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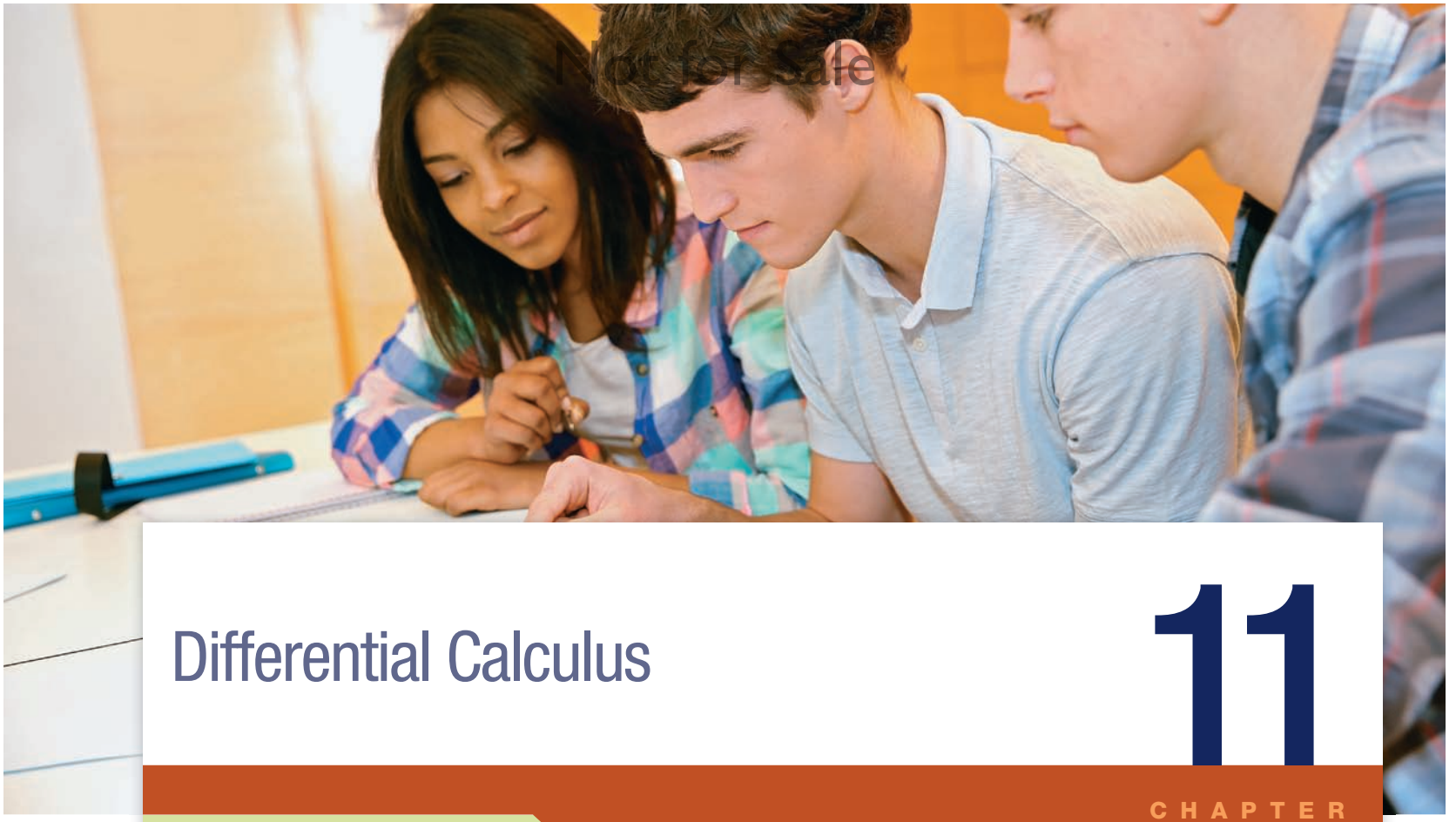

Differential Calculus

11

CHAPTER

CHAPTER OUTLINE

11.1 Limits
11.2 One-Sided Limits and Limits Involving Infinity
11.3 Rates of Change
11.4 Tangent Lines and Derivatives
11.5 Techniques for Finding Derivatives
11.6 Derivatives of Products and Quotients
11.7 The Chain Rule
11.8 Derivatives of Exponential and Logarithmic Functions
11.9 Continuity and Differentiability

CASE STUDY 11
Price Elasticity of Demand

How fast is the number of cell phone subscriptions growing? At what rate is the number of Internet users increasing? How are home prices changing? These questions and many others in the fields of business, finance, health, political science, psychology, sociology, and economics can be answered by using calculus. See Exercise 65 on page 630, Example 12 on page 658, and Exercise 72 on page 660.

The algebraic problems considered in earlier chapters dealt with *static* situations:

What is the revenue when x items are sold?

How much interest is earned in 2 years?

What is the equilibrium price?

Calculus, on the other hand, deals with *dynamic* situations:

At what rate is the economy growing?

How fast is a rocket going at any instant after liftoff?

How quickly can production be increased without adversely affecting profits?

The techniques of calculus will allow us to answer many questions like these that deal with rates of change.

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The key idea underlying the development of calculus is the concept of limit, so we begin by studying limits.

11.1 Limits

We have often dealt with a problem like this: "Find the value of the function $f(x)$ when $x = a$." The underlying idea of "limit," however, is to examine what the function does *near* $x = a$, rather than what it does *at* $x = a$. If you would like to refresh your understanding of functions and functional notation, see Chapter 3.

Example 1 The function

$$f(x) = \frac{2x^2 - 3x - 2}{x - 2}$$

is not defined when $x = 2$. (Why?) What happens to the values of $f(x)$ when x is *very close* to 2?

Solution Evaluate f at several numbers that are very close to $x = 2$, as in the following table:

x	1.99	1.999	2	2.0001	2.001
$f(x)$	4.98	4.998	—	5.0002	5.002

The table suggests that

> as x gets closer and closer to 2 from either direction, the corresponding value of $f(x)$ gets closer and closer to 5.

In fact, by experimenting further, you can convince yourself that the values of $f(x)$ can be made *as close as you want* to 5 by taking values of x close enough to 2. This situation is usually described by saying "The *limit* of $f(x)$ as x approaches 2 is the number 5," which is written symbolically as

$$\lim_{x \to 2} f(x) = 5, \quad \text{or equivalently,} \quad \lim_{x \to 2} \frac{2x^2 - 3x - 2}{x - 2} = 5.$$

The graph of f shown in Figure 11.1 also shows that $\lim_{x \to 2} f(x) = 5$.

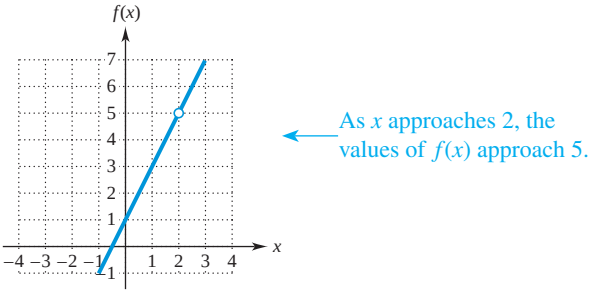

Figure 11.1

Checkpoint 1

Use a calculator to estimate

$$\lim_{x \to 1} \frac{x^3 + x^2 - 2x}{x - 1}$$

by completing the following table:

x	$f(x)$
.9	
.99	
.999	
1.0001	
1.001	
1.01	
1.1	

Answers to Checkpoint exercises are found at the end of the section.

The informal definition of "limit" that follows is similar to the situation in Example 1, but now f is any function, and a and L are fixed real numbers (in Example 1, $a = 2$ and $L = 5$).

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Limit of a Function

Let f be a function, and let a and L be real numbers. Assume that $f(x)$ is defined for all x near $x = a$. Suppose that

> as x takes values very close (but not equal) to a (on both sides of a), the corresponding values of $f(x)$ are very close (and possibly equal) to L

and that

> the values of $f(x)$ can be made as close as you want to L for all values of x that are close enough to a.

Then the number L is the **limit** of the function $f(x)$ as x approaches a, which is written

$$\lim_{x \to a} f(x) = L.$$

This definition is *informal* because the expressions "near," "very close," and "as close as you want" have not been precisely defined. In particular, the tables used in Example 1 and the next set of examples provide strong intuitive evidence, but not a proof, of what the limits must be.

Example 2 If $f(x) = x^2 + x + 1$, what is $\lim_{x \to 3} f(x)$?

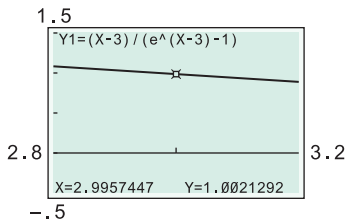
Solution Make a table showing the values of the function at numbers very close to 3:

	x approaches 3 from the left →			3	← x approaches 3 from the right		
x	2.9	2.99	2.9999		3.0001	3.01	3.1
$f(x)$	12.31	12.9301	12.9993 . . .		13.0007 . . .	13.0701	13.71
			$f(x)$ approaches 13 →		← $f(x)$ approaches 13		

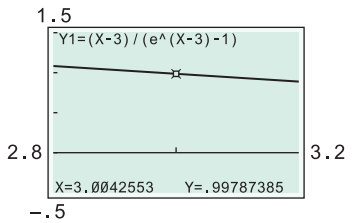
The table suggests that as x approaches 3 from either direction, $f(x)$ gets closer and closer to 13 and, hence, that

$$\lim_{x \to 3} f(x) = 13, \qquad \text{or equivalently,} \qquad \lim_{x \to 3} (x^2 + x + 1) = 13.$$

Note that the function $f(x)$ is defined when $x = 3$ and that $f(3) = 3^2 + 3 + 1 = 13$. So in this case, the limit of $f(x)$ as x approaches 3 is $f(3)$, the value of the function at 3.



(a)



(b)

Figure 11.2

Example 3 Use a graphing calculator to find

$$\lim_{x \to 3} \frac{x - 3}{e^{x-3} - 1}.$$

Solution There are two ways to estimate the limit.

Graphical Method Graph $f(x) = \dfrac{x - 3}{e^{x-3} - 1}$ in a very narrow window near $x = 3$. Use the trace feature to move along the graph and observe the y-coordinates as x gets very close to 3 from either side. Figure 11.2 suggests that $\lim_{x \to 3} f(x) = 1$.

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X	Y1
2.99	1.005
2.999	1.0005
2.9999	1.0001
3	ERROR
3.0001	.99995
3.001	.9995
3.01	.99501

X=3

Figure 11.3

Numerical Method Use the table feature to make a table of values for $f(x)$ when x is very close to 3. Figure 11.3 shows that when x is very close to 3, $f(x)$ is very close to 1. (The table displays "error" at $x = 3$ because the function is not defined when $x = 3$.) Thus, it appears that

$$\lim_{x \to 3} \frac{x - 3}{e^{x-3} - 1} = 1.$$

The function has the limit 1 as x approaches 3, even though $f(3)$ is not defined.

Example 4 Find $\lim_{x \to 4} f(x)$, where f is the function whose rule is

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ 1 & \text{if } x \text{ is not an integer} \end{cases}$$

and whose graph is shown in Figure 11.4.

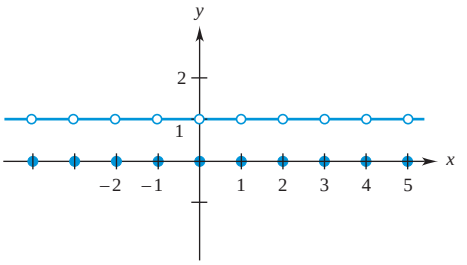

Figure 11.4

Solution The definition of the limit as x approaches 4 involves only values of x that are close, but not equal, to 4—corresponding to the part of the graph on either side of 4, but not at 4 itself. Now, $f(x) = 1$ for all these numbers (because the numbers very near 4, such as 3.99995 and 4.00002, are not integers). Thus, for all x very close to 4, the corresponding value of $f(x)$ is 1, so $\lim_{x \to 4} f(x) = 1$. However, since 4 is an integer, $f(4) = 0$. Therefore, $\lim_{x \to 4} f(x) \neq f(4)$.

Examples 1–4 illustrate the following facts.

Limits and Function Values

If the limit of a function $f(x)$ as x approaches a exists, then there are three possibilities:

1. $f(a)$ is not defined, but $\lim_{x \to a} f(x)$ is defined. (Examples 1 and 3)
2. $f(a)$ is defined and $\lim_{x \to a} f(x) = f(a)$. (Example 2)
3. $f(a)$ is defined, but $\lim_{x \to a} f(x) \neq f(a)$. (Example 4)

Finding Limits Algebraically

As we have seen, tables are very useful for estimating limits. However, it is often more efficient and accurate to find limits algebraically. We begin with two simple functions.

Consider the constant function $f(x) = 5$. To compute $\lim_{x \to a} f(x)$, you must ask "When x is very close to a, what is the value of $f(x)$?" The answer is easy because no matter what

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x is, the value of $f(x)$ is *always* the number 5. As x gets closer and closer to a, the value of $f(x)$ is always 5. Hence,

$$\lim_{x\to a} f(x) = 5, \qquad \text{which is usually written} \qquad \lim_{x\to a} 5 = 5.$$

The same thing is true for *any* constant function.

Limit of a Constant Function

If d is a constant, then, for any real number a,

$$\lim_{x\to a} d = d.$$

Now consider the *identity function*, whose rule is $f(x) = x$. When x is very close to a number a, the corresponding value of $f(x)$ (namely, x itself) is very close to a. So we have the following conclusion.

Limit of the Identity Function

For every real number a,

$$\lim_{x\to a} x = a.$$

The facts in the two preceding boxes, together with the properties of limits that follow, will enable us to find a wide variety of limits.

Properties of Limits

Let a, r, A, and B be real numbers, and let f and g be functions such that

$$\lim_{x\to a} f(x) = A \qquad \text{and} \qquad \lim_{x\to a} g(x) = B.$$

Then the following properties hold:

1. $\lim_{x\to a} [f(x) + g(x)] = A + B = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ *Sum Property*
(The limit of a sum is the sum of the limits.)

2. $\lim_{x\to a} [f(x) - g(x)] = A - B = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ *Difference Property*
(The limit of a difference is the difference of the limits.)

3. $\lim_{x\to a} [f(x) \cdot g(x)] = A \cdot B = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$ *Product Property*
(The limit of a product is the product of the limits.)

4. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{A}{B} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad (B \neq 0)$ *Quotient Property*
(The limit of a quotient is the quotient of the limits, provided that the limit of the denominator is nonzero.)

5. For any real number for which A^r exists,
$\lim_{x\to a} [f(x)]^r = A^r = [\lim_{x\to a} f(x)]^r$. *Power Property*
(The limit of a power is the power of the limit.)

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Although we won't prove these properties (a rigorous definition of limit is needed for that), you should find most of them plausible. For instance, if the values of $f(x)$ get very close to A and the values of $g(x)$ get very close to B when x approaches a, it is reasonable to expect that the corresponding values of $f(x) + g(x)$ will get very close to $A + B$ (Property 1) and that the corresponding values of $f(x)g(x)$ will get very close to AB (Property 3).

Example 5 **Business** The amount of revenue generated (in billions of dollars) over a 5-year period from General Electric can be approximated by the function $f(x) = -4x^2 + 20x + 150$ where $x = 0$ corresponds to the year 2005. Find $\lim_{x\to 5} f(x)$. (Data from: www.morningstar.com.)

Solution

$$\lim_{x\to 5}(-4x^2 + 20x + 150)$$

$$= \lim_{x\to 5}(-4x^2) + \lim_{x\to 5} 20x + \lim_{x\to 5} 150 \qquad \text{Sum property}$$

$$= \lim_{x\to 5}(-4)\cdot\lim_{x\to 5} x^2 + \lim_{x\to 5} 20\cdot\lim_{x\to 5} x + \lim_{x\to 5} 150 \qquad \text{Product property}$$

$$= \lim_{x\to 5}(-4)\cdot[\lim_{x\to 5} x]^2 + \lim_{x\to 5} 20\cdot\lim_{x\to 5} x + \lim_{x\to 5} 150 \qquad \text{Power property}$$

$$= -4\cdot 5^2 + 20\cdot 5 + 150 = 150 \qquad \text{Constant-function limit and identity-function limit}$$

Example 5 shows that $\lim_{x\to 5} f(x) = 150$, where $f(x) = -4x^2 + 20x + 150$. Note that $f(5) = -4\cdot 5^2 + 20\cdot 5 + 150 = 150$. In other words, the limit as x approaches 5 is the value of the function at 5:

$$\lim_{x\to 5} f(x) = f(5).$$

The same analysis used in Example 5 works with any polynomial function and leads to the following conclusion.

Polynomial Limits

If f is a polynomial function and a is a real number, then

$$\lim_{x\to a} f(x) = f(a).$$

In other words, the limit is the value of the function at $x = a$.

This property will be used frequently. ✓2

Checkpoint 2

If $f(x) = 2x^3 - 5x^2 + 8$, find the given limits.

(a) $\lim_{x\to 2} f(x)$

(b) $\lim_{x\to -3} f(x)$

Example 6 Find each limit.

(a) $\lim_{x\to 2}[(x^2 + 1) + (x^3 - x + 3)]$

Solution

$$\lim_{x\to 2}[(x^2 + 1) + (x^3 - x + 3)]$$

$$= \lim_{x\to 2}(x^2 + 1) + \lim_{x\to 2}(x^3 - x + 3) \qquad \text{Sum property}$$

$$= (2^2 + 1) + (2^3 - 2 + 3) = 5 + 9 = 14. \qquad \text{Polynomial limit}$$

(b) $\lim_{x\to -1}(x^3 + 4x)(2x^2 - 3x)$

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Solution $\lim_{x\to -1}(x^3 + 4x)(2x^2 - 3x)$

$= \lim_{x\to -1}(x^3 + 4x) \cdot \lim_{x\to -1}(2x^2 - 3x)$ Product property

$= [(-1)^3 + 4(-1)] \cdot [2(-1)^2 - 3(-1)]$ Polynomial limit

$= (-1 - 4)(2 + 3) = -25.$

(c) $\lim_{x\to -1} 5(3x^2 + 2)$

Solution $\lim_{x\to -1} 5(3x^2 + 2) = \lim_{x\to -1} 5 \cdot \lim_{x\to -1}(3x^2 + 2)$ Product property

$= 5[3(-1)^2 + 2]$ Constant-function limit and polynomial limit

$= 25.$

(d) $\lim_{x\to 9} \sqrt{4x - 11}$

Solution Begin by writing the square root in exponential form.

$$\lim_{x\to 9} \sqrt{4x - 11} = \lim_{x\to 9} [4x - 11]^{1/2}$$

$= [\lim_{x\to 9}(4x - 11)]^{1/2}$ Power property

$= [4 \cdot 9 - 11]^{1/2}$ Polynomial limit

$= [25]^{1/2} = \sqrt{25} = 5.$ ✓3

Checkpoint 3

Use the limit properties to find the given limits.

(a) $\lim_{x\to 4}(3x - 9)$

(b) $\lim_{x\to -1}(2x^2 - 4x + 1)$

(c) $\lim_{x\to 2} \sqrt{3x + 3}$

When a rational function, such as $f(x) = \dfrac{x}{x + 2}$, is defined at $x = a$, it is easy to find the limit of $f(x)$ as x approaches a.

Example 7 Find

$$\lim_{x\to 5} \frac{x}{x + 3}.$$

Solution $\lim_{x\to 5} \dfrac{x}{x + 3} = \dfrac{\lim_{x\to 5} x}{\lim_{x\to 5}(x + 3)}$ Quotient property

$= \dfrac{5}{5 + 3} = \dfrac{5}{8}.$ Polynomial limit

Note that $f(5) = \dfrac{5}{5 + 3} = \dfrac{5}{8}$. So the limit of $f(x)$ as x approaches 5 is the value of the function at 5:

$$\lim_{x\to 5} f(x) = f(5).$$ ✓4

Checkpoint 4

Find the given limits.

(a) $\lim_{x\to 2} \dfrac{2x - 1}{3x + 4}$

(b) $\lim_{x\to -1} \dfrac{x - 2}{3x - 1}$

The argument used in Example 7 works in the general case and proves the following result.

Rational Limits

If f is a rational function and a is a real number such that $f(a)$ is defined, then

$$\lim_{x\to a} f(x) = f(a).$$

In other words, the limit is the value of the function at $x = a$.

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If a rational function is not defined at the number a, the preceding property cannot be used. Different techniques are needed in such cases to find the limit (if one exists), as we shall see in Examples 8 and 9.

The definition of the limit as x approaches a involves only the values of the function when x is *near* a, but not the value of the function *at* a. So two functions that agree for all values of x, except possibly at $x = a$, will necessarily have the same limit when x approaches a. Thus, we have the following fact.

Limit Theorem

If f and g are functions that have limits as x approaches a, and $f(x) = g(x)$ for all x near a, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

Example 8 Find

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2}.$$

Solution The quotient property cannot be used here, because

$$\lim_{x\to 2} (x - 2) = 0.$$

We can, however, simplify the function by rewriting the fraction as

$$\frac{x^2 + x - 6}{x - 2} = \frac{(x + 3)(x - 2)}{x - 2}.$$

When $x \neq 2$, the quantity $x - 2$ is nonzero and may be cancelled, so that

$$\frac{x^2 + x - 6}{x - 2} = x + 3 \qquad \text{for all } x \neq 2.$$

Now the limit theorem can be used:

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x\to 2} (x + 3) = 2 + 3 = 5.$$ ✓5

Checkpoint 5

Find $\lim_{x\to 1} \dfrac{2x^2 + x - 3}{x - 1}$.

Example 9 Find

$$\lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4}.$$

Solution As $x \to 4$, both the numerator and the denominator approach 0, giving the meaningless expression $0/0$. To change the form of the expression, algebra can be used to rationalize the numerator by multiplying both the numerator and the denominator by $\sqrt{x} + 2$. This gives

$$\frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \frac{\sqrt{x}\cdot\sqrt{x} + 2\sqrt{x} - 2\sqrt{x} - 4}{(x - 4)(\sqrt{x} + 2)}$$

$$= \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} = \frac{1}{\sqrt{x} + 2} \qquad \text{for all } x \neq 4.$$

Now use the limit theorem and the properties of limits:

$$\lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x\to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{2 + 2} = \frac{1}{4}.$$ ✓6

Checkpoint 6

Find the given limits.

(a) $\lim_{x\to 1} \dfrac{\sqrt{x} - 1}{x - 1}$

(b) $\lim_{x\to 9} \dfrac{\sqrt{x} - 3}{x - 9}$

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Existence of Limits

It is possible that $\lim_{x\to a} f(x)$ may not exist; that is, there may be no number L satisfying the definition of $\lim_{x\to a} f(x) = L$. This can happen in many ways, two of which are illustrated next.

Example 10 Given that

$$g(x) = \frac{x^2 + 4}{x - 2}, \text{ find } \lim_{x\to 2} g(x).$$

Solution The quotient property cannot be used, since $\lim_{x\to 2}(x - 2) = 0$, and the limit theorem does not apply because $x^2 + 4$ does not factor. So we use the following table and the graph of g in Figure 11.5:

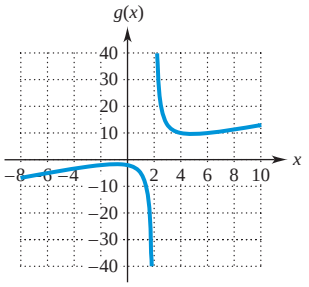

Figure 11.5

	x approaches 2 from the left →				2	← x approaches 2 from the right		
x	1.8	1.9	1.99	1.999	2	2.001	2.01	2.05
$g(x)$	−36.2	−76.1	−796	−7996		8004	804	164
	$g(x)$ gets smaller and smaller →					← $g(x)$ gets larger and larger		

The table and the graph both show that as x approaches 2 from the left, $g(x)$ gets smaller and smaller, but as x approaches 2 from the right, $g(x)$ gets larger and larger. Since $g(x)$ does not get closer and closer to a single real number as x approaches 2 from either side,

$$\lim_{x\to 2} \frac{x^2 + 4}{x - 2} \text{ does not exist.}$$

Checkpoint 7

Let $f(x) = \dfrac{x^2 + 9}{x - 3}$. Find the given limits.

(a) $\lim_{x\to 3} f(x)$

(b) $\lim_{x\to 0} f(x)$

Example 11 What is

$$\lim_{x\to 0} \frac{|x|}{x}?$$

Solution The function $f(x) = \dfrac{|x|}{x}$ is not defined when $x = 0$. Recall the definition of absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Consequently, when $x > 0$,

$$f(x) = \frac{|x|}{x} = \frac{x}{x} = 1,$$

and when $x < 0$,

$$f(x) = \frac{|x|}{x} = \frac{-x}{x} = -1.$$

The graph of f is shown in Figure 11.6. As x approaches 0 from the right, x is always positive, and the corresponding value of $f(x)$ is 1. But as x approaches 0 from the left, x is always negative, and the corresponding value of $f(x)$ is -1. Thus, as x approaches 0 from *both* sides, the corresponding values of $f(x)$ do not get closer and closer to a *single* real number. Therefore, the limit does not exist.*

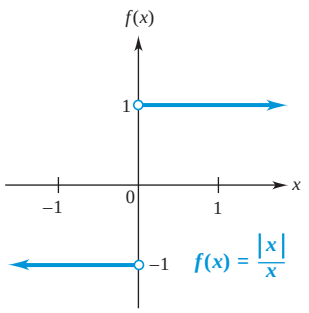

Figure 11.6

*However, the behavior of this function near $x = 0$ can be described by *one-sided limits*, which are discussed in the next section.

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The function f whose graph is shown in Figure 11.7 illustrates various facts about limits that were discussed earlier in this section.

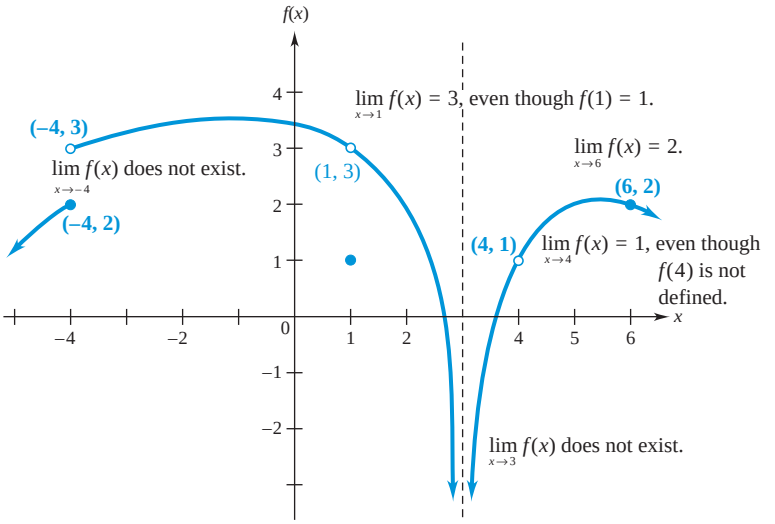
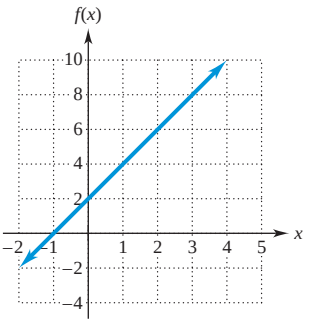


Figure 11.7

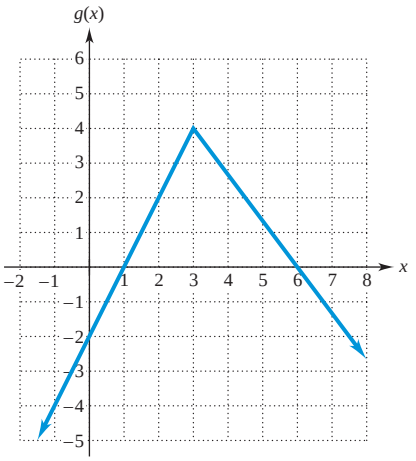
11.1 Exercises

Use the given graph to determine the value of the indicated limits. (See Examples 3, 4, 10, and 11 and Figure 11.7.)

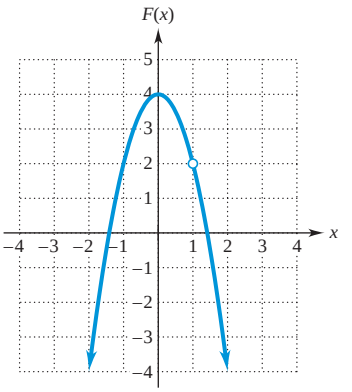
1. **(a)** $\lim_{x\to 3} f(x)$ **(b)** $\lim_{x\to -1} f(x)$



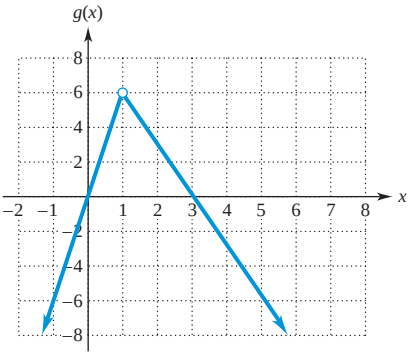
2. **(a)** $\lim_{x\to 3} g(x)$ **(b)** $\lim_{x\to 0} g(x)$



3. **(a)** $\lim_{x\to 0} F(x)$ **(b)** $\lim_{x\to 1} F(x)$



4. **(a)** $\lim_{x\to 1} g(x)$ **(b)** $\lim_{x\to 0} g(x)$



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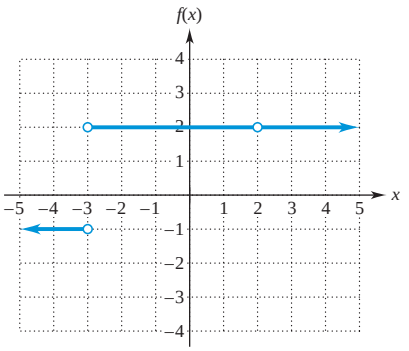
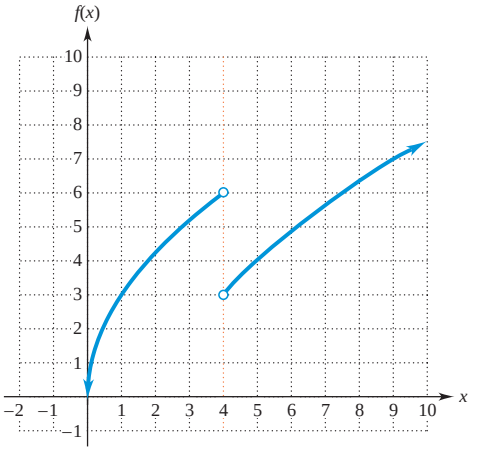
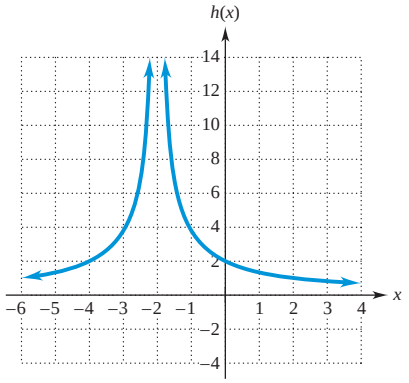
5. **(a)** $\lim_{x\to -3} f(x)$ **(b)** $\lim_{x\to 2} f(x)$



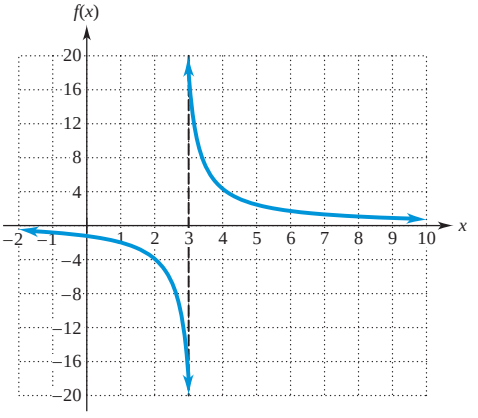
6. **(a)** $\lim_{x\to 4} f(x)$ **(b)** $\lim_{x\to 9} f(x)$



7. **(a)** $\lim_{x\to -2} h(x)$ **(b)** $\lim_{x\to 0} h(x)$



8. **(a)** $\lim_{x\to 3} f(x)$ **(b)** $\lim_{x\to 2} f(x)$



9. Explain why $\lim_{x\to 1} F(x)$ in Exercise 3(b) exists, but $\lim_{x\to -3} f(x)$ in Exercise 5(a) does not.

10. In Exercise 4(a), why does $\lim_{x\to 1} g(x)$ exist even though $g(1)$ is not defined?

Use a calculator to estimate the given limits numerically. (See Examples 1–3.)

11. $\lim_{x\to 1} \dfrac{\ln x}{x-1}$

12. $\lim_{x\to 5} \dfrac{\ln x - \ln 5}{x-5}$

13. $\lim_{x\to 0} (x \ln|x|)$

14. $\lim_{x\to 0} \dfrac{x}{\ln|x|}$

15. $\lim_{x\to 0} \dfrac{e^{3x}-1}{x}$

16. $\lim_{x\to 0} \dfrac{x}{e^x-1}$

17. $\lim_{x\to 3} \dfrac{x^3-3x^2-4x+12}{x-3}$

18. $\lim_{x\to 4} \dfrac{.1x^4-.8x^3+1.6x^2+2x-8}{x-4}$

19. $\lim_{x\to -2} \dfrac{x^4+2x^3-x^2+3x+1}{x+2}$

20. $\lim_{x\to 0} \dfrac{e^{2x}+e^x-2}{e^x-1}$

Suppose $\lim_{x\to 4} f(x) = 25$ ***and*** $\lim_{x\to 4} g(x) = 10$. ***Use the limit properties to find the given limits. (See Examples 6 and 7.)***

21. $\lim_{x\to 4} [f(x) - g(x)]$

22. $\lim_{x\to 4} [g(x)\cdot f(x)]$

23. $\lim_{x\to 4} \dfrac{f(x)}{g(x)}$

24. $\lim_{x\to 4} [3\cdot f(x)]$

25. $\lim_{x\to 4} \sqrt{f(x)}$

26. $\lim_{x\to 4} [g(x)]^3$

27. $\lim_{x\to 4} \dfrac{f(x)+g(x)}{2g(x)}$

28. $\lim_{x\to 4} \dfrac{5g(x)+2}{1-f(x)}$

29. **(a)** Graph the function f whose rule is

$$f(x) = \begin{cases} 3-x & \text{if } x < -2 \\ x+2 & \text{if } -2 \le x < 2. \\ 1 & \text{if } x \ge 2 \end{cases}$$

Use the graph in part (a) to find the following limits:

(b) $\lim_{x\to -2} f(x)$; **(c)** $\lim_{x\to 1} f(x)$; **(d)** $\lim_{x\to 2} f(x)$.

30. **(a)** Graph the function g whose rule is

$$g(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x+2 & \text{if } -1 \le x < 1. \\ 3-x & \text{if } x \ge 1 \end{cases}$$

Use the graph in part (a) to find the following limits:

(b) $\lim_{x\to -1} g(x)$; **(c)** $\lim_{x\to 0} g(x)$; **(d)** $\lim_{x\to 1} g(x)$.

Use algebra and the properties of limits as needed to find the given limits. If the limit does not exist, say so. (See Examples 5–9.)

31. $\lim_{x\to 2} (3x^3 - 4x^2 - 5x + 2)$

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32. $\lim_{x\to -1}(2x^3 - x^2 - 6x + 1)$

33. $\lim_{x\to 3}\dfrac{4x+7}{10x+1}$

34. $\lim_{x\to -2}\dfrac{x+6}{8x-5}$

35. $\lim_{x\to 5}\dfrac{x^2-25}{x-5}$

36. $\lim_{x\to -4}\dfrac{x^2-16}{x+4}$

37. $\lim_{x\to 4}\dfrac{x^2-x-12}{x-4}$

38. $\lim_{x\to 5}\dfrac{x^2-3x-10}{x-5}$

39. $\lim_{x\to 2}\dfrac{x^2-5x+6}{x^2-6x+8}$

40. $\lim_{x\to -2}\dfrac{x^2+3x+2}{x^2-x-6}$

41. $\lim_{x\to 4}\dfrac{(x+4)^2(x-5)}{(x-4)(x+4)^2}$

42. $\lim_{x\to -3}\dfrac{(x+3)(x-3)(x+4)}{(x+8)(x+3)(x-4)}$

43. $\lim_{x\to 3}\sqrt{x^2-3}$

44. $\lim_{x\to 3}\sqrt{x^2-7}$

45. $\lim_{x\to 4}\dfrac{-6}{(x-4)^2}$

46. $\lim_{x\to -2}\dfrac{3x}{(x+2)^3}$

47. $\lim_{x\to 0}\dfrac{[1/(x+3)]-1/3}{x}$

48. $\lim_{x\to 0}\dfrac{[-1/(x+2)]+1/2}{x}$

49. $\lim_{x\to 25}\dfrac{\sqrt{x}-5}{x-25}$

50. $\lim_{x\to 36}\dfrac{\sqrt{x}-6}{x-36}$

51. $\lim_{x\to 5}\dfrac{\sqrt{x}-\sqrt{5}}{x-5}$

52. **(a)** Approximate $\lim_{x\to 0}(1+x)^{1/x}$ to five decimal places. (Evaluate the function at numbers closer and closer to 0 until successive approximations agree in the first five places.)

(b) Find the decimal expansion of the number e to as many places as your calculator can manage.

(c) What do parts (a) and (b) suggest about the exact value of $\lim_{x\to 0}(1+x)^{1/x}$?

Work these exercises.

53. **Business** A company training program has determined that a new employee can process an average of $P(s)$ pieces of work per day after s days of on-the-job training, where

$$P(s) = \frac{105s}{s+7}.$$

Find the given quantities.

(a) $P(1)$ **(b)** $P(13)$ **(c)** $\lim_{s\to 13}P(s)$

54. **Health** The concentration of a drug in a patient's bloodstream h hours after it was injected is given by

$$A(h) = \frac{.3h}{h^2+3}.$$

Find the given quantities.

(a) $A(.5)$ **(b)** $A(1)$ **(c)** $\lim_{h\to 1}A(h)$

55. **Business** An independent driver working for a taxicab company must pay the company \$95 per day and pay for the cost of fuel. The driver's daily cost is approximated by:

$$c(x) = 95 + .20x$$

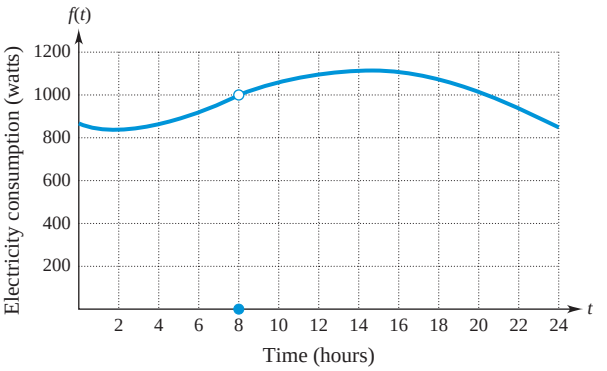
where x is the number of miles driven. The average cost per mile, denoted by $\bar{c}(x)$, is found by dividing $c(x)$ by x. Find and interpret the given quantities.

