the conclusion '$\sim((A \rightarrow B) \vee \sim C)$'. It's hard to make sense of such a complicated statement, and it will likely be difficult to see how it could come from whatever premises may be given in the argument. However, applying DeM to this conclusion yields '$\sim(A \rightarrow B) \cdot \sim\sim C$', and using CE on the first conjunct and DN on the second yields '$\sim(\sim A \vee B) \cdot C$'; using DeM and DN again, we have '$(A \cdot \sim B) \cdot C$'. This statement makes better sense, and it would probably be easier to see how it could be attached in a proof. After attaching the statement, one would simply attach lines *reversing* the steps by which we obtained it from the original conclusion to get that conclusion.

Here is a complete example:

Example 5.5.Z

1.	$\sim(A \cdot B) \rightarrow C$	
2.	$\sim A$	
3.	$C \rightarrow \sim(D \vee B)$	$\therefore \sim(C \rightarrow D)$

Look at the conclusion—where can it come from? You may have no clear idea. So use the tactics explained in the preceding paragraph: from '$\sim(C \rightarrow D)$', by CE we get '$\sim(\sim C \vee D)$', by DeM we get '$\sim\sim C \cdot \sim D$', and by DN we get '$C \cdot \sim D$'—a much clearer goal to shoot for. When we reach this goal, we will simply attach these three steps, in reverse order, to get the original conclusion. So the proof will *end* like this:

$C \cdot \sim D$	
$\sim\sim C \cdot \sim D$	DN
$\sim(\sim C \vee D)$	DeM
$\sim(C \rightarrow D)$	CE

Now what about the first premise in Example 5.5.Z? DeM yields '(~A ∨ ~B) → C', and, given the second premise, we see exactly how to attach the antecedent in this conditional. So the proof goes as follows:

4.	(~A ∨ ~B) → C	1 DeM
5.	~A ∨ ~B	2 Add
6.	C	4,5 MP
7.	~(D ∨ B)	3,6 MP
8.	~D · ~B	7 DeM
9.	~D	8 Simp
10.	C · ~D	6,9 Conj

Now we add the last three steps as just explained:

11.	~~C · ~D	10 DN
12.	~(~C ∨ D)	11 DeM
13.	~(C → D)	12 CE

EXERCISES 5.5

Construct a proof for each of the following arguments:

★**26.** (A → ~B) → C
(~(~A ∨ ~B) ∨ C) → (D ∨ ~~E)   ∴ D ∨ E

★**27.** ~A ∨ B
D ∨ A
D → C   ∴ ~C → B

★**28.** A → B
~C → D
~(B · C)   ∴ ~A ∨ D

★**29.** (A ∨ ~D) → (B ∨ E)
D → B
~H
(E · ~D) → (B ∨ ~J)
(G ∨ ~B) → J
~B   ∴ ~(G ∨ ~B)

★**30.** A → E
C → D
B → ~D
~E ∨ B   ∴ ~(A · C)

★**31.** (A ∨ ~B) ∨ (C · D)
(A ∨ ~B) → (D ∨ H)
(D ∨ H) → ~G
~~G
C → H   ∴ H · ~~G

32. $A \vee (B \rightarrow G)$
$\sim A \vee C$
$C \rightarrow \sim D$
$(B \rightarrow G) \rightarrow E$ $\quad \therefore \sim D \vee E$

33. $(A \rightarrow D) \rightarrow E$ $\quad \therefore E \vee A$

34. $B \leftrightarrow (A \cdot C)$
$A \rightarrow \sim C$ $\quad \therefore \sim B$

35. $E \rightarrow C$
$\sim(A \vee G)$
$(D \rightarrow B) \rightarrow E$
$A \vee \sim C$ $\quad \therefore D \cdot \sim B$

36. $A \rightarrow (B \rightarrow C)$
$(C \cdot D) \rightarrow \sim E$
$(B \rightarrow C) \rightarrow J$
$\sim E \rightarrow (G \vee \sim H)$
$(A \vee (C \cdot D)) \cdot (\sim E \rightarrow \sim C)$ $\quad \therefore J \vee (G \vee \sim H)$

We have now learned five shortcut rules. These rules are summarized in the following box. In the box, for each rule shown, the line means that, given statements of each of the forms above the line, one can attach a statement of the form below the line. Four dots indicate that a segment having the form shown on either side of the dots can be interchanged with a segment having the form on the other side of the dots (as explained in the full statement of each rule earlier). Of course, the rules are all subject to the explanations given in the text.

The First Five Shortcut Rules

Modus Tollens (MT)

$$\frac{p \rightarrow q \qquad \sim q}{\sim p}$$

Constructive Dilemma (CD)

$$\frac{p \vee q \qquad p \rightarrow r \qquad q \rightarrow s}{r \vee s}$$

Double Negation (DN)

$$p :: \sim\sim p$$

Conditional Exchange (CE)

$$\sim p \vee q :: p \rightarrow q$$
$$q \vee \sim p :: p \rightarrow q$$

De Morgan's Theorems (DeM)

$$\sim(p \vee q) :: \sim p \cdot \sim q$$
$$\sim(p \cdot q) :: \sim p \vee \sim q$$

5.6 Shortcut Rules (continued)

As you can probably imagine, there are dozens of possible shortcut rules. In this section, we will learn several. These shortcuts are just as legitimate as the ones introduced in Section 5.5; however, except within the current section itself, proofs in the book will not use the shortcuts that I am about to identify. "But," you may ask, "why not use them once they have been introduced?" The reason is that we seek to minimize the number of rules *you* must learn in order to read the book. The shortcuts I will now present are not required to understand anything that comes later—in this sense, they are optional.

"So," you may ask, "then why introduce the new shortcuts at all?" First, because working with them will give you more experience in constructing proofs. Second, because these new shortcuts can drastically reduce the length of some proofs. And while (except within this section itself) *I* won't use the shortcuts presented here, in constructing proofs for assigned exercises throughout the book, *you* may use any shortcuts you like from among those presented in Section 5.5 or in this section. Indeed, you may even want to create some shortcuts of your own. (You can name them after yourself or your friends or relatives (e.g., "Megan's Rule").) However, be careful: this liberalization does not give you license to attach to a proof sequence any statement you like—not even a statement that (as you may intuit) must somehow follow from earlier lines in the sequence. Each line attached to a proof sequence must be an exact instance of a particular rule, and the rule that warrants the attachment must be correctly cited in the proof. Moreover, each shortcut rule that you create must be justified by a proof such as those already given.

Ten More Shortcut Rules

The next three shortcuts are so obvious that students sometimes take them for granted, but, of course, *doing so is a mistake:* in any proof, you can use only the basic rules or shortcuts that have been explicitly justified. As usual, in stating shortcut rules, the word 'given' is shorthand either for 'given a line in a proof sequence, the entire statement on which has the form' or for 'given lines in a proof sequence, the entire statements on which have the forms', and 'attach' is shorthand for 'attach a line, the entire statement on which has the form'.

The first shortcut is called **tautology.**

RULE **TAUTOLOGY (TAUT):** Given $p \vee p$, one can attach p.

For example, given '$(A \rightarrow B) \vee (A \rightarrow B)$' one can attach '$A \rightarrow B$'. You might wonder how such a rule could ever be used. Tautology is most often useful in proofs that employ one or another kind of dilemma.

Of course, the reverse of tautology (moving from p to $p \vee p$) requires only addition. Parallel moves for conjunction (given $p \cdot p$, attaching p; or given p, attaching $p \cdot p$) are already warranted by simplification and conjunction, respectively.

The justification for tautology (which is very easy) is left as an exercise for you to complete a bit later.

The next rule is **commutation;** it has two parts:

RULE

COMMUTATION (COM): Given $p \cdot q$, one can attach $q \cdot p$.
Given $p \vee q$, one can attach $q \vee p$.

This rule is frequently useful in adjusting statements so that they can be used together in applying some other rule. For example, given '$(A \cdot B) \rightarrow C$' and '$B \cdot A$', one could *not* use modus ponens directly. To use MP on these lines, *without commutation,* one must first simplify each of the conjuncts in '$B \cdot A$' and then rejoin them in the proper order using conjunction—an irritating hassle. However, applying commutation to '$B \cdot A$' gives us the desired antecedent in one step.

The next rule, known as **association,** helps one deal with chains of more than two conjuncts or more than two disjuncts:

RULE

ASSOCIATION (ASSOC): Given $(p \cdot q) \cdot r$, one can attach $p \cdot (q \cdot r)$.
Given $p \cdot (q \cdot r)$, one can attach $(p \cdot q) \cdot r$.
Given $(p \vee q) \vee r$, one can attach $p \vee (q \vee r)$.
Given $p \vee (q \vee r)$, one can attach $(p \vee q) \vee r$.

Notice that this rule can be applied only to chains of conjuncts or to chains of disjuncts—not to mixtures of the two or to conditionals.

The justification for commutation (which is easy) and that for association (which is a bit longer) are left as exercises for you to complete later.

The next shortcut rule is less obvious than the three we have just considered. It is called **distribution,** and it has two 2-part parts:

RULE

DISTRIBUTION (DIST): Given $p \cdot (q \vee r)$, one can attach $(p \cdot q) \vee (p \cdot r)$.
Given $(p \cdot q) \vee (p \cdot r)$, one can attach $p \cdot (q \vee r)$.
Given $p \vee (q \cdot r)$, one can attach $(p \vee q) \cdot (p \vee r)$.
Given $(p \vee q) \cdot (p \vee r)$, one can attach $p \vee (q \cdot r)$.

Here is an example of the second part: Given '$(\sim A \cdot (B \rightarrow C)) \vee (\sim A \cdot (C \rightarrow B))$' one can attach '$\sim A \cdot ((B \rightarrow C) \vee (C \rightarrow B))$'.

The justification for distribution (which is moderately difficult) is left as an exercise for you to complete later.

Here is a proof that uses (among other rules) all four of the shortcut rules we have so far introduced in this section:

Example 5.6.A

1.	E → H	
2.	B ∨ (A · D)	
3.	(A ∨ B) → ((C ∨ E) ∨ G)	
4.	(G → H) · ~C	∴ H
5.	(B ∨ A) · (B ∨ D)	2 Dist
6.	B ∨ A	5 Simp
7.	A ∨ B	6 Comm
8.	(C ∨ E) ∨ G	3,7 MP
9.	C ∨ (E ∨ G)	8 Assoc
10.	~C	4 Simp
11.	E ∨ G	9,10 DS
12.	G → H	4 Simp
13.	H ∨ H	1,11,12 CD
14.	H	13 Taut

Without our new shortcut rules, proving Example 5.6.A would be tedious and would involve several subroutines using indirect proof. (Try it if you would like a challenge!)

EXERCISES 5.6

Justify tautology, commutation, association, and distribution. As in justifying the other shortcut rules, the proofs must be stated in terms of statement forms.

Using shortcut rules where appropriate, construct proofs for the following arguments:

1. D ∨ (~B · A)
B ∨ B ∴ D

2. (A ∨ D) ∨ (B ∨ C)
~B · ~A ∴ ~C → D

3. A · (B ∨ C)
(B · A) → D
C → D ∴ D ∨ ~A

4. (C · C) ∨ (C · D) ∴ D ∨ C

The next four shortcut rules are frequently useful and relatively easy to understand. The first, called **conditional negation,** goes like this:

RULE

CONDITIONAL NEGATION (CN): Given $\sim(p \rightarrow q)$, one can attach $p \cdot \sim q$.
Given $p \cdot \sim q$, one can attach $\sim(p \rightarrow q)$.

For example, given '$\sim((A \rightarrow B) \rightarrow C)$', one can attach '$(A \rightarrow B) \cdot \sim C$'. One way to understand this rule is to recall that a conditional is false if and only if its antecedent is true and its consequent false.

The next shortcut rule, called **transposition,** is closely related to modus tollens:

RULE

TRANSPOSITION (TRANS): Given $p \rightarrow q$, one can attach $\sim q \rightarrow \sim p$.
Given $\sim q \rightarrow \sim p$, one can attach $p \rightarrow q$.

For example, given '$\sim A \rightarrow (C \vee D)$', one can attach '$\sim(C \vee D) \rightarrow \sim\sim A$'.

Exportation is this rule:

RULE

EXPORTATION (EXP): Given $(p \cdot q) \rightarrow r$, one can attach $p \rightarrow (q \rightarrow r)$.
Given $p \rightarrow (q \rightarrow r)$, one can attach $(p \cdot q) \rightarrow r$.

For example, given '$(A \rightarrow B) \rightarrow (C \rightarrow D)$' one can attach '$((A \rightarrow B) \cdot C) \rightarrow D$'.

The final shortcut rule in this group is **hypothetical syllogism.** Like "disjunctive syllogism," the name "hypothetical syllogism" is potentially misleading because the inferences it warrants are not really syllogisms. However, the name is widely used, and, as explained earlier, we will use it also. Here is the rule:

RULE

HYPOTHETICAL SYLLOGISM (HS): Given $p \rightarrow q$ and $q \rightarrow r$, one can attach $p \rightarrow r$.

This rule enables one to work efficiently with linked conditionals. For example, given '$(A \vee B) \rightarrow \sim C$' and '$\sim C \rightarrow (D \cdot E)$', it enables one to attach '$(A \vee B) \rightarrow (D \cdot E)$'.

EXERCISES 5.6

Justify conditional negation, transposition, exportation, and hypothetical syllogism. As in justifying the other shortcut rules, your proofs must be constructed using statement forms.

Using shortcut rules where appropriate, construct proofs for the following arguments:

5. $\sim C \rightarrow \sim(A \cdot B)$
$C \rightarrow D$ $\quad\therefore A \rightarrow (B \rightarrow D)$

6. $\sim(A \rightarrow B)$
$D \rightarrow (C \rightarrow B)$ $\quad\therefore D \rightarrow \sim C$

7. $A \rightarrow \sim B$
$\sim B \vee C$
$C \rightarrow A$
$D \rightarrow B$ $\quad\therefore \sim D$

8. $\sim A \rightarrow \sim D$
$\sim(\sim D \vee \sim B)$
$A \rightarrow C$ $\quad\therefore \sim(C \rightarrow \sim B)$

The two remaining shortcut rules are related to constructive dilemma. The first is called simply **dilemma** (it is also known as proof by cases):

RULE **DILEMMA (DIL):** Given $p \vee q$, $p \rightarrow r$, and $q \rightarrow r$, one can attach r.

For example, here is an argument with this form:

Example 5.6.B
Gertie will invite either Doug or Walt. If Gertie invites Doug, Clyde will be jealous. If Gertie invites Walt, Clyde will be jealous. So Clyde will be jealous.

Exercise 4.6.129 has the same form. Do you see how this rule differs from constructive dilemma?

One special case of dilemma is sometimes useful. (This case resembles the special case of constructive dilemma that we discussed on page 305.) Suppose you have statements like ‘$A \rightarrow B$’ and ‘$\sim A \rightarrow B$’ and that you want to infer ‘B’. Clearly, ‘A’ must be either true or false, and if we had ‘$A \vee \sim A$’, we could use dilemma to attach ‘B’. However, we know already how to get ‘$A \vee \sim A$’ into a proof: suppose ‘A’, conditionalize to get ‘$A \rightarrow A$’, and use CE to get ‘$\sim A \vee A$’.

The final shortcut rule is called **destructive dilemma:**

RULE **DESTRUCTIVE DILEMMA (DD):** Given $\sim r \vee \sim s$, $p \rightarrow r$, and $q \rightarrow s$ one can attach $\sim p \vee \sim q$.

For example, here is an argument that has this form:

Example 5.6.C

Clyde will not be jealous or Clyde will not be angry. If Gertie invites Doug, Clyde will be jealous. If Gertie invites Walt, Clyde will be angry. Therefore, Gertie will not invite Doug or Gertie will not invite Walt.

You should make sure you understand how destructive dilemma differs from dilemma and constructive dilemma.

Exercises 5.6

Justify dilemma and destructive dilemma. As in justifying the other shortcut rules, the proofs must be stated in terms of statement forms.

For each line in the following proofs, identify the line (or lines) and the rule that warrant its having been attached:

★9.
1. $(E \rightarrow B) \cdot C$
2. $(B \cdot C) \rightarrow \sim D$
3. $E \rightarrow D$ $\quad \therefore \sim E$
4. $C \cdot B$
5. $B \cdot C$
6. $\sim D$
7. $(C \cdot B) \rightarrow \sim D$
8. $C \rightarrow (B \rightarrow \sim D)$
9. C
10. $B \rightarrow \sim D$
11. $\sim B \vee \sim D$
12. $E \rightarrow B$
13. $\sim E \vee \sim E$
14. $\sim E$

★10.
1. $\sim E$
2. $(C \vee (B \vee \sim A)) \rightarrow (C \cdot \sim D)$
3. $\sim((C \vee B) \vee \sim A) \rightarrow E$ $\quad \therefore C \cdot \sim D$
4. $\sim\sim((C \vee B) \vee \sim A)$
5. $(C \vee B) \vee \sim A$
6. $C \vee (B \vee \sim A)$
7. $C \cdot \sim D$

★**11.** **1.** $(A \cdot B) \rightarrow D$
2. $B \vee (\sim D \cdot \sim E)$
3. A $\therefore B \leftrightarrow D$
4. $A \rightarrow (B \rightarrow D)$
5. $B \rightarrow D$
┌ **6.** $\sim(B \vee (\sim D \cdot \sim E))$
│ **7.** $\sim B \cdot \sim(\sim D \cdot \sim E)$
│ **8.** $\sim B$
│ **9.** $\sim(\sim D \cdot \sim E)$
│ **10.** $\sim D \cdot \sim E$
└→ **11.** $(\sim D \cdot \sim E) \cdot \sim(\sim D \cdot \sim E)$
12. $B \vee (\sim D \cdot \sim E)$
13. $(B \vee \sim D) \cdot (B \vee \sim E)$
14. $B \vee \sim D$
15. $D \rightarrow B$
16. $B \leftrightarrow D$

★**12.** **1.** B
2. $B \rightarrow \sim D$
3. $C \rightarrow \sim E$ $\therefore A \rightarrow \sim(D \cdot E)$
4. $B \vee C$
5. $\sim D \vee \sim E$
6. $\sim(D \cdot E)$
7. $\sim(D \cdot E) \vee \sim A$
8. $A \rightarrow \sim(D \cdot E)$