(a) $\bar{c}(25)$ **(b)** $\bar{c}(100)$ **(c)** $\lim_{x\to 250}\bar{c}(x)$

56. **Natural Science** The electricity consumption, $f(t)$, of a residence is shown in the accompanying figure for a 24-hour day during which the residence experienced a power outage for an instant. Find the given quantities.

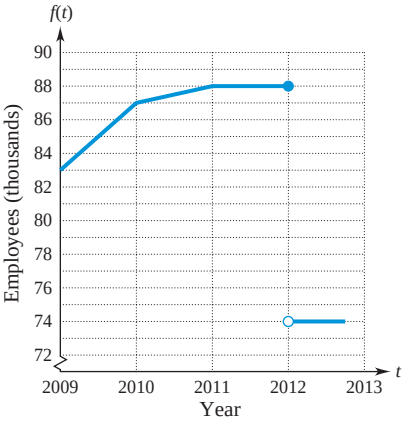
(a) $f(8)$

(b) $\lim_{t\to 8}f(t)$



57. **Business** The number of employees, $f(t)$, at a major airline dropped dramatically in 2012 due to layoffs, as shown in the accompanying figure. Use the graph to find the given limits.

(a) $\lim_{t\to 2010}f(t)$ **(b)** $\lim_{t\to 2011}f(t)$ **(c)** $\lim_{t\to 2012}f(t)$



58. **Economics** A period of U.S. economic expansion began in June 2009, indicated by $t = 0$ in the accompanying figure. Employment figures are indexed to 100 at the start of the expansion. The blue line represents $c(t)$, the employment index during the current economic expansion; the red line represents $a(t)$, the average of the ten previous expansions; and the green line represents $m(t)$, the minimum values from the previous ten

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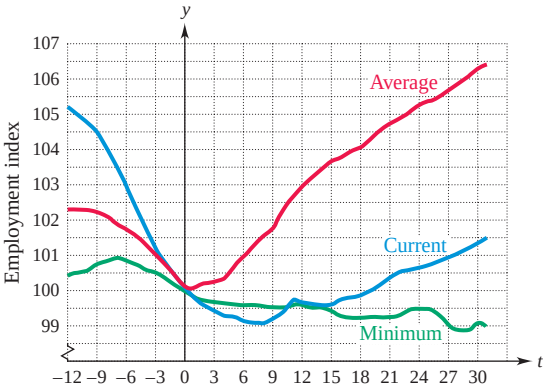
expansions. Use the figure to approximate and interpret the following quantities. (Data from: Federal Reserve Economic Research.)

(a) $\lim_{t\to 6} c(t) - m(t)$ **(b)** $\lim_{t\to 6} c(t) - a(t)$

(c) $\lim_{t\to 12} c(t) - m(t)$ **(d)** $\lim_{t\to 12} c(t) - a(t)$

(e) $\lim_{t\to 27} c(t) - m(t)$ **(f)** $\lim_{t\to 27} c(t) - a(t)$

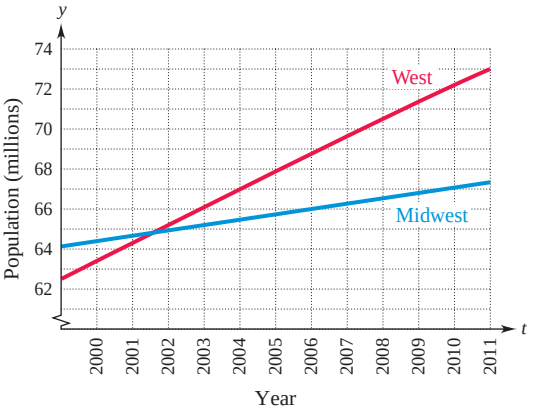


59. Social Science The populations of the Midwestern and Western regions of the United States are shown in the accompanying figure, represented by $M(t)$ and $W(t)$, respectively. Approximate the given limits. (Data from: United States Census Bureau.)

(a) $\lim_{t\to 2000} M(t) - W(t)$

(b) $\lim_{t\to 2010} M(t) - W(t)$

(c) Approximately when are the populations of the two regions equal?



Checkpoint Answers

1. 2.61; 2.9601; 2.996; 3.0004; 3.004; 3.0401; 3.41; the limit appears to be 3.

2. **(a)** 4 **(b)** −91

3. **(a)** 3 **(b)** 7 **(c)** 3

4. **(a)** $\frac{3}{10}$ **(b)** $\frac{3}{4}$ **5.** 5

6. **(a)** $\frac{1}{2}$ **(b)** $\frac{1}{6}$

7. **(a)** Does not exist **(b)** −3

11.2 One-Sided Limits and Limits Involving Infinity

In addition to the limits introduced in Section 11.1, which will be used frequently, there are several other kinds of limits that will appear briefly in Sections 11.9 and 13.3. We begin with one-sided limits.

One-Sided Limits

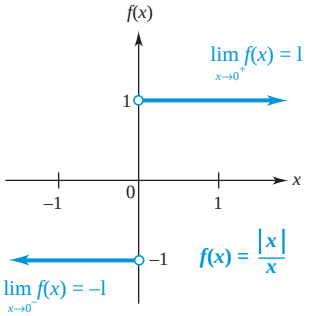


Figure 11.8

Example 11 of Section 11.1 showed that the limit as x approaches 0 of $f(x) = \dfrac{|x|}{x}$ does not exist. However, we can adapt the limit concept to describe the behavior of $f(x)$ near $x = 0$ as follows: First, look at the right half of the graph of $f(x)$ in Figure 11.8.

> As x takes values very close to 0 (with $x > 0$), the corresponding values of $f(x)$ are very close (in fact, equal) to 1.

We express this fact in symbols by writing

$$\lim_{x\to 0^+} f(x) = 1,$$

which is read,

> The limit of $f(x)$ as x approaches 0 from the right is 1.

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Now consider the left half of the graph in Figure 11.8.

As x takes values very close to 0 (with $x < 0$), the corresponding values of $f(x)$ are very close (in fact, equal) to -1.

We express this fact by writing

$$\lim_{x \to 0^-} f(x) = -1,$$

which is read,

The limit of $f(x)$ as x approaches 0 from the left is -1.

The same idea carries over to the general case.

One-Sided Limits

Let f be a function, and let a, K, and M be real numbers. Assume that $f(x)$ is defined for all x near a, with $x > a$. Suppose that

as x takes values very close (but not equal) to a (with $x > a$), the corresponding values of $f(x)$ are very close (and possibly equal) to K,

and that

the values of $f(x)$ can be made as close as you want to K for all values of x (with $x > a$) that are close enough to a.

Then the number K is the **limit of $f(x)$ as x approaches a from the right,** which is written

$$\lim_{x \to a^+} f(x) = K.$$

The statement "M is the **limit of $f(x)$ as x approaches a from the left**," which is written

$$\lim_{x \to a^-} f(x) = M,$$

is defined in a similar fashion. (Just replace K by M and "$x > a$" by "$x < a$" in the previous definition.)

We sometimes refer to a limit of the form $\lim_{x \to a^-} f(x)$ as a **left-hand limit** and a limit of the form $\lim_{x \to a^+} f(x)$ as a **right-hand limit**. The limit $\lim_{x \to a} f(x)$, as defined in Section 11.1, is sometimes called a **two-sided limit**.

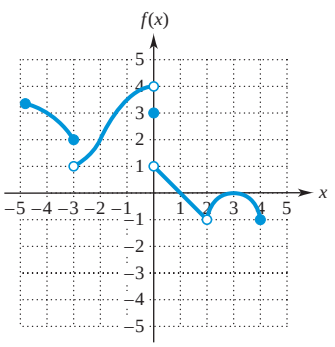

Figure 11.9

Example 1 The graph of a function f is shown in Figure 11.9. Use it to find the following quantities.

(a) $\lim_{x \to -3^-} f(x)$ **(b)** $\lim_{x \to -3^+} f(x)$ **(c)** $\lim_{x \to -3} f(x)$ **(d)** $f(-3)$

(e) For which value(s) of x is $f(x) = -3$?

(f) $\lim_{x \to 2^-} f(x)$ **(g)** $\lim_{x \to 2^+} f(x)$ **(h)** $\lim_{x \to 2} f(x)$ **(i)** $f(2)$

(j) For which value(s) of x is $f(x) = 2$?

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Solution

(a) The graph shows that as x approaches –3 from the left, the corresponding values of $f(x)$ approach 2, therefore $\lim_{x\to -3^-} f(x) = 2$.

(b) As x approaches –3 from the right, the corresponding values of $f(x)$ approach 1, therefore $\lim_{x\to -3^+} f(x) = 1$.

(c) Since the left- and right-hand limits are not equal, the two-sided limit $\lim_{x\to -3} f(x)$ does not exist.

(d) When $x = -3$, the corresponding function value is $f(-3) = 2$.

(e) There are no x-values for which the value of the function is $f(x) = -3$.

In a manner similar to parts (a)–(e), the graph shows the following answers to parts (f)–(j).

(f) $\lim_{x\to 2^-} f(x) = -1$

(g) $\lim_{x\to 2^+} f(x) = -1$

(h) $\lim_{x\to 2} f(x) = -1$

(i) $f(2)$ is undefined.

(j) The x-values corresponding to when $f(x) = 2$ are $x = -3$ and $x = -2$. ✓1

Checkpoint 1

Use Figure 11.9 to find the given quantities.

(a) $\lim_{x\to 0^-} f(x)$

(b) $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 0} f(x)$

(d) $f(0)$

(e) For which value(s) of x is $f(x) = 0$?

In parts (a)–(c) of Example 1, the left- and right-hand limits exist, but are not equal, and the two-sided limit does not exist. In parts (f)–(h), however, all three limits exist and are equal to one another. This case is an example of the following fact.

Two-Sided Limits

Let f be a function, and let a and L be real numbers. Then

$$\lim_{x\to a} f(x) = L \quad \text{exactly when} \quad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L.$$

In other words, f has a two-sided limit at a exactly when it has both a left-hand and right-hand limit at *a and* these two limits are the same.

Although its proof is omitted, the following fact will be used frequently:

All of the properties and theorems about limits in Section 11.1 are valid for left-hand limits and for right-hand limits.

Example 2 Find each of the given limits.

(a) $\lim_{x\to 2^+} \sqrt{4 - x^2}$ and $\lim_{x\to 2^-} \sqrt{4 - x^2}$

Solution Since $f(x) = \sqrt{4 - x^2}$ is not defined when $x > 2$, the right-hand limit (which requires that $x > 2$) does not exist. For the left-hand limit, write the square root in exponential form and apply the appropriate limit properties (see pages 573 and 574).

$$\begin{aligned}
\lim_{x\to 2^-} \sqrt{4 - x^2} &= \lim_{x\to 2^-} (4 - x^2)^{1/2} && \text{Exponential form} \\
&= [\lim_{x\to 2^-} (4 - x^2)]^{1/2} && \text{Power property} \\
&= 0^{1/2} = 0. && \text{Polynomial limit}
\end{aligned}$$

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(b) $\lim_{x\to 3^+} [\sqrt{x-3} + x^2 + 1]$

Solution

$$\lim_{x\to 3^+} [\sqrt{x-3} + x^2 + 1]$$

$$= \lim_{x\to 3^+} [(x-3)^{1/2} + x^2 + 1] \quad \text{Exponential form}$$

$$= \lim_{x\to 3^+} (x-3)^{1/2} + \lim_{x\to 3^+} x^2 + \lim_{x\to 3^+} 1 \quad \text{Sum property}$$

$$= [\lim_{x\to 3^+} (x-3)]^{1/2} + \lim_{x\to 3^+} x^2 + \lim_{x\to 3^+} 1 \quad \text{Power property}$$

$$= 0^{1/2} + 9 + 1 = 10. \quad \text{Polynomial limits}$$ ✓2

Checkpoint 2

Find the given limits.

(a) $\lim_{x\to 3^+} \sqrt{x^2-9}$

(b) $\lim_{x\to -2^-} [\sqrt{x^2-4} + x^2 + 4]$

Example 3 **Business** The cost, $c(x)$, of a New York City taxicab fare during nonpeak hours (assuming no standing time) where x is the distance of the trip (in miles) is \$3 upon entry plus \$.40 per fifth of a mile. Use the accompanying figure to verify that each of the following statements is true. (Data from: www.nyc.gov.)

(a) $c(0) = \$3$ **(b)** $c(1.2) = \$5.40$ **(c)** $\lim_{x\to 1.2^-} c(x) = \$5$

(d) $\lim_{x\to 1.2^+} c(x) = \$5.40$ **(e)** $\lim_{x\to 1.2} c(x)$ does not exist ✓3

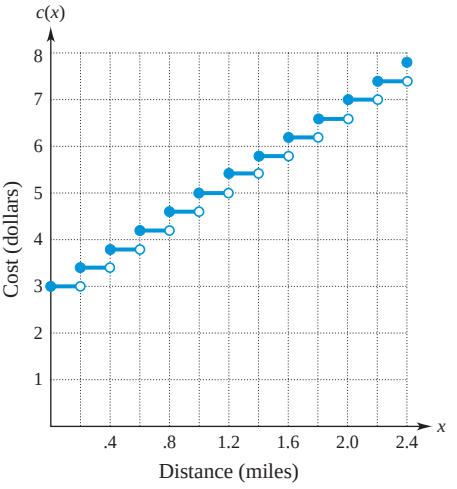

Figure 11.10

Checkpoint 3

Use Figure 11.10 to find the given quantities.

(a) $c(2)$

(b) $\lim_{x\to 2^-} c(x)$

(c) $\lim_{x\to 2^+} c(x)$

(d) $\lim_{x\to 2} c(x)$

Infinite Limits

In the next part of the discussion, it is important to remember that

the symbol ∞, which is usually read "infinity," does *not* represent a real number.

Nevertheless, the word "infinity" and the symbol ∞ are often used as a convenient shorthand to describe the behavior of some functions. Consider the function f whose graph is shown in Figure 11.11 on the following page.

The graph shows that as x approaches 3 (on either side), the corresponding values of $f(x)$ get larger and larger without bound. We express this behavior by writing

$$\lim_{x\to 3} f(x) = \infty,$$

which is read, "The limit of $f(x)$ as x approaches 3 is infinity."

Similarly, as x approaches 1 (on either side), the corresponding values of $f(x)$ get more and more negative (smaller and smaller) without bound. We write

$$\lim_{x\to 1} f(x) = -\infty,$$

which is read, "The limit of $f(x)$ as x approaches 1 is negative infinity."

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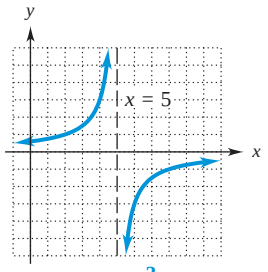
Checkpoint 4

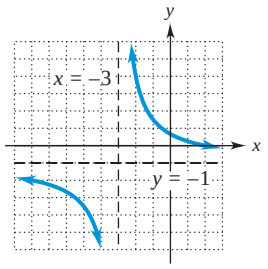
Use "infinite limits" to describe the behavior of the function whose graph is given, near the specified number.

(a) $f(x) = \dfrac{3}{5 - x}$ near $x = 5$



$f(x) = \dfrac{3}{5-x}$

(b) $f(x) = \dfrac{2 - x}{x + 3}$ near $x = -3$



$f(x) = \dfrac{2-x}{x+3}$

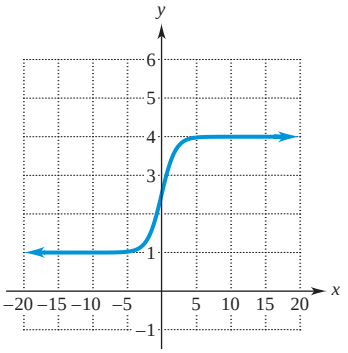


Figure 11.12

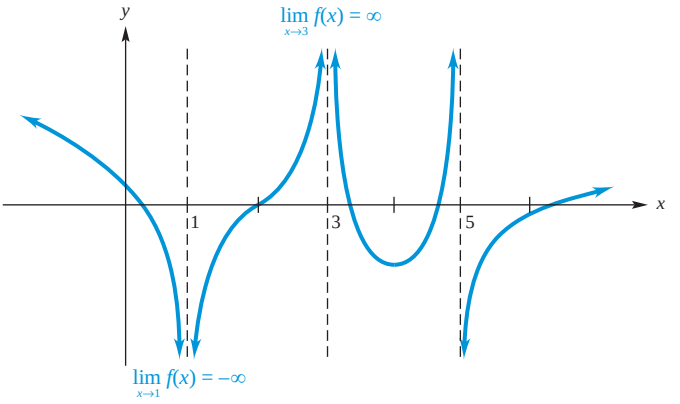


Figure 11.11

The behavior of f near $x = 5$ can be described in a similar fashion with the use of one-sided limits. When x is close to 5, the corresponding values of $f(x)$ get very large on the left side of 5 and very small on the right side of 5, so we write

$$\lim_{x \to 5^-} f(x) = \infty \qquad \text{and} \qquad \lim_{x \to 5^+} f(x) = -\infty$$

and say, "The limit of $f(x)$ as x approaches 5 from the left is infinity" and "The limit of $f(x)$ as x approaches 5 from the right is negative infinity."

In all the situations just described, the language of limits and the word "infinity" are useful for describing the behavior of a function that does not have a limit in the sense discussed in Section 11.1. In particular, this language provides an algebraic way to describe precisely the fact that the function in Figure 11.11 has different types of vertical asymptotes at $x = 1$, $x = 3$, and $x = 5$. ✓4

Limits at Infinity

The word "limit" has been used thus far to describe the values of a function $f(x)$ when x is near a particular number a. Now we consider the behavior of a function when x is very large or very small.

Example 4 Figure 11.12 shows the graph of

$$f(x) = \frac{3}{1 + e^{-x}} + 1.$$

(a) Describe the behavior of f when x is very large.

Solution The right side of the graph appears to coincide with the horizontal line through 4. Actually, as x gets larger and larger, the corresponding values of $f(x)$ are very close (but not equal) to 4, as you can verify with a calculator or by using the trace feature of a graphing calculator (Figure 11.13 on the next page).* We describe this situation by writing

$$\lim_{x \to \infty} f(x) = 4,$$

which is read "The limit of $f(x)$ as x approaches infinity is 4."

(b) Describe the behavior of f when x is very small.

Solution To say that x gets very small means that it moves to the left on the axis (that is, x gets more and more negative). The graph suggests that when x is very small, the

*Because of round-off, a TI-84+ will say that $f(x) = 4$ when $x > 28$. Actually $f(x)$ is always less than 4. Similar remarks apply to other calculators.

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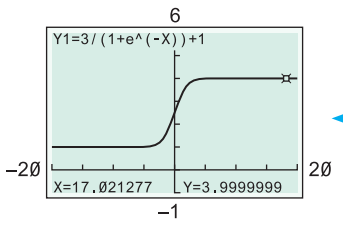
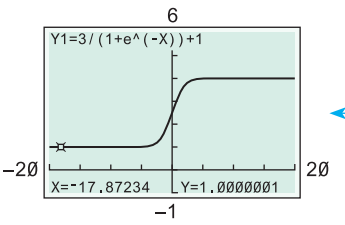


$\lim_{x\to\infty} f(x) = 4$
As x gets larger and larger, the values of $f(x)$ approach 4.

Figure 11.13



$\lim_{x\to-\infty} f(x) = 1$
As x gets more and more negative, the values of $f(x)$ approach 1.

Figure 11.14

corresponding values of $f(x)$ are very close to 1. You can verify this with a calculator or the trace feature on a graphing calculator (Figure 11.14).* We describe this situation by writing

$$\lim_{x\to-\infty} f(x) = 1,$$

which is read, "The limit of $f(x)$ as x approaches negative infinity is 1." ✓5

Once again, the words "infinity" and "negative infinity" and the symbols ∞ and $-\infty$ do *not* denote numbers. They are just a convenient shorthand to express the idea that x takes very large or very small values.

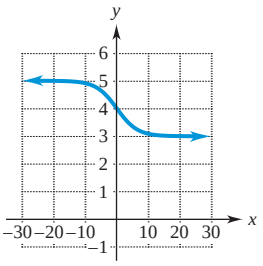
A general definition of the kind of limits illustrated in Example 4 is given next. It is an informal one, as was the definition of limit in Section 11.1.

Checkpoint 5

The graph of a function g is shown. Use the language of limits, as in Example 4, to describe the behavior of g when

(a) x is very small;

(b) x is very large.



Limits at Infinity

Let f be a function that is defined for all large positive values of x, and let L be a real number. Suppose that

> as x takes larger and larger positive values, increasing without bound, the corresponding values of $f(x)$ are very close (and possibly equal) to L

and that

> the values of $f(x)$ can be made arbitrarily close (as close as you want) to L by taking large enough values of x.

Then we say that

the limit of $f(x)$ as x approaches infinity is L,

which is written

$$\lim_{x\to\infty} f(x) = L.$$

Similarly, if f is defined for all small negative values of x and M is a real number, then the statement

$$\lim_{x\to-\infty} f(x) = M,$$

which is read

the limit of $f(x)$ as x approaches negative infinity is M,

means that

> as x takes smaller and smaller negative values, decreasing without bound, the corresponding values of $f(x)$ are very close (and possibly equal) to M

and

> the values of $f(x)$ can be made arbitrarily close (as close as you want) to M by taking small enough values of x.

*Because of round-off, a TI-84+ will say that $f(x) = 1$ when $x < -28$. In fact, $f(x)$ is always more than 1. Similar remarks apply to other calculators.

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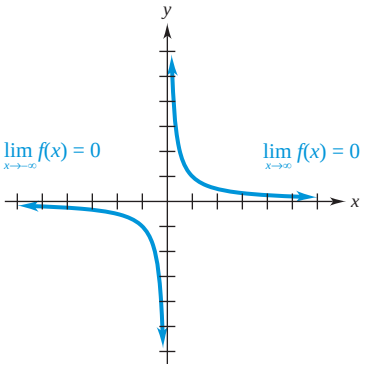

Figure 11.15

Example 5 Find the given limits.

(a) $\lim_{x\to\infty} \frac{1}{x}$.

Solution Make a table of values when x is very large:

x gets larger and larger →

x	50	100	1000	10,000
$1/x$	.02	.01	.001	.0001

→ $\frac{1}{x}$ approaches 0

The table shows that $1/x$ gets very close to 0 as x takes larger and larger values, which suggests that $\lim_{x\to\infty} \frac{1}{x} = 0$. You can reach the same conclusion by examining the right side of the graph of $f(x) = 1/x$ in Figure 11.15.

(b) $\lim_{x\to-\infty} \frac{1}{x}$

Solution Make a table of values when x is very small:

x gets smaller and smaller →

x	−50	−100	−1000	−10,000
$1/x$	−.02	−.01	−.001	−.0001

→ $\frac{1}{x}$ approaches 0

The table shows that $1/x$ gets very close to 0 as x takes smaller and smaller values, which suggests that $\lim_{x\to-\infty} \frac{1}{x} = 0$. The left side of the graph of $f(x) = 1/x$ in Figure 11.15 shows this result graphically.

If the graph of a function approaches a horizontal line very closely when x is very large or very small, we say that this line is a **horizontal asymptote** of the graph. In Example 5, we see that the horizontal line $y = 0$ (the x-axis) is a horizontal asymptote for the graph of $f(x) = \frac{1}{x}$.

Limits at infinity have many of the properties of ordinary limits. Consider, for instance, the constant function $f(x) = 6$. As x gets larger and larger (or smaller and smaller), the corresponding value of $f(x)$ is *always* 6. Hence, $\lim_{x\to\infty} f(x) = 6$ and $\lim_{x\to-\infty} f(x) = 6$. Similarly, for any constant d,

$$\lim_{x\to\infty} d = d \qquad \text{and} \qquad \lim_{x\to-\infty} d = d.$$

Finally, we note the following:

All of the properties of limits in Section 11.1 (page 573) are valid for limits at infinity.

Example 6 Find

$$\lim_{x\to\infty} \frac{4x^2 + 3x}{5x^2 - x + 2}.$$

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Solution Begin by writing the function in a different form. First divide both numerator and denominator by x^2 (which is the highest power of x appearing in either the numerator or denominator).* Then simplify and rewrite the result as follows:

$$\frac{4x^2 + 3x}{5x^2 - x + 2} = \frac{\dfrac{4x^2 + 3x}{x^2}}{\dfrac{5x^2 - x + 2}{x^2}} = \frac{\dfrac{4x^2}{x^2} + \dfrac{3x}{x^2}}{\dfrac{5x^2}{x^2} - \dfrac{x}{x^2} + \dfrac{2}{x^2}} = \frac{4 + \dfrac{3}{x}}{5 - \dfrac{1}{x} + \dfrac{2}{x^2}}$$

$$= \frac{4 + 3 \cdot \dfrac{1}{x}}{5 - \dfrac{1}{x} + 2 \cdot \dfrac{1}{x} \cdot \dfrac{1}{x}}.$$

Now use the limit properties and Example 5 to compute the limit:

$$\lim_{x\to\infty} \frac{4x^2 + 3x}{5x^2 - x + 2} = \lim_{x\to\infty} \frac{4 + 3 \cdot \dfrac{1}{x}}{5 - \dfrac{1}{x} + 2 \cdot \dfrac{1}{x} \cdot \dfrac{1}{x}} \qquad \text{Preceding equation}$$

$$= \frac{\lim\limits_{x\to\infty} \left[4 + 3 \cdot \dfrac{1}{x}\right]}{\lim\limits_{x\to\infty} \left[5 - \dfrac{1}{x} + 2 \cdot \dfrac{1}{x} \cdot \dfrac{1}{x}\right]} \qquad \text{Quotient property}$$

$$= \frac{\lim\limits_{x\to\infty} 4 + \lim\limits_{x\to\infty} \left(3 \cdot \dfrac{1}{x}\right)}{\lim\limits_{x\to\infty} 5 - \lim\limits_{x\to\infty} \dfrac{1}{x} + \lim\limits_{x\to\infty} \left(2 \cdot \dfrac{1}{x} \cdot \dfrac{1}{x}\right)} \qquad \text{Sum and difference properties}$$

$$= \frac{\lim\limits_{x\to\infty} 4 + \lim\limits_{x\to\infty} 3 \cdot \lim\limits_{x\to\infty} \dfrac{1}{x}}{\lim\limits_{x\to\infty} 5 - \lim\limits_{x\to\infty} \dfrac{1}{x} + \lim\limits_{x\to\infty} 2 \cdot \lim\limits_{x\to\infty} \dfrac{1}{x} \cdot \lim\limits_{x\to\infty} \dfrac{1}{x}} \qquad \text{Product property}$$

$$= \frac{4 + 3 \cdot 0}{5 - 0 + 2 \cdot 0 \cdot 0} = \frac{4}{5}. \qquad \text{Constant-function limits and Example 5}$$

Example 7 Find

$$\lim_{x\to-\infty} \frac{2x^2}{3 - x^3}.$$

Solution Divide both numerator and denominator by x^3, the highest power of x appearing in either one.* Then simplify and rewrite:

$$\frac{2x^2}{3 - x^3} = \frac{\dfrac{2x^2}{x^3}}{\dfrac{3 - x^3}{x^3}} = \frac{\dfrac{2x^2}{x^3}}{\dfrac{3}{x^3} - \dfrac{x^3}{x^3}} = \frac{\dfrac{2}{x}}{\dfrac{3}{x^3} - 1} = \frac{2 \cdot \dfrac{1}{x}}{3 \cdot \dfrac{1}{x} \cdot \dfrac{1}{x} \cdot \dfrac{1}{x} - 1}.$$

*In Examples 6 and 7, we are interested only in nonzero values of x, and dividing both numerator and denominator by a nonzero quantity will not change the value of the function.

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Now proceed as in Example 6:

$$\lim_{x\to-\infty}\frac{2x^2}{3-x^3}=\lim_{x\to-\infty}\frac{2\cdot\frac{1}{x}}{3\cdot\frac{1}{x}\cdot\frac{1}{x}\cdot\frac{1}{x}-1}$$ Preceding equation

$$=\frac{\lim\limits_{x\to-\infty}\left[2\cdot\frac{1}{x}\right]}{\lim\limits_{x\to-\infty}\left[3\cdot\frac{1}{x}\cdot\frac{1}{x}\cdot\frac{1}{x}-1\right]}$$ Quotient property

$$=\frac{\lim\limits_{x\to-\infty}\left[2\cdot\frac{1}{x}\right]}{\lim\limits_{x\to-\infty}\left[3\cdot\frac{1}{x}\cdot\frac{1}{x}\cdot\frac{1}{x}\right]-\lim\limits_{x\to-\infty}1}$$ Difference property

$$=\frac{\lim\limits_{x\to-\infty}2\cdot\lim\limits_{x\to-\infty}\frac{1}{x}}{\lim\limits_{x\to-\infty}3\cdot\lim\limits_{x\to-\infty}\frac{1}{x}\cdot\lim\limits_{x\to-\infty}\frac{1}{x}\cdot\lim\limits_{x\to-\infty}\frac{1}{x}-\lim\limits_{x\to-\infty}1}$$ Product property

$$=\frac{2\cdot0}{3\cdot0\cdot0\cdot0-1}=\frac{0}{-1}=0.$$ Constant-function limits and Example 5 ✓6

Checkpoint 6

Find the given limits.

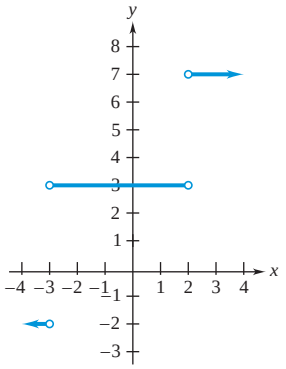
(a) $\lim\limits_{x\to-\infty}\dfrac{5x^3}{2x^3-x+4}$

(b) $\lim\limits_{x\to\infty}\dfrac{10x^3-9x}{28+11x^4}$

The technique in Examples 6 and 7 carries over to any rational function $f(x)=g(x)/h(x)$, where the degree of $g(x)$ is less than or equal to the degree of $h(x)$. These limits provide a mathematical justification for the treatment of horizontal asymptotes in Section 3.7.

11.2 Exercises

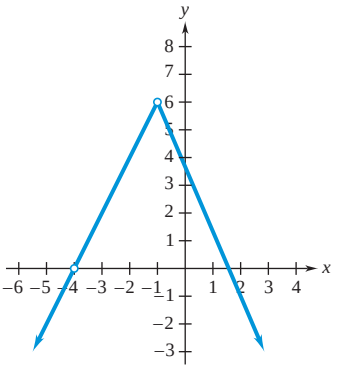
The graph of the function f is shown. Use it to compute the given limits. (See Example 1.)



1. $\lim\limits_{x\to2^-}f(x)$ **2.** $\lim\limits_{x\to2^+}f(x)$

3. $\lim\limits_{x\to-3^-}f(x)$ **4.** $\lim\limits_{x\to-3^+}f(x)$

The graph of the function f is shown. Use it to compute the given limits. (See Example 1.)



5. $\lim\limits_{x\to-1^-}f(x)$ **6.** $\lim\limits_{x\to-1^+}f(x)$

7. $\lim\limits_{x\to-4^-}f(x)$ **8.** $\lim\limits_{x\to-4^+}f(x)$

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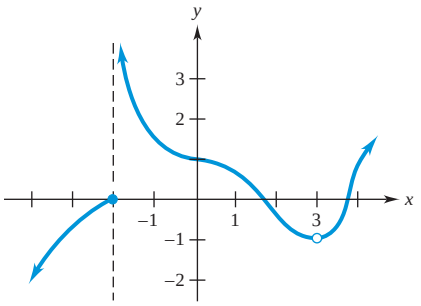
In Exercises 9–12, use the graph of the function f to determine the given limits. (See Example 1 and Figure 11.11.)

(a) $\lim_{x\to -2^-} f(x)$ **(b)** $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 3^-} f(x)$ **(d)** $\lim_{x\to 3^+} f(x)$

9.



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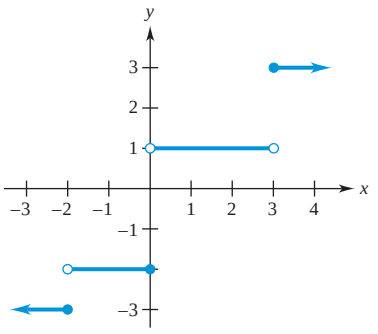

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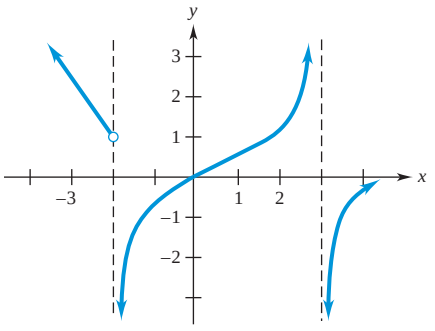

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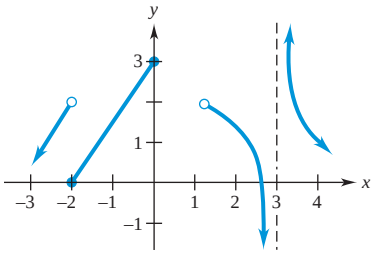


Find the given limits. (See Example 2.)

13. $\lim_{x\to 2^+} \sqrt{x^2 - 4}$

14. $\lim_{x\to 3^-} \sqrt{9 - x^2}$

15. $\lim_{x\to 0^+} \sqrt{x} + x + 1$

16. $\lim_{x\to 1^+} \sqrt{x - 1} + 1$

17. $\lim_{x\to -2^+} (x^3 - x^2 - x + 1)$

18. $\lim_{x\to -2^-} (2x^3 + 3x^2 + 5x + 2)$

19. $\lim_{x\to -5^+} \dfrac{\sqrt{x + 5} + 5}{x^2 - 5}$

20. $\lim_{x\to 1^+} \dfrac{x^3 - 2}{\sqrt{x^3 - 1} + 2}$

21. Graph the function f whose rule is

$$f(x) = \begin{cases} 3 - x & \text{if } x < -2 \\ x + 2 & \text{if } -2 \le x < 2 \\ 1 & \text{if } x = 2 \\ 4 - x & \text{if } x > 2 \end{cases}$$

Use the graph to evaluate the given limits.

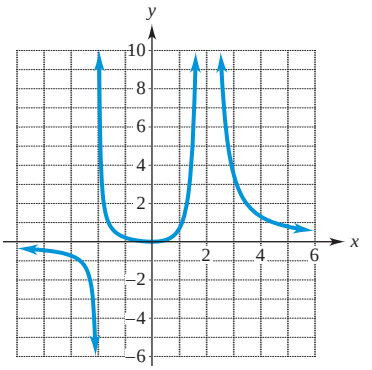
(a) $\lim_{x\to -2^-} f(x)$ **(b)** $\lim_{x\to -2^+} f(x)$ **(c)** $\lim_{x\to -2} f(x)$

22. Let f be the function in Exercise 21. Evaluate the given limits.

(a) $\lim_{x\to 2^-} f(x)$ **(b)** $\lim_{x\to 2^+} f(x)$ **(c)** $\lim_{x\to 2} f(x)$

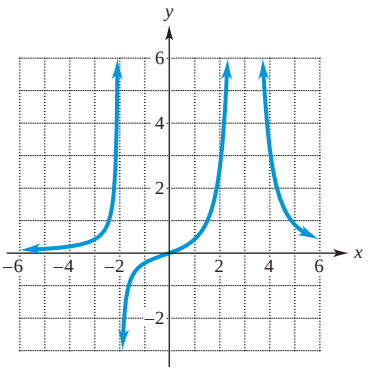
23. For the function f whose graph is shown, find

(a) $\lim_{x\to 2} f(x)$; **(b)** $\lim_{x\to -2^+} f(x)$; **(c)** $\lim_{x\to -2^-} f(x)$.



24. For the function g whose graph is shown, find

(a) $\lim_{x\to 3} g(x)$; **(b)** $\lim_{x\to -2^+} g(x)$; **(c)** $\lim_{x\to -2^-} g(x)$.



Use a calculator to estimate the given limits.

25. $\lim_{x\to 4} \dfrac{-6}{(x - 4)^2}$

26. $\lim_{x\to -2} \dfrac{2x}{(x + 2)^2}$

27. $\lim_{x\to -3^+} \dfrac{2 - x}{x + 3}$

28. $\lim_{x\to -1^+} \dfrac{2}{1 + x}$

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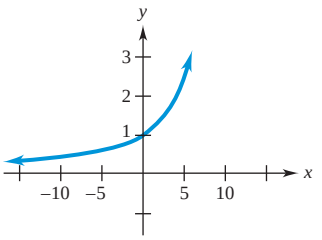
Use the given graph of the function f to find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. (See Example 4.)

29.



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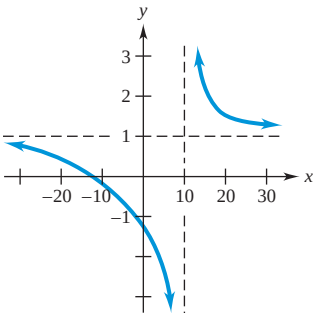

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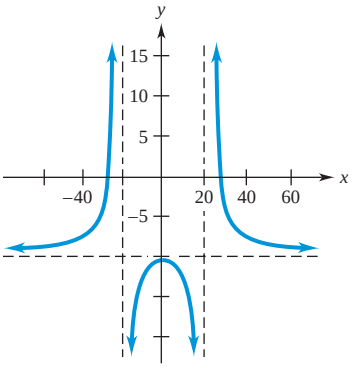

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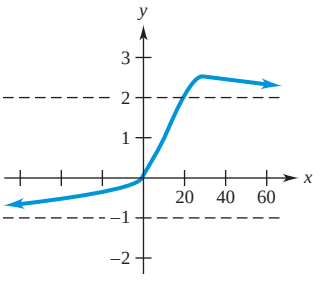

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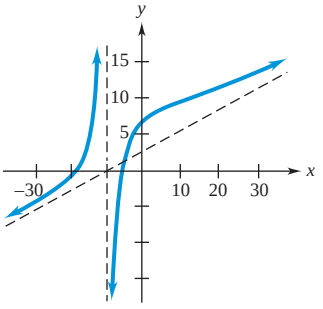

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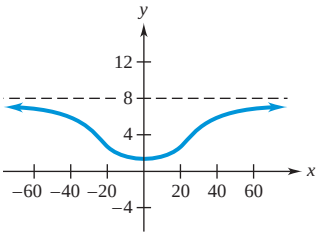

Use a calculator to estimate the limits given numerically. (See Example 5.)

35. $\lim_{x\to\infty} [\sqrt{x^2 + 1} - (x + 1)]$

36. $\lim_{x\to-\infty} [\sqrt{x^2 + x + 1} + x]$

37. $\lim_{x\to-\infty} \dfrac{x^{2/3} - x^{4/3}}{x^3}$

38. $\lim_{x\to\infty} \dfrac{x^{5/4} + x}{2x - x^{5/4}}$

39. $\lim_{x\to-\infty} e^{1/x}$

40. $\lim_{x\to\infty} \dfrac{e^{1/x}}{x}$

41. $\lim_{x\to\infty} \dfrac{\ln x}{x}$

42. $\lim_{x\to\infty} \dfrac{5}{1 + (1.1)^{-x/20}}$

Use the properties of limits to find the given limits. (See Examples 6 and 7.)