★**13.** **1.** $A \rightarrow \sim B$
2. $\sim B \vee C$
3. $C \rightarrow A$
4. $\sim D \vee B$ $\therefore \sim D$
5. $B \rightarrow C$
6. $B \rightarrow A$
7. $B \rightarrow \sim B$
8. $\sim B \vee \sim B$
9. $\sim B$
10. $D \rightarrow B$
11. $\sim D$

★**14.** **1.** (B · C) ∨ A
2. (D · E) ∨ (D · G) ∴ (D · C) ∨ (D · A)
3. D · (E ∨ G)
4. D
5. A ∨ (B · C)
6. (A ∨ B) · (A ∨ C)
7. A ∨ C
8. D · (A ∨ C)
9. (D · A) ∨ (D · C)
10. (D · C) ∨ (D · A)

Extending Some Additional Shortcut Rules to Apply within Lines

Near the end of Section 5.5, we saw that double negation, De Morgan's theorems, and conditional exchange can be restated to allow them to be applied within lines. We referred to the restated rules as *extended* shortcut rules. *Except for hypothetical syllogism, dilemma, and destructive dilemma,* all of the shortcut rules described in this section can be extended to apply within lines. Here is how the rules can be stated: In what follows, 'Segments with these forms are interchangeable' is shorthand for 'If a statement on any line in a proof sequence includes a segment of either of the forms of any one of the following pairs, one can attach a new line containing the same statement, except that that segment is replaced by a segment with the other of the two forms of that pair'. In the following list, the abbreviated name of the rule follows the pair whose interchangeability is warranted by that rule:

Segments with these forms are interchangeable:

$p \cdot q$ and $q \cdot p$	(Comm)
$p \vee q$ and $q \vee p$	(Comm)
$(p \cdot q) \cdot r$ and $p \cdot (q \cdot r)$	(Assoc)
$(p \vee q) \vee r$ and $p \vee (q \vee r)$	(Assoc)
$p \cdot (q \vee r)$ and $(p \cdot q) \vee (p \cdot r)$	(Dist)
$p \vee (q \cdot r)$ and $(p \vee q) \cdot (p \vee r)$	(Dist)
$\sim(p \rightarrow q)$ and $p \cdot \sim q$	(CN)
$p \rightarrow q$ and $\sim q \rightarrow \sim p$	(Trans)
$(p \cdot q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$	(Exp)

Having stated these extended versions of the shortcut rules, you may use them on segments of lines in any proofs you like, although, as with all the shortcut rules in this section, none will appear in proofs printed in later sections of the book.

EXERCISES 5.6

Construct proofs for the following arguments:

15. $A \rightarrow \sim A$ $\quad \therefore \sim A$

16. $A \rightarrow \sim B$
$(C \rightarrow B) \cdot (A \cdot C)$ $\quad \therefore \sim C$

17. $A \cdot ((B \vee C) \cdot D)$
$(C \vee B) \rightarrow (D \rightarrow (E \rightarrow G))$ $\quad \therefore E \rightarrow G$

18. $\sim(C \vee B) \rightarrow \sim(D \vee E)$
$\sim G$
$(\sim A \cdot \sim B) \cdot \sim C$ $\quad \therefore \sim(E \vee G)$

19. $(A \vee C) \rightarrow B$
$(D \vee G) \rightarrow H$
$A \vee D$ $\quad \therefore B \vee H$

20. $(A \rightarrow B) \rightarrow D$
$C \rightarrow B$
$\sim(A \cdot \sim C)$
$\sim(D \cdot E)$ $\quad \therefore D \cdot \sim E$

21. $A \leftrightarrow \sim B$ $\quad \therefore B \vee A$

22. $A \rightarrow B$
$C \rightarrow B$ $\quad \therefore (A \vee C) \rightarrow B$

23. $\sim(A \rightarrow B)$
$C \rightarrow \sim A$ $\quad \therefore \sim(\sim C \rightarrow B)$

In this section, we have learned ten shortcut rules. Using the conventions explained on page 313, and subject to the explanations given in the text, these shortcuts are summarized in the following box.

Ten Further Shortcut Rules

Tautology (Taut)

$p \vee p$
―――
p

Hypothetical Syllogism (HS)

$p \rightarrow q$
$q \rightarrow r$
―――
$p \rightarrow r$

Dilemma (Dil)

$p \vee q$
$p \rightarrow r$
$q \rightarrow r$
―――
r

Destructive Dilemma (DD)

$\sim r \vee \sim s$
$p \rightarrow r$
$q \rightarrow s$
―――
$\sim p \vee \sim q$

Commutation (Comm)

$p \cdot q :: q \cdot p$

$p \vee q :: q \vee p$

Association (Assoc)

$p \cdot (q \cdot r) :: (p \cdot q) \cdot r$

$p \vee (q \vee r) :: (p \vee q) \vee r$

Conditional Negation (CN)

$\sim(p \rightarrow q) :: p \cdot \sim q$

Distribution (Dist)

$p \cdot (q \vee r) :: (p \cdot q) \vee (p \cdot r)$

$p \vee (q \cdot r) :: (p \vee q) \cdot (p \vee r)$

Transposition (Trans)

$p \rightarrow q :: \sim q \rightarrow \sim p$

Exportation (Exp)

$p \rightarrow (q \rightarrow r) :: (p \cdot q) \rightarrow r$

5.7 Strategies and Tactics

As mentioned earlier, the hardest part of constructing a proof is figuring out what lines to attach. We need to give some attention to this problem.

Two kinds of considerations guide the construction of proofs; we can call them strategic (those dictating a broad method) and tactical (those addressing specific kinds of statements). For example, the suggestion to use a backwards approach in looking for proofs (p. 277) is strategic; the suggestion that, given a biconditional, one should attach conditionals (pp. 282–283) is tactical. This is a good place to summarize earlier suggestions of both kinds and to add some new recommendations as well. We will begin with strategy.

Strategies

One can identify five broad strategies for approaching proofs. You needn't choose one over the others, because they all blend together and several may be followed, more or less simultaneously, in any long proof. *First, analyze the argument globally.* Before using any rules, study the conclusion and the premises. What exactly are you trying to prove? What exactly do the premises tell you? You might try simplifying some premises or the conclusion to see what they really say (e.g., by the tactic of "driving in" '∼'s so that they apply to short segments; see p. 311). Are there statement letters in the conclusion that can come only from certain premises? Is it possible to discern an overall pattern in the argument? For example, if one premise is a disjunction, and if one disjunct resembles the conclusion, could the other premises provide the denial of the other disjunct and thereby set up an application of disjunctive syllogism? This approach was used in Example 5.2.G.

Second, analyze the argument backwards. Ultimately, each proof must advance from premises to conclusion, but, as we have seen more than once, often

Chapter 5

the best way to *find* a proof is by working backwards from the conclusion. Ask yourself what final steps could yield the conclusion; its form may suggest an answer: If the conclusion is a *conjunction,* it may be possible to prove each conjunct, one at a time, and then conjoin them. If the conclusion is a *disjunction,* it may be possible to prove one disjunct and get the conclusion by addition; more likely, one may be able to prove a disjunction by constructive dilemma. If the conclusion is a *conditional,* it is almost always advantageous to suppose the antecedent, work to get the consequent, and then conditionalize. At the very least, supposing an antecedent gives you one more statement to work with in your proof. If the conclusion is a *biconditional,* you may need to prove each conditional and then use biconditional exchange.

Third, analyze the argument forwards. After (or even while) examining the conclusion, look again at the premises. Study them carefully to see how they might provide what you need for the conclusion. Sometimes premises connect in promising ways even if, at first, you can't fully see how those steps lead to the conclusion. For example, if one premise in an argument is a disjunction, it may be possible to suppose each disjunct, one at a time, and show that given either supposition, the desired conclusion follows. It's natural to begin looking for a proof by trying to see what you can do with the premises, but reaching any destination usually depends on a firm and prior grasp of where you are trying to go, so resist the inclination to focus exclusively on the premises.

Fourth, if nothing else presents itself, use indirect proof. Given any valid argument, supposing the denial of the conclusion will ultimately yield a contradiction, so every proof can be structured as an indirect proof. Indirect proofs are often longer and less creative than direct proofs, but, with enough patience, they always work. I once had a student who used *only* indirect proofs. Given any argument, after supposing the denial of the conclusion, he systematically subjected the premises and the supposition to one rule after another. His proofs usually included many redundant lines—sometimes two or three *times* as many as were actually required, but in the end he was always successful. As you may realize from our proofs for the various shortcut rules, indirect proof is a very powerful way of approaching arguments.

Finally, if all else fails, use "random walk." Suppose you exhaust all the global, backwards, and forwards leads and are put off by the idea of a long, boring, systematic quest for a contradiction. This may occur if you are stumped at the beginning of, or even at a later point in, a proof. If you feel lucky, try using rules more or less at random, both on the premises and (in backwards mode) on the conclusion. Your attempts may not be *completely* random: intuition or prior experience may guide you one way or another. But sometimes, even without a clear sense of what you are doing or why, you may stumble onto possibilities you had originally overlooked. The "random-walk" approach has worked for me on many occasions, but it usually reflects desperation and it often fails; it should be undertaken only for personal amusement or as a last resort.