43. $\lim_{x\to\infty} \dfrac{10x^2 - 5x + 8}{3x^2 + 8x - 29}$

44. $\lim_{x\to\infty} \dfrac{44x^2 - 12x + 82}{11x^2 - 35x - 37}$

45. $\lim_{x\to-\infty} \dfrac{5x + 9x^2 - 17}{21x + 5x^3 + 12x^2}$

46. $\lim_{x\to-\infty} \dfrac{11x + 21}{7 + 67x - x^2}$

47. $\lim_{x\to-\infty} \dfrac{8x^5 + 7x^4 - 10x^3}{25 - 4x^5}$

48. $\lim_{x\to\infty} \dfrac{7x^4 + 3x^3 - 12x^2 + x - 76}{x^4 - 33x^3 - 17x^2 + 6}$

49. $\lim_{x\to-\infty} \dfrac{(4x - 1)(x + 2)}{5x^2 - 3x + 1}$

50. $\lim_{x\to\infty} \dfrac{(3x - 1)(4x + 3)}{6x^2 - x + 7}$

51. $\lim_{x\to\infty} \left(\dfrac{1}{4x^2}\right)$

52. $\lim_{x\to-\infty} (7x^2 + 2)^{-2}$

53. $\lim_{x\to-\infty} \left(\dfrac{18x}{x - 9} - \dfrac{3x}{x + 4}\right)$

54. $\lim_{x\to\infty} \left(\dfrac{2x}{x^2 + 2} + \dfrac{17x}{x^3 - 11}\right)$

Use the definition of absolute value to find the given limits.

55. $\lim_{x\to\infty} \dfrac{x}{|x|}$

56. $\lim_{x\to-\infty} \dfrac{x}{|x|}$

57. $\lim_{x\to-\infty} \dfrac{x}{|x| + 1}$

58. Use the change-of-base formula for logarithms (in Section 4.3) to show that $\lim_{x\to\infty} \dfrac{\ln x}{\log x} = \ln 10$.

Work these exercises.

59. **Business** A metropolitan waterworks normally charges a flat monthly rate of \$20 for up to 4000 gallons of water

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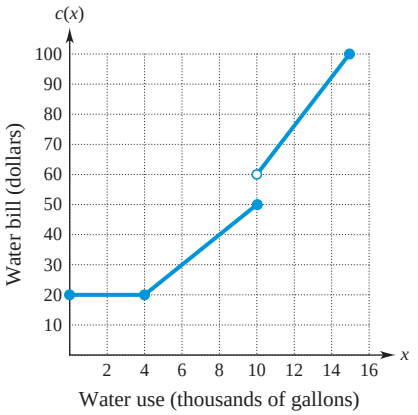
consumption and a rate of $5 per thousand gallons of water for consumption in excess of 4000 gallons. To encourage conservation during a severe drought, the waterworks imposed an excessive use surcharge and an additional rate increase for residential customers who consumed more than 10,000 gallons of water in a month. Use the accompanying graph, which shows the monthly cost, $c(x)$, of consuming x gallons of water to find the given quantities.

(a) $\lim_{x\to 4^-} c(x)$ **(b)** $\lim_{x\to 4^+} c(x)$ **(c)** $\lim_{x\to 4} c(x)$

(d) $\lim_{x\to 10^-} c(x)$ **(e)** $\lim_{x\to 10^+} c(x)$ **(f)** $\lim_{x\to 10} c(x)$

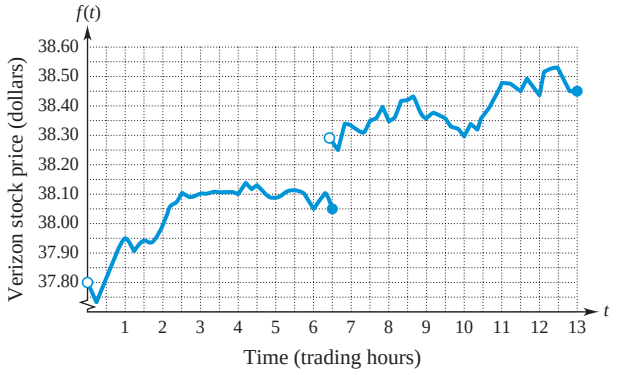
(g) What rate does the waterworks charge for consumption in excess of 10,000 gallons?

(h) What is the amount of the surcharge for customers who consume more than 10,000 gallons of water?



60. Finance The opening price of a stock typically differs somewhat from the closing price the previous day. The accompanying graph shows the stock price for Verizon Communications, $f(t)$, over two trading days where $t = 0$ corresponds to the opening of trading on February 16, 2012. Each trading day lasts 6.5 hours, beginning at 9:30 a.m. and concluding at 4:00 p.m. Eastern time. Since t represents the time (in hours) that the market has been open for trading, $t = 13$ represents the close of trading on February 17, 2012. Approximate and interpret the given limits. (Data from: www.morningstar.com.)

(a) $\lim_{t\to 6.5^-} f(t)$ **(b)** $\lim_{t\to 6.5^+} f(t)$ **(c)** $\lim_{t\to 6.5} f(t)$

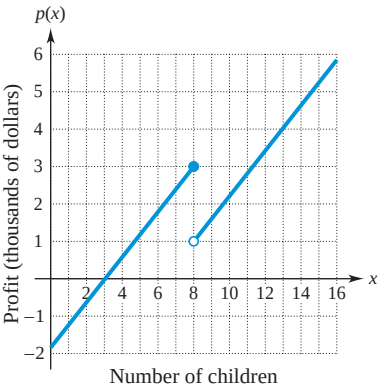


61. Business According to the Virginia Department of Social Services, an individual daycare provider in Virginia may care for up to eight children (ages two to four). If a daycare provider hires an additional worker, it increases capacity but also incurs additional expense. The accompanying graph approximates the monthly profit, $p(x)$, from providing care for x children. Find the given quantities.

(a) $\lim_{x\to 8^-} p(x)$ **(b)** $\lim_{x\to 8^+} p(x)$ **(c)** $\lim_{x\to 8} p(x)$

(d) How many children must be cared for to break even (zero profit)?

(e) How many children must be cared for to make hiring a second provider beneficial?



62. Business The cost of a first-class stamp increased slowly between 1919 and 1968 and has increased more rapidly since 1968. The cost (in cents) can be approximated by

$$c(x) = \begin{cases} .08x + .56 & \text{if } 19 \le x \le 68 \\ .89x - 54.52 & \text{if } x > 68 \end{cases},$$

where $x = 0$ corresponds to the year 1900. Find the given limits. (Data from: United States Postal Service.)

(a) $\lim_{x\to 68^-} c(x)$ **(b)** $\lim_{x\to 68^+} c(x)$ **(c)** $\lim_{x\to 68} c(x)$

63. Business An aquarium parking garage charges $3 for up to and including one hour and $2 for each additional hour (or fraction of an hour), up to a daily maximum of $15. Let $c(t)$ represent the daily cost of parking, where t is the number of hours parked. Find the given quantities.

(a) $c(1.5)$ **(b)** $c(2)$ **(c)** $c(2.5)$

(d) $c(3)$ **(e)** $c(8)$ **(f)** $\lim_{t\to 1.5} c(t)$

(g) $\lim_{t\to 2^-} c(t)$ **(h)** $\lim_{t\to 2^+} c(t)$ **(i)** $\lim_{t\to 2} c(t)$

(j) $\lim_{t\to 8} c(t)$

64. Natural Science If you travel from Birmingham, Alabama to Atlanta, Georgia at an average speed of r miles per hour, the time (in hours) it takes to complete the trip is given by $T(r) = \dfrac{150}{r}$. Find and interpret the given limits.

(a) $\lim_{r\to 75} T(r)$ **(b)** $\lim_{r\to \infty} T(r)$ **(c)** $\lim_{r\to 0^+} T(r)$

65. Business The cost of printing a certain paperback book using a publish-on-demand service is $c(x) = 6000 + 5x$, where x is the number of books printed. The average cost per book, denoted by $\bar{c}(x)$, is found by dividing $c(x)$ by x. Find and interpret the given quantities.

(a) $\bar{c}(100)$ **(b)** $\bar{c}(1000)$ **(c)** $\lim_{x\to \infty} \bar{c}(x)$

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66. Health The concentration of a drug in a patient's bloodstream h hours after it was injected is given by

$$A(h) = \frac{.17h}{h^2 + 2}.$$

Find and interpret $\lim_{h\to\infty} A(h)$.

67. Natural Science Researchers have developed a mathematical model that can be used to estimate the number of teeth, $N(t)$, at time t (days of incubation) for *Alligator mississippiensis*[*]:

$$N(t) = 71.8e^{-8.96e^{-.0685t}}.$$

(a) Find $N(65)$, the number of teeth of an alligator that hatched after 65 days.

(b) Find $\lim_{t\to\infty} N(t)$, and use this value as an estimate of the number of teeth of a newborn alligator. Does this estimate differ significantly from the estimate of part (a)?

[*]Kulesa, P., G. Cruywagen, et al. "On a Model Mechanism for the Spatial Patterning of Teeth Primordia in the Alligator," *Journal of Theoretical Biology*, Vol. 180, 1996, pp. 287–296.

Checkpoint Answers

1. **(a)** 4 **(b)** 1 **(c)** Does not exist **(d)** 3 **(e)** $x = 1$ and $x = 3$

2. **(a)** 0 **(b)** 8

3. **(a)** \$7.00 **(b)** \$6.60 **(c)** \$7.00 **(d)** Does not exist

4. **(a)** $\lim_{x\to 5^-} \frac{3}{5-x} = \infty$ and $\lim_{x\to 5^+} \frac{3}{5-x} = -\infty$

(b) $\lim_{x\to -3^-} \frac{2-x}{x+3} = -\infty$ and $\lim_{x\to -3^+} \frac{2-x}{x+3} = \infty$

5. **(a)** $\lim_{x\to-\infty} g(x) = 5$ **(b)** $\lim_{x\to\infty} g(x) = 3$

6. **(a)** $\frac{5}{2}$ **(b)** 0

11.3 Rates of Change

One of the main applications of calculus is determining how one variable changes in relation to another. A person in business wants to know how profit changes with respect to changes in advertising, while a person in medicine wants to know how a patient's reaction to a drug changes with respect to changes in the dose.

We begin the discussion with a familiar situation. A driver makes the 168-mile trip from Cleveland to Columbus, Ohio, in 3 hours. The following table shows how far the driver has traveled from Cleveland at various times:

Time (in hours)	0	.5	1	1.5	2	2.5	3
Distance (in miles)	0	22	52	86	118	148	168

If f is the function whose rule is

$$f(x) = \text{distance from Cleveland at time } x,$$

then the table shows, for example, that $f(2) = 118$ and $f(3) = 168$. So the distance traveled from time $x = 2$ to $x = 3$ is $168 - 118$—that is, $f(3) - f(2)$. In a similar fashion, we obtain the other entries in the following chart:

Time Interval	**Distance Traveled**
$x = 2$ to $x = 3$	$f(3) - f(2) = 168 - 118 = 50$ miles
$x = 1$ to $x = 3$	$f(3) - f(1) = 168 - 52 = 116$ miles
$x = 0$ to $x = 2.5$	$f(2.5) - f(0) = 148 - 0 = 148$ miles
$x = .5$ to $x = 1$	$f(1) - f(.5) = 52 - 22 = 30$ miles
$x = a$ to $x = b$	$f(b) - f(a)$ miles

The last line of the chart shows how to find the distance traveled in any time interval $(0 \le a < b \le 3)$.

Since distance = average speed × time,

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time interval}}.$$

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In the preceding chart, you can compute the length of each time interval by taking the difference between the two times. For example, from $x = 1$ to $x = 3$ is a time interval of length $3 - 1 = 2$ hours; hence, the average speed over this interval is $116/2 = 58$ mph. Similarly, we have the following information:

Time Interval	Average Speed $= \dfrac{\text{Distance Traveled}}{\text{Time Interval}}$
$x = 2$ to $x = 3$	$\dfrac{f(3) - f(2)}{3 - 2} = \dfrac{168 - 118}{3 - 2} = \dfrac{50}{1} = 50$ mph
$x = 1$ to $x = 3$	$\dfrac{f(3) - f(1)}{3 - 1} = \dfrac{168 - 52}{3 - 1} = \dfrac{116}{2} = 58$ mph
$x = 0$ to $x = 2.5$	$\dfrac{f(2.5) - f(0)}{2.5 - 0} = \dfrac{148 - 0}{2.5 - 0} = \dfrac{148}{2.5} = 59.2$ mph
$x = .5$ to $x = 1$	$\dfrac{f(1) - f(.5)}{1 - .5} = \dfrac{52 - 22}{1 - .5} = \dfrac{30}{.5} = 60$ mph
$x = a$ to $x = b$	$\dfrac{f(b) - f(a)}{b - a}$ mph

The last line of the chart shows how to compute the average speed over any time interval ($0 \le a < b \le 3$). ✓1

Now, speed (miles per hour) is simply the *rate of change* of distance with respect to time, and what was done for the distance function f in the preceding discussion can be done with any function.

Quantity	Meaning for the Distance Function	Meaning for an Arbitrary Function f
$b - a$	Time interval = change in time from $x = a$ to $x = b$	Change in x from $x = a$ to $x = b$
$f(b) - f(a)$	Distance traveled = corresponding change in distance as time changes from a to b	Corresponding change in $f(x)$ as x changes from a to b
$\dfrac{f(b) - f(a)}{b - a}$	**Average speed** = average rate of change of distance with respect to time as time changes from a to b	**Average rate of change** of $f(x)$ with respect to x as x changes from a to b (where $a < b$)

Checkpoint 1

Find the average speed

(a) from $t = 1.5$ to $t = 2$;

(b) from $t = s$ to $t = r$.

Example 1 If $f(x) = x^2 + 4x + 5$, find the average rate of change of $f(x)$ with respect to x as x changes from -2 to 3.

Solution This is the situation described in the last line of the preceding chart, with $a = -2$ and $b = 3$. The average rate of change is

$$\frac{f(3) - f(-2)}{3 - (-2)} = \frac{(3)^2 + 4(3) + 5 - [(-2)^2 + 4(-2) + 5]}{3 - (-2)} = \frac{26 - 1}{5}$$

$$= \frac{25}{5} = 5. \quad ✓2$$

Checkpoint 2

Find the average rate of change of $f(x)$ in Example 1 when x changes from

(a) 0 to 4;

(b) 2 to 7.

Example 2 **Finance** Suppose $D(t)$ is the value of the Dow Jones Industrial Average (DJIA) at time t. Figure 11.16 on the following page shows the values of $D(t)$ for the end of the given year over a twelve-year period. (Data from: www.morningstar.com.)

Approximate the average rate of change of the DJIA with respect to time over the given intervals.

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Year	DJIA
2000	10,787
2001	10,022
2002	8342
2003	10,454
2004	10,783
2005	10,718
2006	12,463
2007	13,265
2008	8776
2009	10,428
2010	11,578
2011	12,218

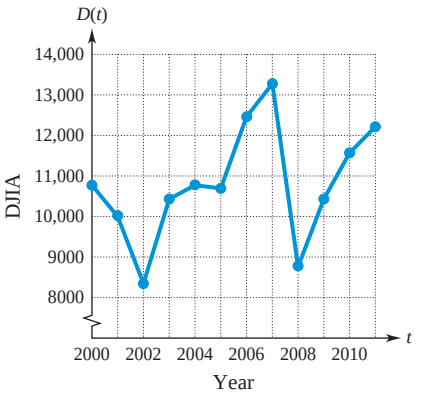

Figure 11.16

(a) From the end of 2007 to the end of 2008

Solution The table says that the DJIA stood at 13,265 at the end of 2007 and 8776 at the end of 2008; that is $D(2007) = 13{,}265$ and $D(2008) = 8776$. The average rate of change over this period is

$$\frac{D(2008) - D(2007)}{2008 - 2007} = \frac{8776 - 13{,}265}{2008 - 2007} = \frac{-4489}{1} = -4489.$$

The negative number implies that the DJIA was decreasing at an average rate of 4489 index points per year.

(b) From the end of 2008 to the end of 2011

Solution The table shows that $D(2011) = 12{,}218$, and we saw earlier that $D(2008) = 8776$. The average rate of change is

$$\frac{D(2011) - D(2008)}{2011 - 2008} = \frac{12{,}218 - 8776}{2011 - 2008} = \frac{3442}{3} \approx 1147.3.$$

Thus the Dow was increasing at an average rate of 1147.3 index points per year during this period. ✓3

Checkpoint 3

Use Figure 11.16 (and the corresponding table of values) to find the average rate of change of the DJIA with respect to time over the given interval.

(a) From the end of 2005 to the end of 2011

(b) From the end of 2006 to the end of 2011

Example 2 also illustrates the geometric interpretation of the average rate of change. In part (b), for instance,

$$\text{average rate of change from 2008 to 2011} = \frac{D(2011) - D(2008)}{2011 - 2008} = \frac{12{,}218 - 8776}{2011 - 2008} \approx 1147.3 \frac{\text{index points}}{\text{year}}.$$

This last number is precisely the *slope* of the line from (2008, 8776) to (2011, 12,218) on the graph of D, as shown in red in Figure 11.17. The same thing is true in general.

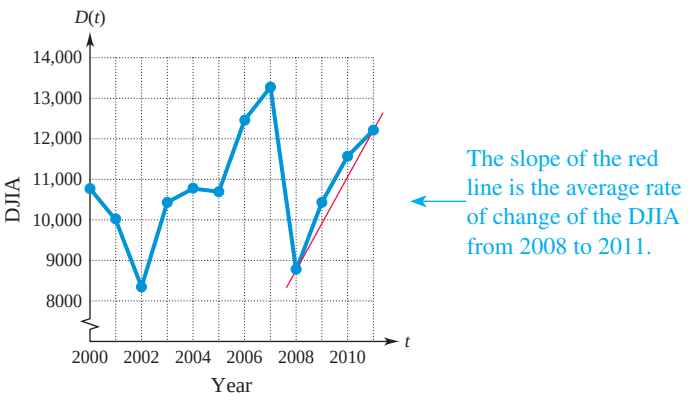

Figure 11.17

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Geometric Meaning of Average Rate of Change

If f is a function, then the average rate of change of $f(x)$ with respect to x, as x changes from a to b, namely,

$$\frac{f(b) - f(a)}{b - a},$$

is the slope of the line joining the points $(a, f(a))$ and $(b, f(b))$ on the graph of f. Examples are shown in Figure 11.18.

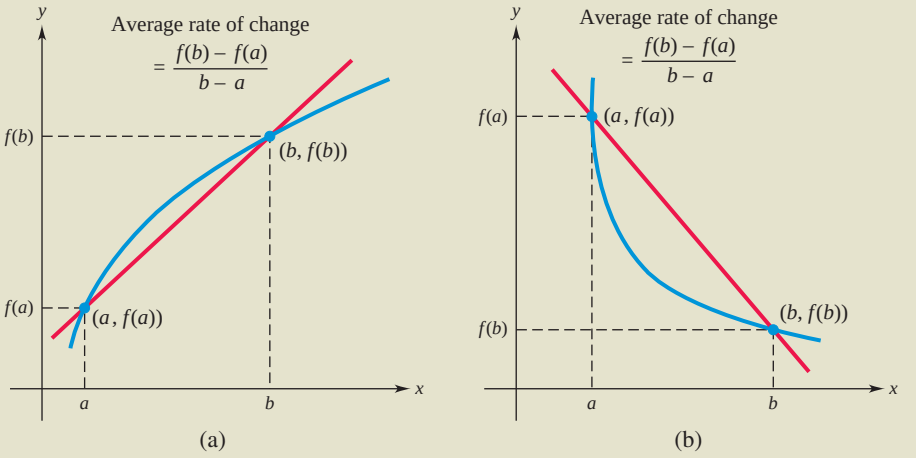

Figure 11.18

Instantaneous Rate of Change

Suppose a car is stopped at a traffic light. When the light turns green, the car begins to move along a straight road. Assume that the distance traveled by the car is given by the function

$$s(t) = 2t^2 \qquad (0 \le t \le 30),$$

where time t is measured in seconds and the distance $s(t)$ at time t is measured in feet. We know how to find the *average* speed of the car over any time interval, so we now turn to a different problem: determining the *exact* speed of the car at a particular instant—say, $t = 10$.*

The intuitive idea is that the exact speed at $t = 10$ is very close to the average speed over a very short time interval near $t = 10$. If we take shorter and shorter time intervals near $t = 10$, the average speeds over these intervals should get closer and closer to the exact speed at $t = 10$. In other words,

> the exact speed at $t = 10$ is the limit of the average speeds over shorter and shorter time intervals near $t = 10$.

The following chart illustrates this idea:

Intervals	Average Speed
$t = 10$ to $t = 10.1$	$\frac{s(10.1) - s(10)}{10.1 - 10} = \frac{204.02 - 200}{.1} = 40.2$ ft/sec
$t = 10$ to $t = 10.01$	$\frac{s(10.01) - s(10)}{10.01 - 10} = \frac{200.4002 - 200}{.01} = 40.02$ ft/sec
$t = 10$ to $t = 10.001$	$\frac{s(10.001) - s(10)}{10.001 - 10} = \frac{200.040002 - 200}{.001} = 40.002$ ft/sec

*As distance is measured in feet and time in seconds here, speed is measured in feet per second. It may help to know that 15 mph is equivalent to 22 ft/sec and 60 mph to 88 ft/sec.

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The chart suggests that the exact speed at $t = 10$ is 40 ft/sec. We can confirm this intuition by computing the average speed from $t = 10$ to $t = 10 + h$, where h is any very small nonzero number. (The chart does this for $h = .1$, $h = .01$, and $h = .001$.) The average speed from $t = 10$ to $t = 10 + h$ is

$$\begin{aligned}\frac{s(10+h)-s(10)}{(10+h)-10} &= \frac{s(10+h)-s(10)}{h}\\ &= \frac{2(10+h)^2-2\cdot 10^2}{h}\\ &= \frac{2(100+20h+h^2)-200}{h}\\ &= \frac{200+40h+2h^2-200}{h}\\ &= \frac{40h+2h^2}{h} = \frac{h(40+2h)}{h} \qquad (h \neq 0)\\ &= 40+2h.\end{aligned}$$

Saying that the time interval from 10 to $10 + h$ gets shorter and shorter is equivalent to saying that h gets closer and closer to 0. Hence, the exact speed at $t = 10$ is the limit, as h approaches 0, of the average speeds over the intervals from $t = 10$ to $t = 10 + h$; that is,

$$\begin{aligned}\lim_{h\to 0}\frac{s(10+h)-s(10)}{h} &= \lim_{h\to 0}(40+2h)\\ &= 40 \text{ ft/sec}.\end{aligned}$$

In the preceding example, the car moved in one direction along a straight line. Now suppose that an object is moving back and forth along a number line, with its position at time t given by the function $s(t)$. The **velocity** of the object is the rate of motion in the direction in which it is moving. **Speed** is the absolute value of velocity. For example, a velocity of -30 ft/sec indicates a speed of 30 ft/sec in the negative direction, while a velocity of 40 ft/sec indicates a speed of 40 ft/sec in the positive direction.

Hereafter, we deal with **average velocity** and **instantaneous velocity** instead of average speed and exact speed, respectively. When the term "velocity" is used alone, it means instantaneous velocity.

Let a be a fixed number. By replacing 10 by a in the preceding discussion, we see that the average velocity of the object from time $t = a$ to time $t = a + h$ is the quotient

$$\frac{s(a+h)-s(a)}{(a+h)-a} = \frac{s(a+h)-s(a)}{h}.$$

The instantaneous velocity at time a is the limit of this quotient as h approaches 0.

Velocity

If an object moves along a straight line, with position $s(t)$ at time t, then the **velocity** of the object at $t = a$ is

$$\lim_{h\to 0}\frac{s(a+h)-s(a)}{h},$$

provided that this limit exists.

Example 3 **Natural Science** The distance, in feet, of an object from a starting point is given by $s(t) = 2t^2 - 5t + 40$, where t is time in seconds.

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(a) Find the average velocity of the object from 2 seconds to 4 seconds.

Solution The average velocity is

$$\frac{s(4) - s(2)}{4 - 2} = \frac{52 - 38}{2} = \frac{14}{2} = 7$$

feet per second.

(b) Find the instantaneous velocity at 4 seconds.

Solution For $t = 4$, the instantaneous velocity is

$$\lim_{h \to 0} \frac{s(4 + h) - s(4)}{h}$$

feet per second. We have

$$\begin{aligned} s(\mathbf{4 + h}) &= 2(\mathbf{4 + h})^2 - 5(\mathbf{4 + h}) + 40 \\ &= 2(16 + 8h + h^2) - 20 - 5h + 40 \\ &= 32 + 16h + 2h^2 - 20 - 5h + 40 \\ &= 2h^2 + 11h + 52 \end{aligned}$$

and

$$s(4) = 2(4)^2 - 5(4) + 40 = 52.$$

Thus,

$$s(4 + h) - s(4) = (2h^2 + 11h + 52) - 52 = 2h^2 + 11h,$$

and the instantaneous velocity at $t = 4$ is

$$\begin{aligned} \lim_{h \to 0} \frac{2h^2 + 11h}{h} &= \lim_{h \to 0} \frac{h(2h + 11)}{h} \\ &= \lim_{h \to 0} (2h + 11) = 11 \text{ ft/sec.} \end{aligned}$$

✓4

Checkpoint 4

In Example 3, if $s(t) = t^2 + 3$, find

(a) the average velocity from 1 second to 5 seconds;

(b) the instantaneous velocity at 5 seconds.

Example 4 **Health** The velocity of blood cells is of interest to physicians; a velocity slower than normal might indicate a constriction, for example. Suppose the position of a red blood cell in a capillary is given by

$$s(t) = 1.2t + 5,$$

where $s(t)$ gives the position of a cell in millimeters from some reference point and t is time in seconds. Find the velocity of this cell at time $t = a$.

Solution We have $s(a) = 1.2a + 5$. To find $s(a + h)$, substitute $a + h$ for the variable t in $s(t) = 1.2t + 5$:

$$s(a + h) = 1.2(a + h) + 5.$$

Now use the definition of velocity:

$$\begin{aligned} v(a) &= \lim_{h \to 0} \frac{s(a + h) - s(a)}{h} \\ &= \lim_{h \to 0} \frac{1.2(a + h) + 5 - (1.2a + 5)}{h} \\ &= \lim_{h \to 0} \frac{1.2a + 1.2h + 5 - 1.2a - 5}{h} = \lim_{h \to 0} \frac{1.2h}{h} = 1.2 \text{ mm/sec.} \end{aligned}$$

The velocity of the blood cell at $t = a$ is 1.2 millimeters per second, regardless of the value of a. In other words, the blood velocity is a constant 1.2 millimeters per second at any time. ✓5

Checkpoint 5

Repeat Example 4 with $s(t) = .3t - 2$.

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The ideas underlying the concept of the velocity of a moving object can be extended to any function f. In place of average velocity at time t, we have the average rate of change of $f(x)$ with respect to x as x changes from one value to another. Taking limits leads to the following definition.

Instantaneous Rate of Change

The **instantaneous rate of change** for a function f when $x = a$ is

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

provided that this limit exists.

Example 5 **Business** A company determines that the cost (in hundreds of dollars) of manufacturing x cases of computer mice is

$$C(x) = -.2x^2 + 8x + 40 \qquad (0 \le x \le 20).$$

(a) Find the average rate of change of cost for manufacturing between 5 and 10 cases.

Solution Use the formula for average rate of change. The cost to manufacture 5 cases is

$$C(5) = -.2(5^2) + 8(5) + 40 = 75,$$

or \$7500. The cost to manufacture 10 cases is

$$C(10) = -.2(10^2) + 8(10) + 40 = 100,$$

or \$10,000. The average rate of change of cost is

$$\frac{C(10) - C(5)}{10 - 5} = \frac{100 - 75}{5} = 5.$$

Thus, on the average, cost is increasing at the rate of \$500 per case when production is increased from 5 to 10 cases.

(b) Find the instantaneous rate of change with respect to the number of cases produced when 5 cases are produced.

Solution The instantaneous rate of change when $x = 5$ is given by

$$\begin{aligned}
&\lim_{h \to 0} \frac{C(5 + h) - C(5)}{h} \\
&= \lim_{h \to 0} \frac{[-.2(5 + h)^2 + 8(5 + h) + 40] - [-.2(5^2) + 8(5) + 40]}{h} \\
&= \lim_{h \to 0} \frac{[-5 - 2h - .2h^2 + 40 + 8h + 40] - [75]}{h} \\
&= \lim_{h \to 0} \frac{6h - .2h^2}{h} && \text{Combine terms.} \\
&= \lim_{h \to 0} (6 - .2h) && \text{Divide by } h. \\
&= 6. && \text{Calculate the limit.}
\end{aligned}$$

When 5 cases are manufactured, the cost is increasing at the rate of \$600 per case.

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The rate of change of the cost function is called the **marginal cost.*** Similarly, **marginal revenue** and **marginal profit** are the rates of change of the revenue and profit functions, respectively. Part (b) of Example 5 shows that the marginal cost when 5 cases are manufactured is $600 per case.

Example 6 **Health** The total number of bachelor's degrees (in thousands) earned in the health professions in year t can be approximated by

$$f(t) = 52e^{.094t} \qquad (3 \le t \le 10),$$

where $t = 0$ corresponds to the year 2000. (Data from: *The Statistical Abstract of the United States*: 2012.)

(a) Find the rate of change of the number of health professions bachelor's degrees in 2004.

Solution Since $t = 0$ corresponds to the year 2000, we must find the instantaneous rate of change of $f(t)$ at $t = 4$. The algebraic techniques used in the preceding examples will not work with an exponential function, but the rate of change can be approximated by a graphing calculator or spreadsheet program. The average rate of change from 4 to $4 + h$ is:

$$\text{average rate of change} = \frac{f(4+h) - f(4)}{h} = \frac{52e^{.094(4+h)} - 52e^{.094(4)}}{h}.$$

X	Y1
-.001	7.1188
-1E-4	7.1191
-1E-5	7.1191
Ø	ERROR
1E-5	7.1191
1E-4	7.1191
.001	7.1194

X=-.001

Figure 11.19

To approximate the instantaneous rate of change, we evaluate this quantity for very small values of h, as shown in Figure 11.19 (in which X is used in place of h, and Y1 is the average rate of change). The table suggests that at $t = 4$, the instantaneous rate of change can be approximated by:

$$\text{instantaneous rate of change} = \lim_{h \to 0} \frac{52e^{.094(4+h)} - 52e^{.094(4)}}{h} \approx 7.119.$$

Thus, the number of bachelor's degrees in the health professions was increasing at a rate of about 7119 degrees per year in 2004.

(b) Find the rate of change in 2009.

Solution The instantaneous rate of change of $f(t)$ when $t = 9$ is

$$\lim_{h \to 0} \frac{f(9+h) - f(9)}{h} = \lim_{h \to 0} \frac{52e^{.094(9+h)} - 52e^{.094(9)}}{h} \approx 11.391,$$

X	Y1
-.001	11.39
-1E-4	11.39
-1E-5	11.391
Ø	ERROR
1E-5	11.391
1E-4	11.391
.001	11.391

X=-.001

Figure 11.20

as shown in Figure 11.20. In 2009, therefore, the number of health professions degrees was increasing at a rate of about 11,391 degrees per year.

*Marginal cost for linear cost functions was discussed in in Section 3.3.

11.3 Exercises

Find the average rate of change for the given functions. (See Example 1.)

1. $f(x) = x^2 + 2x$ between $x = 0$ and $x = 6$
2. $f(x) = -4x^2 - 6$ between $x = 1$ and $x = 7$
3. $f(x) = 2x^3 - 4x^2 + 6$ between $x = -1$ and $x = 2$
4. $f(x) = -3x^3 + 2x^2 - 4x + 2$ between $x = 0$ and $x = 2$
5. $f(x) = \sqrt{x}$ between $x = 1$ and $x = 9$
6. $f(x) = \sqrt{3x - 2}$ between $x = 2$ and $x = 6$
7. $f(x) = \dfrac{1}{x - 1}$ between $x = -2$ and $x = 0$
8. $f(x) = .4525\, e^{1.556\sqrt{x}}$ between $x = 4$ and $x = 4.5$

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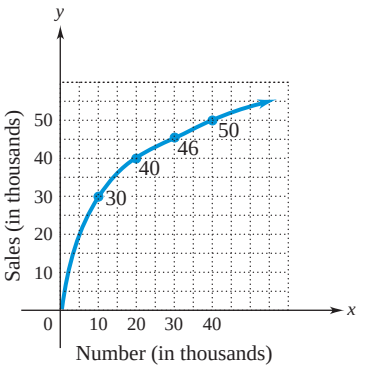
Work these exercises. (See Example 2.)

9. Business The accompanying graph shows the total sales, in thousands of dollars, from the distribution of x thousand catalogs. Find and interpret the average rate of change of sales with respect to the number of catalogs distributed for the given changes in x.



(a) 10 to 20 **(b)** 20 to 30 **(c)** 30 to 40

(d) What is happening to the average rate of change of sales as the number of catalogs distributed increases?

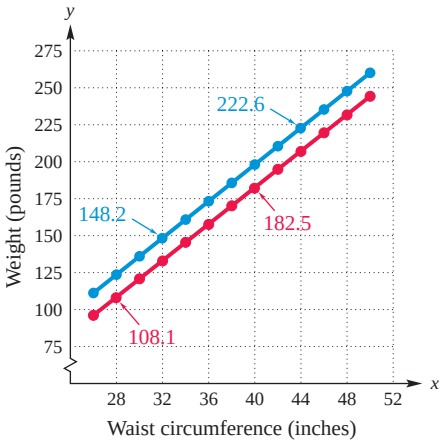
(e) Explain why what you have found in part (d) might happen.

10. Health The accompanying graph shows the relationship between waist circumference (in inches) and weight (in pounds) for American males (blue) and females (red). Find the average rate of change in weight for changes in waist circumference from

(a) 32 to 44 inches for males;

(b) 28 to 40 inches for females.

(c) Do we need to calculate the rate of change at any other points for males or females? Explain. (Data from: National Health and Nutrition Examination Study (NHANES) 2005–2006.)

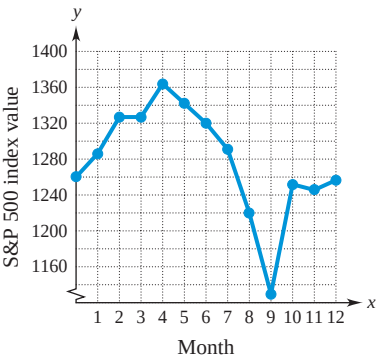


11. Finance The accompanying graph gives the value of the S&P 500 Index for 12 months, where $x = 1$ corresponds to January, 2011. (Data from: www.morningstar.com.)

(a) Which time period has the greater average rate of change: month 2 to month 3, month 7 to month 9, or month 9 to month 10?

(b) Which time period has the greater average rate of change: month 3 to month 4, or month 9 to month 10?

(c) Which time period has the greater average rate of change in absolute value: month 4 to month 7, or month 7 to month 9?

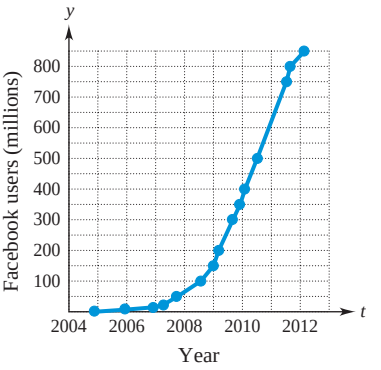


12. Business The accompanying figure shows the number of Facebook users from its launch in 2004 until 2012. (Data from: Facebook.)

(a) Approximately when did Facebook reach 350 million users?

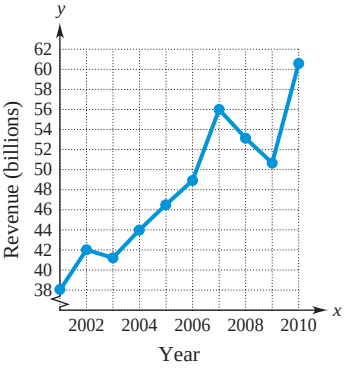
(b) Approximately how many Facebook users were there at the start of 2011?

(c) Which time period has the largest rate of change: [2004, 2009] or [2009, 2012] or [2004, 2012]?



13. Business The accompanying graph shows the revenue (in billions of dollars) generated by the automaker BMW from 2001 to 2010. Find and interpret the approximate average rate of change in the revenue for each period. (Data from: www.morningstar.com.)

(a) 2004 to 2007 **(b)** 2007 to 2009 **(c)** 2004 to 2009



14. Social Science The accompanying graph shows projected enrollment (in millions) in high school, $h(t)$, and college, $c(t)$, in

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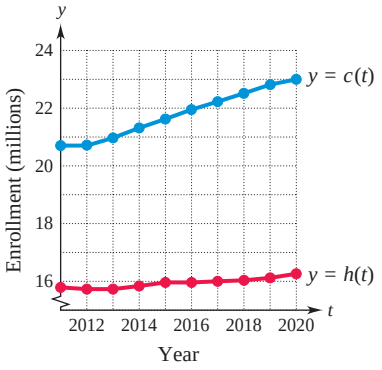
the United States from 2011 to 2020. (Data from: *The Statistical Abstract of the United States*: 2012.)



(a) Find the projected average rate of change in college enrollment over the period 2016 to 2020.

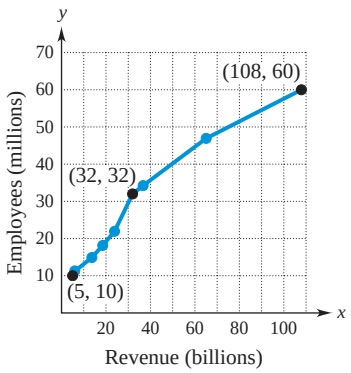
(b) Find the projected average rate of change in high school enrollment over the period 2015 to 2016.

15. Business The accompanying figure shows the number of full-time employees of Apple, Inc. (in millions) as a function of the revenue generated by Apple, Inc. (in billions). Find and interpret the average rate of change of employees with respect to revenue for the following changes in revenue. (Data from: www.mergentonline.com.)

(a) $5 billion to $32 billion

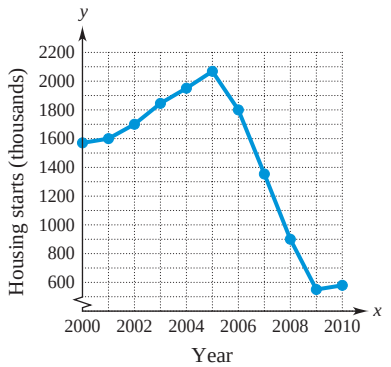
(b) $32 billion to $108 billion

(c) $5 billion to $108 billion



16. Business The accompanying figure shows the number of privately owned housing units started (in thousands) in the United States. Find and interpret the average rate of change for each period. (Data from: *The Statistical Abstract of the United States*: 2012.)