Tactics

Now let us examine some tactical suggestions for dealing with statements of specific forms. We have already noted (1) that it is often useful to "drive in" '∼'s as a means of simplifying compound statements and (2) that biconditionals should usually be subjected to biconditional exchange to obtain conditionals that can be used with other rules. Here are some additional tactical suggestions: (3) Conditional exchange is frequently useful, to obtain either a conditional from a disjunction or a disjunction from a conditional. Most of our rules involve conditionals or disjunctions, and attaching one or the other can often help. In connection with this tactic, remember that any ordinary disjunction (say, '$A \vee B$') can yield a conditional by double negation ('$\sim\sim A \vee B$') followed by conditional exchange ('$\sim A \rightarrow B$'). (4) If a premise or other statement in a proof is a conjunction, simplify each conjunct, as it often helps to have each one available alone. (5) Given a disjunction (either as a premise or as a statement subsequently attached in the course of a proof), check to see whether you can apply constructive dilemma or disjunctive syllogism. (6) Similarly, if any statement is a conditional, look for ways to use modus ponens or modus tollens. (7) Perhaps because addition is seldom used in ordinary life, it is frequently overlooked. So if you need a disjunction, try proving one disjunct and adding the other.

Often enough, you will encounter proofs in which none of these suggestions carry you far; indeed, proofs are too diverse for any short list of tactics to be widely applicable. However, the suggestions afford at least some fruitful ways of applying rules to individual statements. Other ways may occur to you as you gain experience.

Examples of Proof Strategies and Tactics

The proofs that follow are not particularly difficult, but they illustrate the kind of strategic and tactical thinking that should guide your approach.

Here is a first example:

Example 5.7.A

$A \leftrightarrow B$
$\sim B \vee C$
$A \quad\quad \therefore C$

Global and backwards analysis of Example 5.7.A: I notice immediately that the conclusion—'C'—will almost certainly come from the second premise, since it alone includes that statement letter. *Forwards analysis:* I recognize that the second premise, by conditional exchange, yields '$B \rightarrow C$'. Moreover, I see that using biconditional exchange on the first premise can provide '$A \rightarrow B$'. Of course, the third premise is the antecedent of this conditional, which opens the way for modus ponens (twice), and I'm on my way.

Here is a second example:

Example 5.7.B

1.	∼D ∨ A	
2.	∼(D → ∼D)	
3.	A → C	
4.	∼B → ∼(D → C)	∴ ∼(C → ∼B)

Global analysis of Example 5.7.B: I recognize that applying conditional exchange to the first premise yields 'D → A'; if I can get 'D', then applying modus ponens will yield 'A' and then again (by the third premise) 'C'. *Backwards analysis:* I look carefully at the conclusion—it's strange and complex. What can it mean? To find out, we drive in the '∼'. Use conditional exchange on the conditional inside the conclusion and then De Morgan's theorems, we get the conjunction '∼∼C · ∼∼B', which, by double negation twice, yields 'C · B'. So now the question is how to get 'C' and 'B', since having those will enable me to reverse the steps we just recounted and reach the conclusion. A few sentences back, we saw one way to get 'C', but it required having 'D'. What to do? *Forwards analysis:* About now, the odd second premise attracts my attention; what does it mean? Again, we can simplify it by using conditional exchange, De Morgan's theorems, and double negation, yielding 'D'. Now what about 'B'? From 'C', by addition and conditional exchange, we have 'D → C'. Double negation and modus tollens on line 4 then yields '∼∼B'. But that's it! All that remains is to assemble the foregoing steps in the proper order:

5.	D → A	1 CE
6.	∼(∼D ∨ ∼D)	2 CE
7.	∼∼D · ∼∼D	6 DeM
8.	D · D	7 DN (twice)
9.	D	8 Simp
10.	A	5,9 MP
11.	C	3,10 MP
12.	∼D ∨ C	11 Add
13.	D → C	12 CE
14.	∼∼(D → C)	13 DN
15.	∼∼B	4,14 MT
16.	B	15 DN
17.	C · B	11,16 Conj
18.	∼∼C · ∼∼B	17 DN (twice)
19.	∼(∼C ∨ ∼B)	18 DeM
20.	∼(C → ∼B)	19 CE

The next example illustrates several of the tactics mentioned on p. 325.

Example 5.7.C

1. ~(A → B)
2. ~(C ∨ D)
3. (C ∨ B) ∨ E ∴ E

Global analysis of Example 5.7.C: We see that 'E' must come from the third premise. If only that premise were a conditional and, somehow, the first two premises could be made to yield the antecedent, MP would provide the conclusion. Fortunately, by DN and CE, one can get a conditional from the third premise:

4.	~~(C ∨ B) ∨ E	3 DN
5.	~(C ∨ B) → E	4 CE

If we drive the '~' inside the parentheses in the antecedent, the preceding statement is a bit clearer:

6.	(~C · ~B) → E	5 DeM

Now we have a conditional leading to the desired conclusion, but how can we get the antecedent? Where can it come from? *Forwards analysis:* The antecedent must come from the first two premises. However, you may not see exactly how. Start off with a bit of "*random walk*," guided only by the tactical suggestion to move '~'s inside parentheses:

7.	~(~A ∨ B)	1 CE
8.	~~A · ~B	7 DeM
9.	~C · ~D	2 DeM

Now we see exactly what must be done:

10.	~C	9 Simp
11.	~B	8 Simp
12.	~C · ~B	10,11 Conj

Line 12 is the antecedent we seek, so we get our desired conclusion:

13.	E	6,12 MP

Chapter 5

Next, look at this short argument:

Example 5.7.D

1. A → B
2. A ∨ B ∴ B

Global and backwards analysis: I see nothing. *Forwards analysis:* I still see nothing. However, use the magic of *indirect proof,* and it couldn't be easier:

┌	3. ~B	
│	4. ~A	1,3 MT
│	5. A	2,3 DS
└→	6. A · ~A	4,5 Conj
	7. B	3–6 IP

Now it's time for some serious work on proofs.

Exercises 5.7

Construct proofs for the following arguments:

★**1.** A ∨ (B → C)
C → D
(B → D) → E
~A ∴ E · ~A

★**2.** A → (B · C)
A ∨ D
(C ∨ D) → E ∴ E

★**3.** D · ~(~C → A) ∴ ~(D → A)

★**4.** A → (C ∨ D)
(~D · A) → B ∴ (~D · (C → ~B)) → ~A

★**5.** ~(D ∨ C) → A
(D ∨ C) → (B → J)
~(A · ~G) ∴ B → (J ∨ G)

6. A → B
B → C
~C ∴ ~A

7. E
A → ~C
A · B
C ∨ D ∴ D · E

8. $(B \lor C) \to \sim(A \to D)$
$(A \to D) \lor \sim E$
B
$B \to \sim G$ $\quad \therefore \sim G \cdot \sim E$

9. $A \lor ((B \to C) \cdot D)$ $\quad \therefore \sim(\sim D \cdot \sim A)$

10. $A \to (B \to C)$ $\quad \therefore (D \to (A \cdot B)) \to (D \to C)$

11. $\sim M \to G$
$\sim G$
$M \to C$
$C \to B$ $\quad \therefore B$

12. $A \to \sim B$
$B \lor D$
$\sim D \lor \sim C$
$E \to C$ $\quad \therefore A \to \sim E$

13. $A \to (B \lor E)$
$\sim G$
$(\sim C \lor \sim A) \to D$
$(E \lor D) \to G$ $\quad \therefore (B \lor D) \cdot C$

14. $A \to B$
$C \to A$
$\sim B \to C$ $\quad \therefore B$

15. $A \leftrightarrow B$
$B \leftrightarrow C$ $\quad \therefore A \leftrightarrow C$

16. $A \to (B \cdot \sim C)$ $\quad \therefore A \to \sim(B \to C)$

17. $A \to (B \to C)$
$C \to (D \cdot E)$ $\quad \therefore A \to (B \to D)$

18. $(A \to B) \to (B \to C)$
$C \to \sim B$ $\quad \therefore \sim B$

19. $\sim B \lor (B \to \sim A)$
$(B \to \sim A) \to (\sim C \cdot D)$
$\sim B \to \sim C$ $\quad \therefore C \to \sim C$

20. $(A \to B) \lor (C \to B)$ $\quad \therefore (A \to C) \to (A \to B)$

21. $\sim A \lor (D \cdot B)$
$C \to (B \cdot D)$
$A \lor C$ $\quad \therefore B \cdot D$

22. $\sim(G \lor \sim D)$
$A \to E$
$B \lor (D \to C)$
$\sim(C \cdot E)$
$\sim A \to G$ $\quad \therefore B \cdot D$

23. $(\sim A \lor \sim B) \cdot \sim C$
$\sim\sim(D \to E)$
$\sim(C \lor (A \cdot B)) \to ((D \to E) \to G)$ $\quad \therefore G$

24. $\sim\sim E \cdot \sim D$
$(A \to B) \to (C \vee G)$
$\sim E \vee (A \to B)$ $\quad \therefore \sim G \to C$

25. $A \to (B \to D)$
$A \cdot (H \to G)$
$A \to B$
$D \to (G \to H)$ $\quad \therefore G \leftrightarrow H$

26. $C \vee (\sim B \cdot \sim A)$
$\sim(\sim E \cdot D)$ $\quad \therefore (B \to C) \cdot (D \to E)$

27. $A \to (C \to (D \cdot \sim E))$
$H \to (A \cdot (C \vee \sim B))$
$H \cdot G$
C $\quad \therefore \sim(D \to E)$

28. $\sim(D \vee A)$
$E \to (G \to H)$
$(E \vee D) \cdot (D \vee C)$
$C \to (G \vee \sim H)$ $\quad \therefore G \leftrightarrow H$

29. $(A \cdot \sim B) \to C$
$(B \vee C) \to \sim C$
$\sim(A \to B)$ $\quad \therefore D$

30. $A \to (B \to E)$
$\sim(D \vee \sim B)$
$(C \cdot \sim D) \to A$
$C \vee \sim B$ $\quad \therefore E$

31. B
$B \to \sim D$
$C \to \sim E$ $\quad \therefore \sim(D \cdot E)$

32. $A \to (C \vee \sim D)$
$B \leftrightarrow (D \cdot E)$
$E \to \sim C$ $\quad \therefore \sim(A \cdot B)$

33. $A \to (B \cdot C)$
$C \to (E \to I)$
$\sim(D \to I)$ $\quad \therefore (A \cdot G) \to (\sim I \cdot \sim E)$

34. $\sim(A \vee B) \vee \sim(C \vee D)$
$(E \vee G) \to D$ $\quad \therefore A \to \sim E$

35. $A \to \sim(C \cdot D)$
$\sim(C \to (E \vee \sim J))$
$\sim D \to \sim(E \to G)$ $\quad \therefore A \to \sim G$

36. $(\sim D \cdot B) \vee C$
$\sim G$
$(C \cdot \sim D) \to A$
$(E \vee D) \to G$
$\sim A \vee E$ $\quad \therefore \sim(A \vee \sim B)$

37. $A \vee (B \cdot C)$
$(A \rightarrow D) \cdot (D \rightarrow C)$ $\quad \therefore \sim C \rightarrow C$

38. $\sim(A \cdot C) \rightarrow \sim B$
$D \vee B$
$\sim(D \cdot H)$
$\sim A \cdot (H \vee G)$ $\quad \therefore G \cdot \sim(A \vee H)$

39. $E \vee B$
$\sim(G \cdot B)$
$(C \cdot E) \rightarrow A$
$G \vee E$ $\quad \therefore D \rightarrow (A \vee \sim C)$

40. $A \rightarrow D$
$\sim C \rightarrow \sim D$
$(C \vee D) \rightarrow \sim C$ $\quad \therefore A \rightarrow E$

41. $\sim(D \vee C) \rightarrow A$
$(D \vee C) \rightarrow (B \rightarrow J)$
$\sim(A \cdot \sim G)$ $\quad \therefore B \rightarrow (J \vee G)$

42. $\sim(D \rightarrow G)$
$(E \vee \sim J) \rightarrow C$
$(D \cdot J) \rightarrow (B \vee A)$
$A \rightarrow (\sim H \vee \sim D)$ $\quad \therefore H \rightarrow (\sim B \rightarrow C)$

43. $C \rightarrow (N \leftrightarrow R)$
$N \leftrightarrow H$
$\sim R$ $\quad \therefore C \rightarrow \sim H$

44. $A \rightarrow \sim B$
$\sim(B \cdot D) \rightarrow D$
$\sim A \leftrightarrow C$
$B \rightarrow \sim C$ $\quad \therefore \sim B \cdot D$

45. $(C \vee D) \rightarrow (E \rightarrow A)$
$(E \rightarrow (E \cdot A)) \rightarrow G$
$G \rightarrow ((\sim B \vee \sim\sim B) \rightarrow (C \cdot B))$ $\quad \therefore C \leftrightarrow G$

46. $M \rightarrow (B \cdot \sim G)$
$D \rightarrow G$
$\sim M \vee (H \rightarrow D)$ $\quad \therefore B \vee (M \rightarrow \sim H)$

47. $E \rightarrow (D \rightarrow (B \cdot C))$
$\sim D \rightarrow (G \vee A)$
$\sim(E \rightarrow G)$ $\quad \therefore (B \rightarrow \sim C) \rightarrow A$

48. $(A \cdot B) \rightarrow C$
$B \rightarrow (A \vee C)$
$B \rightarrow (E \vee D)$
$E \rightarrow (C \rightarrow D)$ $\quad \therefore B \rightarrow (C \cdot D)$

Symbolize and construct a proof for each of the following arguments:

★**49.** There are only three possibilities: either your sister is mad, or she is telling lies, or she is telling the truth. You know she does not tell lies, and she is obviously not mad, so we must conclude that she is telling the truth. (C. S. Lewis, *The Lion, the Witch and the Wardrobe*) See 1.3.18, 4.2.63, 4.6.68.

★**50.** If the first, not the second; but the first. Therefore, not if the first, the second. See 4.6.80.

★**51.** If 'There is no truth' is true, it is false. Thus, it is false. (Thomas Aquinas, *Summa Theologica*) See 1.4.74, 4.2.54, 4.6.67, 4.6.124.

★**52.** If God does not exist, it is false that if I pray, God answers my prayers. I do not pray. Therefore, God exists. (W. D. Hart, quoted by Dorothy Edgington, "Do Conditionals Have Truth-Conditions?" in *A Philosophical Companion to First-Order Logic*) See 4.6.93.

★**53.** Either I'm not a bad influence on the young, or I am a bad influence but without realizing it. If I am a bad influence without realizing it, you ought not to have brought me to court. If I'm not a bad influence, I should not have been brought to court. Therefore, I ought not to have been brought to court. (Plato, "Apology") See 4.6.97.

★**54.** If everything is merely contingent, at one time there was nothing in existence. If this were true, even now there would be nothing in existence—which is absurd. Therefore, not all beings are merely contingent; there must exist something the existence of which is necessary. If a necessary being exists, God exists. Therefore, God exists. (Thomas Aquinas, *Summa Theologica*) See 4.2.56, 4.6.C, 4.6.71.

★**55.** Substance absolutely infinite is indivisible. If it were divisible, the parts into which it would be divided will or will not retain the nature of substance absolutely infinite. If they retain it, there will be a plurality of substances possessing the same nature, which is absurd. If the second case is supposed, substance absolutely infinite can cease to be, which is also absurd. (Baruch Spinoza, *Ethics*) See 4.6.75.

56. The constitution is either a superior paramount law, unchangeable by ordinary means, or it is on a level with ordinary legislative acts, and, like other acts, is alterable when the legislature shall please to alter it. If the former, a legislative act contrary to the constitution is not law; if the latter, written constitutions are absurd attempts, on the part of the people, to limit a power, in its own nature illimitable. Certainly, as all those who have framed written constitutions have known, written constitutions can limit the power of the sovereign. Therefore, legislative acts contrary to the constitution are not law. (Supreme Court, *Marbury* v. *Madison,* 1803) See 4.6.101.

57. Suppose I buy a new Porsche. That would impress Hattie, but it would take all my money, and I'd have to get a second job. That would mean I couldn't spend time with Hattie, and she would go out with Biff. If Hattie went out with Biff, I'd be miserable. So if I bought a new Porsche, I'd only be miserable. See 5.3.A.

58. Iron in the mantle of the earth, is moving toward the earth's center. If no other factors modify the earth's moment of inertia, this will decrease the earth's moment of inertia. If the moment of inertia of a rotating body decreases, if its speed is not otherwise affected, its speed of rotation increases. If the speed of the earth's rotation increases, days get shorter. In fact, days are getting longer, not shorter; the earth is not speeding up but slowing down. Thus, either the earth's moment of inertia is being changed by factors other than the sinking of iron, or its speed is being affected by factors other than its change in moment of inertia. (Harold C. Urey, "The Origin of the Earth," *Scientific American*) See 4.6.99.

5.8 Other Uses for Truth-functional Proofs

Our system of proofs can be used to establish more than just the validity of arguments. We can also construct proofs to establish that one statement implies another, that two statements are equivalent, or that a statement is tautological or self-contradictory. In this section, we will see how.