(a) 2002 to 2006 **(b)** 2006 to 2008 **(c)** 2000 to 2010



17. Explain the difference between the average rate of change of $y = f(x)$ as x changes from a to b and the instantaneous rate of change of y at $x = a$.

18. If the instantaneous rate of change of $f(x)$ with respect to x is positive when $x = 1$, is f increasing or decreasing there?

Exercises 19–21 deal with a car moving along a straight road, as discussed on pages 596–597. At time t seconds, the distance of the car (in feet) from the starting point is $s(t) = 2.2t^2$. Find the instantaneous velocity of the car at

19. $t = 5$ **20.** $t = 20$

21. What was the average velocity of the car during the first 30 seconds?

An object moves along a straight line; its distance (in feet) from a fixed point at time t seconds is $s(t) = t^2 + 4t + 3$. Find the instantaneous velocity of the object at the given times. (See Example 3.)

22. $t = 6$ **23.** $t = 1$ **24.** $t = 10$

25. Physical Science A car is moving along a straight test track. The position (in feet) of the car, $s(t)$, at various times t is measured, with the following results:

t (seconds)	0	2	4	6	8	10
$s(t)$ (feet)	0	10	14	20	30	36

Find and interpret the average velocities for the following changes in t:

(a) 0 to 2 seconds; **(b)** 2 to 4 seconds;

(c) 4 to 6 seconds; **(d)** 6 to 8 seconds.

(e) Estimate the instantaneous velocity at 4 seconds

(i) by using the formula for estimating the instantaneous rate (with $h = 2$) and

(ii) by averaging the answers for the average velocity in the 2 seconds before and the 2 seconds afterwards (that is, the answers to parts (b) and (c)).

(f) Estimate the instantaneous velocity at 6 seconds, using the two methods in part (e).

In Exercises 26–30, find (a) $f(a + h)$; (b) $\dfrac{f(a + h) - f(a)}{h}$; (c) the instantaneous rate of change of f when $a = 5$. (See Examples 3–5.)

26. $f(x) = x^2 + x$ **27.** $f(x) = x^2 - x - 1$

28. $f(x) = x^2 + 2x + 2$ **29.** $f(x) = x^3$

30. $f(x) = x^3 - x$

Solve Exercises 31–34 by algebraic methods. (See Examples 3–5.)

31. Business The revenue (in thousands of dollars) from producing x units of an item is

$$R(x) = 10x - .002x^2.$$

(a) Find the average rate of change of revenue when production is increased from 1000 to 1001 units.

(b) Find the marginal revenue when 1000 units are produced.

(c) Find the additional revenue if production is increased from 1000 to 1001 units.

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(d) Compare your answers for parts (a) and (c). What do you find?

32. Business Suppose customers in a hardware store are willing to buy $N(p)$ boxes of nails at p dollars per box, as given by

$$N(p) = 80 - 5p^2 \qquad (1 \le p \le 4).$$

(a) Find the average rate of change of demand for a change in price from \$2 to \$3.

(b) Find the instantaneous rate of change of demand when the price is \$2.

(c) Find the instantaneous rate of change of demand when the price is \$3.

(d) As the price is increased from \$2 to \$3, how is demand changing? Is the change to be expected?

33. Health Epidemiologists in College Station, Texas, estimate that t days after the flu begins to spread in town, the percentage of the population infected by the flu is approximated by

$$p(t) = t^2 + t \qquad (0 \le t \le 5).$$

(a) Find the average rate of change of p with respect to t over the interval from 1 to 4 days.

(b) Find the instantaneous rate of change of p with respect to t at $t = 3$.

34. Health The maximum recommended heart rate (in beats per minute) is given by $h(x) = 220 - x$ where x represents the age of the individual. (Data from: American Heart Association.)

(a) Find and interpret the average rate of change of $h(x)$ with respect to x for $18 \le x \le 28$ years.

(b) Find and interpret the instantaneous rate of change of $h(x)$ with respect to x at $x = 23$ years.

(c) Explain why your answers in parts (a) and (b) are the same in this case.

Use technology to work Exercises 35–38. (See Example 6.)

35. Social Science The number of bachelor's degrees (in thousands) earned in the social sciences and history in year x can be approximated by $f(x) = 24 \ln x + 117$, where $x = 3$ corresponds to the year 2003. Estimate the rate at which the number of social science and history degrees was changing in the given year. (Data from: *The Statistical Abstract of the United States*: 2012.)

(a) 2004 **(b)** 2012

36. Business The amount of revenue (in billions of dollars) generated by the bookstore industry in year x can be approximated by $f(x) = -.1x^2 + 1.22x + 3.47$, where $x = 0$ corresponds to the year 2000. Estimate the rate at which revenue was changing in the given year. (Data from: *The Statistical Abstract of the United States*: 2012.)

(a) 2006 **(b)** 2011

37. Health The metabolic rate of a person who has just eaten a meal tends to go up and then, after some time has passed, returns to a resting metabolic rate. This phenomenon is known as the *thermic effect of food*. Researchers have indicated that the thermic effect of food (in kJ/hr) for a particular person is

$$F(t) = -10.28 + 175.9te^{-t/1.3},$$

where t is the number of hours that have elapsed since eating a meal.*

(a) Graph the given function in a window with $0 \le x \le 6$ and $-20 \le y \le 100$.

(b) Find the average rate of change of the thermic effect of food during the first hour after eating.

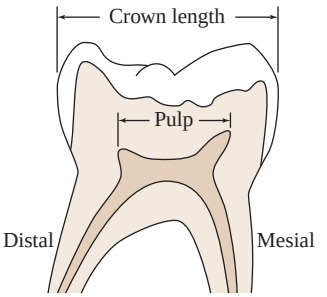
(c) Use a graphing calculator to find the instantaneous rate of change of the thermic effect of food exactly 1 hour after eating.

(d) Use a graphing calculator to estimate when the function stops increasing and begins to decrease.

38. Health The mesiodistal crown length (as shown in the accompanying diagram) of deciduous mandibular first molars in fetuses is related to the postconception age of the tooth as

$$L(t) = -.01t^2 + .788t - 7.048,$$

where $L(t)$ is the crown length, in millimeters, of the molar t weeks after conception.†



(a) Find the average rate of growth in mesiodistal crown length during weeks 22 through 28.

(b) Find the instantaneous rate of growth in mesiodistal crown length when the tooth is exactly 22 weeks of age.

(c) Graph the given function in a window with $0 \le x \le 50$ and $0 \le y \le 9$.

(d) Does a function that increases and then begins to decrease make sense for this particular application? What do you suppose is happening during the first 11 weeks? Does this function accurately model crown length during those weeks?

Checkpoint Answers

1. **(a)** 64 mph **(b)** $\dfrac{f(r) - f(s)}{r - s}$ mph

2. **(a)** 8 **(b)** 13

3. **(a)** Changing at the average rate of 250 index points per year
(b) Changing at the average rate of −49 index points per year

4. **(a)** 6 ft per second **(b)** 10 ft per second

5. The velocity is .3 millimeter per second.

*Reed, G. and J. Hill, "Measuring the Thermic Effect of Food," *American Journal of Clinical Nutrition*, Vol. 63, 1996, pp. 164–169.)

†Harris, E. F., J. D. Hicks, and B. D. Barcroft, "Tissue Contributions to Sex and Race: Differences in Tooth Crown Size of Deciduous Molars," *American Journal of Physical Anthropology,* Vol. 115, 2001, pp. 223–237.

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11.4 Tangent Lines and Derivatives

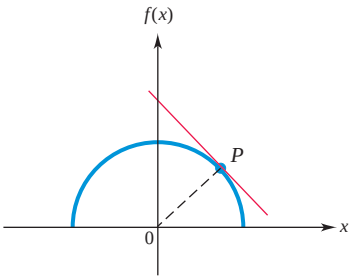

Figure 11.21

We now develop a geometric interpretation of the rates of change considered in the previous section. In geometry, a tangent line to a circle at a point P is defined to be the line through P that is perpendicular to the radius OP, as in Figure 11.21 (which shows only the top half of the circle). If you think of this circle as a road on which you are driving at night, then the tangent line indicates the direction of the light beam from your headlights as you pass through the point P. This analogy suggests a way of extending the idea of a tangent line to any curve: the tangent line to the curve at a point P indicates the "direction" of the curve as it passes through P. Using this intuitive idea of direction, we see, for example, that the lines through P_1 and P_3 in Figure 11.22 appear to be tangent lines, whereas the lines through P_2 and P_4 do not.

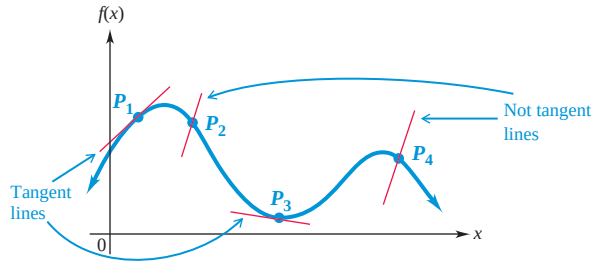

Figure 11.22

We can use these ideas to develop a precise definition of the tangent line to the graph of a function f at the point R. As shown in Figure 11.23, choose a second point S on the graph and draw the line through R and S; this line is called a **secant line.** You can think of this secant line as a rough approximation of the tangent line.

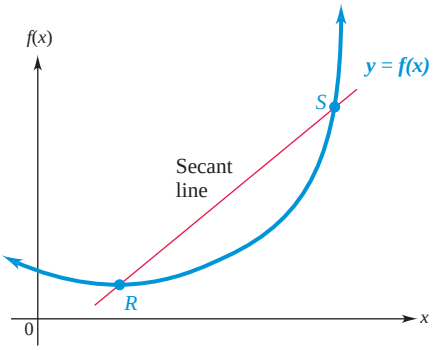

Figure 11.23

Now suppose that the point S slides down the curve closer to R. Figure 11.24 on the next page shows successive positions S_2, S_3, and S_4 of the point S. The closer S gets to R, the better the secant line RS approximates our intuitive idea of the tangent line at R.

In particular, the closer S gets to R, the closer the slope of the secant line gets to the slope of the tangent line. Informally, we say that

$$\text{the slope of tangent line at } R = \text{the limit of the slope of secant line } RS \text{ as } S \text{ gets closer and closer to } R.$$

In order to make this statement more precise, suppose the first coordinate of R is a. Then the first coordinate of S can be written as $a + h$ for some number h. (In Figure 11.24, h is the distance on the x-axis between the two first coordinates.) Thus, R has coordinates

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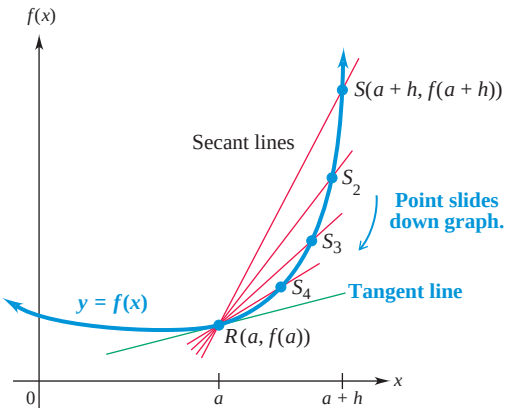

Figure 11.24

$(a, f(a))$ and S has coordinates $(a + h, f(a + h))$, as shown in Figure 11.24. Consequently, the slope of the secant line RS is

$$\frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}.$$

Now, as S moves closer to R, their first coordinates move closer to each other, that is, h gets smaller and smaller. Hence,

$$\begin{aligned} \text{slope of tangent line at } R &= \text{limit of slope of secant line } RS \\ &\quad \text{as } S \text{ gets closer and closer to } R \\ &= \text{limit of } \frac{f(a + h) - f(a)}{h} \\ &\quad \text{as } h \text{ gets closer and closer to } 0 \\ &= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \end{aligned}$$

This intuitive development suggests the following formal definition.

Tangent Line

The **tangent line** to the graph of $y = f(x)$ at the point $(a, f(a))$ is the line through this point having slope

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

provided that this limit exists. If this limit does not exist, then there is no tangent line at the point.

The slope of the tangent line at a point is also called the **slope of the curve** at that point. Since the slope of a line indicates its direction (see the table on page 78), the slope of the tangent line at a point indicates the direction of the curve at that point. For a review of slope and equations of lines, see Section 2.2.

Example 1 Consider the graph of $y = x^2 + 2$.

(a) Find the slope of the tangent line to the graph when $x = -1$.

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Solution Use the preceding definition with $f(x) = x^2 + 2$ and $a = -1$. The slope of the tangent line is calculated as follows:

$$\begin{aligned}\text{Slope of tangent} &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\\ &= \lim_{h\to 0}\frac{[(-1+h)^2+2]-[(-1)^2+2]}{h}\\ &= \lim_{h\to 0}\frac{[1-2h+h^2+2]-[1+2]}{h}\\ &= \lim_{h\to 0}\frac{-2h+h^2}{h} = \lim_{h\to 0}(-2+h) = -2.\end{aligned}$$

So the slope of the tangent line is -2.

(b) Find the equation of the tangent line.

Solution When $x = -1$, then $y = (-1)^2 + 2 = 3$. So the tangent line passes through the point $(-1, 3)$ and has slope -2. Its equation can be found with the point–slope form of the equation of a line (see Chapter 2):

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form}\\ y - 3 &= -2[x - (-1)] && \text{Substitute.}\\ y - 3 &= -2(x + 1) && \text{Simplify.}\\ y - 3 &= -2x - 2 && \text{Distributive property}\\ y &= -2x + 1. && \text{Solve for } y.\end{aligned}$$

Figure 11.25 shows a graph of $f(x) = x^2 + 2$ along with a graph of the tangent line at $x = -1$. ✓1

✓ Checkpoint 1

Let $f(x) = x^2 + 2$.
Find the equation of the tangent line to the graph at the point where $x = 1$.

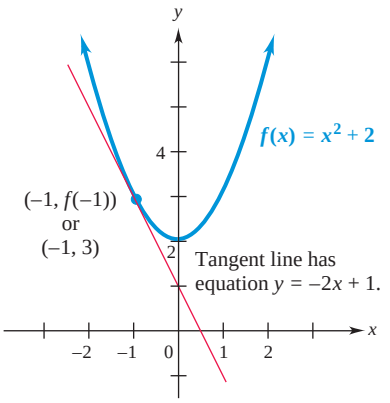

Figure 11.25

TECHNOLOGY TIP When finding the equation of a tangent line algebraically, you can confirm your answer with a graphing calculator by graphing both the function and the tangent line on the same screen to see if the tangent line appears to be correct.

Once a function has been graphed, many graphing calculators can draw the tangent line at any specified point. Most also display the slope of the tangent line, and some actually display its equation. Look for TANGENT or TANLN in the MATH, DRAW, or SKETCH menu. ✓2

✓ Checkpoint 2

Use a graphing calculator to confirm your answer to Checkpoint 1 by graphing $f(x) = x^2 + 2$ and the tangent line at the point where $x = 1$ on the same screen.

In Figure 11.25, the tangent line $y = -2x + 1$ appears to coincide with the graph of $f(x) = x^2 + 2$ near the point $(-1, 3)$. This becomes more obvious when both $f(x)$ and the

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tangent line are graphed in a very small window on a graphing calculator (Figure 11.26). The graph and the tangent line now appear virtually identical near $(-1, 3)$. The table of values in Figure 11.27 (in which Y_1 is $f(x)$ and Y_2 the tangent line) confirms the fact that the tangent line is a good approximation of the function when x is very close to -1.

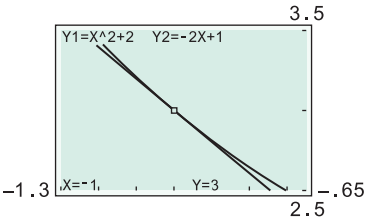

Figure 11.26

X	Y1	Y2
-1.1	3.21	3.2
-1.05	3.1025	3.1
-1.02	3.0404	3.04
-1	3	3
-.97	2.9409	2.94
-.94	2.8836	2.88
-.9	2.81	2.8

X=-.9

Figure 11.27

The same thing is true in the general case.

Tangent Line

If it exists, the tangent line to the graph of a function f at $x = a$ is a good approximation of the function f near $x = a$.

Suppose the graph of a function f is a straight line. The fact in the preceding box suggests that the tangent line to f at any point should be the graph of f itself (because it is certainly the best possible linear approximation of f). The next example shows that this is indeed the case.

Example 2 Let a be any real number. Find the equation of the tangent line to the graph of $f(x) = 7x + 3$ at the point where $x = a$.

Solution According to the definition, the slope of the tangent line is

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{[7(a+h)+3]-[7a+3]}{h}$$

$$= \lim_{h\to 0}\frac{[7a+7h+3]-7a-3}{h}$$

$$= \lim_{h\to 0}\frac{7h}{h} = \lim_{h\to 0} 7 = 7.$$

Hence, the equation of the tangent line at the point $(a, f(a))$ is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - f(a) &= 7(x - a) \\ y &= 7x - 7a + f(a) \\ y &= 7x - 7a + 7a + 3 \\ y &= 7x + 3. \end{aligned}$$

Thus, the tangent line *is* the graph of $f(x) = 7x + 3$.

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Secant lines and tangent lines (or, more precisely, their slopes) are the geometric analogues of the average and instantaneous rates of change studied in the previous section, as summarized in the following chart:

Quantity	Algebraic Interpretation	Geometric Interpretation
$\dfrac{f(a+h)-f(a)}{h}$	Average rate of change of f from $x=a$ to $x=a+h$	Slope of the secant line through $(a, f(a))$ and $(a+h, f(a+h))$
$\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$	Instantaneous rate of change of f at $x=a$	Slope of the tangent line to the graph of f at $(a, f(a))$

The Derivative

If $y=f(x)$ is a function and a is a number in its domain, then we shall use the symbol $f'(a)$ to denote the special limit

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$

provided that it exists. In other words, to each number a, we can assign the number $f'(a)$ obtained by calculating the preceding limit. This process defines an important new function.

Derivative

The **derivative** of the function f is the function denoted f' whose value at the number x is defined to be the number

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

provided that this limit exists.

The derivative function f' has as its domain all the points at which the specified limit exists, and the value of the derivative function at the number x is the number $f'(x)$. Using x instead of a here is similar to the way that $g(x)=2x$ denotes the function that assigns to each number a the number $2a$.

If $y=f(x)$ is a function, then its derivative is denoted either by f' or by y'. If x is a number in the domain of $y=f(x)$ such that $y'=f'(x)$ is defined, then the function f is said to be **differentiable** at x. The process that produces the function f' from the function f is called **differentiation.**

The derivative function may be interpreted in many ways, two of which were already discussed:

1. The derivative function f' gives the *instantaneous rate of change* of $y=f(x)$ with respect to x. This instantaneous rate of change can be interpreted as marginal cost, marginal revenue, or marginal profit (if the original function represents cost, revenue, or profit, respectively) or as velocity (if the original function represents displacement along a line). From now on, we will use "rate of change" to mean "instantaneous rate of change."
2. The derivative function f' gives the *slope* of the graph of f at any point. If the derivative is evaluated at $x=a$, then $f'(a)$ is the slope of the tangent line to the curve at the point $(a, f(a))$.

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Example 3 Use the graph of the function $f(x)$ in Figure 11.28 to answer the given questions.

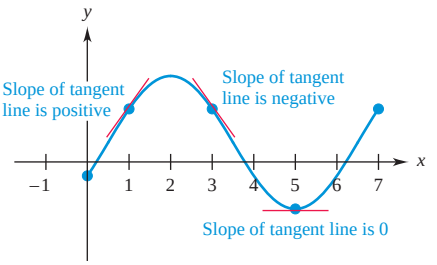


Figure 11.28

(a) Is $f'(3)$ positive or negative?

Solution We know that $f'(3)$ is the slope of the tangent line to the graph at the point where $x = 3$. Figure 11.28 shows that this tangent line slants downward from left to right, meaning that its slope is negative. Hence, $f'(3) < 0$.

(b) Which is larger, $f'(1)$ or $f'(5)$?

Solution Figure 11.28 shows that the tangent line to the graph at the point where $x = 1$ slants upward from left to right, meaning that its slope, $f'(1)$, is a positive number. The tangent line at the point where $x = 5$ is horizontal, so that it has slope 0. (That is, $f'(5) = 0$). Therefore, $f'(1) > f'(5)$.

(c) For what values of x is $f'(x)$ positive?

Solution On the graph, find the points where the tangent line has positive slope (slants upward from left to right). At each such point, $f'(x) > 0$. Figure 11.28 shows that this occurs when $0 < x < 2$ and when $5 < x < 7$. ✓3

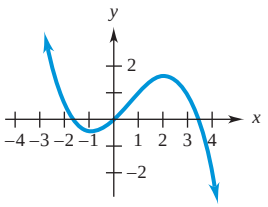
Checkpoint 3

The graph of a function g is shown. Determine whether the given numbers are positive, negative, or zero.

(a) $g'(0)$

(b) $g'(-1)$

(c) $g'(3)$



Example 4 **Natural Science** A student brings a cold soft drink to a 50-minute math class but is too busy during class to drink it. If $C(t)$ represents the temperature of the soft drink (in degrees Fahrenheit) t minutes after the start of class, interpret the meaning of the following statements, including units.

(a) Interpret $C(0) = 38$.

Solution Since $C(t)$ represents the temperature of the drink at time t, $C(0) = 38$ means that the temperature of the drink was 38°F at the start of class.

(b) Interpret $C(50) - C(0) = 30$.

Solution Since $C(b) - C(a)$ represents the change in temperature from time $t = a$ to time $t = b$ minutes, $C(50) - C(0) = 30$ means that the drink warmed up by 30°F between the start and end of class.

(c) Interpret $\dfrac{C(50) - C(0)}{50 - 0} = .6$.

Solution Since $\dfrac{C(b) - C(a)}{b - a}$ represents the average rate of change of $C(t)$ from time $t = a$ to time $t = b$ minutes, $\dfrac{C(50) - C(0)}{50 - 0} = .6$ means that the drink warmed up at an average rate of .6°F per minute during class.

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(d) Interpret $C'(0) = 1.5$.

Solution Since $C'(t)$ represents the instantaneous rate of change of the temperature of the drink with respect to time, $C'(0) = 1.5$ means that, at the start of the class, the temperature is changing at an instantaneous rate of 1.5°F per minute. Since the average rate of change over a very small time interval is a good *approximation* of the instantaneous rate of change, we can think of $C'(0) = 1.5$ as saying that, at the start of class, the drink will warm up by *approximately* 1.5°F in the next minute. ✓4

✓Checkpoint 4

For the function $C(t)$ in Example 4, interpret the following, including units.

(a) $C(50) = 68$

(b) $C(50) - C(25) = 7.6$

(c) $\dfrac{C(50) - C(25)}{50 - 25} = .3$

(d) $C'(50) = .2$

The rule of a derivative function can be found by using the definition of the derivative and the following four-step procedure.

Finding $f'(x)$ from the Definition of the Derivative

Step 1 Find $f(x + h)$.

Step 2 Find $f(x + h) - f(x)$.

Step 3 Divide by h to get $\dfrac{f(x + h) - f(x)}{h}$.

Step 4 Treat x as a constant and let $h \to 0$.

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \text{ if this limit exists.}$$

Example 5 Let $f(x) = x^3 - 4x$.

(a) Find the derivative $f'(x)$.

Solution By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

Step 1 Find $f(x + h)$.
Replace x with $x + h$ in the rule of $f(x)$:

$$\begin{aligned} f(\boldsymbol{x}) &= \boldsymbol{x}^3 - 4\boldsymbol{x} \\ f(\boldsymbol{x + h}) &= (\boldsymbol{x + h})^3 - 4(\boldsymbol{x + h}) \\ &= (x^3 + 3x^2h + 3xh^2 + h^3) - 4(x + h) \\ &= x^3 + 3x^2h + 3xh^2 + h^3 - 4x - 4h. \end{aligned}$$

Step 2 Find $f(x + h) - f(x)$.
Since $f(x) = x^3 - 4x$,

$$\begin{aligned} f(x + h) - f(x) &= (x^3 + 3x^2h + 3xh^2 + h^3 - 4x - 4h) - (x^3 - 4x) \\ &= x^3 + 3x^2h + 3xh^2 + h^3 - 4x - 4h - x^3 + 4x \\ &= 3x^2h + 3xh^2 + h^3 - 4h. \end{aligned}$$

Step 3 Form and simplify the quotient $\dfrac{f(x + h) - f(x)}{h}$:

$$\begin{aligned} \frac{f(x + h) - f(x)}{h} &= \frac{3x^2h + 3xh^2 + h^3 - 4h}{h} \\ &= \frac{h(3x^2 + 3xh + h^2 - 4)}{h} \\ &= 3x^2 + 3xh + h^2 - 4. \end{aligned}$$

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Step 4 Find the limit as h approaches 0 of the result in Step 3, treating x as a constant:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2 - 4)$$
$$= 3x^2 - 4.$$

Therefore, the derivative of $f(x) = x^3 - 4x$ is $f'(x) = 3x^2 - 4$.

(b) Calculate and interpret $f'(1)$.

Solution The procedure in part (a) works for *every* x and $f'(x) = 3x^2 - 4$. Hence, when $x = 1$,

$$f'(1) = 3\cdot 1^2 - 4 = -1.$$

The number -1 is the slope of the tangent line to the graph of $f(x) = x^3 - 4x$ at the point where $x = 1$ (that is, at $(1, f(1)) = (1, -3)$).

(c) Find the equation of the tangent line to the graph of $f(x) = x^3 - 4x$ at the point where $x = 1$.

Solution By part (b), the point on the graph where $x = 1$ is $(1, -3)$, and the slope of the tangent line is $f'(1) = -1$. Therefore, the equation is

$$y - (-3) = (-1)(x - 1) \qquad \text{Point–slope form}$$
$$y = -x - 2. \qquad \text{Slope–intercept form}$$

Both $f(x)$ and the tangent line are shown in Figure 11.29. ✓5

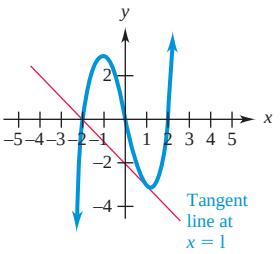

Figure 11.29

✓ Checkpoint 5

Let $f(x) = -5x^2 + 4$. Find the given expressions.

(a) $f(x + h)$

(b) $f(x + h) - f(x)$

(c) $\dfrac{f(x+h)-f(x)}{h}$

(d) $f'(x)$

(e) $f'(3)$

(f) $f'(0)$

CAUTION

1. In Example 5(a), note that $f(x + h) \neq f(x) + h$ because, by Step 1,

$$f(x + h) = x^3 + 3x^2h + 3xh^2 + h^3 - 4x - 4h,$$

but

$$f(x) + h = (x^3 - 4x) + h = x^3 - 4x + h.$$

2. In Example 5(b), do not confuse $f(1)$ and $f'(1)$. $f(1)$ is the value of the original function $f(x) = x^3 - 4x$ at $x = 1$, namely, -3, whereas $f'(1)$ is the value of the derivative function $f'(x) = 3x^2 - 4$ at $x = 1$, namely, -1.

Example 6

Let $f(x) = 1/x$. Find $f'(x)$.

Solution

Step 1 $f(x + h) = \dfrac{1}{x + h}$.

Step 2
$$f(x + h) - f(x) = \frac{1}{x + h} - \frac{1}{x}$$
$$= \frac{x - (x + h)}{x(x + h)} \qquad \text{Find a common denominator.}$$
$$= \frac{x - x - h}{x(x + h)} \qquad \text{Simplify the numerator.}$$
$$= \frac{-h}{x(x + h)}.$$

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Step 3 $$\frac{f(x+h)-f(x)}{h} = \frac{\dfrac{-h}{x(x+h)}}{h}$$

$$= \frac{-h}{x(x+h)} \cdot \frac{1}{h} \qquad \text{Invert and multiply.}$$

$$= \frac{-1}{x(x+h)}.$$

Step 4 $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{-1}{x(x+h)}$$

$$= \frac{-1}{x(x+0)} = \frac{-1}{x(x)} = \frac{-1}{x^2}.$$ ✓6

✓Checkpoint 6

Let $f(x) = -5/x$. Find the given expressions.

(a) $f(x+h)$

(b) $f(x+h)-f(x)$

(c) $\dfrac{f(x+h)-f(x)}{h}$

(d) $f'(x)$

(e) $f'(-1)$

Example 7

Let $g(x) = \sqrt{x}$. Find $g'(x)$.

Solution

Step 1 $g(x+h) = \sqrt{x+h}$.

Step 2 $g(x+h) - g(x) = \sqrt{x+h} - \sqrt{x}$.

Step 3 $$\frac{g(x+h)-g(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h}.$$

At this point, in order to be able to divide by h, multiply both numerator and denominator by $\sqrt{x+h}+\sqrt{x}$, that is, rationalize the *numerator*:

$$\frac{g(x+h)-g(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x+h}+\sqrt{x}}.$$

Step 4 $$g'(x) = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Example 8

Business A sales representative for a textbook-publishing company frequently makes a 4-hour drive from her home in a large city to a university in another city. Let $s(t)$ represent her position t hours into the trip (where position is measured in miles on the highway, with position 0 corresponding to her home). Then $s(t)$ is given by

$$s(t) = -5t^3 + 30t^2.$$

(a) How far from home will she be after 1 hour? after $1\frac{1}{2}$ hours?

Solution

Her distance from home after 1 hour is

$$s(1) = -5(1)^3 + 30(1)^2 = 25,$$

or 25 miles. After $1\frac{1}{2}$ (or $\frac{3}{2}$) hours, it is

$$s\left(\frac{3}{2}\right) = -5\left(\frac{3}{2}\right)^3 + 30\left(\frac{3}{2}\right)^2 = \frac{405}{8} = 50.625,$$

or 50.625 miles.

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(b) How far apart are the two cities?

Solution Since the trip takes 4 hours and the distance is given by $s(t)$, the university city is $s(4) = 160$ miles from her home.

(c) What is her velocity 1 hour into the trip? $1\frac{1}{2}$ hours into the trip?

Solution Velocity is the instantaneous rate of change in position with respect to time. We need to find the value of the derivative $s'(t)$ at $t = 1$ and $t = 1\frac{1}{2}$. ✓7

✓Checkpoint 7

Go through the four steps to find $s'(t)$, the velocity of the car at any time t, in Example 8.

From Checkpoint 7, $s'(t) = -15t^2 + 60t$. At $t = 1$, the velocity is

$$s'(1) = -15(1)^2 + 60(1) = 45,$$

or 45 miles per hour. At $t = 1\frac{1}{2}$, the velocity is

$$s'\left(\frac{3}{2}\right) = -15\left(\frac{3}{2}\right)^2 + 60\left(\frac{3}{2}\right) = 56.25,$$

about 56 miles per hour.

Existence of the Derivative

The definition of the derivative includes the phrase "provided that this limit exists." If the limit used to define $f'(x)$ does not exist, then, of course, the derivative does not exist at that x. For example, a derivative cannot exist at a point where the function itself is not defined. If there is no function value for a particular value of x, there can be no tangent line for that value. This was the case in Example 6: There was no tangent line (and no derivative) when $x = 0$.

Derivatives also do not exist at "corners" or "sharp points" on a graph. For example, the function graphed in Figure 11.30 is the *absolute-value function*, defined by

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

and written $f(x) = |x|$. The graph has a "corner point" when $x = 0$.

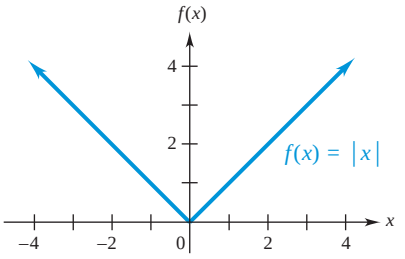

Figure 11.30

By the definition of derivative, the derivative at any value of x is given by

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{|x+h|-|x|}{h}$$

provided that this limit exists. To find the derivative at 0 for $f(x) = |x|$, replace x with 0 and $f(x)$ with $|0|$ to get

$$\lim_{h\to 0}\frac{|0+h|-|0|}{h} = \lim_{h\to 0}\frac{|h|}{h}.$$

In Example 11 of Section 11.1 (with x in place of h), we showed that

$$\lim_{h\to 0}\frac{|h|}{h} \text{ does not exist.}$$

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Therefore, there is no derivative at 0. However, the derivative of $f(x) = |x|$ *does* exist for all values of x other than 0.

Since a vertical line has an undefined slope, the derivative cannot exist at any point where the tangent line is vertical, as at x_5 in Figure 11.31. This figure summarizes various ways that a derivative can fail to exist.

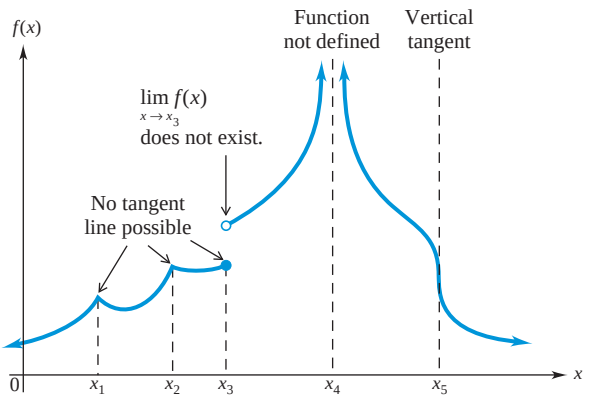

Figure 11.31

Derivatives and Technology

Many computer programs (such as Mathematica and Maple) and a few graphing calculators (such as the TI-89) can find symbolic formulas for the derivatives of most functions. Although other graphing calculators cannot find the rule of a derivative function, most of them can approximate the numerical value of the derivative function at any number where it is defined by using the **numerical derivative** feature. (See Exercise 50 at the end of this section for an explanation of computational technique used by the numerical derivative feature.)

Example 9 **Social Science** A psychology experiment found that the number of facts $N(t)$ remembered t days after the facts were memorized was given by

$$N(t) = \frac{10.003}{1 + .0003e^{.8t}}.$$

Use a graphing calculator to find an approximate value of $N'(10)$ and interpret the answer.

Solution The numerical derivative feature is labeled nDeriv, or d/dx, or nDer and is usually in the MATH or CALC menu or one of its submenus. Most calculators require that we use x as the independent variable instead of t, so we will write the function as $N(x) = \dfrac{10.003}{1 + .0003e^{.8x}}$. Check your instruction manual for the correct syntax, but on many graphing calculators, entering either

$$\text{nDeriv}(10.003/(1 + .0003e^{.8x}), x, 10) \quad \text{or}$$
$$\text{d/dx}(10.003/(1 + .0003e^{.8x}), x, 10)$$

produces an (approximate) value of $N'(10)$. Answers will vary slightly depending on the calculator, but you are likely to get an answer close to -1.9943694066.

The negative sign implies that the number of facts remembered is decreasing over time. One interpretation of $N'(10) \approx -2$ is that 10 days after the facts were memorized, they are being forgotten at a rate of about 2 facts per day. In other words, 10 days after the facts were memorized, approximately 2 facts will be forgotten during the next day. ✓8

Checkpoint 8

If $f(x) = x^{2/3} - 8x^3 + 4^x$, use a graphing calculator to find the (approximate) value of

(a) $f'(2)$;

(b) $f'(3.2)$.

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TECHNOLOGY TIP On many graphing calculators, if you have the function $f(x)$ stored in the function memory as, say, Y_1, you can use Y_1 with the nDeriv key instead of typing in the rule of $f(x)$.

Example 10 **Finance** If \$15,000 is deposited in a money market account that pays 1.2% per year compounded monthly, then the amount in the account after n months is

$$A(n) = 15{,}000\left(1 + \frac{.012}{12}\right)^n = 15{,}000(1.001)^n.$$

Use a graphing calculator to find an approximate value of $A'(120)$ and interpret the answer.

Solution Using the numerical derivative feature on a typical graphing calculator gives an approximate value of $A'(120) = 16.90$. This means that at time $t = 120$ months, the instantaneous rate of change of the account balance with respect to time is 16.90 dollars per month. We can also describe this by saying that 10 years (120 months) after making the deposit, the balance on the account will increase by *approximately* \$16.90 in the next month. Note that the *actual* change in the balance over the next one month is given by:

$$A(121) - A(120) = 15{,}000(1.001)^{121} - 15{,}000(1.001)^{120} = \$16.91.$$

Checkpoint 9

For the account in Example 10, suppose that one year has passed since the initial deposit. Use a derivative to approximate how much the balance will change in the next month.

CAUTION Because of the approximation methods used, the nDeriv key may display an answer at numbers where the derivative is not defined. For instance, we saw earlier that the derivative of $f(x) = |x|$ is not defined when $x = 0$, but the nDeriv key on most calculators produces 1 or 0 or -1 as $f'(0)$.

11.4 Exercises

Find $f'(x)$ for each function. Then find $f'(2)$, $f'(0)$, and $f'(-3)$. (See Examples 5–7.)

1. $f(x) = 8x + 6$
2. $f(x) = 9 - 2x$
3. $f(x) = -4x^2 + 11x$
4. $f(x) = x^2 - 3x + 7$
5. $f(x) = x^3$
6. $f(x) = x^3 - 6x$
7. $f(x) = \dfrac{-2}{x}$
8. $f(x) = \dfrac{4}{x - 1}$
9. $f(x) = 2\sqrt{x}$
10. $f(x) = 4\sqrt{x}$

The derivatives of each of the given functions were found in Examples 5–8. Use them to find the equation of the tangent line to the graph of the function at the given point. (See Examples 1 and 2.)

11. $g(x) = \sqrt{x}$ at $x = 9$
12. $f(x) = x^3 - 4x$ at $x = 2$
13. $f(x) = 1/x$ at $x = -2$
14. $s(t) = -5t^3 + 30t^2$ at $t = 2$

For each of the given functions, (a) find the slope of the tangent line to the graph at the given point; (b) find the equation of the tangent line. (See Examples 1 and 2.)

15. $f(x) = x^2 + 3$ at $x = 2$
16. $g(x) = 1 - 2x^2$ at $x = -1$
17. $h(x) = \dfrac{7}{x}$ at $x = 3$
18. $f(x) = \dfrac{-3}{x}$ at $x = -4$
19. $g(x) = 5\sqrt{x}$ at $x = 9$
20. $g(x) = \sqrt{x + 1}$ at $x = 15$ (*Hint:* In Step 3, multiply numerator and denominator by $\sqrt{15 + h} + \sqrt{15}$.)

Use the fact that $f'(c)$ is the slope of the tangent line to the graph of $f(x)$ at $x = c$ to work these exercises. (See Example 3.)

21. In the graph on the next page of the function f, at which of the labeled x-values is
    (a) $f(x)$ the largest? **(b)** $f(x)$ the smallest?
    (c) $f'(x)$ the smallest? **(d)** $f'(x)$ the closest to 0?