Implication

Suppose one wants to prove that '$A \rightarrow ((B \rightarrow C) \rightarrow \sim A)$' implies '$A \rightarrow \sim(B \rightarrow C)$'. Here is one way to approach the problem: Recall that, in any valid argument, the conjunction of the premises implies the conclusion (p. 223). As this suggests, one way of showing that the first of these statements implies the second is to think of them as constituting an argument in which the first is the single premise and the second is the conclusion. Constructing a proof for the validity of this argument is then equivalent to showing that the first statement implies the second. Here is one such proof:

Example 5.8.A

1.	$A \rightarrow ((B \rightarrow C) \rightarrow \sim A)$	$\therefore A \rightarrow \sim(B \rightarrow C)$
┌2.	A	
│3.	$\sim\sim A$	2 DN
│4.	$(B \rightarrow C) \rightarrow \sim A$	1,2 MP
└→5.	$\sim(B \rightarrow C)$	3,4 MT
6.	$A \rightarrow \sim(B \rightarrow C)$	2–5 CP

EXERCISES 5.8

For each of the following pairs of statements, construct a proof to show that the statement on the left implies the one on the right:

★**1.**	$(A \rightarrow B) \cdot A$	B
★**2.**	$\sim C$	$C \rightarrow (A \vee B)$
★**3.**	$\sim(A \rightarrow B)$	$A \cdot \sim B$
4.	A	$B \vee A$
5.	$(A \vee B) \cdot \sim B$	A
6.	$(A \rightarrow B) \cdot \sim(B \cdot C)$	$\sim(A \cdot C)$
7.	$\sim C \cdot D$	$\sim(C \cdot A) \cdot (C \rightarrow A)$
8.	$A \rightarrow (B \rightarrow C)$	$(A \cdot B) \rightarrow C$
9.	$(A \cdot B) \rightarrow C$	$A \rightarrow (B \rightarrow C)$

Equivalence

Since equivalence is mutual implication, to show that two statements are equivalent, one assumes the first and proves the second and then assumes the second and proves the first. This shows that each statement implies the other; that is, together, the two proofs show that the statements are equivalent. For example, consider again the two statements in Example 5.8.A. Not only does '$A \rightarrow ((B \rightarrow C) \rightarrow \sim A)$' imply '$A \rightarrow \sim(B \rightarrow C)$', as the example shows, but the reverse is also true; that is, '$A \rightarrow \sim(B \rightarrow C)$' implies '$A \rightarrow ((B \rightarrow C) \rightarrow \sim A)$'. Here is a proof:

Example 5.8.B

1.	$A \rightarrow \sim(B \rightarrow C)$	$\therefore A \rightarrow ((B \rightarrow C) \rightarrow \sim A)$
2.	A	
3.	$B \rightarrow C$	
4.	$\sim\sim(B \rightarrow C)$	3 DN
5.	$\sim A$	1,4 MT
6.	$(B \rightarrow C) \rightarrow \sim A$	3–5 CP
7.	$A \rightarrow ((B \rightarrow C) \rightarrow \sim A)$	2–6 CP

Since, as Examples 5.8.A and 5.8.B show, '$A \rightarrow \sim(B \rightarrow C)$' and '$A \rightarrow ((B \rightarrow C) \rightarrow \sim A)$' imply one another, they are equivalent.

EXERCISES 5.8

For each of the following pairs of statements, construct a pair of proofs to show that they are equivalent:

★**10.**	$(A \vee \sim C) \rightarrow B$	$(A \rightarrow B) \cdot (\sim B \rightarrow C)$
11.	$A \rightarrow B$	$\sim(A \cdot \sim B)$
12.	$A \rightarrow \sim A$	$\sim A$
13.	$A \rightarrow (B \cdot \sim B)$	$\sim A$
14.	$(A \leftrightarrow B) \cdot \sim B$	$\sim(A \vee B)$
15.	$(A \rightarrow B) \cdot C$	$\sim(C \rightarrow A) \vee (B \cdot C)$

Tautologies

Next, we learn to construct a proof establishing that a given statement is a tautology. Recall that any statement implies every tautology (p. 225); thus, any argument with a tautologous conclusion is valid. Another way of putting this is to say that the validity of an argument with a tautologous conclusion does not depend on the premises of the argument. Accordingly, to show that some statement is a tautology, it is sufficient to construct a proof *without premises* in which the statement in question is the conclusion. But how can one construct a proof without premises? How could such a proof begin? The only possible way is with a supposition. As an example, consider again Example 4.4.14: '$((A \cdot B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$'. How can our system of proofs establish that this statement is a tautology? Here is one way:

Example 5.8.C

1.	$(A \cdot B) \rightarrow C$	
2.	A	
3.	B	
4.	$A \cdot B$	2,3 Conj
5.	C	1,4 MP
6.	$B \rightarrow C$	3–5 CP
7.	$A \rightarrow (B \rightarrow C)$	2–6 CP
8.	$((A \cdot B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$	1–7 CP

In general, to establish that a conditional statement is a tautology, (1a) suppose its antecedent, then (1b) suppose the antecedent of its consequent (if there is one), then (1c) suppose the antecedent of the consequent of the consequent (if there is one), and so on; next (2) work to attach the final consequent, and finally, (3) conditionalize your way back out. Actually, we have already seen an ultra-simple application of this idea. Recall the proof of Example 5.3.G:

Example 5.8.D

1.	A	
2.	$A \rightarrow A$	1 CP

Chapter 5

Similar proofs can establish that any statement of the form $p \rightarrow p$ is a tautology. Such proofs are sometimes useful in introducing statements of the form $\sim p \vee p$ into a longer proof.

Now you must be cautioned against a common mistake. Consider, again, establishing that the statement '$((A \cdot B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$' is a tautology (the same tautology we dealt with in Example 5.8.C). Students often begin such a proof *incorrectly* as follows:

Example 5.8.E

1. $((A \cdot B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$
2. $(A \cdot B) \rightarrow C$
3. A
4. B
5. $A \cdot B$ — 3,4 Conj

Do you see why this "proof" is incorrect? Perhaps because we typically begin proofs of valid arguments by numbering the premises as given, if students are asked to establish that some statement is a tautology, they often begin (incorrectly) by numbering, as part of the proof, the very tautology they are trying to prove. But this would be like including the conclusion of an argument as a premise, and that is so even if, as may be the case in Example 5.8.E, line 1 (i.e., the tautology in question) is never actually used in the proof that follows. So resist the inclination to start with something other than the suppositions on which the proof is to be based.

We now have a way of constructing proofs to demonstrate the tautological nature of tautologous *conditionals,* but what about proofs of tautologies that are *not* conditionals? With such tautologies, how could we begin a proof without premises? The answer is that it is often best to use indirect proof: suppose the denial of the tautology in question, and derive a contradiction. Once again, this yields a proof without premises which establishes that the statement in question is a tautology. As an example, consider the following proof of '$A \vee \sim A$':

Example 5.8.F

1. $\sim(A \vee \sim A)$
2. $\sim A \cdot \sim\sim A$ — 1 DeM
3. $A \vee \sim A$ — 1–2 IP

Notice that line 2, '$\sim A \cdot \sim\sim A$', is a contradiction, concluding the indirect proof.

Exercises 5.8

Construct proofs showing that each of the following statements is a tautology:

★**16.** $A \to (B \to A)$
★**17.** $A \to (B \to (A \cdot B))$
★**18.** $(A \to B) \to (\sim B \to (\sim A \lor \sim C))$
19. $(A \to B) \to ((B \to C) \to (A \to C))$
20. $(A \leftrightarrow B) \to ((A \to C) \leftrightarrow (B \to C))$
21. $((A \to B) \to A) \to A$
22. $(A \to B) \lor (\sim C \to \sim B)$
23. $A \to ((A \cdot B) \leftrightarrow B)$
24. $((B \to C) \to (A \cdot D)) \to ((A \to \sim D) \to (B \to (\sim E \to \sim C)))$

Self-Contradictions

Finally, we can also use our system of proofs to demonstrate that a statement is a self-contradiction. Since we know that a contradiction is implied only by other contradictions (p. 225), and since each line in a proof is implied by the lines above it, we can prove that a given line is self contradictory by deriving an obvious self-contradiction (i.e., something of the form $p \cdot \sim p$) from it. As an example, we will show that '$(A \leftrightarrow B) \cdot \sim(\sim A \lor B)$' is self-contradictory. We begin by taking this statement as a premise, and we then derive an obvious self-contradiction:

Example 5.8.G

1.	$(A \leftrightarrow B) \cdot \sim(\sim A \lor B)$	
2.	$A \leftrightarrow B$	1 Simp
3.	$A \to B$	2 BE
4.	$\sim(\sim A \lor B)$	1 Simp
5.	$\sim(A \to B)$	4 CE
6.	$(A \to B) \cdot \sim(A \to B)$	3,5 Conj

Notice that the last line is of the form $p \cdot \sim p$ and so is a self-contradiction.

EXERCISES 5.8

Construct proofs showing that the following statements are self-contradictions:

★**25.** $\sim((A \cdot \sim A) \to \sim B)$
26. $A \cdot (A \to (B \cdot \sim B))$
27. $((A \to B) \cdot A) \cdot \sim B$
28. $((A \to B) \cdot \sim(B \cdot C)) \cdot (A \cdot C)$
29. $\sim(A \to B) \cdot \sim(B \to C)$
30. $\sim(((A \to A) \to A) \to A)$

5.9 The Nature of Truth-functional Proofs

We now have two independent ways of approaching truth-functional logic: truth tables and the construction of proofs. It is essential to be clear about the relation between these two approaches and, if you have studied syllogisms, about how truth-functional logic compares to syllogistic logic.

We can best understand the relation between truth tables and proofs by appealing to two concepts that are important both in logic and in linguistics: syntax and semantics. **Syntax** is the grammar of a language; it encompasses rules for deciding whether a group of symbols is correctly put together or, using the term most common in this context, well formed. Syntax also includes rules for changing the order of sequences of symbols. For example, English syntax assures us that a dependent clause can be moved from the beginning to the end of a sentence without altering the meaning of the sentence.

Semantics is the meaning of the symbols in a language. Thus, an English dictionary contains part of the semantics of the English language. Here is an important point: truth is an inherently semantic concept. Why is this so? For the simple reason that to know whether a sentence is true, one must understand it. By contrast, one can determine that a sentence is grammatically correct (well formed) even without understanding it. For example, here is the first sentence from Lewis Carroll's famous poem "Jabberwocky":

Example 5.9.A

'Twas brillig, and the slithy toves
Did gyre and gimble in the wabe:
All mimsy were the borogoves,
And the mome raths outgrabe.

It's easy to see that this meaningless sentence is grammatical.

Now remember the definition of 'valid': an argument is valid if, given the premises, it is impossible for the conclusion to be false. As this definition reveals, there is a subtle, but essential, connection between validity and truth, so that validity, like truth, is a semantic concept. Using truth tables or Venn diagrams, we test for validity *directly* by literally looking to see whether a given argument can have true premises and a false conclusion. Other concepts, such as *implication, equivalence, tautologicality,* and *contradictoriness* are also defined in terms of 'truth'; that is, they, too, are semantic concepts. But what of proofs?

In constructing proofs, we begin with assumptions (premises or suppositions), and, following certain rules, we attach other statements until we can attach the statement we seek. At no point do we concern ourselves with interpreting the symbols—the meanings of the symbols and the truth-values of the statements they represent are never taken into account. The rules we follow in constructing proofs are transformation rules that enable us to modify strings of symbols to produce other strings of symbols. In this respect, the rules we use to construct proofs resemble the rules of grammar that enable us to move clauses in compound sentences without ever asking what the clauses mean or whether they are true. For example, assuming that Example 5.9.A is true, English grammar assures us that these sentences also will be true:

Example 5.9.B

The slithy toves gyred and gimbled in the wabe. It was brillig, the borogoves were all mimsy, and the mome raths were outgrabe.

As this discussion suggests, rules for constructing proofs are *syntactic* rules. When we say that a proof can be constructed for an argument, we are saying something about the grammar of the statements that compose the argument.

Here is a crucial question: How can we be sure that the *syntactic* construction of proofs and the *semantically* based truth tables will yield the same results? Given some argument, could it happen that truth tables show that the argument is valid, but that no proof can be constructed for it? Or could the reverse happen—that we can construct proofs for arguments that truth tables show to be invalid? In short, how do we know that proofs work?

We say that a system of proofs is **complete** for truth-functional validity if a proof can be constructed within that system for every valid truth-functional argument; we say that a system is **consistent** for truth-functional validity if proofs can be constructed within the system only for valid truth-functional arguments. Thus, if our system of proofs is complete and consistent for validity, proofs can be constructed for all and only valid truth-functional arguments. In other words, our system is complete and consistent if and only if the arguments that truth tables show to be valid are exactly the arguments for which proofs can be constructed. Proving that systems of proofs are complete and consistent is difficult; such proofs, called "meta-proofs," go beyond the scope of this book. However, the proofs *can* be given, and our system *is* complete and consistent.[2] This being true, we know that the provable truth-functional arguments are exactly the valid ones. In other words, because of the meta-proofs, we know that the semantically defined valid arguments and syntactically defined provable arguments are exactly the same.

In a sense, it is misleading to describe what we have done in this chapter simply as proving validity, implication, equivalence, tautologicality, or contradictoriness. Such talk obscures the gulf between syntactic proofs and these semantic concepts—a gulf that can be bridged only by way of the meta-proofs that we mention, but do not provide. More precisely, using our method, we can construct proofs that, thanks to the completeness and consistency of our system of proofs, establish the validity of particular arguments. (Similar remarks apply to the other semantic concepts mentioned at the beginning of this paragraph.)

If you have read Chapter 3, you may recall that our method of proofs for syllogisms can also be proven to be complete and consistent. Because both systems for constructing proofs are complete and consistent, truth-functional and syllogistic logic can be approached either semantically, through the quest for refutations, or syntactically, by way of proofs.

While the semantic and syntactic approaches establish the validity of the same arguments, there are differences between them. Most importantly, semantic procedures for finding refutations report both positive and negative results: if an argument is valid, truth tables and Venn diagrams show it; if the argument is invalid, the same processes show it also. By contrast, the construction of proofs is *not* a decision procedure. If we seek a proof of an invalid argument (whether truth functional or syllogistic), we will look forever for a proof that simply does not exist—the unsuccessful quest for a proof never establishes that there is no such proof. So, in this respect, our semantic procedures for finding refutations are superior to the construction of proofs: they always terminate conclusively either with a refutation or with the assurance that no refutation is possible. However, in another sense, proofs are more essential to logic than the semantic quest for refutations.

The last sentence of the previous paragraph requires some explanation. For both syllogisms and truth-functional arguments, we have ways of symbolizing arguments, processes for determining whether any given argument can be refuted, and systems for constructing proofs. However, there are two problems: First, there are many arguments that are neither syllogistic nor truth functional; second, it is conceptually unsatisfying to have two independent systems for dealing with the two classes of arguments—it would be nice if the two systems could, somehow, be brought together into one more powerful system of logic.

In Chapter 6, we move to a new level of generality—a level that encompasses syllogisms, truth-functional arguments, and many other arguments as well. But this increased generality and power comes at a price: *within this new level, there is no systematic way of finding refutations* and, therefore, no decision procedure. Indeed, one proof that there is no decision procedure for quantificational logic is given as Exercise 16 in Section 5.4, and you may have demonstrated its validity without really understanding what it was about. We can always try to refute invalid arguments by the trial-and-error methods of Section 1.6, but if arguments are approached on this new level of generality, there is no *systematic* way of refuting particular invalid arguments. Consequently, on this new level, the syntactic construction of proofs is the only effective way of appraising validity. Thus, as we observed earlier, proofs are absolutely vital to any really adequate system of logic.

What Have I Learned in This Chapter?

- In this chapter, we have learned a system for constructing proofs for valid truth-functional arguments. The rules in our system operate by forming instances of statement forms. Thus, in Section 5.1, we began by examining with some care the relation between statements and instances.
- Our system of proof uses eight basic rules. In Section 5.2, we learned the first six of these rules: modus ponens, simplification, conjunction, disjunctive syllogism, addition, and biconditional exchange.
- Sections 5.3 and 5.4 introduced the basic rules of conditional proof and indirect proof, respectively.
- While our basic rules are adequate to construct a proof for any valid truth-functional argument, it is much more efficient to have available additional, shortcut rules. In Section 5.5, we introduced the shortcut rules called double negation, De Morgan's theorems, conditional exchange, modus tollens, and constructive dilemma. These rules are used, where appropriate, throughout this book. Each rule was justified by proving it by means of statement forms. We also expanded double negation, De Morgan's theorems, and conditional exchange to allow those rules to be used *within* lines.
- In Section 5.6, we introduced more shortcut rules. These rules can be used in homework, but they do not appear in the proofs printed in later sections of the book.
- The most difficult thing about proofs is deciding, of all the possible lines one could attach, which lines are likely to bring the proof closer to the desired conclusion. In Section 5.7, we discussed some of the strategic and tactical considerations that help decide which lines to attach. We distinguished five main strategies: global analysis, backwards analysis, forwards analysis, indirect proof, and random walk.
- In Section 5.8, we learned to use proofs to establish implication and equivalence, as well as to classify statements as tautologous and contradictory.
- In Section 5.9, we discussed the nature of truth-functional proofs in relation to truth tables, on the one hand, and to similar techniques developed for syllogisms, on the other. Since our system of proofs for truth-functional logic is complete and consistent, it is a genuine alternative to the use of truth tables.

How Can I Apply What I Have Learned?

A. From contemporary sources, such as editorials, letters to editors, or nonfiction books or essays, locate a valid truth-functional argument. Symbolize it, construct a proof to demonstrate its validity, and decide whether its premises are true or false.

B. In a one-page paper, use a proof or a truth table to establish the validity or invalidity of one of the following arguments, and then appraise the soundness of the argument:

1. If existence is a multitude of different things, it must be limited in number. This is because any multitude has a certain number of members, neither more nor less. But if existence is a multitude, it must also be unlimited in number; because two things are two only when they are separable; but in order that they may be separated, there must be something between them; and so too between this intermediate and each of the two, and so *ad infinitum.* Therefore, if existence is a multitude it must be both limited and unlimited in number, but this is impossible. Therefore, existence is not a multitude. (G. S. Kirk and J. E. Raven, *The Presocratic Philosophers*)
2. If God exists, he is all loving and all powerful. If God is all loving, he cares about mortals. If God is all powerful, there can be no evil in the world if God cares about mortals. Therefore, if God exists, there is no evil in the world.
3. Senator Orrin Hatch argues that life does not begin at conception. Why? Because an embryo conceived in a petri dish is not human life until it is put into a mother's womb. Two embryos, same age, same stage of development, but through Hatch's view, the situation of each determines its status as a human life. Is the embryo in a mother's womb? Yes, it is a human life. Is it in a petri dish? No, it is not a human life. (Deena King, letter to the editor, *Deseret News*)

C. Make up a valid truth-functional argument with plausible premises for one of the claims that follow. Then expand the argument into a one-page paper. To do this, begin with a short paragraph stating your conclusion, then make each premise a thesis statement of a paragraph in which you explain or justify that premise, and, finally, write a short paragraph that restates the conclusion and show how it follows from the premises. The claims are as follows:

1. All firearms should be registered with the government.
2. Firearms should be subject to no further governmental controls than are already in place.
3. Same-sex marriages should be prohibited by law.
4. Subject to the same provisions that regulate ordinary marriages, same-sex marriages should be legally recognized.

LSAT

D. The questions that follow are similar to questions that have appeared on the analytical thinking section of the LSAT. In each case, select the best response and then compare your responses with the explanations given after all the questions.