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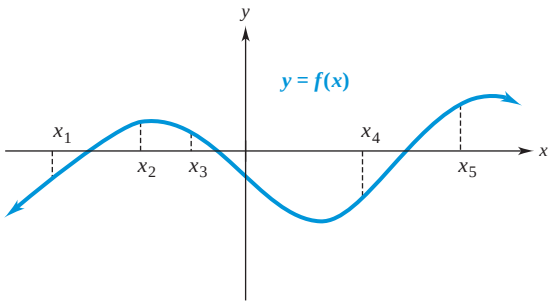
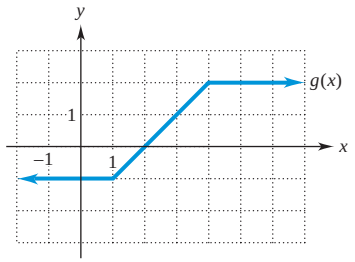
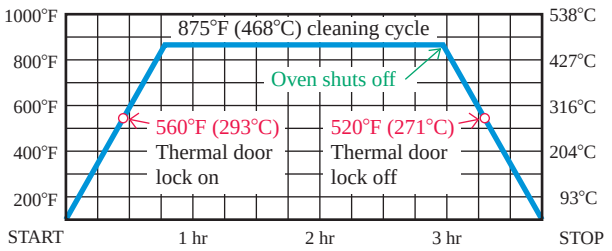


22. Sketch the graph of the *derivative* of the function g whose graph is shown. (*Hint:* Consider the slope of the tangent line at each point along the graph of g. Are there any points where there is no tangent line?)



23. Sketch the graph of a function g with the property that $g'(x) > 0$ for $x < 0$ and $g'(x) < 0$ for $x > 0$. Many correct answers are possible.

24. Physical Science The accompanying graph shows the temperature in an oven during a self-cleaning cycle.* (The open circles on the graph are not points of discontinuity, but merely the times when the thermal door lock turns on and off.) The oven temperature is 100° when the cycle begins and 600° after half an hour. Let $T(x)$ be the temperature (in degrees Fahrenheit) after x hours.



(a) Find and interpret $T'(.5)$.

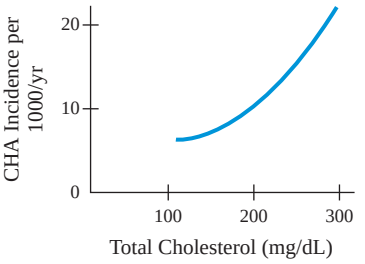
(b) Find and interpret $T'(2)$.

(c) Find and interpret $T'(3.5)$.

25. At what x-values does the derivative of the function graphed in Exercise 24 fail to exist?

*Whirlpool Use and Care Guide, Self-Cleaning Electric Range.

26. Health The graph shows how the risk of coronary heart attack (CHA) rises as blood cholesterol increases.†



(a) Approximate the average rate of change of the risk of coronary heart attack as blood cholesterol goes from 100 to 300 mg/dL.

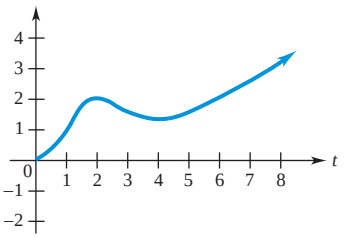
(b) Is the rate of change when blood cholesterol is 100 mg/dL higher or lower than the average rate of change in part (a)? What feature of the graph shows this?

(c) Do part (b) when blood cholesterol is 300 mg/dL.

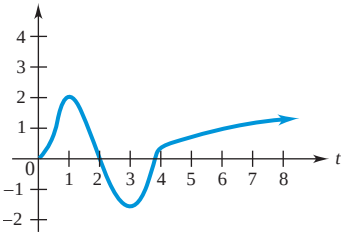
(d) Do part (b) when blood cholesterol is 200 mg/dL.

In Exercises 27 and 28, tell which graph, (a) or (b), represents velocity and which represents distance from a starting point. (Hint: Consider where the derivative is zero, positive, or negative.)

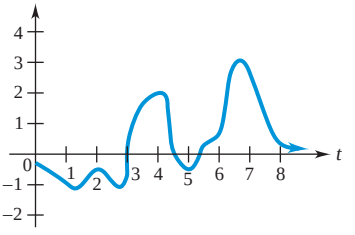
27. (a)



(b)



28. (a)



†John C. LaRosa, et al., "The Cholesterol Facts: A Joint Statement by the American Heart Association and the National Heart, Lung, and Blood Institute," *Circulation* 81, no. 5 (May 1990): 1722.

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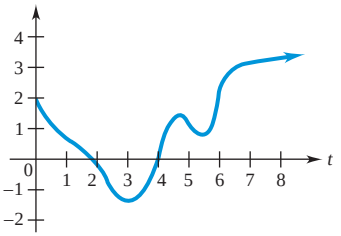
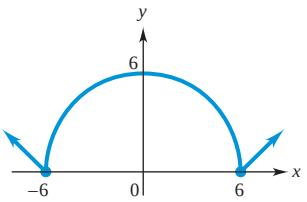
(b)

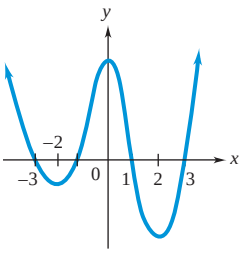


Find all points where the functions whose graphs are shown do not have derivatives. (See Figure 11.31 and the preceding discussion.)

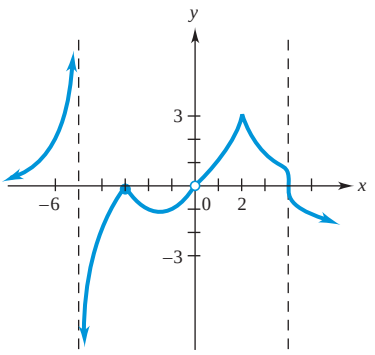
29.



30.



31.



32. **(a)** Sketch the graph of $g(x) = \sqrt[3]{x}$ for $-1 \le x \le 1$.

(b) Explain why the derivative of $g(x)$ is not defined at $x = 0$. (*Hint:* What is the slope of the tangent line at $x = 0$?)

Work these exercises. (See Example 4.)

33. **Natural Science** A tourist drops a stone off the Cliffs of Moher, Ireland and it falls into the Atlantic Ocean. Let $s(t)$ represent the position of the stone (in feet above sea level) t seconds after it was dropped. Explain what the equations $s(7) = 0$ and $s'(7) = -224$ mean in practical terms.

34. **Natural Science** Kayakers like to paddle Slippery Rock Creek, Pennsylvania when the river is at a safe level between 1 and 4 feet high (the river level is measured at a fixed location). Let $H(t)$ represent yesterday's river level (in feet) t hours after midnight. Due to a flash flood, it happened that $H(13) = 3.5$ and $H'(13) = 3$. Explain what this means in practical terms for a kayaker.

35. **Natural Science** On a hiking trail up Mount LeConte, Tennessee, the trailhead has an elevation of 3830 feet and the summit has an elevation of 6953 feet (above sea level). Let $T(x)$ represent the record high temperature on the trail (in degrees Fahrenheit) at an elevation x feet above sea level.

(a) What are the units for $T'(x)$?

(b) Would you expect the sign of $T'(x)$ to be positive, negative, or zero? Why?

36. **Social Science** A polling firm determines that the percent of likely voters who support a certain mayoral candidate can be represented by a function $P(t)$ where t is measured in weeks since the candidate entered the race. Interpret the given equations, including units.

(a) $P'(2) = .5$ **(b)** $P'(6) = 3$

37. **Finance** You plan to take out a 30-year fixed-rate mortgage for \$300,000. Let $P(r)$ be your monthly payment if the interest rate is $r\%$ per year, compounded monthly. Interpret the given equations, including units.

(a) $P(5) = 1{,}610.46$ **(b)** $P'(5) = 183.37$

38. **Finance** You plan to take out a 30-year fixed-rate mortgage at an interest rate of 5.25% per year, compounded monthly. Let $P(A)$ be your monthly payment if A is the initial amount of the loan. Would you expect the sign of $P'(A)$ to be positive, negative, or zero? Why?

39. **Finance** You plan to take out a 30-year fixed-rate mortgage and you want your monthly payment to be \$1200. The amount of the loan you can afford depends on the interest rate you will have to pay. Let $A(r)$ be the amount of the money you can borrow if the interest rate is $r\%$ per year, compounded monthly. Interpret the given equations, including units.

(a) $A(5) = 223{,}537.94$ **(b)** $A'(5) = -25{,}449.21$

40. **Finance** A credit card statement includes information on how long it will take to pay off the card if you make only the minimum payments and how long it will take if you make larger payments. Let $M(p)$ represent the number of months it will take to pay off your credit card if you make a monthly payment of \$$p$ (assuming no additional charges are made).

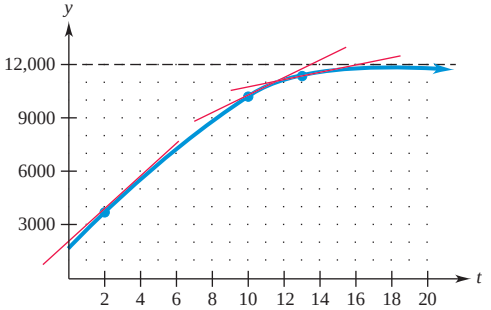
(a) Explain what $M(25) = 50$ means.

(b) Would you expect $M'(p)$ to be positive, negative, or zero? Why?

(c) What are the units for $M'(p)$?

(d) Explain what $M'(25) = -3$ means.

41. **Natural Science** In one research study, the population of a certain shellfish in an area at time t was closely approximated by the accompanying graph. Estimate and interpret the derivative at each of the marked points.



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42. Health The eating behavior of a typical human during a meal can be described by

$$I(t) = 27 + 72t - 1.5t^2,$$

where t is the number of minutes since the meal began and $I(t)$ represents the amount (in grams) that the person has eaten at time t.*

(a) Find the rate of change of the intake of food for a person 5 minutes into a meal, and interpret it.

(b) Verify that the rate at which food is consumed is zero 24 minutes after the meal starts.

(c) Comment on the assumptions and usefulness of this function after 24 minutes. On the basis of your answer, determine a logical range for the function.

Use numerical derivatives to work these exercises. (See Examples 9 and 10.)

43. Business The amount of money (in billions of U.S. dollars) that China spends on research and development can be approximated by the function $f(t) = 26.67e^{.166t}$, where $t = 0$ corresponds to the year 2000. (Data from: National Science Foundation, *National Patterns of R&D Resources*, annual report.)

(a) Estimate the value of the derivative in the years 2000, 2006, and 2012.

(b) What does your answer in part (a) say about the rate at which research and development spending is increasing in China?

44. Business The revenue (in millions) generated by Southwest Airlines can be approximated by the function $R(x) = 1.073x^3 + 52.763x^2 + 137.72x + 5366.4$, where $x = 0$ corresponds to the year 2000. Estimate the derivative for the years 2010 and 2014. (Data from: www.morningstar.com.)

45. Social Science The number of licensed drivers in the United States (in millions) can be approximated by the function $g(x) = 180.532x^{.068}$, where $x = 4$ corresponds to the year 2004. Estimate and interpret the value of the derivative in 2014. (Data from: www.fhwa.dot.gov.)

46. Health The temperature of a patient (in degrees Fahrenheit) during a 12-hour hospital shift is modeled by $P(t) = 98.6 + 2.3te^{-.25t}$, where t is measured in hours since the start of the shift.

(a) Use a calculator to sketch $y = P(t)$, and give a general description of the graph in terms of what it means for the patient.

(b) Find and interpret $P(1)$ and $P'(1)$.

(c) Find and interpret $P(8)$ and $P'(8)$.

Use technology for Exercises 47–50.

47. (a) Graph the (numerical) derivative of

$$f(x) = .5x^5 - 2x^3 + x^2 - 3x + 2 \text{ for } -3 \le x \le 3.$$

(b) Graph $g(x) = 2.5x^4 - 6x^2 + 2x - 3$ on the same screen.

(c) How do the graphs of $f'(x)$ and $g(x)$ compare? What does this suggest that the derivative of $f(x)$ is?

48. Repeat Exercise 47 for $f(x) = (x^2 + x + 1)^{1/3}$ (with $-6 \le x \le 6$) and $g(x) = \dfrac{2x + 1}{3(x^2 + x + 1)^{2/3}}$.

49. By using a graphing calculator to compare graphs, as in Exercises 47 and 48, decide which of the given functions *could* be the derivative of $y = \dfrac{4x^2 + x}{x^2 + 1}$.

(a) $f(x) = \dfrac{2x + 1}{2x}$ **(b)** $g(x) = \dfrac{x^2 + x}{2x}$

(c) $h(x) = \dfrac{2x + 1}{x^2 + 1}$ **(d)** $k(x) = \dfrac{-x^2 + 8x + 1}{(x^2 + 1)^2}$

50. If f is a function such that $f'(x)$ is defined, then it can be proved that

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h}.$$

Consequently, when h is very small, say, $h = .001$,

$$f'(x) \approx \frac{f(x + h) - f(x - h)}{2h} = \frac{f(x + .001) - f(x - .001)}{.002}.$$

(a) In Example 7, we saw that the derivative of $f(x) = \sqrt{x}$ is the function $f'(x) = \dfrac{1}{2\sqrt{x}}$. Make a table in which the first column lists $x = 1$, 6, 11, 16, and 21; the second column lists the corresponding values of $f'(x)$; and the third column lists the corresponding values of

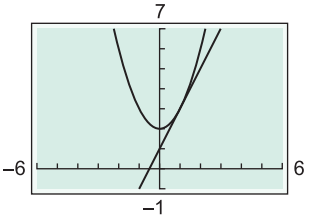
$$\frac{f(x + .001) - f(x - .001)}{.002}.$$

(b) How do the second and third columns of the table compare? (If you use the table feature on a graphing calculator, the entries in the table will be rounded off, so move the cursor over each entry to see it fully displayed at the bottom of the screen.) Your answer to this question may explain why most graphing calculators use the method in the third column to compute numerical derivatives.

Checkpoint Answers

1. $y = 2x + 1$

2.



3. (a) Positive **(b)** Zero **(c)** Negative

4. (a) At the end of class, the drink was 68°F.

*Kissileff, H. R. and J. L. Guss, "Microstructure of Eating Behaviour in Humans," *Appetite*, Vol. 36, No. 1, Feb. 2001, pp. 70–78.

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(b) The drink warmed up by 7.6°F between $t = 25$ and $t = 50$ minutes.

(c) The average rate of change of the temperature of the drink between $t = 25$ and $t = 50$ minutes was .3°F/min.

(d) At the end of class, the drink will warm up by approximately .2°F in the next minute.

5. **(a)** $-5x^2 - 10xh - 5h^2 + 4$

(b) $-10xh - 5h^2$ **(c)** $-10x - 5h$ **(d)** $-10x$

(e) -30 **(f)** 0

6. **(a)** $\dfrac{-5}{x + h}$ **(b)** $\dfrac{5h}{x(x + h)}$ **(c)** $\dfrac{5}{x(x + h)}$

(d) $\dfrac{5}{x^2}$ **(e)** 5

7. $s'(t) = -15t^2 + 60t$

8. **(a)** -73.29 **(b)** -128.2

9. \$15.17

11.5 Techniques for Finding Derivatives

In the previous section, the derivative of a function was defined as a special limit. The mathematical process of finding this limit, called *differentiation*, resulted in a new function that was interpreted in several different ways. Using the definition to calculate the derivative of a function is a very involved process, even for simple functions. In this section, we develop rules that make the calculation of derivatives much easier. Keep in mind that even though the process of finding a derivative will be greatly simplified with these rules, *the interpretation of the derivative will not change.*

In addition to y' and $f'(x)$, there are several other commonly used notations for the derivative.

Notations for the Derivative

The derivative of the function $y = f(x)$ may be denoted in any of the following ways:

$$f'(x), \quad y', \quad \frac{dy}{dx}, \quad \frac{d}{dx}[f(x)], \quad D_x y, \quad \text{or} \quad D_x[f(x)].$$

The dy/dx notation for the derivative is sometimes referred to as *Leibniz notation*, named after one of the coinventors of calculus, Gottfried Wilhelm Leibniz (1646–1716). (The other was Sir Isaac Newton (1642–1727).)

For example, the derivative of $y = x^3 - 4x$, which we found in Example 5 of the last section to be $y' = 3x^2 - 4$, can also be written in the following ways:

$$\frac{dy}{dx} = 3x^2 - 4;$$

$$\frac{d}{dx}(x^3 - 4x) = 3x^2 - 4;$$

$$D_x(x^3 - 4x) = 3x^2 - 4.$$

A variable other than x may be used as the independent variable. For example, if $y = f(t)$ gives population growth as a function of time, then the derivative of y with respect to t could be written as

$$f'(t), \quad \frac{dy}{dt}, \quad \frac{d}{dt}[f(t)], \quad \text{or} \quad D_t[f(t)].$$

In this section, the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

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is used to develop some rules for finding derivatives more easily than by the four-step process of the previous section. The first rule tells how to find the derivative of a constant function, such as $f(x) = 5$. Since $f(x + h)$ is also 5, $f'(x)$ is by definition

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{5 - 5}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0.$$

The same argument works for $f(x) = k$, where k is a constant real number, establishing the following rule.

Constant Rule

If $f(x) = k$, where k is any real number, then

$$\boldsymbol{f'(x) = 0.}$$

(The derivative of a constant function is 0.)

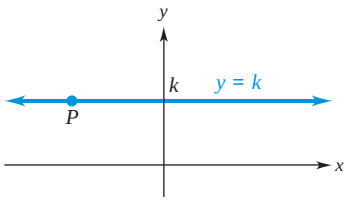

Figure 11.32

Figure 11.32 illustrates the constant rule; it shows a graph of the horizontal line $y = k$. At any point P on this line, the tangent line at P is the line itself. Since a horizontal line has a slope of 0, the slope of the tangent line is 0. This finding agrees with the constant rule: The derivative of a constant is 0.

Example 1 Find and label the derivative of the given function.

(a) $f(x) = 25$

Solution Since 25 is a constant, we can write $f'(x) = 0$, $\frac{d}{dx}[f(x)] = 0$, or $D_x[f(x)] = 0$.

(b) $y = \pi$

Solution Since π is a constant with value equal to 3.14159265 . . . , we have $y' = 0$, $dy/dx = 0$, or $D_x y = 0$.

(c) $y = 4^3$

Solution Since 4^3 is the constant 64, we have $y' = 0$, $dy/dx = 0$, or $D_x y = 0$. ✓1

Checkpoint 1

Find the derivative of the given function.

(a) $y = -4$

(b) $f(x) = \pi^3$

(c) $y = 0$

Functions of the form $y = x^n$, where n is a fixed real number, are common in applications. We now find the derivative functions when $n = 2$ and $n = 3$:

$f(x) = x^2$

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x + h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$f'(x) = \lim_{h\to 0} (2x + h) = 2x$$

$f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x + h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$f'(x) = \lim_{h\to 0} (3x^2 + 3xh + h^2) = 3x^2.$$

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Similar calculations show that

$$\frac{d}{dx}(x^4) = 4x^3 \quad \text{and} \quad \frac{d}{dx}(x^5) = 5x^4.$$

The pattern here (the derivative is the product of the exponent on the original function and a power of x that is one less) suggests that the derivative of $y = x^n$ is $y' = nx^{n-1}$, which is indeed the case. (See Exercise 77 at the end of this section.)

Furthermore, the pattern holds even when n is not a positive integer. For instance, in Example 6 of the last section we showed that

$$\text{if } f(x) = \frac{1}{x}, \quad \text{then } f'(x) = -\frac{1}{x^2}.$$

This is the same result that the preceding pattern produces:

$$\text{If } f(x) = \frac{1}{x} = x^{-1}, \quad \text{then } f'(x) = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}.$$

Similarly, Example 7 in the last section showed that the derivative of $g(x) = \sqrt{x}$ is $g'(x) = \dfrac{1}{2\sqrt{x}}$, and the preceding pattern gives the same answer:

$$\text{If } g(x) = \sqrt{x} = x^{1/2}, \quad \text{then } g'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2}\cdot\frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}.$$

Consequently, the following statement should be plausible.

Power Rule

If $f(x) = x^n$ for any number n, then

$$f'(x) = nx^{n-1}.$$

(The derivative of $f(x) = x^n$ is found by multiplying the exponent n on the original function by a power of x that is one less.)

Example 2 For each of the given functions, find the derivative.

(a) $y = x^8$

Solution The exponent for x is $n = 8$. Using the power rule, we multiply by 8 and decrease the power on x by 1 to obtain $y' = 8x^7$.

(b) $y = x$

Solution The exponent for x is $n = 1$. Using the power rule, we multiply by 1 and decrease the power on x by 1 to obtain $y' = 1x^0$. Recall that $x^0 = 1$ if $x \neq 0$. So we have $y' = 1$.

(c) $y = t^{3/2}$

Solution The exponent for t is $n = \frac{3}{2}$. Using the power rule, we multiply by $\frac{3}{2}$ and decrease the power on t by 1 to obtain $D_t(y) = \frac{3}{2}t^{3/2-1} = \frac{3}{2}t^{1/2}$.

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(d) $y = \sqrt[3]{x}$

Solution First we rewrite $y = \sqrt[3]{x}$ as $y = x^{\frac{1}{3}}$. Using the power rule, we multiply by $\frac{1}{3}$ and decrease the power on x by 1 to obtain

$$\frac{dy}{dx} = \frac{1}{3}x^{1/3-1} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} = \frac{1}{3\sqrt[3]{x^2}}.$$

(e) $y = \dfrac{1}{x^3}$

Solution Rewrite $y = \dfrac{1}{x^3}$ as $y = x^{-3}$. Using the power rule, we multiply by -3 and decrease the power on x by 1 to obtain $\dfrac{dy}{dx} = -3x^{-4} = -\dfrac{3}{x^4}$. ✓2

✓Checkpoint 2

(a) If $y = x^4$, find y'.
(b) If $y = x^{17}$, find y'.
(c) If $y = x^{-2}$, find dy/dx.
(d) If $y = t^{-5}$, find dy/dt.
(e) If $y = t^{5/4}$, find y'.

The next rule shows how to find the derivative of the product of a constant and a function.

Constant Times a Function

Let k be a real number. If $g'(x)$ exists, then the derivative of $f(x) = k \cdot g(x)$ is

$$\boldsymbol{f'(x) = k \cdot g'(x).}$$

(The derivative of a constant times a function is the constant times the derivative of the function.)

Example 3

(a) If $y = 8x^4$, find y'.

Solution Since the derivative of $g(x) = x^4$ is $g'(x) = 4x^3$ and $y = 8x^4 = 8g(x)$,

$$y' = 8g'(x) = 8(4x^3) = 32x^3.$$

(b) If $y = -\dfrac{3}{4}t^{12}$, find dy/dt.

Solution $\dfrac{dy}{dt} = -\dfrac{3}{4}\left[\dfrac{dy}{dt}(t^{12})\right] = -\dfrac{3}{4}(12t^{11}) = -9t^{11}.$

(c) Find $D_x(15x)$.

Solution $D_x(15x) = 15 \cdot D_x(x) = 15(1) = 15.$

(d) If $y = 6/x^2$, find y'.

Solution Replace $\dfrac{6}{x^2}$ by $6 \cdot \dfrac{1}{x^2}$, or $6x^{-2}$. Then

$$y' = 6(-2x^{-3}) = -12x^{-3} = -\frac{12}{x^3}.$$

(e) Find $D_x(10x^{3/2})$, and use a graphing calculator to confirm your answer numerically and graphically.

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Solution $D_x(10x^{3/2}) = 10\left(\frac{3}{2}x^{1/2}\right) = 15x^{1/2}$.

To confirm this result numerically, make a table of values for $y_1 = 15x^{1/2}$ and $y_2 =$ the numerical derivative of $10x^{3/2}$; check your instruction manual for the correct syntax for y_2, which is probably one of the following:

$$\text{nDeriv}(10x^{3/2}, x), \quad \text{Nderiv}(10x^{3/2}, x, x),$$
$$\text{d/dx}(10x^{3/2}), \quad \text{or} \quad \text{d/dx}(10x^{3/2}, x).$$

Figure 11.33(a) indicates that the corresponding values are identical to three decimal places. To confirm this result graphically, graph y_1 and y_2 on the same screen and verify that the graphs appear to be the same. (See Figure 11.33(b)).

Checkpoint 3

Find the derivative of the given function.

(a) $y = 12x^3$

(b) $f(t) = 30t^7$

(c) $y = -35t$

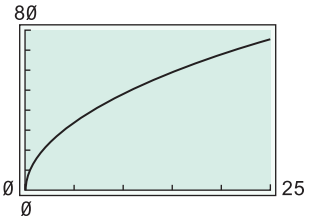
(d) $y = 5\sqrt{x}$

(e) $y = -10/t$

X	Y1	Y2
5	33.541	33.541
6	36.742	36.742
7	39.686	39.686
8	42.426	42.426
9	45	45
10	47.434	47.434
11	49.749	49.749

X=11

(a)



(b)

Figure 11.33

Confirming your calculations numerically or graphically, as in part (e) of Example 3, is a good way to detect algebraic errors. If you compute the rule of the derivative $f'(x)$, but its graph differs from the graph of the numerical derivative of $f(x)$, then you have made a mistake. If the two graphs appear to be identical, then you are *probably* correct. (The fact that two graphs appear identical on a calculator screen does not *prove* that they really are identical.)

The final rule in this section is for the derivative of a function that is a sum or difference of functions.

Sum-or-Difference Rule

If $f(x) = u(x) + v(x)$, and if $u'(x)$ and $v'(x)$ exist, then

$$\boldsymbol{f'(x) = u'(x) + v'(x).}$$

If $f(x) = u(x) - v(x)$, and if $u'(x)$ and $v'(x)$ exist, then

$$\boldsymbol{f'(x) = u'(x) - v'(x).}$$

(The derivative of a sum or difference of two functions is the sum or difference of the derivatives of the functions.)

For a proof of this rule, see Exercise 78 at the end of this section. This rule also works for sums and differences with more than two terms.

Example 4 Find the derivatives of the given functions.

(a) $y = 6x^3 + 15x^2$

Solution Let $u(x) = 6x^3$ and $v(x) = 15x^2$. Then

$$y = u(x) + v(x) \quad \text{and}$$
$$y' = u'(x) + v'(x)$$
$$= 6(3x^2) + 15(2x) = 18x^2 + 30x.$$

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(b) $p(t) = 8t^4 - 6\sqrt{t} + \dfrac{5}{t}$

Solution Rewrite $p(t)$ as $p(t) = 8t^4 - 6t^{1/2} + 5t^{-1}$; then $p'(t) = 32t^3 - 3t^{-1/2} - 5t^{-2}$, which also may be written as $p'(t) = 32t^3 - \dfrac{3}{\sqrt{t}} - \dfrac{5}{t^2}$. ✓4

(c) $f(x) = 5\sqrt[3]{x^2} + 4x^{-2} + 7$

Solution Rewrite $f(x)$ as $f(x) = 5x^{2/3} + 4x^{-2} + 7$. Then

$$D_x[f(x)] = \frac{10}{3}(x^{-1/3}) - 8x^{-3},$$

or

$$D_x[f(x)] = \frac{10}{3\sqrt[3]{x}} - \frac{8}{x^3}.$$ ✓5

✓ Checkpoint 4

Find the derivatives of the given functions.

(a) $y = -10x^3 - 6x + 12$

(b) $y = 5t^{8/7} + 5t^{1/2}$

(c) $f(t) = -2\sqrt{t} + 4/t$

✓ Checkpoint 5

Use a graphing calculator to confirm your answer to part (c) by graphing $D_x[f(x)]$ and the numerical derivative of $f(x)$ on the same screen.

The rules developed in this section make it possible to find the derivative of a function more directly than with the definition of the derivative, so that applications of the derivative can be dealt with more effectively. The examples that follow illustrate some business applications.

Marginal Analysis

In business and economics, the rates of change of such variables as cost, revenue, and profit are important considerations. Economists use the word *marginal* to refer to rates of change: for example, *marginal cost* refers to the rate of change of cost with respect to the number of items produced. Since the derivative of a function gives the rate of change of the function, a marginal cost (or revenue, or profit) function is found by taking the derivative of the cost (or revenue, or profit) function. Roughly speaking, the marginal cost at some level of production x is the cost of producing the $(x + 1)$st item, as we now show. (Similar statements could be made for revenue or profit.)

Look at Figure 11.34, where $C(x)$ represents the cost of producing x units of some item. Then the cost of producing $x + 1$ units is $C(x + 1)$. The cost of the $(x + 1)$st unit is, therefore, $C(x + 1) - C(x)$. This quantity is shown on the graph in Figure 11.34.

Now, if C is the cost function, then the marginal cost C' represents the slope of the tangent line at any point $(x, C(x))$. The graph in Figure 11.35 shows the cost function C and the tangent line at point $P = (x, C(x))$. We know that the slope of the tangent line at P is $C'(x)$ and that the slope can be computed using the triangle PQR in Figure 11.35:

$$C'(x) = \text{slope} = \frac{QR}{PR} = \frac{QR}{1} = QR.$$

So the length of the line segment QR is the number $C'(x)$.

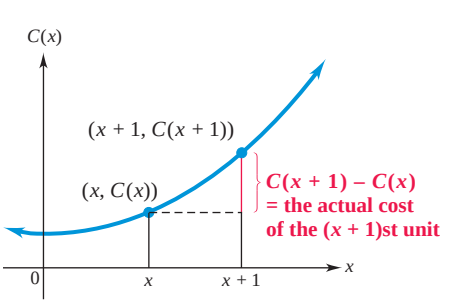

Figure 11.34

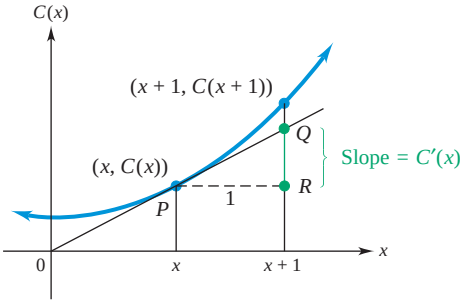

Figure 11.35

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Superimposing the graphs from Figures 11.34 and 11.35, as in Figure 11.36, shows that $C'(x)$ is indeed very close to $C(x + 1) - C(x)$. Therefore, we have the conclusion in the following box.

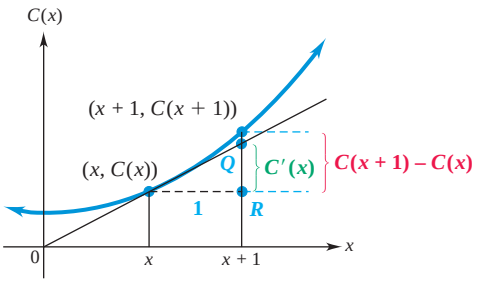

Figure 11.36

Marginal Cost

If $C(x)$ is the cost function, then the marginal cost (rate of change of cost) is given by the derivative $C'(x)$:

$$C'(x) \approx \text{cost of making one more item after } x \text{ items have been made.}$$

The marginal revenue $R'(x)$ and marginal profit $P'(x)$ are interpreted similarly.

Example 5 **Business** A land developer has purchased 115 acres in the Blue Ridge Mountains to subdivide and sell as one-acre lots for mountain homes. Construction of gravel roads to access the lots is more expensive at higher elevations on the property. The cost of access roads (in thousands of dollars) for x lots can be approximated by

$$C(x) = 150 + 9x + .5x^{1.6} \qquad (0 \le x \le 115).$$

Find and interpret the marginal cost for the given values of x.

(a) $x = 25$

Solution To find the marginal cost, first find the derivative of the cost function:

$$C'(x) = 9 + .5(1.6x^{.6}) = 9 + .8x^{.6}.$$

When $x = 25$,

$$C'(25) = 9 + .8(25)^{.6} = 14.519.$$

After 25 access roads have been constructed, the cost of constructing one more access road will be *approximately* \$14,519.

Note that the *actual* cost of constructing one more access road is:

$$\begin{aligned} C(26) - C(25) &= (150 + 9(26) + .5(26)^{1.6}) - (150 + 9(25) + .5(25)^{1.6}) \\ &\approx 14.585. \end{aligned}$$

So the actual cost to construct one additional road is \$14,585.

(b) $x = 50$

Solution After 50 access roads have been constructed, the cost of constructing an additional access road will be *approximately*

$$C'(50) = 9 + .8(50)^{.6} = 17.365,$$

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or $17,365. Compare this result with that for part (a). The cost of accessing one additional lot is almost $3000 more after 50 lots have road access than the cost of accessing one additional lot when only 25 lots have road access. Management must be careful to keep track of marginal costs. If the marginal cost of producing an extra unit exceeds the revenue received from selling it, the company will lose money on that unit. ✓6

✓Checkpoint 6

A cost function is given by $C(x) = 2790 + 30x + 12\sqrt{x}$. Find the marginal cost at the given production levels.

(a) $x = 1$

(b) $x = 4$

Demand Functions

The **demand function**, defined by $p = f(x)$, relates the number of units, x, of an item that consumers are willing to purchase at the price p. (Demand functions were also discussed in Section 3.3.) The total revenue $R(x)$ is related to the price per unit and the amount demanded (or sold) by the equation

$$R(x) = xp = x \cdot f(x).$$

Example 6 **Business** The demand function for a certain product is given by

$$p = \frac{50{,}000 - x}{25{,}000}.$$

Find the marginal revenue when $x = 10{,}000$ units and p is in dollars.

Solution From the function p, the revenue function is given by

$$\begin{aligned} R(x) &= xp \\ &= x\left(\frac{50{,}000 - x}{25{,}000}\right) \\ &= \frac{50{,}000x - x^2}{25{,}000} = 2x - \frac{1}{25{,}000}x^2. \end{aligned}$$

The marginal revenue is

$$R'(x) = 2 - \frac{2}{25{,}000}x.$$

When $x = 10{,}000$, the marginal revenue is

$$R'(10{,}000) = 2 - \frac{2}{25{,}000}(10{,}000) = 1.2,$$

or $1.20 per unit. Thus, the next unit sold (at sales of 10,000) will produce additional revenue of about $1.20. ✓7

✓Checkpoint 7

Suppose the demand function for x units of an item is

$$p = 5 - \frac{x}{1000},$$

where p is the price in dollars. Find

(a) the marginal revenue;

(b) the marginal revenue at $x = 500$;

(c) the marginal revenue at $x = 1000$.

In economics, the demand function is written in the form $p = f(x)$, as in Example 6. From the perspective of a consumer, it is probably more reasonable to think of the quantity demanded as a function of price. Mathematically, these two viewpoints are equivalent. In Example 6, the demand function could have been written from the consumer's viewpoint as

$$x = 50{,}000 - 25{,}000p.$$

Example 7 **Business** Suppose that the cost function for the product in Example 6 is given by

$$C(x) = 2100 + .25x \qquad (0 \le x \le 30{,}000).$$

Find the marginal profit from the production of the given numbers of units.

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(a) 15,000

Solution From Example 6, the revenue from the sale of x units is

$$R(x) = 2x - \frac{1}{25{,}000}x^2.$$

Since profit P is given by $P = R - C$,

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= \left(2x - \frac{1}{25{,}000}x^2\right) - (2100 + .25x) \\ &= 2x - \frac{1}{25{,}000}x^2 - 2100 - .25x \\ &= 1.75x - \frac{1}{25{,}000}x^2 - 2100. \end{aligned}$$

The marginal profit from the sale of x units is

$$P'(x) = 1.75 - \frac{2}{25{,}000}x = 1.75 - \frac{1}{12{,}500}x.$$

At $x = 15{,}000$, the marginal profit is

$$P'(15{,}000) = 1.75 - \frac{1}{12{,}500}(15{,}000) = .55,$$

or \$.55 per unit. This means that the next unit sold (at sales of 15,000) will produce additional profit of about 55¢.

(b) 21,875

Solution When $x = 21{,}875$, the marginal profit is

$$P'(21{,}875) = 1.75 - \frac{1}{12{,}500}(21{,}875) = 0.$$

(c) 25,000

Solution When $x = 25{,}000$, the marginal profit is

$$P'(25{,}000) = 1.75 - \frac{1}{12{,}500}(25{,}000) = -.25,$$

or $-$\$.25 per unit.

As shown by parts (b) and (c), if more than 21,875 units are sold, the marginal profit is negative. This indicates that increasing production beyond that level will *reduce* profit. ✓8

✓Checkpoint 8

For a certain product, the cost is $C(x) = 1300 + .80x$ and the revenue is $R(x) = 6x - \dfrac{2x^2}{11{,}000}$ for x units.

(a) Find the profit $P(x)$.

(b) Find $P'(12{,}000)$.

(c) Find $P'(25{,}000)$.

(d) Interpret the results of parts (b) and (c).

The final example shows a medical application of the derivative as the rate of change of a function.

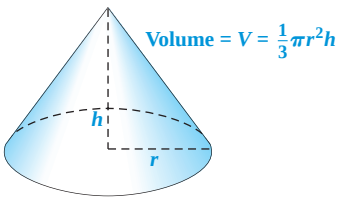

Figure 11.37

Example 8 **Health** A tumor has the approximate shape of a cone. (See Figure 11.37.) The radius of the tumor is fixed by the bone structure at 2 centimeters, but the tumor is growing along the height of the cone. The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height of the cone. Find the rate of change of the volume of the tumor with respect to the height.

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Checkpoint 9

A balloon is spherical. The formula for the volume of a sphere is $V = (4/3)\pi r^3$, where r is the radius of the sphere. Find the given quantities.

(a) dV/dr

(b) The rate of change of the volume when $r = 3$ inches

Solution To emphasize that the rate of change of the volume is found with respect to the height, we use the symbol dV/dh for the derivative. For this tumor, r is fixed at 2 cm. By substituting 2 for r,

$$V = \frac{1}{3}\pi r^2 h \quad \text{becomes} \quad V = \frac{1}{3}\pi \cdot 2^2 \cdot h, \quad \text{or} \quad V = \frac{4}{3}\pi h.$$

Since $4\pi/3$ is constant,

$$\frac{dV}{dh} = \frac{4\pi}{3} \approx 4.2 \text{ cu cm per cm.}$$

For each additional centimeter that the tumor grows in height, its volume will increase by approximately 4.2 cubic centimeters. ✓9

11.5 Exercises

Find the derivatives of the given functions. (See Examples 1–4.)

1. $f(x) = 4x^2 - 3x + 5$

2. $g(x) = 8x^2 + 2x - 12$

3. $y = 2x^3 + 3x^2 - 6x + 2$

4. $y = 4x^3 + 24x + 4$

5. $g(x) = x^4 + 3x^3 - 8x - 7$

6. $f(x) = 6x^6 - 3x^4 + x^3 - 3x + 9$

7. $f(x) = 6x^{1.5} - 4x^{.5}$

8. $f(x) = -2x^{2.5} + 8x^{.5}$

9. $y = -15x^{3/2} + 2x^{1.9} + x$

10. $y = 18x^{1.6} - 4x^{3.1} + x^4$

11. $y = 24t^{3/2} + 4t^{1/2}$

12. $y = -24t^{5/2} - 6t^{1/2}$

13. $y = 8\sqrt{x} + 6x^{3/4}$

14. $y = -100\sqrt{x} - 11x^{2/3}$

15. $g(x) = 6x^{-5} - 2x^{-1}$

16. $y = 8x^{-2} + 6x^{-1} + 1$

17. $y = 10x^{-2} + 6x^{-4} + 3x$

18. $y = 3x^{-7} + 4x^{-5} - 9x^{-4} + 21x^{-1}$

19. $f(t) = \dfrac{-4}{t} + \dfrac{8}{t^2}$

20. $f(t) = \dfrac{8}{t} - \dfrac{5}{t^2}$

21. $y = \dfrac{12 - 7x + 6x^3}{x^4}$

22. $y = \dfrac{102 - 6x + 17x^4}{x^6}$

23. $g(x) = 5x^{-1/2} - 7x^{1/2} + 101x$

24. $f(x) = -8x^{-1/2} - 8x^{1/2} + 12x$

25. $y = 7x^{-3/2} + 8x^{-1/2} + x^3 - 9$

26. $y = 2x^{-3/2} + 9x^{-1/2} + x^{-4} - x^2$

27. $y = \dfrac{19}{\sqrt[4]{x}}$

28. $y = \dfrac{-5}{\sqrt[3]{x}}$

29. $y = \dfrac{-8t}{\sqrt[3]{t^2}}$

30. $g(t) = \dfrac{10t}{\sqrt[3]{t^8}}$

Find each of the given derivatives. (See Examples 1–4.)

31. $\dfrac{dy}{dx}$ if $y = 8x^{-5} - 9x^{-4} + 9x^4$

32. $\dfrac{dy}{dx}$ if $y = -3x^{-2} - 4x^{-5} + 5x^{-7}$

33. $D_x\left(9x^{-1/2} + \dfrac{2}{x^{3/2}}\right)$

34. $D_x\left(\dfrac{8}{\sqrt[4]{x}} - \dfrac{3}{\sqrt{x^3}}\right)$

35. $f'(-2)$ if $f(x) = 6x^2 - 2x$

36. $f'(3)$ if $f(x) = 9x^3 - 6x^2$

37. $f'(4)$ if $f(t) = 2\sqrt{t} - \dfrac{3}{\sqrt{t}}$

38. $f'(8)$ if $f(t) = -5\sqrt[3]{t} + \dfrac{6}{\sqrt[3]{t}}$

39. If $f(x) = -\dfrac{(3x^2 + x)^2}{7}$, which of the following is *closest* to $f'(1)$?

(a) -12 **(b)** -9 **(c)** -6
(d) -3 **(e)** 0 **(f)** 3

40. If $g(x) = -3x^{3/2} + 4x^2 - 9x$, which of the following is *closest* to $g'(4)$?

(a) 3 **(b)** 6 **(c)** 9
(d) 12 **(e)** 15 **(f)** 18

Find the slope and the equation of the tangent line to the graph of each function at the given value of x.

41. $f(x) = x^4 - 2x^2 + 1; x = 1$

42. $g(x) = -x^5 + 4x^2 - 2x + 2; x = 2$

43. $y = 4x^{1/2} + 2x^{3/2} + 1; x = 4$

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44. $y = -x^{-3} + 5x^{-1} + x; x = 2$

Work these exercises. (See Examples 5 and 7.)

45. Business The profit in dollars from the sale of x expensive watches is

$$P(x) = .03x^2 - 4x + 3x^{.8} - 5000.$$

Find the marginal profit for the given values of x.

(a) $x = 100$ **(b)** $x = 1000$

(c) $x = 5000$ **(d)** $x = 10{,}000$

46. Business The total cost to produce x handcrafted weather vanes is

$$C(x) = 100 + 12x + .1x^2 + .001x^3.$$

Find the marginal cost for the given values of x.

(a) $x = 0$ **(b)** $x = 10$

(c) $x = 30$ **(d)** $x = 50$

47. Business Often, sales of a new product grow rapidly at first and then slow down with time. This is the case with the sales represented by the function

$$S(t) = 10{,}000 - 10{,}000t^{-.2} + 100t^{.1},$$

where t represents time in years. Find the rate of change of sales for the given values of t.

(a) $t = 1$ **(b)** $t = 10$

48. Business The revenue equation (in billions of dollars) for corn production in the United States is approximated by

$$R(x) = .016x^2 + 1.034x + 11.338,$$

where x is in billions of bushels. (Data from: www.usda.gov.)

(a) Find the marginal-revenue function.

(b) Find the marginal revenue for the production of 10 billion bushels.

(c) Find the marginal revenue for the production of 20 billion bushels.

49. Business The revenue equation (in hundreds of millions of dollars) for barley production in the United States is approximated by

$$R(x) = .0608x^2 + 1.4284x + 2.3418,$$

where x is in hundreds of millions of bushels. (Data from: www.usda.gov.)

(a) Find the marginal-revenue function.

(b) Find the marginal revenue for the production of 500,000,000 bushels.

(c) Find the marginal revenue for the production of 700,000,000 bushels.

50. Business The total cost in dollars to produce x hundred tee shirts imprinted with a college logo can be approximated by

$$C(x) = 1500 + 1000x^{.9} \qquad (0 \le x \le 50).$$

Find the marginal cost for the given production levels.

(a) 1500 shirts **(b)** 2000 shirts

51. Business The revenue generated from the sale of x picnic tables is given by

$$R(x) = 20x - \frac{x^2}{500}.$$

(a) Find the marginal revenue when $x = 1000$ units.

(b) Determine the actual revenue from the sale of the 1001st item.

(c) Compare the answers to parts (a) and (b). How are they related?

52. Business The profit (in dollars) from producing x hundred kegs of beer is given by $P(x) = 1500 + 30x - 2x^2$. Find the marginal profit at the given production levels. In each case, decide whether the firm should increase production of beer.

(a) 700 **(b)** 650

(c) 1200 **(d)** 1800

53. Business The demand for a certain item is given by $D(p) = -p^2 + 5p + 1$, where p represents the price of an item in dollars.

(a) Find the rate of change of demand with respect to price.

(b) Find and interpret the rate of change of demand when the price is 12 dollars.

54. Business The profit in dollars from the sale of x hundred linear feet of custom wood molding can be approximated by

$$P(x) = 400x - 490\sqrt{x} - 600 \qquad (5 \le x \le 25).$$

Find the marginal profit for the given values of x.

(a) $x = 10$ **(b)** $x = 15$

Work these exercises. (See Example 6.)

55. Business The annual demand equation for handcrafted violins by a certain violin maker can be approximated by $p = 24 - x$ where p is the price in thousands of dollars and x is the quantity of violins demanded. Find and interpret the marginal revenue for each of the given production levels.

(a) $x = 5$ **(b)** $x = 10$ **(c)** $x = 12$

56. Business The cost (in thousands of dollars) of making x violins described in the previous exercise can be approximated by $C(x) = 15 + 1.2x$. Find the marginal profit for each of the given production levels.

(a) $x = 5$ **(b)** $x = 10$ **(c)** $x = 12$

57. Business Assume that a demand equation is given by $x = 5000 - 100p$. Find the marginal revenue for the given production levels (values of x). (*Hint:* Solve the demand equation for p and use $R(x) = xp$.)

(a) 1000 units **(b)** 2500 units

(c) 3000 units

58. Business Suppose that, for the situation in Exercise 57, the cost of producing x units is given by $C(x) = 3000 - 20x + .03x^2$. Find the marginal profit for each of the given production levels.

(a) 500 units **(b)** 815 units

(c) 1000 units

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Work these exercises.

59. Business An insurance company typically offers discounts for purchasing larger amounts of life insurance coverage. For a female under 30 years of age who qualifies for the lowest premiums, the annual cost (in dollars) of a life insurance policy can be approximated by

$$C(x) = 3\sqrt[4]{x^3},$$

where x is the amount of life insurance coverage (in thousands of dollars). Find and interpret the given quantities. (Data from: www.abendowment.org/insurance/life20_rate.asp.)

(a) $C(100)$ and $C'(100)$ **(b)** $C(1000)$ and $C'(1000)$

60. Business For a male 50 years of age who qualifies for the lowest premiums, the annual cost (in dollars) of a life insurance policy can be approximated by $C(x) = 5.60x^{.88}$, where x is the amount of life insurance coverage (in thousands of dollars). Find and interpret the given quantities. (Data from: www.abendowment.org/insurance/life20_rate.asp.)

(a) $C(100)$ and $C'(100)$ **(b)** $C(1000)$ and $C'(1000)$

61. Business A marketing firm is interested in the percentage of the total United States population that is age 0–17 years. This percentage can be estimated by $P(x) = 26x^{-.03}$, where $x = 0$ corresponds to the year 2000 and the estimate is valid through the year 2025. Find and interpret the given quantities. (Data from: www.childstats.gov.)

(a) $P(14)$ and $P'(14)$ **(b)** $P(24)$ and $P'(24)$

62. Business A marketing firm is interested in the percentage of the total United States population that is age 65 years or older. This percentage can be estimated by $P(x) = .01x^2 - .01x + 12$, where $x = 0$ corresponds to the year 2000 and the estimate is valid through the year 2025. Find and interpret the given quantities. (Data from: www.childstats.gov.)

(a) $P(14)$ and $P'(14)$ **(b)** $P(24)$ and $P'(24)$

63. Social Science The women's 100-meter dash was introduced to the Olympic Games in 1928. Between 1928 and 2012, the winning time (in seconds) can be approximated by $W(x) = .000178x^2 - .0404x + 13.068$, where $x = 0$ corresponds to the year 1900. (Data from: www.Olympic.org.)

(a) Would you expect the derivative of $W(x)$ to be positive, negative, or zero? Why?

(b) Find and interpret the rate of change of $W(x)$ in 1928.

(c) Find and interpret the rate of change of $W(x)$ in 2012.

64. Finance The total amount (in billions of dollars) of currency (in notes) circulating in the United States can be approximated by $M(x) = .52x^3 - 7.52x^2 + 62.37x + 559.79$ where $x = 0$ corresponds to the year 2000. Find the derivative of $M(x)$ and use it to find the rate of change of the currency in circulation in the given years. (Data from: U.S. Federal Reserve.)

(a) 2001 **(b)** 2011

65. Economics The Case-Shiller Composite 20 Home Price Index can be approximated by $C(x) = .114x^4 - 2.8x^3 + 19.89x^2 - 30.56x + 118.14$, where $x = 0$ corresponds to the year 2000. Find and interpret the rate of change of this home price index in the given years. (Data from: Standard & Poor's.)

(a) 2008 **(b)** 2011

66. Social Science According to the U.S. Census Bureau, the number of Americans (in thousands) who are expected to be over 100 years old in year x is approximated by the function $f(x) = .27x^2 + 3.52x + 51.78$, where $x = 0$ corresponds to the year 2000 and the formula is valid through 2045.

(a) Find a formula giving the rate of change in the number of Americans over 100 years old.

(b) What is the rate of change in the number of Americans expected to be over 100 years old in the year 2020?

(c) Is the number of Americans expected to be over 100 years old in 2020 increasing or decreasing?

67. Natural Science A short length of blood vessel has a cylindrical shape. The volume of a cylinder is given by $V = \pi r^2 h$. Suppose an experimental device is set up to measure the volume of blood in a blood vessel of fixed length 80 mm as the radius changes.

(a) Find dV/dr.

Suppose a drug is administered that causes the blood vessel to expand. Evaluate dV/dr for the following values of r, and interpret your answers:

(b) 4 mm; **(c)** 6 mm; **(d)** 8 mm.

68. Health The birth weight (in pounds) of an infant can be approximated by the function

$$g(x) = .0011x^{2.394},$$

where x represents the number of completed weeks of gestation.

(a) Find the function $g'(x)$.

(b) Evaluate $g'(32)$.

(c) Evaluate $g'(40)$.

(d) Interpret the results of parts (b) and (c).

69. Health The body mass index (BMI) is a number that can be calculated for any individual as follows: Multiply weight (in pounds) by 703 and divide by the person's height (in inches) squared. That is,

$$\text{BMI} = \frac{703w}{h^2},$$

where w is in pounds and h is in inches. The National Heart, Lung, and Blood Institute uses the BMI to determine whether a person is "overweight" ($25 \le \text{BMI} < 30$) or "obese" ($\text{BMI} \ge 30$).

(a) Calculate the BMI for a male who weighs 220 pounds and is $6'2''$ tall.

(b) How much weight would the person in part (a) have to lose until he reaches a BMI of 24.9 and is no longer "overweight"?

(c) For a 125-lb female, what is the rate of change of BMI with respect to height? (*Hint:* Take the derivative of the function $f(h) = \dfrac{703(125)}{h^2}$.)

(d) Calculate and interpret the meaning of $f'(65)$.

70. Health To increase the velocity of the air flowing through the trachea when a human coughs, the body contracts the windpipe, producing a more effective cough. Tuchinsky determined that

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the velocity V of the air flowing through the trachea during a cough is given by

$$V = C(R_0 - R)R^2,$$

where C is a constant based on individual body characteristics, R_0 is the radius of the windpipe before the cough, and R is the radius of the windpipe during the cough.*

(a) Think of V as a function of R. Multiply out the rule for V and find $\frac{dV}{dR}$. (*Hint:* C and R_0 are constants here; if you cannot see what to do, it may help to consider the case when $C = 2$ and $R_0 = .15$.)

(b) It can be shown that the maximum velocity of the cough occurs when $\frac{dV}{dR} = 0$. Find the value of R that maximizes the velocity.† (Remember that R must be positive.)

Physical Science *We saw earlier that the velocity of a particle moving in a straight line is given by*

$$\lim_{h\to 0}\frac{s(t+h)-s(t)}{h},$$

where $s(t)$ gives the position of the particle at time t. This limit is the derivative of $s(t)$, so the velocity of a particle is given by $s'(t)$. If $v(t)$ represents velocity at time t, then $v(t) = s'(t)$. For each of the given position functions, find (a) $v(t)$; (b) the velocity when $t = 0$, $t = 5$, and $t = 10$.

71. $s(t) = 8t^2 + 3t + 1$

72. $s(t) = 10t^2 - 5t + 6$

73. $s(t) = 2t^3 + 6t^2$

74. $s(t) = -t^3 + 3t^2 + t - 1$

75. Physical Science If a rock is dropped from a 144-foot-high building, its position (in feet above the ground) is given by $s(t) = -16t^2 + 144$, where t is the time in seconds since it was dropped.

(a) What is its velocity 1 second after being dropped? 2 seconds after being dropped?

(b) When will it hit the ground?

(c) What is its velocity upon impact?

76. Physical Science A ball is thrown vertically upward from the ground at a velocity of 64 feet per second. Its distance from the ground at t seconds is given by $s(t) = -16t^2 + 64t$.

(a) How fast is the ball moving 2 seconds after being thrown? 3 seconds after being thrown?

(b) How long after the ball is thrown does it reach its maximum height?

(c) How high will the ball go?

77. Perform each step and give reasons for your results in the following proof that the derivative of $y = x^n$ is $y' = n \cdot x^{n-1}$ (we prove this result only for positive integer values of n, but it is valid for all values of n):

(a) Recall the binomial theorem from algebra:

$$(p+q)^n = p^n + n\cdot p^{n-1}q + \frac{n(n-1)}{2}p^{n-2}q^2 + \cdots + q^n.$$

Evaluate $(x+h)^n$.

(b) Find the quotient $\frac{(x+h)^n - x^n}{h}$.

(c) Use the definition of the derivative to find y'.

78. Perform each step and give reasons for your results in the following proof that the derivative of $y = f(x) + g(x)$ is

$$y' = f'(x) + g'(x).$$

(a) Let $s(x) = f(x) + g(x)$. Show that

$$s'(x) = \lim_{h\to 0}\frac{[f(x+h)+g(x+h)] - [f(x)+g(x)]}{h}.$$

(b) Show that

$$s'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right].$$

(c) Finally, show that $s'(x) = f'(x) + g'(x)$.

Use a graphing calculator or computer to graph each function and its derivative on the same screen. Determine the values of x where the derivative is (a) positive, (b) zero, and (c) negative. (d) What is true of the graph of the function in each case?

79. $g(x) = 6 - 4x + 3x^2 - x^3$

80. $k(x) = 2x^4 - 3x^3 + x$

Checkpoint Answers

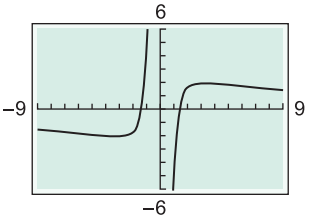
1. **(a)** 0 **(b)** 0 **(c)** 0

2. **(a)** $y' = 4x^3$ **(b)** $y' = 17x^{16}$ **(c)** $dy/dx = -2/x^3$
(d) $dy/dt = -5/t^6$ **(e)** $y' = \frac{5}{4}t^{1/4}$

3. **(a)** $36x^2$ **(b)** $210t^6$ **(c)** -35
(d) $(5/2)x^{-1/2}$, or $5/(2\sqrt{x})$ **(e)** $10t^{-2}$, or $10/t^2$

4. **(a)** $y' = -30x^2 - 6$
(b) $y' = \frac{40}{7}t^{1/7} + \frac{5}{2}t^{-1/2}$, or $y' = \frac{40}{7}t^{1/7} + \frac{5}{2t^{1/2}}$
(c) $f'(t) = \frac{-1}{\sqrt{t}} - \frac{4}{t^2}$

5. Both graphs look like this:



6. **(a)** \$36 per unit **(b)** \$33 per unit

*Philip Tuchinsky, "The Human Cough," *UMAP Module 211* (Lexington, MA: COMAP, Inc, 1979): 1–9.

†Interestingly, Tuchinsky also states that X-rays indicate that the body naturally contracts the windpipe to this radius during a cough.

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7. **(a)** $R'(x) = 5 - \dfrac{x}{500}$

(b) \$4 **(c)** \$3

8. **(a)** $P(x) = 5.2x - \dfrac{2x^2}{11{,}000} - 1300$

(b) .84 **(c)** −3.89

(d) Profit is increasing by \$.84 per unit at 12,000 units in part (b) and decreasing by \$3.89 per unit at 25,000 units in part (c).

9. **(a)** $4\pi r^2$

(b) 36π cubic inches per inch

11.6 Derivatives of Products and Quotients

In the last section, we saw that the derivative of the sum of two functions can be obtained by taking the sum of the derivatives. What about products? Is the derivative of a product of two functions equal to the product of their derivatives? For example, if

$$u(x) = 2x + 4 \quad \text{and} \quad v(x) = 3x^2,$$

then the product of u and v is

$$f(x) = (2x + 4)(3x^2) = 6x^3 + 12x^2.$$

Using the rules of the last section, we have

$$u'(x) = 2, \quad v'(x) = 6x, \quad \text{and} \quad f'(x) = 18x^2 + 24x,$$

so that

$$u'(x) \cdot v'(x) = 12x \quad \text{and} \quad f'(x) = 18x^2 + 24x.$$

Obviously, these two functions are *not* the same, which shows that the derivative of the product is *not* equal to the product of the derivatives. ✓1

The correct rule for finding the derivative of a product is as follows.

✓ Checkpoint 1

Let $u(x) = x^2$ and $v(x) = x^3$.

(a) Find $f(x) = u(x) \cdot v(x)$.

(b) Find $f'(x) = [u(x) \cdot v(x)]'$.

(c) Find $u'(x)$.

(d) Find $v'(x)$.

(e) Find $u'(x) \cdot v'(x)$.

(f) Check that $[u(x) \cdot v(x)]' \neq u'(x) \cdot v'(x)$.

Product Rule

If $f(x) = u(x) \cdot v(x)$, and if both $u'(x)$ and $v'(x)$ exist, then

$$\boldsymbol{f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x).}$$

(The derivative of a product of two functions is the first function times the derivative of the second, plus the second function times the derivative of the first.)

To sketch the method used to prove the product rule, let

$$f(x) = u(x) \cdot v(x).$$

Then $f(x + h) = u(x + h) \cdot v(x + h)$, and, by definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{u(x + h) \cdot v(x + h) - u(x) \cdot v(x)}{h}.$$

Now subtract and add $u(x + h) \cdot v(x)$ in the numerator, giving

$$f'(x) = \lim_{h \to 0} \frac{u(x + h) \cdot v(x + h) \; \mathbf{- u(x + h) \cdot v(x) + u(x + h) \cdot v(x)} - u(x) \cdot v(x)}{h}.$$

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Factor $u(x + h)$ from the first two terms of the numerator, and factor $v(x)$ from the last two terms. Then use the properties of limits (Section 11.1) as follows:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{u(x+h)[v(x+h) - v(x)] + v(x)[u(x+h) - u(x)]}{h} \\ &= \lim_{h\to 0} u(x+h)\left[\frac{v(x+h) - v(x)}{h}\right] + \lim_{h\to 0} v(x)\left[\frac{u(x+h) - u(x)}{h}\right] \\ &= \lim_{h\to 0} u(x+h) \cdot \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} + \lim_{h\to 0} v(x) \cdot \lim_{h\to 0} \frac{u(x+h) - u(x)}{h}. \quad (*) \end{aligned}$$

If u' and v' both exist, then

$$\lim_{h\to 0} \frac{u(x+h) - u(x)}{h} = u'(x) \qquad \text{and} \qquad \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} = v'(x).$$

The fact that u' exists can be used to prove that

$$\lim_{h\to 0} u(x+h) = u(x),$$

and since no h is involved in $v(x)$,

$$\lim_{h\to 0} v(x) = v(x).$$

Substituting these results into equation (*) gives

$$f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x),$$

the desired result.

Example 1 Let $f(x) = (2x + 4)(3x^2)$. Use the product rule to find $f'(x)$.

Solution Here, f is given as the product of $u(x) = 2x + 4$ and $v(x) = 3x^2$. By the product rule and the fact that $u'(x) = 2$ and $v'(x) = 6x$,

$$\begin{aligned} f'(x) &= u(x) \cdot \boldsymbol{v'(x)} + v(x) \cdot u'(x) \\ &= (2x + 4)\boldsymbol{(6x)} + (3x^2)\boldsymbol{(2)} \\ &= 12x^2 + 24x + 6x^2 = 18x^2 + 24x. \end{aligned}$$

This result is the same as that found at the beginning of the section. ✓2

Checkpoint 2

Use the product rule to find the derivatives of the given functions.

(a) $f(x) = (5x^2 + 6)(3x)$

(b) $g(x) = (8x)(4x^2 + 5x)$

Example 2 Find the derivative of $y = (\sqrt{x} + 3)(x^2 - 5x)$.

Solution Let $u(x) = \sqrt{x} + 3 = x^{1/2} + 3$ and $v(x) = x^2 - 5x$. Then

$$\begin{aligned} y' &= u(x) \cdot \boldsymbol{v'(x)} + v(x) \cdot \boldsymbol{u'(x)} \\ &= (x^{1/2} + 3)\boldsymbol{(2x - 5)} + (x^2 - 5x)\left(\boldsymbol{\frac{1}{2}x^{-1/2}}\right) \\ &= 2x^{3/2} - 5x^{1/2} + 6x - 15 + \frac{1}{2}x^{3/2} - \frac{5}{2}x^{1/2} \\ &= \frac{5}{2}x^{3/2} + 6x - \frac{15}{2}x^{1/2} - 15. \end{aligned}$$

✓3

Checkpoint 3

Find the derivatives of the given functions.

(a) $f(x) = (x^2 - 3)(\sqrt{x} + 5)$

(b) $g(x) = (\sqrt{x} + 4)(5x^2 + x)$

We could have found the derivatives in Examples 1 and 2 by multiplying out the original functions. The product rule then would not have been needed. In the next section, however, we shall see products of functions where the product rule is essential.

What about *quotients* of functions? To find the derivative of the quotient of two functions, use the next rule.

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Quotient Rule

If $f(x) = \dfrac{u(x)}{v(x)}$, if all indicated derivatives exist, and if $v(x) \neq 0$, then

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}.$$

(The derivative of a quotient is the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.)

The proof of the quotient rule is similar to that of the product rule and is omitted here.

CAUTION Just as the derivative of a product is *not* the product of the derivatives, the derivative of a quotient is *not* the quotient of the derivatives. If you are asked to take the derivative of a product or a quotient, it is essential that you recognize that the function contains a product or quotient and then use the appropriate rule.

Example 3 Find $f'(x)$ if

$$f(x) = \frac{2x - 1}{4x + 3}.$$

Solution Let $u(x) = 2x - 1$, with $u'(x) = 2$. Also, let $v(x) = 4x + 3$, with $v'(x) = 4$. Then, by the quotient rule,

$$\begin{aligned} f'(x) &= \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2} \\ &= \frac{(4x + 3)(2) - (2x - 1)(4)}{(4x + 3)^2} \\ &= \frac{8x + 6 - 8x + 4}{(4x + 3)^2} \\ f'(x) &= \frac{10}{(4x + 3)^2}. \end{aligned}$$

Checkpoint 4

Find the derivatives of the given functions.

(a) $f(x) = \dfrac{3x + 7}{5x + 8}$

(b) $g(x) = \dfrac{2x + 11}{5x - 1}$

CAUTION In the second step of Example 3, we had the expression

$$\frac{(4x + 3)(2) - (2x - 1)(4)}{(4x + 3)^2}.$$

Students often incorrectly "cancel" the $4x + 3$ in the numerator with one factor of the denominator. Because the numerator is a *difference* of two products, however, you must multiply and combine terms *before* looking for common factors in the numerator and denominator.

Example 4 Find

$$D_x\left(\frac{x - 2x^2}{4x^2 + 1}\right).$$

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Solution Use the quotient rule:

$$D_x\left(\frac{x-2x^2}{4x^2+1}\right) = \frac{(4x^2+1)D_x(x-2x^2) - (x-2x^2)D_x(4x^2+1)}{(4x^2+1)^2}$$
$$= \frac{(4x^2+1)(1-4x) - (x-2x^2)(8x)}{(4x^2+1)^2}$$
$$= \frac{4x^2 - 16x^3 + 1 - 4x - 8x^2 + 16x^3}{(4x^2+1)^2}$$
$$= \frac{-4x^2 - 4x + 1}{(4x^2+1)^2}. \checkmark_5$$

Checkpoint 5

Find each derivative. Write your answer with positive exponents.

(a) $D_x\left(\frac{x^{-2}-1}{x^{-1}+2}\right)$

(b) $D_x\left(\frac{2+x^{-1}}{x^3+1}\right)$

Example 5 Find

$$D_x\left(\frac{(3-4x)(5x+1)}{7x-9}\right).$$

Solution This function has a product within a quotient. Instead of multiplying the factors in the numerator first (which is an option), we can use the quotient rule together with the product rule. Use the quotient rule first to get

$$D_x\left(\frac{(3-4x)(5x+1)}{7x-9}\right)$$
$$= \frac{(7x-9)D_x[(3-4x)(5x+1)] - [(3-4x)(5x+1)]D_x(7x-9)}{(7x-9)^2}.$$

Now use the product rule to find $D_x[(3-4x)(5x+1)]$ in the numerator:

$$= \frac{(7x-9)[(3-4x)5 + (5x+1)(-4)] - (3+11x-20x^2)(7)}{(7x-9)^2}$$
$$= \frac{(7x-9)(15-20x-20x-4) - (21+77x-140x^2)}{(7x-9)^2}$$
$$= \frac{(7x-9)(11-40x) - 21 - 77x + 140x^2}{(7x-9)^2}$$
$$= \frac{-280x^2 + 437x - 99 - 21 - 77x + 140x^2}{(7x-9)^2}$$
$$= \frac{-140x^2 + 360x - 120}{(7x-9)^2}. \checkmark_6$$

Checkpoint 6

Find each derivative.

(a) $D_x\left(\frac{(3x-1)(4x+2)}{2x}\right)$

(b) $D_x\left(\frac{5x^2}{(2x+1)(x-1)}\right)$

Average Cost

Suppose $y = C(x)$ gives the total cost of manufacturing x items. As mentioned earlier, the average cost per item is found by dividing the total cost by the number of items. The rate of change of average cost, called the *marginal average cost*, is the derivative of the average cost.

Average Cost

If the total cost of manufacturing x items is given by $C(x)$, then the **average cost per item** is

$$\overline{C}(x) = \frac{C(x)}{x}.$$

The **marginal average cost** is the derivative of the average-cost function $\overline{C}'(x)$.

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A company naturally would be interested in making the average cost as small as possible. We will see in the next chapter that this can be done by using the derivative of $C(x)/x$. The derivative often can be found with the quotient rule, as in the next example.

Example 6 **Business** The total cost (in dollars) to manufacture x mobile phones is given by

$$C(x) = \frac{50x^2 + 30x + 4}{x + 2} + 80{,}000.$$

(a) Find the average cost per phone.

Solution The average cost is given by the total cost divided by the number of items:

$$\begin{aligned}\overline{C}(x) = \frac{C(x)}{x} = \frac{1}{x}C(x) &= \frac{1}{x}\left(\frac{50x^2 + 30x + 4}{x + 2} + 80{,}000\right) \\ &= \frac{50x^2 + 30x + 4}{x^2 + 2x} + \frac{80{,}000}{x} \\ &= \frac{50x^2 + 30x + 4}{x^2 + 2x} + 80{,}000x^{-1}.\end{aligned}$$

X	Y1
5000	65.986
10000	57.993
100000	50.799

X=

Figure 11.38

(b) Find the average cost per phone for each of the following production levels:

$$5000; \qquad 10{,}000; \qquad 100{,}000.$$

Solution Evaluate $\overline{C}(x)$ at each of the numbers, either by hand or by using technology, as in Figure 11.38. Note that the average cost per phone is \$65.99 when 5000 phones are produced, and that reduces to \$50.80 when 100,000 are produced.

(c) Find the marginal average cost.

Solution The marginal average cost is the derivative of the average-cost function. Using the sum rule and the quotient rule yields

$$\begin{aligned}\overline{C}'(x) &= \frac{(x^2 + 2x)(100x + 30) - (50x^2 + 30x + 4)(2x + 2)}{(x^2 + 2x)^2} + (-1)80{,}000x^{-2} \\ &= \frac{(100x^3 + 230x^2 + 60x) - (100x^3 + 160x^2 + 68x + 8)}{(x^2 + 2x)^2} - \frac{80{,}000}{x^2} \\ &= \frac{70x^2 - 8x - 8}{(x^2 + 2x)^2} - \frac{80{,}000}{x^2}.\end{aligned}$$

(d) Find the marginal average cost at production levels of 500 phones and 1000 phones.

Solution Evaluating the marginal average-cost function at $x = 500$ yields

$$\overline{C}'(500) = \frac{70(500)^2 - 8(500) - 8}{(500^2 + 2(500))^2} - \frac{80{,}000}{500^2} \approx -.32.$$

Therefore, at a production level of 500 phones, if an additional phone is produced, the *average cost per phone* is decreased by approximately 32 cents per phone.

Similarly, $\overline{C}'(1000) \approx -.08$ means that, at a production level of 1000 phones, if an additional phone is produced, the average cost per phone is decreased by approximately 8 cents per phone. ✓7

✓Checkpoint 7

A cost function is given by

$$C(x) = \frac{x^2 + 2x + 5}{x + 1} + 1000.$$

(a) Find the average cost.

(b) Find the marginal average cost.

Example 7 **Business** The cost (in hundreds of dollars) incurred by a flower shop to produce x hundred flower arrangements for delivery on Valentine's day is given by

$$C(x) = x^2 + 3x + 18.$$

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(a) Find the marginal cost at a production level of 200 flower arrangements.

Solution Since the marginal cost function is $C'(x) = 2x + 3$, the marginal cost at a production level of 200 arrangements is $C'(2) = 2(2) + 3 = 7$. At this production level, the cost of producing an *additional* hundred arrangements is approximately \$700.

(b) Find the average cost at a production level of 200 flower arrangements.

Solution The average cost is

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{x^2 + 3x + 18}{x} = x + 3 + \frac{18}{x}.$$

Evaluating at $x = 2$ yields $\overline{C}(2) = 2 + 3 + \frac{18}{2} = 14$. At a production level of 200 arrangements, the average cost is \$1400 per hundred arrangements (or \$14 per arrangement).

(c) Find the marginal average cost at production levels of 200 and 500 flower arrangements.

Solution The marginal average cost is the derivative of the average-cost function:

$$\overline{C}'(x) = \left(x + 3 + \frac{18}{x}\right)' = 1 - \frac{18}{x^2}.$$

Evaluating at $x = 2$ yields $\overline{C}'(2) = 1 - \frac{18}{2^2} = -3.5$. At this production level, if an additional hundred arrangements are produced, the average cost will be *decreased* by approximately \$350 per hundred arrangements. Increasing production will lower the average cost.

Similarly, $\overline{C}'(5) = 1 - \frac{18}{5^2} = .28$. At this higher production level, if an additional hundred arrangements are produced, the average cost will be *increased* by approximately \$28 per hundred arrangements. It would not make good business sense to increase production, since it would raise the average cost.

(d) Find the level of production at which the marginal average cost is zero.

Solution Set the derivative $\overline{C}'(x) = 0$ and solve for x:

$$\begin{aligned}
1 - \frac{18}{x^2} &= 0 \\
\frac{x^2 - 18}{x^2} &= 0 && \text{Find a common denominator.} \\
x^2 - 18 &= 0 && \text{A rational function can be zero only when the numerator is zero.} \\
x^2 &= 18 \\
x &= \pm\sqrt{18} \approx \pm 4.24.
\end{aligned}$$

You cannot make a negative number of flower arrangements, so $x = 4.24$. The production of 424 flower arrangements will yield a marginal average cost of zero dollars.

Checkpoint 8

The total cost (in hundreds of dollars) to produce x thousand items is given by

$$C(x) = 2x^2 + 15x + 50.$$

(a) Find $\overline{C}'(4)$.

(b) At a production level of 4000 items, if an additional 1000 items are produced, will the average cost per item increase or decrease?

11.6 Exercises

Use the product rule to find the derivatives of the given functions. (See Examples 1 and 2.) (Hint for Exercises 6–9: Write the quantity as a product.)

1. $y = (x^2 - 2)(3x + 2)$

2. $y = (2x^2 + 3)(3x + 5)$

3. $y = (6x^3 + 2)(5x - 3)$

4. $y = (2x^2 + 4x - 3)(5x^3 + 2x + 2)$

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5. $y = (x^4 - 2x^3 + 2x)(4x^2 + x - 3)$

6. $y = (3x - 4)^2$

7. $y = (5x^2 + 12x)^2$

8. $y = (3x^2 - 8)^2$

9. $y = (4x^3 + 6x^2)^2$

Use the quotient rule to find the derivatives of the given functions. (See Examples 3 and 4.)

10. $y = \dfrac{x + 1}{2x - 1}$

11. $y = \dfrac{3x - 5}{x - 3}$

12. $f(x) = \dfrac{7x + 1}{3x + 6}$

13. $f(t) = \dfrac{t^2 - 4t}{t + 3}$

14. $y = \dfrac{4x + 11}{x^2 - 3}$

15. $g(x) = \dfrac{3x^2 + x}{2x^3 - 2}$

16. $k(x) = \dfrac{-x^2 + 6x}{4x^3 + 3}$

17. $y = \dfrac{x^2 - 4x + 2}{x + 3}$

18. $y = \dfrac{x^2 + 7x - 2}{x - 2}$

19. $r(t) = \dfrac{\sqrt{t}}{3t + 4}$

20. $y = \dfrac{5x + 8}{\sqrt{x}}$

21. $y = \dfrac{9x - 8}{\sqrt{x}}$

22. $y = \dfrac{2 - 7x}{5 - x}$

23. $y = \dfrac{59}{4x + 11}$

24. $f(t) = \dfrac{2t^2 + t}{t - 9}$

Find the derivative of each of the given functions. (See Example 5.)

25. $f(p) = \dfrac{(6p - 7)(11p - 1)}{4p + 3}$

26. $f(t) = \dfrac{(8t - 3)(2t + 5)}{t - 7}$

27. $g(x) = \dfrac{x^3 - 8}{(3x + 9)(2x - 1)}$

28. $f(x) = \dfrac{x^3 - 1}{(4x - 1)(3x + 7)}$

29. Find the error in the following work:

$$D_x\left(\frac{2x + 5}{x^2 - 1}\right) = \frac{(2x + 5)(2x) - (x^2 - 1)2}{(x^2 - 1)^2}$$
$$= \frac{4x^2 + 10x - 2x^2 + 2}{(x^2 - 1)^2}$$
$$= \frac{2x^2 + 10x + 2}{(x^2 - 1)^2}.$$

30. Find the error in the following work:

$$D_x\left(\frac{x^2 - 4}{x^3}\right) = x^3(2x) - (x^2 - 4)(3x^2)$$
$$= 2x^4 - 3x^4 + 12x^2 = -x^4 + 12x^2.$$

Find the equation of the tangent line to the graph of $f(x)$ at the given point.

31. $f(x) = \dfrac{x}{x - 2}$ at $(3, 3)$

32. $f(x) = \dfrac{x}{x^2 + 2}$ at $(2, \frac{1}{3})$

33. $f(x) = \dfrac{(x - 1)(2x + 3)}{x - 5}$ at $(9, 42)$

34. $f(x) = \dfrac{(x + 3)^2}{x + 2}$ at $(-4, -\frac{1}{2})$

Work these exercises. (See Examples 6 and 7.)

35. Business A hotdog vendor must pay a monthly fee to operate a food cart in the city park. The cost in dollars of selling x hundred hotdogs in a month can be approximated by

$$C(x) = 200 + 75x.$$

(a) Find the average-cost function.

(b) Find and interpret the average cost at sales levels of 1000 hotdogs and 5000 hotdogs.

(c) Find the marginal average-cost function.

(d) Find and interpret the marginal average cost at sales levels of 1000 hotdogs and 5000 hotdogs.

36. Business For the vendor in the previous exercise, the revenue generated by selling x hundred hotdogs is $R(x) = 400x$ and so the profit function (revenue minus cost) is given by

$$P(x) = 325x - 200.$$

(a) Find and interpret the average profit at sales levels of 2000 hotdogs and 4000 hotdogs.

(b) Find and interpret the marginal average profit at sales levels of 2000 hotdogs and 4000 hotdogs.

37. Business The Student Government Association is making Mother's Day gift baskets to sell at a fund-raiser. If the SGA makes a larger quantity of baskets, it can purchase materials in bulk. The total cost (in hundreds of dollars) of making x gift baskets can be approximated by

$$C(x) = \frac{10x + 1}{x + 100}.$$

(a) Find the marginal cost function and the marginal cost at $x = 20$ and $x = 40$.

(b) Find the average-cost function and the average cost at $x = 20$ and $x = 40$.

(c) Find the marginal average-cost function and the marginal average cost at $x = 20$ and $x = 40$.

38. Business According to inventory records at a shoe store, the number of pairs of sandals (in hundreds) in stock can be approximated by

$$S(x) = \frac{2x + 175}{x + 10},$$

where x is the number of days since the start of the season. For the given times, find the number of pairs of sandals in stock and the rate at which sandals are being sold.

(a) The start of the season

(b) Sixty days after the start of the season

39. Health During the course of an illness, a patient's temperature (in degrees Fahrenheit) x hours after the start of the illness is given by

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$$T(x) = \frac{10x}{x^2 + 5} + 98.6.$$

(a) Find dT/dx.

Evaluate dT/dx at the following times, and interpret your answer:

(b) $x = 0$; **(c)** $x = 1$;

(d) $x = 3$; **(e)** $x = 9$.