Questions 1–5

Specimens of the following minerals are arranged in a circular display: beryl, garnet, jade, opal, quartz, and topaz. The following conditions apply:

The topaz is next to the jade.
The quartz is next to the beryl or to the opal.
The jade is not next to the beryl.
If the quartz is next to the topaz, then the opal is not next to the jade.

1. Which one of the following arrangements would not violate the specified conditions:
   - **a.** Beryl, topaz, opal, jade, quartz, garnet
   - **b.** Beryl, garnet, jade, topaz, opal, quartz
   - **c.** Beryl, garnet, jade, quartz, topaz, opal
   - **d.** Beryl, quartz, topaz, jade, opal, garnet
   - **e.** Beryl, opal, garnet, quartz, jade, topaz
2. If the garnet is between the topaz and the opal, the quartz must be between
   - **a.** The opal and the jade
   - **b.** The jade and the topaz
   - **c.** The opal and the beryl
   - **d.** The beryl and the jade
   - **e.** The topaz and the quartz
3. If the quartz is between the garnet and the opal, the beryl must be next to
   - **a.** The garnet
   - **b.** The opal
   - **c.** The jade
   - **d.** The quartz
   - **e.** The topaz
4. If the quartz is between the topaz and the beryl, the garnet must be between
   - **a.** The opal and the jade
   - **b.** The opal and the beryl
   - **c.** The opal and the topaz
   - **d.** The jade and the beryl
   - **e.** The jade and the topaz

5. If the opal is next to the jade, in addition to jade, which of the following is a complete and accurate list of the minerals that could be next to the topaz?

   a. Quartz
   b. Beryl
   c. Beryl or garnet
   d. Garnet or quartz
   e. Beryl or garnet or quartz

Questions 6–10

An athletic council comprises students from Cornell, Dartmouth, Harvard, and Yale. The council must have a president, a vice president, a secretary, and a treasurer, as well as coordinators for football and basketball. These officers must be selected in accordance with the following criteria:

> No one can hold more than one office. At least one officer must be from each school, and no school can have more than two officers.
> The president and vice president must be from different schools.
> The secretary and the treasurer must be from the same school.
> If the football coordinator is from Yale, the basketball coordinator must be from Harvard.
> The football coordinator cannot be from the same school as the president.

6. If the president is from Cornell and the secretary is from Harvard, which of the following must be true?

   a. Both coordinators are from the same school.
   b. Exactly one coordinator is from Harvard.
   c. At least one coordinator is from Yale.
   d. The coordinators are not from the same school.
   e. The football coordinator is from Dartmouth.

7. If the basketball coordinator is from the same school as the president or vice president, which of the following must be false?

   a. The vice president is from the same school as one coordinator.
   b. The vice president is from the same school as the football coordinator.
   c. The president is from the same school as the basketball coordinator.
   d. The vice president is from the same school as the basketball coordinator.
   e. The coordinators are from different schools.

8. If the vice president is from Dartmouth and the football coordinator is from Yale, which of the following is a complete and accurate list of the schools the president could be from?

- **a.** Harvard
- **b.** Cornell
- **c.** Yale
- **d.** Cornell or Harvard
- **e.** Cornell or Harvard or Yale

9. If the coordinators are from the same school, and if the treasurer is not from Yale, which of the following must be true?

- **a.** Either the president or the vice president is from Cornell.
- **b.** Either the president or the vice president is from Yale.
- **c.** The president is from Yale.
- **d.** The president is from Cornell.
- **e.** The president is from Dartmouth.

10. If the basketball coordinator is not from Harvard, and if the president is from Dartmouth, which of the following must be false?

- **a.** The two coordinators are from the same school.
- **b.** The football coordinator and the vice president are from the same school.
- **c.** The basketball coordinator and the president are from the same school.
- **d.** The football coordinator and the vice president are from Yale.
- **e.** The football coordinator and the vice president are from Cornell.

LSAT EXPLANATIONS: These sets of questions are similar to questions on the analytical reasoning part of the LSAT. Formal logic alone is generally not sufficient to answer such questions efficiently, but the solution to each question is found by a kind of deduction or proof, and the proofs are often truth-functional in nature. Moreover, in some cases, as in the questions in the first set, truth-functional notation can help by expressing facts succinctly, thereby saving the need for repeated rereading.

For the first five questions, let the first letter of each name represent the mineral sample, and then represent the relation 'next to' by writing letters together. The initial conditions can be expressed succinctly in truth-functional notation as follows:

TJ

QB ∨ QO

~(JB)
QT → ~(OJ)

Notice that, in each pair of letters, the order is immaterial: saying that T is next to J (i.e., 'TJ') is the same as saying that J is next to T (i.e. 'JT'). The main advantage of this array is that it provides a graphic summary of the initial conditions. Now look at the specific questions:

The answer to (1) must be (B). (A) and (C) violate the first condition, (D) violates the fourth condition, and (E) violates the second.

To answer each of questions (2) through (5), draw a circle, and, taking account of the initial conditions and of the conditions given in each question, place letters representing the samples around the circle at intervals of about one-sixth of the circumference of the circle. In question (2), from the first initial condition, we know that TJ. From the given condition, we also know that OGT, so we place these letters on the circle. Only B and Q remain, but the third condition tells us that ~(JB), so B must be next to O and Q must be between B and J. Accordingly, the correct answer is (D).

The given condition in (3) assures us that G, Q, and O must be together. (Again, the order doesn't matter, so long as Q is in the middle). Sketch the trio on a circle opposite T and J, which (as we know from the first initial condition) must be together. B must be between the TJ pair and the GQO trio. The only question is, Does it comes next to T or next to J? But the third initial condition makes the second of these alternatives impossible. Of course, B will also be next to G or to O, but since, depending on the order of the trio, either of these conditions could hold, all we know for sure is that B is next to T. Thus, the answer is (E).

In (4), the given condition ensures that B, Q, and T will be together, and the first initial condition tells us that J will be next to T, so the only question is whether O is next to B or to J. But since Q is next to T, the fourth initial condition tells us that O cannot be next to J. The correct answer is therefore (A).

The given condition in (5), together with the fourth initial condition, ensures that Q and T cannot be together. This restriction excludes (A), (D), and (E). However, there is no way of deciding whether B or G is next to T, so the correct answer is (C).

For questions (6) through (10), let each school and officer be represented by the first letter of the name or title—I'll use this convention in these explanations. The only initial condition that can profitably be represented symbolically is the fourth, for which you might write 'Yf → Hb'. For each question, it may help you visualize things if you write two columns, with letters representing schools in one and letters representing titles in the other.

For question (6), the given condition is 'Cp · Hs'. We know that f cannot be from C (since f and p are not from the same school) or from H (since s and, therefore, t are from H and no more than two officers are from any one school). But f cannot be from Y, because then (from the fourth initial condition) b would have to be from H (which would mean that there were three officers from H). So f must be from D. The correct answer is thus (E).

Question (7) illustrates the need for careful reading. Here, if f were from the same school as b, three officers would be from the same school. Hence, (E) must be *true,* and you might quickly select that as the right answer, but in this question we are looking for the choice that must be *false.* The best approach is to suppose each possible answer in turn until you get a contradiction (a series of indirect proofs!). No apparent contradiction emerges from supposing (A). In the case of (B), v and f are from the same school, which means (from the given condition) that p and b are from the same school, but since s and t are also (by the third initial condition), that leaves one school unrepresented—a contradiction. Hence, (B) must be false, so that is the correct answer.

In (8), we are given 'Dv · Yf'. But the fourth initial condition tells us that 'Yf → Hb', so, by modus ponens, 'Hb'. Since s and t must be together, they must be from C (the only school left with space for two officers). So p cannot be from C (C would have too many officers), from D (p can't be with v, who is already from D) or from Y (p can't be with f, who is already from Y). The correct answer is therefore (A).

In (9), b and f are from the same school, and it can't be Y (because, from the initial conditions, 'Yf → Hb'). So, too, s and t are from the same school, and it can't be Y (from the given conditions). The correct answer is obviously (B); the others could be, but are not necessarily, true.

Finally, in (10), the given conditions tell us that '~Hb · Dp'. As in question (7), we suppose successive possible answers and look for a contradiction. Since '~Hb' is given, we can infer '~Yf' from the fourth initial condition by modus tollens. Because choice (D) is 'Yf · Yv', this must be false. Hence, the correct answer is (D).

Where Do I Go From Here?

Having mastered the construction of proofs in truth-functional logic, you can continue your study with either of the following topics:

(1) *Inductive logic.* If you are interested in studying inductive logic, you can go next to Chapter 7.

(2) *Quantificational logic.* To continue your work in deductive logic, go on to Chapter 6. There we will study a system of logic that encompasses both truth-functional arguments and syllogisms and goes far beyond them both. This system, which we will call quantificational logic, was created in the late nineteenth century by the German philosopher and logician Gottlob Frege.

Endnotes

1. Irving M. Copi, *Symbolic Logic,* 5th ed. (New York: Macmillan Publishing Co., Inc. 1979) pp. 248ff.
2. Gordon Dahl, Wendy Grow, and David Sundahl, "An Intuitive Axiomatic Development of Truth-Functional Logic," *Aporia: A Student Journal of Philosophy,* 1992, *2*:35–46.