40. Business The number of visitors (in thousands) who have entered a theme park on a typical day can be approximated by

$$V(x) = \frac{8x}{2x + 7},$$

where x is the number of hours since the park opened. At the given times, find the number of visitors who have entered the park and the rate at which visitors are entering.

(a) 2 hours after the park opens

(b) 6 hours after the park opens

Work these exercises.

41. Natural Science Murrell's formula for calculating the total amount of rest, in minutes, required after performing a particular type of work activity for 30 minutes is given by the formula

$$R(w) = 30\frac{w - 4}{w - 1.5},$$

where w is the work expended in kilocalories per min (kcal/min).*

(a) A value of 5 for w indicates light work, such as riding a bicycle on a flat surface at 10 miles per hour. Find $R(5)$.

(b) A value of 7 for w indicates moderate work, such as mowing grass with a push mower on level ground. Find $R(7)$.

(c) Find $R'(5)$ and $R'(7)$ and compare your answers. Explain whether these answers do or do not make sense.

42. Health When a certain drug is introduced into a muscle, the muscle responds by contracting. The amount of contraction, s, in millimeters, is related to the concentration of the drug, x, in milliliters, by

$$s(x) = \frac{x}{m + nx},$$

where m and n are constants.

(a) Find $s'(x)$.

(b) Evaluate $s'(x)$ when $x = 50$, $m = 10$, and $n = 3$.

(c) Interpret your results for part (b).

43. Business The average number of vehicles waiting in line to enter a parking lot can be modeled by the function

$$f(x) = \frac{x^2}{2(1 - x)},$$

where x is a quantity between 0 and 1 known as the traffic intensity.† Find the rate of change of the number of vehicles waiting with respect to the traffic intensity for the following values of the intensity:

(a) $x = .1$ **(b)** $x = .6$.

44. Natural Science Using data collected by zoologist Reto Zach, researchers can estimate the work done by a crow to break open a whelk (a large marine snail) by the function

$$W = \left(1 + \frac{20}{H - .93}\right)H,$$

where H is the height of the whelk (in meters) when it is dropped.‡

(a) Find $\dfrac{dW}{dH}$.

(b) The amount of work is minimized when $\dfrac{dW}{dH} = 0$. Find the value of H that minimizes W.

(c) Interestingly, Zach observed the crows dropping the whelks from an average height of 5.23 meters. What does this imply?

45. Business Cashiers at a grocery store are required to learn codes for 80 produce items sold by the pound. The number of codes that a typical cashier has memorized after working x shifts can be approximated by

$$M(x) = \frac{80x^2}{x^2 + 75}.$$

(a) Find $M'(x)$, the rate at which the cashier is memorizing codes after x shifts.

(b) At what rate is the cashier memorizing codes after 1 shift? 3 shifts? 5 shifts? 8 shifts? 20 shifts?

(c) Interpret your results for part (b).

46. Natural Science The time it takes (in hours) to travel on the Garden State Parkway from the Delaware Memorial Bridge to the George Washington Bridge (assuming a 1-minute waiting time per toll plaza) can be approximated by

$$T(v) = \frac{117 + .3v}{v},$$

where v is the average velocity (in mph) while driving, excluding periods when stopped at toll plazas.

(a) Find and interpret $T(60)$.

(b) Find and interpret $T'(60)$.

Checkpoint Answers

1. (a) $f(x) = u(x)v(x) = x^5$ **(b)** $f'(x) = [u(x)v(x)]' = 5x^4$

(c) $u'(x) = 2x$ **(d)** $v'(x) = 3x^2$

(e) $u'(x)v'(x) = 6x^3$

(f) $5x^4 \neq 6x^3$, so the derivative of the product is not equal to the product of the derivatives

2. (a) $45x^2 + 18$ **(b)** $96x^2 + 80x$

*Mark Sanders and Ernest McCormick, *Human Factors in Engineering and Design*, Seventh Edition (New York: McGraw-Hill, 1993), pp. 243–246.

†F. Mannering and W. Kilareski, *Principles of Highway Engineering and Traffic Control*, Second Edition (John Wiley and Sons, 1997).

‡Brian Kellar and Heather Thompson, "Whelk-come to Mathematics," *Mathematics Teacher* 92, no. 6 (September 1999): 475–481.

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3. **(a)** $\frac{5}{2}x^{3/2} + 10x - \frac{3}{2}x^{-1/2}$

(b) $\frac{25}{2}x^{3/2} + 40x + \frac{3}{2}x^{1/2} + 4$

4. **(a)** $\frac{-11}{(5x + 8)^2}$ **(b)** $\frac{-57}{(5x - 1)^2}$

5. **(a)** $\frac{-1 - 4x - x^2}{x^2 + 4x^3 + 4x^4}$ **(b)** $-\frac{6x^4 + 4x^3 + 1}{x^2(x^3 + 1)^2}$

6. **(a)** $\frac{6x^2 + 1}{x^2}$ **(b)** $\frac{-5x^2 - 10x}{(2x + 1)^2(x - 1)^2}$

7. **(a)** $\overline{C}(x) = \frac{x^2 + 2x + 5}{x^2 + x} + \frac{1000}{x}$

(b) $\overline{C}'(x) = -\frac{x^2 + 10x + 5}{(x^2 + x)^2} - \frac{1000}{x^2}$

8. **(a)** -1.125 **(b)** Decrease

11.7 The Chain Rule

Many of the most useful functions for applications are created by combining simpler functions. Viewing complex functions as combinations of simpler functions often makes them easier to understand and use.

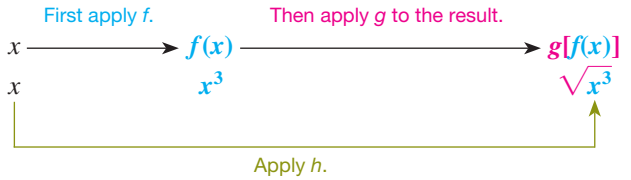
Composition of Functions

Consider the function h whose rule is $h(x) = \sqrt{x^3}$. To compute $h(4)$, for example, you first find $4^3 = 64$ and then take the square root: $\sqrt{64} = 8$. So the rule of h may be rephrased as follows:

First apply the function $f(x) = x^3$.

Then apply the function $g(x) = \sqrt{x}$ to the preceding result.

The same idea can be expressed in functional notation like this:



So the rule of h may be written as $h(x) = g[f(x)]$, where $f(x) = x^3$ and $g(x) = \sqrt{x}$. We can think of the functions g and f as being "composed" to create the function h. Here is a formal definition of this idea.

Composite Function

Let f and g be functions. The **composite function,** or **composition,** of g and f is the function whose values are given by $g[f(x)]$ for all x in the domain of f such that $f(x)$ is in the domain of g.

Example 1 Let $f(x) = 2x - 1$ and $g(x) = \sqrt{3x + 5}$. Find each of the following.

(a) $g[f(4)]$

Solution Find $f(4)$ first:

$$f(4) = 2 \cdot 4 - 1 = 8 - 1 = 7.$$

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Then

$$g[f(4)] = g[7] = \sqrt{3 \cdot 7 + 5} = \sqrt{26}.$$

(b) $f[g(4)]$

Solution Since $g(4) = \sqrt{3 \cdot 4 + 5} = \sqrt{17}$,

$$f[g(4)] = 2 \cdot \sqrt{17} - 1 = 2\sqrt{17} - 1.$$

(c) $f[g(-2)]$

Solution This composite function does not exist, since -2 is not in the domain of g. ✓1

Checkpoint 1

Let $f(x) = 3x - 2$ and $g(x) = (x - 1)^5$. Find the following.

(a) $g[f(2)]$

(b) $f[g(2)]$

Example 2 Let $f(x) = 4x + 1$ and $g(x) = 2x^2 + 5x$. Find each of the following.

(a) $g[f(x)]$

Solution Use the given functions:

$$\begin{aligned} g[f(x)] &= g[4x + 1] \\ &= 2(4x + 1)^2 + 5(4x + 1) \\ &= 2(16x^2 + 8x + 1) + 20x + 5 \\ &= 32x^2 + 16x + 2 + 20x + 5 \\ &= 32x^2 + 36x + 7. \end{aligned}$$

(b) $f[g(x)]$

Solution By the definition of a composite function, with f and g interchanged, we have

$$\begin{aligned} f[g(x)] &= f[2x^2 + 5x] \\ &= 4(2x^2 + 5x) + 1 \\ &= 8x^2 + 20x + 1. \end{aligned}$$ ✓2

Checkpoint 2

Let $f(x) = \sqrt{x + 4}$ and $g(x) = x^2 + 5x + 1$. Find $f[g(x)]$ and $g[f(x)]$.

As Example 2 shows, $f[g(x)]$ usually is *not* equal to $g[f(x)]$. In fact, it is rare to find two functions f and g for which $f[g(x)] = g[f(x)]$.

It is often necessary to write a given function as the composite of two other functions, as is illustrated in the next example.

Example 3

(a) Express the function $h(x) = (x^3 + x^2 - 5)^4$ as the composite of two functions.

Solution One way to do this is to let $f(x) = x^3 + x^2 - 5$ and $g(x) = x^4$; then

$$g[f(x)] = g[x^3 + x^2 - 5] = (x^3 + x^2 - 5)^4 = h(x).$$

(b) Express the function $h(x) = \sqrt{4x^2 + 5}$ as the composite of two functions in two different ways.

Solution One way is to let $f(x) = 4x^2 + 5$ and $g(x) = \sqrt{x}$, so that

$$g[f(x)] = g[4x^2 + 5] = \sqrt{4x^2 + 5} = h(x).$$

Another way is to let $k(x) = 4x^2$ and $t(x) = \sqrt{x + 5}$; then

$$t[k(x)] = t[4x^2] = \sqrt{4x^2 + 5} = h(x).$$ ✓3

Checkpoint 3

Express the given function as a composite of two other functions.

(a) $h(x) = (7x^2 + 5)^4$

(b) $h(x) = \sqrt{15x^2 + 1}$

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The Chain Rule

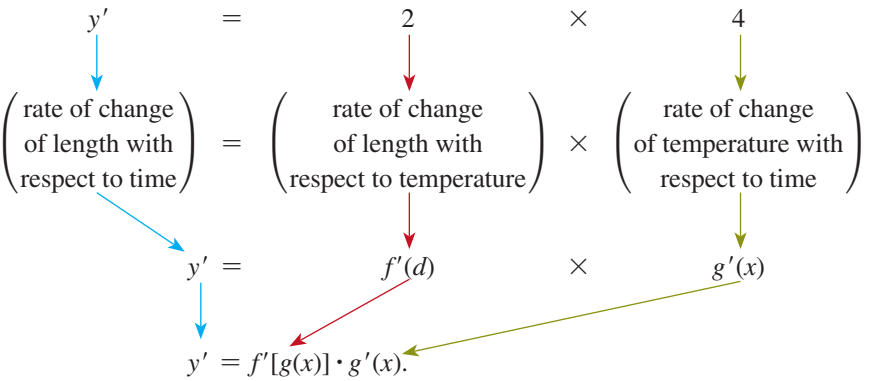
The product and quotient rules tell how to find the derivative of fg and f/g from the derivatives of f and g. To develop a way to find the derivative of the composite function $f[g(x)]$ from the derivatives of f and g, we consider an example.

Metal expands or contracts as the temperature changes, and the temperature changes over a period of time. Think of the length y of a metal bar as a function of the temperature d, say, $y = f(d)$, and the temperature d as a function of time x, say, $d = g(x)$. Then

$$y = f(d) = f(g(x)).$$

So the composite function $f(g(x))$ gives the length y as a function of time x.

Suppose that the length of the bar is increasing at the rate of 2 mm for every degree increase in temperature and that the temperature is increasing at the rate of 4° per hour. In functional notation, this means that $f'(d) = 2$ and $g'(x) = 4$. During the course of an hour, the length of the bar will increase by $2 \cdot 4 = 8$ mm. So the rate of change of the length y with respect to time x is 8 mm/hr; that is, $y' = 8$. Consequently,



This example shows how the derivative of the composite function y is related to the derivatives of f and g. The same result holds for any composite function.

Chain Rule

If f and g are functions and $y = f[g(x)]$, then

$$\boldsymbol{y' = f'[g(x)] \cdot g'(x),}$$

provided that $f'[g(x)]$ and $g'(x)$ exist.

(To find the derivative of $f[g(x)]$, find the derivative of $f(x)$, replace each x with $g(x)$, and multiply the result by the derivative of $g(x)$.)

Example 4 Use the chain rule to find $D_x\sqrt{15x^2 + 1}$.

Solution Write $\sqrt{15x^2 + 1}$ as $(15x^2 + 1)^{1/2}$. Let $f(x) = x^{1/2}$ and $g(x) = 15x^2 + 1$. Then $\sqrt{15x^2 + 1} = f[g(x)]$ and

$$D_x(15x^2 + 1)^{1/2} = f'[g(x)] \cdot g'(x).$$

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Here, $f'(x) = \frac{1}{2}x^{-1/2}$, with $f'[g(x)] = \frac{1}{2}[g(x)]^{-1/2}$, and $g'(x) = 30x$ so that

$$D_x\sqrt{15x^2 + 1} = \frac{1}{2}[g(x)]^{-1/2} \cdot g'(x)$$
$$= \frac{1}{2}(15x^2 + 1)^{-1/2} \cdot (30x)$$
$$= \frac{15x}{(15x^2 + 1)^{1/2}}. \checkmark_4$$

Checkpoint 4

Let $y = \sqrt{2x^4 + 3}$.
Find dy/dx.

The chain rule can also be stated in the Leibniz notation for derivatives. If y is a function of u, say, $y = f(u)$, and u is a function of x, say, $u = g(x)$, then

$$f'(u) = \frac{dy}{du} \quad \text{and} \quad g'(x) = \frac{du}{dx}.$$

Now y can be considered a function of x, namely, $y = f(u) = f(g(x))$. According to the chain rule, the derivative of y is

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = f'(u) \cdot g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Thus, we have the following alternative version of the chain rule.

The Chain Rule (Alternative Form)

If y is a function of u, say, $y = f(u)$, and if u is a function of x, say, $u = g(x)$, then $y = f(u) = f[g(x)]$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

provided that dy/du and du/dx exist.

One way to remember the chain rule is to *pretend* that dy/du and du/dx are fractions, with du "canceling out."

Example 5 Find dy/dx if $y = (3x^2 - 5x)^7$.

Solution Let $y = u^7$, where $u = 3x^2 - 5x$. Then $\frac{dy}{du} = 7u^6$ and $\frac{du}{dx} = 6x - 5$. Hence,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= 7u^6 \cdot (6x - 5).$$

Replacing u with $3x^2 - 5x$ gives

$$\frac{dy}{dx} = 7(3x^2 - 5x)^6(6x - 5). \checkmark_5$$

Checkpoint 5

Use the chain rule to find dy/dx if $y = 10(2x^2 + 1)^4$.

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Each of the functions in Examples 4 and 5 has the form $y = u^n$, with u a function of x:

$$y = (15x^2 + 1)^{1/2} \quad \text{and} \quad y = (3x^2 - 5x)^7.$$
$$y = u^{1/2} \qquad\qquad y = u^7.$$

Their derivatives are

$$y' = \frac{1}{2}(15x^2 + 1)^{-1/2}(30x) \quad \text{and} \quad y' = 7(3x^2 - 5x)^6(6x - 5)$$
$$y' = \frac{1}{2}u^{-1/2}\,u' \qquad\qquad y' = 7u^6\,u'.$$

Thus, these functions are examples of the following special case of the chain rule.

Generalized Power Rule

Let u be a function of x, and let $y = u^n$, for any real number n. Then

$$\boldsymbol{y' = n \cdot u^{n-1} \cdot u',}$$

provided that u' exists.

(The derivative of $y = u^n$ is found by decreasing the exponent on u by 1 and multiplying the result by the exponent n and by the derivative of u with respect to x.)

Example 6

(a) Use the generalized power rule to find the derivative of $y = (3 + 5x)^2$.

Solution Let $u = 3 + 5x$ and $n = 2$. Then $u' = 5$. By the generalized power rule,

$$y' = n \cdot u^{n-1} \cdot u'$$
$$= 2 \cdot (3 + 5x)^{2-1} \cdot \frac{d}{dx}(3 + 5x)$$
$$= 2(3 + 5x)^{2-1} \cdot 5$$
$$= 10(3 + 5x)$$
$$= 30 + 50x.$$

(b) Find y' if $y = (3 + 5x)^{-3/4}$.

Solution Use the generalized power rule with $n = -\frac{3}{4}$, $u = 3 + 5x$, and $u' = 5$:

$$y' = -\frac{3}{4}(3 + 5x)^{(-3/4)-1}(5)$$
$$= -\frac{15}{4}(3 + 5x)^{-7/4}.$$

Checkpoint 6

Find dy/dx for the given functions.

(a) $y = (2x + 5)^6$

(b) $y = (4x^2 - 7)^3$

(c) $f(x) = \sqrt{3x^2 - x}$

(d) $g(x) = (2 - x^4)^{-3}$

Example 7

Find the derivative of the given function.

(a) $y = 2(7x^2 + 5)^4$

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Solution Let $u = 7x^2 + 5$. Then $u' = 14x$, and

$$y' = 2 \cdot 4(7x^2 + 5)^{4-1} \cdot \frac{d}{dx}(7x^2 + 5)$$
$$= 2 \cdot 4(7x^2 + 5)^3(14x)$$
$$= 112x(7x^2 + 5)^3.$$

(b) $y = \sqrt{9x + 2}$

Solution Write $y = \sqrt{9x + 2}$ as $y = (9x + 2)^{1/2}$. Then

$$y' = \frac{1}{2}(9x + 2)^{-1/2}(9) = \frac{9}{2}(9x + 2)^{-1/2}.$$

The derivative also can be written as

$$y' = \frac{9}{2(9x + 2)^{1/2}} \quad \text{or} \quad y' = \frac{9}{2\sqrt{9x + 2}}.$$ ✓7

✓ Checkpoint 7

Find dy/dx for the given functions.

(a) $y = 12(x^2 + 6)^5$

(b) $y = 8(4x^2 + 2)^{3/2}$

CAUTION

(a) A common error is to forget to multiply by $g'(x)$ when using the generalized power rule. Remember, the generalized power rule is an example of the chain rule, so the derivative must involve a "chain," or product, of derivatives.

(b) Another common mistake is to write the derivative as $n[g'(x)]^{n-1}$. Remember to leave $g(x)$ unchanged and then to multiply by $g'(x)$.

Sometimes both the generalized power rule and either the product or quotient rule are needed to find a derivative, as the next two examples show.

Example 8 Find the derivative of $y = 4x(3x + 5)^5$.

Solution Write $4x(3x + 5)^5$ as the product

$$y = 4x \cdot (3x + 5)^5.$$

According to the product rule,

$$y' = 4x \cdot \frac{d}{dx}(3x + 5)^5 + (3x + 5)^5 \cdot \frac{d}{dx}(4x).$$

Use the generalized power rule to find the derivative of $(3x + 5)^5$;

Derivative of $(3x + 5)^5$ Derivative of $4x$

$$y' = 4x[5(3x + 5)^4 \cdot 3] + (3x + 5)^5(4)$$
$$= 60x(3x + 5)^4 + 4(3x + 5)^5$$
$$= 4(3x + 5)^4[15x + (3x + 5)^1]$$
$$= 4(3x + 5)^4(18x + 5).$$ ✓8

Factor out the greatest common factor, $4(3x + 5)^4$.

✓ Checkpoint 8

Find the derivatives of the given functions.

(a) $y = 9x(x + 7)^2$

(b) $y = -3x(4x^2 + 7)^3$

Example 9 Find the derivative of

$$y = \frac{(3x + 2)^7}{x - 1}.$$

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Solution Use the quotient rule and the generalized power rule:

$$\frac{dy}{dx} = \frac{(x-1)\cdot\frac{d}{dx}(3x+2)^7 - (3x+2)^7\frac{d}{dx}(x-1)}{(x-1)^2}$$ Quotient rule

$$= \frac{(x-1)[7(3x+2)^6\cdot 3] - (3x+2)^7(1)}{(x-1)^2}$$ Generalized power rule

$$= \frac{21(x-1)(3x+2)^6 - (3x+2)^7}{(x-1)^2}$$

$$= \frac{(3x+2)^6[21(x-1) - (3x+2)]}{(x-1)^2}$$ Factor out the greatest common factor, $(3x+2)^6$.

$$= \frac{(3x+2)^6[21x - 21 - 3x - 2]}{(x-1)^2}$$ Simplify inside brackets.

$$= \frac{(3x+2)^6(18x-23)}{(x-1)^2}.$$ ✓9

✓Checkpoint 9

Find the derivatives of the given functions.

(a) $y = \dfrac{(5x+2)^3}{11x}$

(b) $y = \dfrac{(x-12)^7}{6x+1}$

Applications

Some applications requiring the use of the chain rule or the generalized power rule are illustrated in the next three examples.

Example 10 **Business** During a week-long promotion, the profit generated by an online sporting goods retailer from the sale of n official basketballs is given by

$$P(n) = \frac{45n^2}{3n+10}.$$

Sales are approximately constant at a rate of 25 basketballs per day, therefore $\dfrac{dn}{dt} = 25$. How fast is profit changing 4 days after the start of the promotion?

Solution We want to find $\dfrac{dP}{dt}$, the rate of change of profit with respect to time. By the chain rule,

$$\frac{dP}{dt} = \frac{dP}{dn}\frac{dn}{dt}.$$

First find $\dfrac{dP}{dn}$ as follows:

$$\frac{dP}{dn} = \frac{(3n+10)(90n) - (45n^2)(3)}{(3n+10)^2}$$ Quotient rule

$$= \frac{270n^2 + 900n - 135n^2}{(3n+10)^2}$$

$$= \frac{135n^2 + 900n}{(3n+10)^2}.$$

With sales at 25 basketballs per day, 4 days after the start of the promotion, n = (4 days)(25 basketballs/day) = 100 basketballs. So after 4 days,

$$\frac{dP}{dn} = \frac{135(100)^2 + 900(100)}{(3(100)+10)^2} \approx 14.9844.$$

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We are given that $\frac{dn}{dt} = 25$, so we have

$$\frac{dP}{dt} = \frac{dP}{dn}\frac{dn}{dt} = (14.9844)(25) \approx 374.61.$$

After 4 days, profit from basketballs is increasing at a rate of \$374.61 per day. ✓10

✓ Checkpoint 10

Suppose the profit generated by the sale of n baseballs for the retailer in Example 10 is given by $P(n) = \frac{4n^2}{n+5}$ and baseballs are selling at a rate of 30 per day. How fast is profit changing after 1 day?

Example 11 **Business** A generous aunt deposits \$20,000 in an account to be used by her newly born niece to attend college. The account earns interest at the rate of r percent per year, compounded monthly. At the end of 18 years, the balance in the account is given by

$$A = 20{,}000\left(1 + \frac{r}{1200}\right)^{216}.$$

Find the rate of change of A with respect to r if $r = 1.5$, 2.5, or 3.

Solution First find dA/dr, using the generalized power rule:

$$\frac{dA}{dr} = (216)(20{,}000)\left(1 + \frac{r}{1200}\right)^{215}\left(\frac{1}{1200}\right) = 3600\left(1 + \frac{r}{1200}\right)^{215}.$$

If $r = 1.5$, we obtain

$$\frac{dA}{dr} = 3600\left(1 + \frac{1.5}{1200}\right)^{215} = 4709.19,$$

or \$4709.19 per percentage point. If $r = 2.5$, we obtain

$$\frac{dA}{dr} = 3600\left(1 + \frac{2.5}{1200}\right)^{215} = 5631.55,$$

or \$5631.55 per percentage point. If $r = 3$, we obtain

$$\frac{dA}{dr} = 3600\left(1 + \frac{3}{1200}\right)^{215} = 6158.07,$$

or \$6158.07 per percentage point. This means that when the interest rate is 3%, an increase of 1% in the interest rate will produce an increase in the balance of approximately \$6158.07. ✓11

✓ Checkpoint 11

If the initial donation in Example 11 is \$25,000, find the rate of change of A with respect to r if $r = 2$.

The chain rule can be used to develop the formula for the **marginal-revenue product,** an economic concept that approximates the change in revenue when a manufacturer hires an additional employee. Start with $R = px$, where R is total revenue from the daily production of x units and p is the price per unit. The demand function is $p = f(x)$, as before. Also, x can be considered a function of the number of employees, n. Since $R = px$, and x—and therefore, p—depends on n, R can also be considered a function of n. To find an expression for dR/dn, use the product rule for derivatives on the function $R = px$ to get

$$\frac{dR}{dn} = p \cdot \frac{dx}{dn} + x \cdot \frac{dp}{dn}. \qquad (*)$$

By the chain rule,

$$\frac{dp}{dn} = \frac{dp}{dx} \cdot \frac{dx}{dn}.$$

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Substituting for dp/dn in equation (*) yields

$$\frac{dR}{dn} = p \cdot \frac{dx}{dn} + x\left(\frac{dp}{dx} \cdot \frac{dx}{dn}\right)$$
$$= \left(p + x \cdot \frac{dp}{dx}\right)\frac{dx}{dn}. \qquad \text{Factor out } \frac{dx}{dn}.$$

The expression for dR/dn gives the marginal-revenue product.

Example 12 **Business** Find the marginal-revenue product dR/dn (in dollars per employee) when $n = 20$ if the demand function is $p = 600/\sqrt{x}$ and $x = 5n$.

Solution As shown previously,

$$\frac{dR}{dn} = \left(p + x \cdot \frac{dp}{dx}\right)\frac{dx}{dn}.$$

Find dp/dx and dx/dn. From

$$p = \frac{600}{\sqrt{x}} = 600x^{-1/2},$$

we have the derivative

$$\frac{dp}{dx} = -300x^{-3/2}.$$

Also, from $x = 5n$, we have

$$\frac{dx}{dn} = 5.$$

Then, by substitution,

$$\frac{dR}{dn} = \left[\frac{600}{\sqrt{x}} + x(-300x^{-3/2})\right]5 = \left[\frac{600}{\sqrt{x}} - \frac{300}{\sqrt{x}}\right]5 = \frac{1500}{\sqrt{x}}.$$

If $n = 20$, then $x = 100$, and

$$\frac{dR}{dn} = \frac{1500}{\sqrt{100}}$$
$$= 150.$$

This means that hiring an additional employee when production is at a level of 100 items will produce an increase in revenue of \$150. ✓12

✓Checkpoint 12

Find the marginal-revenue product at $n = 10$ if the demand function is $p = 1000/x^2$ and $x = 8n$. Interpret your answer.

11.7 Exercises

Let $f(x) = 2x^2 + 3x$ and $g(x) = 4x - 1$. Find each of the given values. (See Example 1.)

1. $f[g(5)]$
2. $f[g(-4)]$
3. $g[f(5)]$
4. $g[f(-4)]$

Find $f[g(x)]$ and $g[f(x)]$ for the given functions. (See Example 2.)

5. $f(x) = 8x + 12; g(x) = 2x + 3$
6. $f(x) = -6x + 9; g(x) = 5x + 7$
7. $f(x) = -x^3 + 2; g(x) = 4x + 2$
8. $f(x) = 2x; g(x) = 6x^2 - x^3 - 1$
9. $f(x) = \frac{1}{x}; g(x) = x^2$
10. $f(x) = \frac{2}{x^4}; g(x) = 2 - x$
11. $f(x) = \sqrt{x + 2}; g(x) = 8x^2 - 6x$

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12. $f(x) = 9x^2 - 11x;\ g(x) = 2\sqrt{x+2}$

Write each function as a composition of two functions. (There may be more than one way to do this.) (See Example 3.)

13. $y = (4x+3)^5$ **14.** $y = (x^2+2)^{1/5}$

15. $y = \sqrt{6+3x^2}$ **16.** $y = \sqrt{x+3} - \sqrt[3]{x+3}$

17. $y = \dfrac{\sqrt{x}+3}{\sqrt{x}-3}$ **18.** $y = \dfrac{2x}{\sqrt{x}+5}$

19. $y = (x^{1/2}-3)^3 + (x^{1/2}-3) + 5$

20. $y = (x^2+5x)^{1/3} - 2(x^2+5x)^{2/3} + 7$

Find the derivative of each of the given functions. (See Examples 4–7.)

21. $y = (7x-12)^5$ **22.** $y = (13x-3)^4$

23. $y = 7(4x+3)^4$ **24.** $y = -4(3x-2)^7$

25. $y = -4(10x^2+8)^5$ **26.** $y = -3(x^4+7x^2)^3$

27. $y = 11(3x+5)^{3/2}$ **28.** $y = 13(4x-2)^{2/3}$

29. $y = -7(4x^2+9x)^{3/2}$ **30.** $y = 11(5x^2+6x)^{3/2}$

31. $y = 8\sqrt{4x+7}$ **32.** $y = -3\sqrt{7x-1}$

33. $y = -2\sqrt{x^2+4x}$ **34.** $y = 4\sqrt{2x^2+3}$

Use the product or quotient rule or the generalized power rule to find the derivative of each of the given functions. (See Examples 8 and 9.)

35. $y = (x+1)(x-3)^2$ **36.** $y = (2x+1)^3(x-5)$

37. $y = 5(x+3)^2(2x-1)^5$ **38.** $y = -9(x+4)^2(2x-3)^3$

39. $y = (3x+1)^3\sqrt{x}$ **40.** $y = (3x+5)^2\sqrt{x+1}$

41. $y = \dfrac{1}{(x-2)^3}$ **42.** $y = \dfrac{-3}{(4x+1)^2}$

43. $y = \dfrac{(2x-7)^2}{4x-1}$ **44.** $y = \dfrac{(x-4)^2}{5x+2}$

45. $y = \dfrac{x^2+5x}{(3x+1)^3}$ **46.** $y = \dfrac{4x^3-x}{(x-3)^2}$

47. $y = (x^{1/2}+2)(2x+3)$ **48.** $y = (4-x^{2/3})(x+1)^{1/2}$

Consider the following table of values of the functions f and g and their derivatives at various points:

x	1	2	3	4
$f(x)$	2	4	1	3
$f'(x)$	−6	−7	−8	−9
$g(x)$	2	3	4	1
$g'(x)$	2/7	3/7	4/7	5/7

Find each of the given derivatives.

49. **(a)** $D_x(f[g(x)])$ at $x = 1$ **(b)** $D_x(f[g(x)])$ at $x = 2$

50. **(a)** $D_x(g[f(x)])$ at $x = 1$ **(b)** $D_x(g[f(x)])$ at $x = 2$

51. If $f(x) = (2x^2+3x+1)^{50}$, then which of the following is closest to $f'(0)$?

(a) 1 **(b)** 50 **(c)** 100
(d) 150 **(e)** 200 **(f)** 250

52. The graphs of $f(x) = 3x+5$ and $g(x) = 4x-1$ are straight lines.

(a) Show that the graph of $f[g(x)]$ is also a straight line.

(b) How are the slopes of the graphs of $f(x)$ and $g(x)$ related to the slope of the graph of $f[g(x)]$?

Work these exercises. (See Examples 10 and 11.)

53. **Business** Suppose the demand for a certain brand of vacuum cleaner is given by

$$D(p) = \frac{-p^2}{100} + 500,$$

where p is the price in dollars. If the price, in terms of the cost c, is expressed as

$$p(c) = 2c - 10,$$

find the demand in terms of the cost.

54. **Natural Science** An oil well off the Gulf Coast is leaking, spreading oil in a circle over the surface. At any time t, in minutes, after the beginning of the leak, the radius of the circular oil slick on the surface is $r(t) = t^2$ feet. Let $A(r) = \pi r^2$ represent the area of a circle of radius r. Find and interpret $A[r(t)]$.

55. **Business** You are driving on a business trip to another city. Let $D(g)$ represent the distance (in miles) you can drive on g gallons of gas. Let $T(D)$ represent the time it takes (in hours) to drive a distance of D miles. Let $f(g) = T[D(g)]$.

(a) What does $f(g) = T[D(g)]$ represent?

(b) Explain what $D(4) = 120$ means in practical terms.

(c) Explain what $T'(120) = \dfrac{1}{60}$ means in practical terms.

(d) Explain what $D'(4) = 30$ means in practical terms.

(e) Find and interpret $f'(4)$.

56. **Business** A contractor has determined formulas for the following functions. Let $H(d)$ be the number of hours that a tile-setter will work for d dollars. Let $A(t)$ be the area (in square feet) that a tile-setter can tile in t hours. Let $f(d) = A[H(d)]$.

(a) What does $f(d) = A[H(d)]$ represent?

(b) Explain what $f'(900) = .4$ means in practical terms.

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57. Business The cost of producing x bags of dog food is given by

$$C(x) = 600 + \sqrt{200 + 25x^2 - x} \qquad (0 \le x \le 5000).$$

Find the marginal-cost function.

58. Business The cost (in hundreds of dollars) of producing x smart phones is given by

$$C(x) = 900 + \sqrt{5x^2 + 4x + 1200}.$$

(a) Find the marginal-cost function.

(b) Find and interpret $C'(200)$.

59. Business The revenue function from the sale of x earbuds for portable music players is given by

$$R(x) = 15\sqrt{500x - 2x^2} \qquad (0 \le x \le 200).$$

(a) Find the marginal-revenue function.

(b) Evaluate the marginal-revenue function at $x = 40$, 80, 120, and 160.

(c) Explain the significance of the answers found in part (b).

60. Business The profit (in thousands of dollars) of producing x thousand pairs of running shoes is given by

$$P(x) = 45\sqrt[3]{x^3 + 246x + 500} - 500 \qquad (0 \le x \le 100).$$

(a) Is the company profitable or losing money when producing 2000 pairs of running shoes? 6000 pairs?

(b) Find the marginal-profit function.

61. Business A tablet computer is purchased for $500. Its value in dollars after t years can be approximated by

$$V(t) = \frac{500}{(2t + 1)^2} \qquad (0 \le t \le 4).$$

(a) Find the value of the tablet computer 2 years after purchase.

(b) At time $t = 2$ years, use a derivative to estimate how much the value will change within the next year.

62. Health The percent of patients who report being symptom-free t days after the onset of a certain illness can be approximated by

$$F(t) = 100 - \frac{200}{\sqrt{t^2 + 4}} \qquad (0 \le t \le 7).$$

Find the rate at which patients are recovering after the given number of days.

(a) 2 days **(b)** 5 days

63. Business The demand function for tickets to the student theatrical production is given by

$$p = \sqrt{900 - 2x} \qquad (0 \le x \le 450),$$

where x is the number of tickets demanded and p is the price in dollars.

(a) Find the revenue as a function of x.

(b) Find the marginal-revenue function.

(c) Find and interpret the marginal revenue at $x = 250$.

64. Business The demand (in thousands) for a certain smart phone app as a function of the price $\$p$ is given by

$$x = 30(8 - .5p^2)^{1/2} \qquad (0 \le p \le 4).$$

(a) Find the demand when the price is $2.

(b) Find the rate of change of demand with respect to price when the price is $2.

65. Finance An amount of $15,000 is deposited in an account with an interest rate of r percent per year, compounded monthly. At the end of 8 years, the balance in the account is given by

$$A = 15{,}000\left(1 + \frac{r}{1200}\right)^{96}.$$

Find the rate of change of A with respect to r (written as a percentage) for the given interest rates.

(a) 2.5% **(b)** 3.25% **(c)** 4%

66. Finance An amount of $8000 is invested in a 5-year certificate of deposit with an interest rate of r percent per year, compounded quarterly. On the maturity date, the value of the certificate of deposit is given by

$$A = 8000\left(1 + \frac{r}{400}\right)^{20}.$$

(a) Would you expect the rate of change of A with respect to r to be positive, negative, or zero? Why?

(b) Find $\dfrac{dA}{dr}$.

(c) Find and interpret $\dfrac{dA}{dr}$ at an interest rate of 2%. (Use $r = 2$ not $r = .02$, since r is written as a percentage.)

67. Finance An amount of $25,000 is invested in a money market account with an interest rate of r percent per year, compounded monthly. At the end of 2 years, the balance on the account is given by

$$A = 25{,}000\left(1 + \frac{r}{1200}\right)^{24}.$$

Find and interpret the rate of change of A with respect to r (written as a percentage) for the given interest rates.

(a) 1% **(b)** 2% **(c)** 4%

68. Finance The price (present value) of a zero coupon bond with a face value of $5000, a maturity date of 10 years, and an interest rate of r percent compounded annually is given by

$$P = \frac{5000}{\left(1 + \dfrac{r}{100}\right)^{10}}.$$

(a) Find the price of this bond, assuming an interest rate of 4%. (Use $r = 4$ not $r = .04$, since r is written as a percentage.)

(b) Would you expect the rate of change of P with respect to r to be positive, negative, or zero? Why?

(c) Find $\dfrac{dP}{dr}$.

(d) Find and interpret $\dfrac{dP}{dr}$ at an interest rate of 4%.

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Work these exercises. (See Example 12.)

69. Business Find the marginal-revenue product for a manufacturer with 10 workers if the demand function is $p = \dfrac{200 - x}{4}$ and if $x = 6n$.

70. Business Find the marginal-revenue product for a manufacturer with 5 workers if the demand function is $p = \dfrac{210}{\sqrt{x}}$ and if $x = 45n$.

Work the following exercises.

71. Business A cost function is given by $C(x) = \sqrt{6x + 5}$, where x is the number of items produced.

(a) Find the average-cost function.

(b) Find the marginal average-cost function.

72. Social Science The percent of the adult population of a city who have heard about a scandal in the city government t days after it occurs can be approximated by

$$F(t) = \frac{85t}{\sqrt{t^2 + 2}}.$$

(a) What percent of the population is aware of the scandal after 1 day?

(b) Find $F'(t)$.

(c) Find and interpret $F'(1)$.

73. Natural Science To test an individual's use of calcium, a researcher injects a small amount of radioactive calcium into the person's bloodstream. The calcium remaining in the bloodstream is measured each day for several days. Suppose the amount of the calcium remaining in the bloodstream, in milligrams per cubic centimeter, t days after the initial injection is approximated by

$$C(t) = \frac{1}{2}(2t + 1)^{-1/2}.$$

Find the rate of change of C with respect to time for each of the given number of days.

(a) $t = 0$ **(b)** $t = 4$

(c) $t = 6$ **(d)** $t = 7.5$

74. Health The strength of a person's reaction to a certain drug is given by

$$R(Q) = Q\left(C - \frac{Q}{3}\right)^{1/2},$$

where Q represents the quantity of the drug given to the patient and C is a constant. The derivative $R(Q)$ is called the *sensitivity* to the drug.

(a) Find $R'(Q)$.

(b) Find $R'(Q)$ if $Q = 87$ and $C = 59$.

75. Social Science Studies show that after t hours on the job, the number of items a supermarket cashier can scan per minute is given by

$$F(t) = 60 - \frac{150}{\sqrt{8 + t^2}}.$$

(a) Find $F'(t)$, the rate at which the cashier's speed is increasing.

(b) At what rate is the cashier's speed increasing after 5 hours? 10 hours? 20 hours? 40 hours?

(c) Are your answers in part (b) increasing or decreasing with time? Is this reasonable? Explain.

76. Natural Science The volume and surface area of a jawbreaker candy of any radius r are given by the formulas

$$V(r) = \frac{4}{3}\pi r^3 \quad \text{and} \quad S(r) = 4\pi r^2,$$

respectively. It is estimated that the radius of a jawbreaker while in a person's mouth is

$$r(t) = 6 - \frac{3}{17}t,$$

where $r(t)$ is in mm and t is in min.*

(a) What is the life expectancy of a jawbreaker?

(b) Find $\dfrac{dV}{dt}$ and $\dfrac{dS}{dt}$ when $t = 17$, and interpret your answer.

(c) Construct an analogous experiment using some other type of food, or verify the results of this experiment.

Checkpoint Answers

1. **(a)** 243 **(b)** 1

2. $f[g(x)] = \sqrt{x^2 + 5x + 5}$;

$g[f(x)] = (\sqrt{x + 4})^2 + 5\sqrt{x + 4} + 1$

$= x + 5 + 5\sqrt{x + 4}$

3. There are several correct answers, including

(a) $h(x) = f[g(x)]$, where $f(x) = x^4$ and $g(x) = 7x^2 + 5$;

(b) $h(x) = f[g(x)]$, where $f(x) = \sqrt{x}$ and $g(x) = 15x^2 + 1$.

4. $\dfrac{dy}{dx} = \dfrac{4x^3}{\sqrt{2x^4 + 3}}$

5. $160x(2x^2 + 1)^3$

6. **(a)** $12(2x + 5)^5$ **(b)** $24x(4x^2 - 7)^2$

(c) $\dfrac{6x - 1}{2\sqrt{3x^2 - x}}$ **(d)** $\dfrac{12x^3}{(2 - x^4)^4}$

7. **(a)** $120x(x^2 + 6)^4$ **(b)** $96x(4x^2 + 2)^{1/2}$

8. **(a)** $9(x + 7)(3x + 7)$ **(b)** $-21(4x^2 + 7)^2(4x^2 + 1)$

9. **(a)** $\dfrac{11(5x + 2)^2(10x - 2)}{121x^2}$ **(b)** $\dfrac{(x - 12)^6(36x + 79)}{(6x + 1)^2}$

10. \$117.55 per day

11. \$6437.32 per percentage point

12. −\$1.25; hiring an additional employee will produce a decrease in revenue of \$1.25.

*Roger Guffey, "The Life Expectancy of a Jawbreaker: An Application of the Composition of Functions," *Mathematics Teacher* 92, no. 2 (February 1999): 125–127.

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11.8 Derivatives of Exponential and Logarithmic Functions

The exponential function $f(x) = e^x$ and the logarithmic function to the base e, $g(x) = \ln x$, were studied in Chapter 4. (Recall that $e \approx 2.71828$.) We are interested in exponential and logarithmic functions because they arise naturally in a variety of applications in business, economics, and the social and life sciences. In this section, we shall find the derivatives of these functions. In order to do that, we must first find a limit that will be needed in our calculations.

We claim that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Although a rigorous proof of this fact is beyond the scope of this book, a graphing calculator provides both graphical and numerical support, as shown in Figure 11.39.*

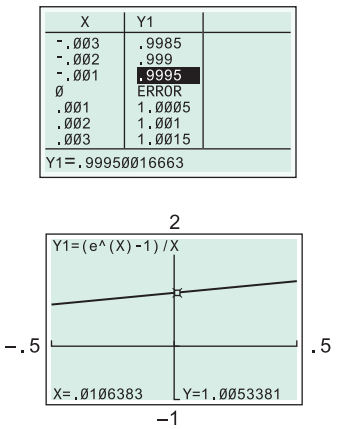

Figure 11.39

To find the derivative of $f(x) = e^x$, we use the definition of the derivative function,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided that this limit exists. (Remember that h is the variable here and x is treated as a constant.)

For $f(x) = e^x$, we see that

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} && \text{Product property of exponents} \\ &= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} && \text{Factor out } e^x. \\ &= \lim_{h\to 0} e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h}, && \text{Product property of limits} \end{aligned}$$

provided that the last two limits exist. But h is the variable here and x is constant; therefore,

$$\lim_{h\to 0} e^x = e^x.$$

Combining this fact with our previous work, we see that

$$f'(x) = \lim_{h\to 0} e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x.$$

In other words, *the exponential function $f(x) = e^x$ is its own derivative.*

Example 1 Find each derivative.

(a) $y = x^3 e^x$

Solution The product rule and the fact that $f(x) = e^x$ is its own derivative show that

$$\begin{aligned} y' &= x^3 \cdot D_x(e^x) + e^x \cdot D_x(x^3) \\ &= x^3 \cdot e^x + e^x \cdot 3x^2 = e^x(x^3 + 3x^2). \end{aligned}$$

*As usual, the variable X is used in place of h on a calculator.

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(b) $y = (2e^x + x)^5$

Solution By the generalized power, sum, and constant rules,

$$\begin{aligned} y' &= 5(2e^x + x)^4 \cdot D_x(2e^x + x) \\ &= 5(2e^x + x)^4 \cdot [D_x(2e^x) + D_x(x)] \\ &= 5(2e^x + x)^4 \cdot [2D_x(e^x) + 1] \\ &= 5(2e^x + x)^4(2e^x + 1). \end{aligned}$$ ✓1

Checkpoint 1

Differentiate the given expressions.

(a) $(2x^2 - 1)e^x$

(b) $(1 - e^x)^{1/2}$

Example 2 Find the derivative of $y = e^{x^2-3x}$.

Solution Let $f(x) = e^x$ and $g(x) = x^2 - 3x$. Then

$$y = e^{x^2-3x} = e^{g(x)} = f[g(x)],$$

and $f'(x) = e^x$ and $g'(x) = 2x - 3$. By the chain rule,

$$\begin{aligned} y' &= f'[g(x)] \cdot g'(x) \\ &= e^{g(x)} \cdot (2x - 3) \\ &= e^{x^2-3x} \cdot (2x - 3) = (2x - 3)e^{x^2-3x}. \end{aligned}$$

The argument used in Example 2 can be used to find the derivative of $y = e^{g(x)}$ for any differentiable function g. Let $f(x) = e^x$; then $y = e^{g(x)} = f[g(x)]$. By the chain rule,

$$y' = f'[g(x)] \cdot g'(x) = e^{g(x)} \cdot g'(x) = g'(x)e^{g(x)}.$$

We summarize these results as follows.

Derivatives of e^x and $e^{g(x)}$

If $y = e^x$, then $y' = e^x$.

If $y = e^{g(x)}$, then $y' = g'(x) \cdot e^{g(x)}$.

CAUTION Notice the difference between the derivative of a variable to a constant power, such as $D_x x^3 = 3x^2$, and the derivative of a constant to a variable power, like $D_x e^x = e^x$. Remember, $D_x e^x \neq xe^{x-1}$.

Example 3 Find derivatives of the given functions.

(a) $y = e^{7x}$

Solution Let $g(x) = 7x$, with $g'(x) = 7$. Then

$$y' = g'(x)e^{g(x)} = 7e^{7x}.$$

(b) $y = 5e^{-3x}$

Solution Here, let $g(x) = -3x$ and $g'(x) = -3$, so that

$$y' = 5(-3e^{-3x}) = -15e^{-3x}.$$

(c) $y = 8e^{4x^3}$

Solution Let $g(x) = 4x^3$, with $g'(x) = 12x^2$. Then

$$y' = 8(12x^2e^{4x^3}) = 96x^2e^{4x^3}.$$ ✓2

Checkpoint 2

Find the derivative of the given function.

(a) $y = 4e^{11x}$

(b) $y = -5e^{(-9x+2)}$

(c) $y = e^{-x^3}$

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Example 4 Let

$$y = \frac{250{,}000}{2 + 15e^{-4x}}. \text{ Find } y'.$$

Solution Use the quotient rule:

$$y' = \frac{(2 + 15e^{-4x})(0) - 250{,}000(-60e^{-4x})}{(2 + 15e^{-4x})^2}$$

$$= \frac{15{,}000{,}000e^{-4x}}{(2 + 15e^{-4x})^2}.$$ ✓3

Checkpoint 3

Find the derivative of the given function.

(a) $y = \dfrac{e^x}{12 + x}$

(b) $y = \dfrac{9000}{15 + 3e^x}$

Example 5 **Natural Science** A 100-gram sample of a radioactive substance decays exponentially. The amount left after t years is given by $A(t) = 100e^{-.12t}$. Find the rate of change of the amount after 3 years.

Solution The rate of change is given by the derivative dA/dt:

$$\frac{dA}{dt} = 100(-.12e^{-.12t}) = -12e^{-.12t}.$$

After 3 years ($t = 3$), the rate of change is

$$\frac{dA}{dt} = -12e^{-.12(3)} = -12e^{-.36} \approx -8.4$$

grams per year.

Example 6 If $y = 10^x$, find y'.

Solution By one of the basic properties of logarithms (page 202), we have $10 = e^{\ln 10}$. Therefore,

$$y = 10^x = (e^{\ln 10})^x = e^{(\ln 10)x},$$

and

$$y' = \frac{d}{dx}[(\ln 10)x] \cdot e^{(\ln 10)x}$$ By the preceding box, with $g(x) = (\ln 10)x$

$$= (\ln 10)e^{(\ln 10)x}$$ Remember that ln 10 is a constant.

$$= (\ln 10)(e^{\ln 10})^x$$

$$= (\ln 10)10^x.$$ $e^{\ln 10} = 10$ ✓4

Checkpoint 4

Adapt the solution of Example 6 to find the derivative of $y = 25^x$.

Derivatives of Logarithmic Functions

To find the derivative of $g(x) = \ln x$, we use the definitions and properties of natural logarithms that were developed in Section 4.3:

$$g(x) = \ln x \quad \text{means} \quad e^{g(x)} = x,$$

and for all $x > 0$, $y > 0$, and every real number r,

$$\ln xy = \ln x + \ln y, \quad \ln \frac{x}{y} = \ln x - \ln y, \quad \text{and} \quad \ln x^r = r \ln x.$$

Note that we must have $x > 0$ and $y > 0$ because logarithms of zero and negative numbers are not defined.

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Differentiating with respect to x on each side of $e^{g(x)} = x$ shows that

$$D_x(e^{g(x)}) = D_x(x)$$
$$g'(x) \cdot e^{g(x)} = 1.$$

Because $e^{g(x)} = x$, this last equation becomes

$$g'(x) \cdot x = 1$$
$$g'(x) = \frac{1}{x}$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \text{for all } x > 0.$$

Example 7

(a) Assume that $x > 0$, and use properties of logarithms to find the derivative of $y = \ln 6x$.

Solution
$$y' = \frac{d}{dx}(\ln 6x)$$
$$= \frac{d}{dx}(\ln 6 + \ln x) \qquad \text{Product rule for logarithms}$$
$$= \frac{d}{dx}(\ln 6) + \frac{d}{dx}(\ln x). \qquad \text{Sum rule for derivatives}$$

Be careful here: $\ln 6$ is a *constant* ($\ln 6 \approx 1.79$), so its derivative is 0 (*not* $\frac{1}{6}$). Hence,

$$y' = \frac{d}{dx}(\ln 6) + \frac{d}{dx}(\ln x) = 0 + \frac{1}{x} = \frac{1}{x}.$$

(b) Assume that $x > 0$, and use the chain rule to find the derivative of $y = \ln 6x$.

Solution Let $f(x) = \ln x$ and $g(x) = 6x$, so that $y = \ln 6x = \ln g(x) = f(g(x))$. Then, by the chain rule,

$$y' = f'[g(x)] \cdot g'(x) = \frac{1}{g(x)} \cdot \frac{d}{dx}(6x) = \frac{1}{6x} \cdot 6 = \frac{1}{x}.$$

The argument used in Example 6(b) applies equally well in the general case. The derivative of $y = \ln g(x)$, where $g(x)$ is a function and $g(x) > 0$, can be found by letting $f(x) = \ln x$, so that $y = f[g(x)]$, and applying the chain rule:

$$y' = f'[g(x)] \cdot g'(x) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}.$$

We summarize these results as follows.

Derivatives of ln *x* and ln *g*(*x*)

If $y = \ln x$, then $y' = \dfrac{1}{x}$ $\quad (x > 0)$.

If $y = \ln g(x)$, then $y' = \dfrac{g'(x)}{g(x)}$ $\quad (g(x) > 0)$.

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Example 8 Find the derivatives of the given functions.

(a) $y = \ln(3x^2 - 4x)$

Solution Let $g(x) = 3x^2 - 4x$, so that $g'(x) = 6x - 4$. From the second formula in the preceding box,

$$y' = \frac{g'(x)}{g(x)} = \frac{6x - 4}{3x^2 - 4x}.$$

(b) $y = 3x \ln x^2$

Solution Since $3x \ln x^2$ is the product of $3x$ and $\ln x^2$, use the product rule:

$$\begin{aligned} y' &= (3x)\left(\frac{d}{dx}\ln x^2\right) + (\ln x^2)\left(\frac{d}{dx}(3x)\right) \\ &= 3x\left(\frac{2x}{x^2}\right) + (\ln x^2)(3) && \text{Take derivatives.} \\ &= 6 + 3\ln x^2 \\ &= 6 + \ln (x^2)^3 && \text{Property of logarithms} \\ &= 6 + \ln x^6. && \text{Property of exponents} \end{aligned}$$

(c) $y = \ln[(x^2 + x + 1)(4x - 3)^5]$

Solution Here, we use the properties of logarithms before taking the derivative (the same thing could have been done in part (b) by writing $\ln x^2$ as $2 \ln x$):

$$\begin{aligned} y &= \ln[(x^2 + x + 1)(4x - 3)^5] \\ &= \ln(x^2 + x + 1) + \ln(4x - 3)^5 && \text{Property of logarithms} \\ &= \ln(x^2 + x + 1) + 5\ln(4x - 3); && \text{Property of logarithms} \\ y' &= \frac{2x + 1}{x^2 + x + 1} + 5 \cdot \frac{4}{4x - 3} && \text{Take derivatives.} \\ &= \frac{2x + 1}{x^2 + x + 1} + \frac{20}{4x - 3}. \end{aligned}$$

Checkpoint 5

Find y' for the given functions.

(a) $y = \ln(7 + x)$

(b) $y = \ln(4x^2)$

(c) $y = \ln(8x^3 - 3x)$

(d) $y = x^2 \ln x$

The function $y = \ln(-x)$ is defined for all $x < 0$ (since $-x > 0$ when $x < 0$.) Its derivative can be found by applying the derivative rule for $\ln g(x)$ with $g(x) = -x$:

$$\begin{aligned} y' &= \frac{g'(x)}{g(x)} \\ &= \frac{-1}{-x} \\ &= \frac{1}{x}. \end{aligned}$$

This is the same as the derivative of $y = \ln x$, with $x > 0$. Since

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0, \end{cases}$$

we can combine two results into one in the manner that follows.

Derivative of ln |x|

If $y = \ln|x|$, then $y' = \dfrac{1}{x}$ $\quad (x \neq 0)$.

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Example 9 Let $y = e^x \ln|x|$. Find y'.

Solution Use the product rule:

$$y' = e^x \cdot \frac{1}{x} + \ln|x| \cdot e^x = e^x\left(\frac{1}{x} + \ln|x|\right).$$ ✓6

Checkpoint 6

Find the derivative of each function.

(a) $y = e^{x^2} \ln|x|$

(b) $y = x^2/\ln|x|$

Example 10 If $f(x) = \log x$, find $f'(x)$.

Solution By the change-of-base theorem (page 204),

$$f(x) = \log x = \frac{\ln x}{\ln 10} = \frac{1}{\ln 10} \cdot \ln x.$$

Since $1/\ln 10$ is a constant,

$$f'(x) = \frac{1}{\ln 10} \cdot \frac{d}{dx}(\ln x) = \frac{1}{\ln 10} \cdot \frac{1}{x} = \frac{1}{(\ln 10)x}.$$

Often, a population or the sales of a certain product will start growing slowly, then grow more rapidly, and then gradually level off. Such growth can frequently be approximated by a *logistic function* of the form

$$f(x) = \frac{c}{1 + ae^{kx}}$$

for appropriate constants a, c, and k. (Logistic functions were introduced in Section 4.2.)

Example 11 **Finance** If a person borrows \$120,000 on a 30-year fixed-rate mortgage at 5.7% interest per year, compounded monthly, then the required monthly payment is \$696. However, the borrower would like to pay off the loan early by making larger monthly payments. The actual number of months it will take to pay off the mortgage if a payment of p dollars is made each month is approximated by

$$f(p) = 211 \ln\left(\frac{p}{p - 570}\right) \qquad (p \geq 696).$$

Find $f(710)$ and $f'(710)$ and explain what your answers mean for the borrower.

Solution Evaluating $f(p)$ at $p = 710$ yields:

$$f(710) = 211 \ln\left(\frac{710}{710 - 570}\right) = 211 \ln\left(\frac{710}{140}\right) \approx 342.6.$$

Making monthly payments of \$710 means that the borrower will pay off the loan in 343 months instead of 360 months (the payment in the final month will be somewhat smaller).

Prior to taking the derivative, we may choose to use properties of logarithms to rewrite $f(p)$ as follows:

$$f(p) = 211 \ln\left(\frac{p}{p - 570}\right) = 211[\ln p - \ln(p - 570)] = 211 \ln(p) - 211 \ln(p - 570).$$

Differentiating $f(p)$ with respect to p yields

$$f'(p) = 211\left(\frac{1}{p}\right) - 211\left(\frac{1}{p - 570}\right) = \frac{211}{p} - \frac{211}{p - 570}.$$

By evaluating the derivative at $p = 710$ we find that

$$f'(710) = \frac{211}{710} - \frac{211}{710 - 570} = \frac{211}{710} - \frac{211}{140} \approx -1.2.$$

This means that, when $p = \$710$, increasing the monthly payment by 1 dollar will shorten the length of time needed to pay off the loan by approximately 1 month. ✓7

Checkpoint 7

If $f(x) = 6 \ln\left(\frac{x}{7 - 3x}\right)$, find $f'(2)$.

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Example 12 **Business** The number of cell phone subscriptions in the United States (in millions) in year t can be approximated by

$$f(t) = \frac{320}{1 + 34e^{-.28t}},$$

where $t = 0$ corresponds to the year 1990. Find the rate of change of cell phone subscriptions in 1990, 2004, and 2014. (Data from: www.worldbank.org.)

Solution Use the quotient rule to find the derivative of $f(t)$:

$$f'(t) = \frac{(1 + 34e^{-.28t})(0) - 320(34(-.28)e^{-.28t})}{(1 + 34e^{-.28t})^2}$$
$$= \frac{3046.4e^{-.28t}}{(1 + 34e^{-.28t})^2}.$$

The rate of change in 1990 ($t = 0$) is

$$f'(0) = \frac{3046.4e^{-.28(0)}}{(1 + 34e^{-.28(0)})^2} \approx 2.49.$$

Similarly, we compute that $f'(14) \approx 21.55$ and $f'(24) \approx 3.39$. This means that, in 1990, cell phone subscriptions were increasing at a rate of 2.49 million per year. By 2004, the rate of change had jumped dramatically to 21.55 million per year. By 2014, the rate of increase had slowed to 3.39 million per year. ✓8

✓Checkpoint 8

Suppose that a fish population is given by

$$f(x) = \frac{4000}{1 + e^{.5x}},$$

where x is time in years. Find the rate of change of the population when

(a) $x = 0$

(b) $x = 3$

(c) Is the population increasing or decreasing?

11.8 Exercises

Find the derivatives of the given functions. (See Examples 1–4 and 7–9.)

1. $y = e^{5x}$

2. $y = e^{-4x}$

3. $f(x) = 5e^{2x}$

4. $f(x) = 4e^{-2x}$

5. $g(x) = -4e^{-7x}$

6. $g(x) = 6e^{x/2}$

7. $y = e^{x^2}$

8. $y = e^{-x^2}$

9. $f(x) = e^{x^3/3}$

10. $y = e^{4-x^3}$

11. $y = -3e^{3x^2+5}$

12. $y = 4e^{2x^2-4}$

13. $y = \ln(-10x^2 + 7x)$

14. $y = \ln(11 - 2x)$

15. $y = \ln\sqrt{5x + 1}$

16. $y = \ln\sqrt{x - 8}$

17. $f(x) = \ln[(5x + 9)(x^2 + 1)]$

18. $y = \ln[(4x + 1)(5x - 2)]$

19. $y = \ln\left(\frac{4 - x}{3x + 8}\right)$

20. $f(x) = \ln\left(\frac{7x - 1}{12x + 5}\right)$

21. $y = x^4e^{-3x}$

22. $y = xe^{3x}$

23. $y = (6x^2 - 5x)e^{-3x}$

24. $y = (x - 5)^2e^{3x}$

25. $y = \ln(8x^3 - 3x)^{1/2}$

26. $y = \ln(x^5 + 8x^2)^{3/2}$

27. $y = x\ln(9 - x^4)$

28. $y = -4x\ln(x + 12)$

29. $y = x^4\ln|x|$

30. $y = (2x^3 - 1)\ln|x|$

31. $y = \frac{-5\ln|x|}{5 - 2x}$

32. $y = \frac{3\ln|x|}{3x + 4}$

33. $y = \sqrt{\ln(x - 3)}$

34. $y = [\ln(x + 1)]^4$

35. $y = \frac{e^x - 1}{\ln|x|}$

36. $y = \frac{e^x}{\ln|x|}$

37. $y = \frac{e^x - e^{-x}}{x}$

38. $y = \frac{e^x + e^{-x}}{x}$

39. $f(x) = e^{3x+2}\ln(4x - 5)$

40. $y = e^{x^3}\ln|x|$

41. $y = \frac{700}{7 - 10e^{.4x}}$

42. $f(x) = \frac{2400}{3 + 8e^{.2x}}$

43. $y = \frac{500}{12 + 5e^{-.5x}}$

44. $y = \frac{10{,}000}{9 + 4e^{-.2x}}$

45. $y = \ln(\ln|x|)$

46. $y = 5e^{e^x}$

Find the derivatives of the given functions. (See Examples 6 and 10.) The chain rule is needed in Exercises 48–52.

47. $y = 8^x$

48. $y = 8^{x^2}$

49. $y = 15^{2x}$

50. $f(x) = 2^{-x}$

51. $g(x) = \log 6x$

52. $y = \log(x^2 + 1)$

For Exercises 53–56, find the equation of the tangent line to the graph of f at the given point.

53. $f(x) = \frac{e^x}{2x + 1}$ at $(0, 1)$

54. $f(x) = \frac{x^2}{e^x}$ when $x = 1$

55. $f(x) = \ln(1 + x^2)$ at $(0, 0)$

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56. $f(x) = \ln(2x^2 - 7x)$ when $x = -1$

57. If $f(x) = e^{2x}$, find $f'[\ln(1/4)]$.

58. If $g(x) = 3e \ln[\ln x]$, find $g'(e)$.

Work these exercises. (See Example 5.)

59. **Business** The gross national income of China (in billions of U.S. dollars) can be approximated by

$$P(t) = 1048e^{.165t},$$

where $t = 0$ corresponds to the year 2000. At what rate is Chinese gross national income changing in the given year? (Data from: www.worldbank.org.)

(a) 2010 **(b)** 2014

60. **Business** The amount of money (in billions of U.S. dollars) that China spends on research and development can be approximated by $f(t) = 26.67e^{.166t}$, where $t = 0$ corresponds to the year 2000. Find the rate of change of Chinese research and development spending in 2012. (Data from: National Science Foundation, *National Patterns of R&D Resources*, annual report.)

61. **Finance** Total United States reserves (in billions of dollars), including gold, can be approximated by

$$R(x) = 87e^{.166x} \qquad (4 \le x \le 10),$$

where $x = 4$ corresponds to the year 2004. Find the amount of U.S. reserves and the rate of change of U.S. reserves in 2010.(Data from: www.worldbank.org.)

62. **Finance** The United States experienced a recession during 2007–2009. The percent of nonperforming loans in year x can be approximated by

$$N(x) = .038e^{.537x} \qquad (5 \le x \le 9),$$

where $x = 5$ corresponds to the year 2005. Find the percentage of nonperforming loans and the rate of change of this percentage in 2009. (Data from: www.worldbank.org.)

63. **Social Science** The number of doctorates (in thousands) awarded in the field of science in the United States is approximated by the function

$$g(x) = 17.6e^{.043x},$$

where $x = 2$ corresponds to the year 2002. (Data from: www.nsf.gov/statistics/infbrief/nsf09307.)

(a) Find $g(10)$. **(b)** Find $g'(10)$.

(c) Interpret your answers for parts (a) and (b).

64. **Health** The preschool mortality rate (number of deaths under age 5 per 1000 people) in Russia can be approximated by

$$M(t) = 23e^{-.068t},$$

where $t = 0$ corresponds to the year 2000. Assuming this model remains accurate, find the Russian preschool mortality rate and how this mortality rate will be changing in 2016. (Data from: www.worldbank.org.)

65. **Finance** The United States government debt as a percent of gross domestic product (GDP) can be approximated by

$$D(x) = 36.42e^{.081x} \qquad (6 \le x \le 12),$$

where $x = 0$ corresponds to the year 2000. Find how U.S. government debt as a percent of GDP was changing in 2011. (Data from: U.S. Bureau of the Public Debt.)

66. **Business** A business provides a $50,000 life insurance policy for each of its employees. The annual cost in dollars of a policy (which depends on the age of the employee) can be approximated by $C(x) = 4.55e^{.075x}$, where x is the age of the employee in years ($18 \le x \le 65$). Find and interpret the given quantities. (Data from: Internal Revenue Code section 79, Table 01.)

(a) $C(20)$ and $C'(20)$ **(b)** $C(60)$ and $C'(60)$

67. **Health** The temperature of a certain patient (in degrees Fahrenheit) t hours after the onset of an infection can be approximated by

$$P(t) = 98.6 + te^{-.2t}.$$

Find how the patient's temperature is changing at the given times.

(a) $t = 1$ **(b)** $t = 12$

68. **Business** The number of hourly hits (in thousands) on a company's website x hours after a news story breaks can be approximated by

$$H(x) = 1 + 30xe^{-.5x}.$$

Find the rate of change of hits to this website for the given number of hours after the story breaks.

(a) 1 hour **(b)** 12 hours

Work these exercises. (See Examples 11 and 12.)

69. **Natural Science** The population of a bed of clams in the Great South Bay off Long Island is approximated by the logistic function

$$G(t) = \frac{52{,}000}{1 + 12e^{-.52t}},$$

where t is measured in years. Find the clam population and its rate of growth after

(a) 1 year **(b)** 4 years **(c)** 10 years.

(d) What happens to the rate of growth over time?

70. **Business** The number of cell phone subscriptions per 100 people in Spain can be approximated by

$$f(t) = \frac{112}{1 + 510e^{-.6t}},$$

where $t = 0$ corresponds to the year 1990. Find the number of cell phone subscriptions per 100 people in Spain and the rate of change of this number in the given year. (Data from: www.worldbank.org.)

(a) 1990 **(b)** 2002 **(c)** 2014

71. **Health** The function

$$V(t) = \frac{1100}{(1 + 1023e^{-.02415t})^4},$$

where t is in months and $V(t)$ is in cubic centimeters, models the relationship between the volume of a breast tumor and the amount of time it has been growing.*

*Data from John Spratt et al., "Decelerating Growth and Human Breast Cancer," *Cancer* 71, no. 6 (1993): 2013–2019.

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(a) Find the tumor volume at 240 months.

(b) Assuming that the shape of a tumor is spherical, find the radius of the tumor from part (a). (*Hint:* The volume of a sphere is given by the formula $v = \frac{4}{3}\pi r^3$.)

(c) If a tumor of volume .5 cm^3 is detected, according to the formula, how long has it been growing? What does this imply?

(d) Calculate the rate of change of tumor volume at 240 months. Interpret this rate.

72. Business The percent of the population of the United States who are Internet users is approximated by

$$f(t) = \frac{80}{1 + 48e^{-.4t}},$$

where $t = 0$ corresponds to the year 1990. Find the percent of the population who are Internet users and the rate at which this percentage is changing in the given year. (Data from: www.worldbank.org.)

(a) 1990 **(b)** 2000 **(c)** 2015

73. Education The number of bachelor's degrees (in thousands) earned in the social sciences and history in year x can be approximated by

$$f(x) = 24 \ln x + 117,$$

where $x = 3$ corresponds to the year 2003. Find the rate at which the number of social science and history degrees is changing in the given year. (Data from: *The Statistical Abstract of the United States*: 2012.)

(a) 2012 **(b)** 2016

74. Business The producer price index (PPI) for net output of the construction sand and gravel mining industry can be approximated by

$$f(x) = 94 \ln(x) + 61,$$

where $x = 5$ corresponds to the year 2005. Find the rate of change of the PPI for this industry in 2014. (Data from: United States Bureau of Labor Statistics.)

Work the following exercises.

75. Business Suppose the demand function for x thousand of a certain item is

$$p = 100 + \frac{50}{\ln x} \qquad (x > 1),$$

where p is in dollars.

(a) Find the marginal revenue.

(b) Find the revenue from the next thousand items at a demand of 8000 ($x = 8$).

76. Finance A person owes $4000 on a credit card that charges an annual interest rate of 15.9%, compounded monthly. If the card holder makes no additional charges and makes a payment of p dollars each month, the number of months it will take to pay off the debt is given by

$$f(p) = 76 \ln\left(\frac{p}{p - 53}\right) \qquad (p > 53).$$

(a) Find and interpret $f(78)$.

(b) Would you expect $f'(p)$ to be positive or negative?

(c) Find and interpret $f'(78)$.

77. Finance An individual deposits $250 in an account that earns 2.8% interest per year, compounded monthly. The number of months needed for the balance in the account to reach an amount of A dollars is given by

$$f(A) = 429 \ln\left(\frac{A}{250}\right).$$

(a) Find and interpret $f(300)$.

(b) Would you expect $f'(A)$ to be positive or negative?

(c) Find and interpret $f'(300)$.

78. Physical Science The Richter scale provides a measure of the magnitude of an earthquake. One of the largest Richter numbers M ever recorded for an earthquake was 8.9, from the 1933 earthquake in Japan. The following formula shows a relationship between the amount of energy released, E, and the Richter number:

$$M = \frac{2 \ln E - 2 \ln .007}{3 \ln 10}.$$

Here, E is measured in kilowatt-hours (kWh).*

(a) For the 1933 earthquake in Japan, what value of E gives a Richter number $M = 8.9$?

(b) If the average household uses 247 kilowatt-hours per month, for how many months would the energy released by an earthquake of this magnitude power 10 million households?

(c) Find the rate of change of the Richter number M with respect to energy when $E = 70{,}000$ kWh.

(d) What happens to $\frac{dM}{dE}$ as E increases?

79. Social Science A child is waiting at a street corner for a gap in traffic so that she can safely cross the street. A mathematical model for traffic shows that if the child's expected waiting time is to be at most 1 minute, then the maximum traffic flow (in cars per hour) is given by

$$f(x) = \frac{67{,}338 - 12{,}595 \ln x}{x},$$

where x is the width of the street in feet.† Find the maximum traffic flow and the rate of change of the maximum traffic flow with respect to street width when x is the given number.

(a) 30 feet **(b)** 40 feet

80. Health One measure of whether a dialysis patient has been adequately dialyzed is the urea reduction ratio (URR). It is generally agreed that a patient has been adequately dialyzed when the URR exceeds a value of .65. The value of the URR can be

*Christopher Bradley, "Media Clips," *Mathematics Teacher* 93, no. 4 (April 2000): 300–303.

†Edward A. Bender, *An Introduction to Mathematical Modeling*, Dover Publications, Inc., 2000.

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calculated for a particular patient from the following formula by Gotch:

$$\text{URR} = 1 - e^{-.0056t+.04} - \frac{8t(1 - e^{-.0056t+.04})}{126t + 900}.$$

Here, t is measured in minutes.*

(a) Find the value of the URR after this patient receives dialysis for 180 minutes. Has the patient received adequate dialysis?

(b) Find the value of the URR after the patient receives dialysis for 240 minutes. Has the patient received adequate dialysis?

(c) Use the numerical derivative feature on a graphing calculator to compute the instantaneous rate of change of the URR when the time on dialysis is 240 minutes. Interpret this rate.

81. Natural Science The age–weight relationship of female Arctic foxes caught in Svalbard, Norway, can be estimated by the function

$$M(t) = 3102e^{-e^{-.022(t-56)}},$$

where t is the age of the fox in days and $M(t)$ is the weight of the fox in grams.†

(a) Estimate the weight of a female fox that is 200 days old.

(b) Estimate the rate of change in weight of an Arctic fox that is 200 days old.

(c) Use a graphing calculator to graph $M(t)$, and then describe the growth pattern.

(d) Use the table function on a graphing calculator or a spreadsheet to develop a chart that shows the estimated weight and growth rate of female foxes for days 50, 100, 150, 200, 250, and 300.

*Edward Kessler, Nathan Ritchey, et al., "Urea Reduction Ratio and Urea Kinetic Modeling: A Mathematical Analysis of Changing Dialysis Parameters," *American Journal of Nephrology* 18 (1998): 471–477.

†Pal Prestrud and Kjell Nilssen, "Growth, Size, and Sexual Dimorphism in Arctic Foxes," *Journal of Mammalogy* 76, no. 2 (May 1995): 522–530.

Checkpoint Answers

1. **(a)** $(2x^2 - 1)e^x + 4xe^x$ **(b)** $\dfrac{-e^x}{2(1 - e^x)^{1/2}}$

2. **(a)** $y' = 44e^{11x}$ **(b)** $y' = 45e^{(-9x+2)}$ **(c)** $y' = -3x^2e^{-x^3}$

3. **(a)** $y' = \dfrac{e^x(11 + x)}{(12 + x)^2}$ **(b)** $y' = \dfrac{-27{,}000e^x}{(15 + 3e^x)^2}$

4. $y' = (\ln 25)25^x$

5. **(a)** $y' = \dfrac{1}{7 + x}$ **(b)** $y' = \dfrac{2}{x}$ **(c)** $y' = \dfrac{24x^2 - 3}{8x^3 - 3x}$ **(d)** $y' = x(1 + 2\ln x)$

6. **(a)** $y' = e^{x^2}\left(\dfrac{1}{x} + 2x\ln|x|\right)$ **(b)** $y' = \dfrac{2x\ln|x| - x}{(\ln|x|)^2}$

7. 21

8. **(a)** −500 fish per year **(b)** About −298 fish per year **(c)** Decreasing

11.9 Continuity and Differentiability

Intuitively speaking, a function is **continuous** at a point if you can draw the graph of the function near that point without lifting your pencil from the paper. Conversely, a function is **discontinuous** at a point if the pencil *must* be lifted from the paper in order to draw the graph on both sides of the point.

Looking at some graphs that have points of discontinuity will clarify the idea of continuity at a point. Each of the three graphs in Figure 11.40 is continuous *except* at $x = 2$. As

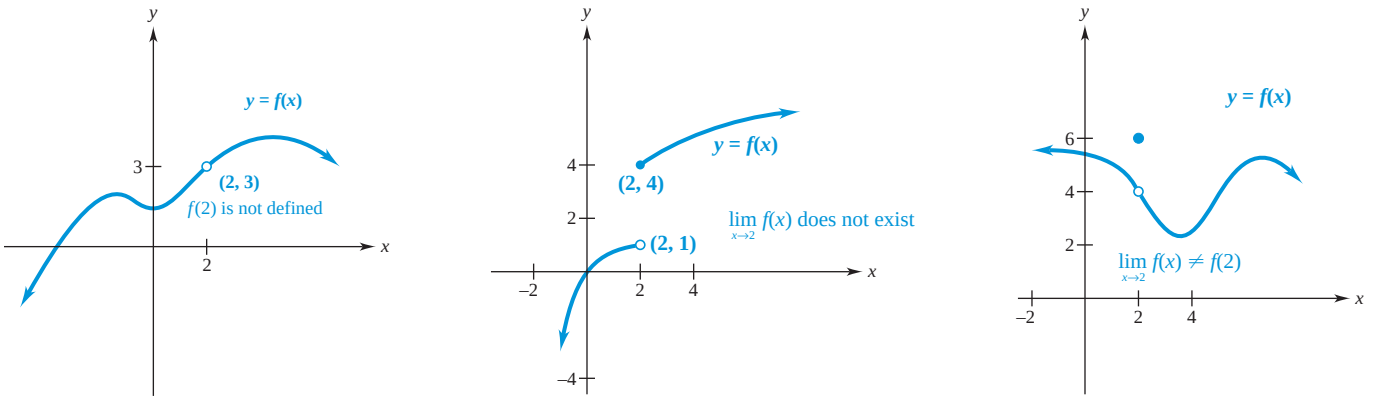

Figure 11.40

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usual, an open circle indicates that the point is not part of the graph. You cannot draw these graphs without lifting your pencil at $x = 2$, at least for an instant.

However, there are different reasons why each function is discontinuous at $x = 2$:

1. In graph (a), $f(2)$ is not defined (because there is a hole in the graph).
2. In graph (b), $\lim_{x\to 2} f(x)$ does not exist (because the left-hand limit does not equal the right-hand limit—see Section 11.2).
3. In graph (c), $f(2)$ is defined and $\lim_{x\to 2} f(x)$ does exist, but $\lim_{x\to 2} f(x) \neq f(2)$ (because $\lim_{x\to 2} f(x) = 4$ whereas $f(2) = 6$).

By phrasing each of these considerations positively, we obtain the following definition.

Continuity at a Point

A function f is **continuous** at $x = c$ if

(a) $f(c)$ is defined;

(b) $\lim_{x\to c} f(x)$ exists;

(c) $\lim_{x\to c} f(x) = f(c)$.

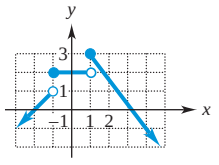
If f is not continuous at $x = c$, it is **discontinuous** there.

The idea behind condition (c) is this: To draw the graph near $x = c$ without lifting your pencil, it must be the case that when x is very close to c, $f(x)$ must be very close to $f(c)$—otherwise, you would have to lift the pencil at $x = c$, as happens in Figure 11.40 at $x = 2$. ✓1

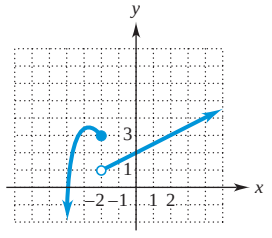
✓Checkpoint 1

Find any points of discontinuity for the given functions.

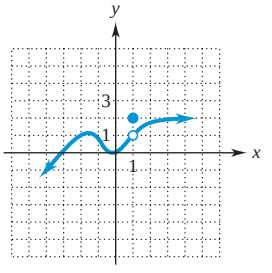
(a)



(b)



(c)



Example 1 Tell why the given functions are discontinuous at the indicated point.

(a) $f(x)$ in Figure 11.41 at $x = 3$

Solution The open circle on the graph in Figure 11.41 at the point where $x = 3$ means that $f(3)$ is not defined. Because of this, part (a) of the definition fails.

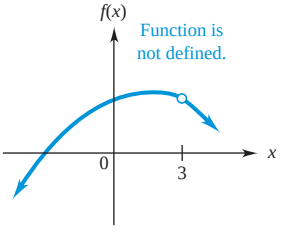


Figure 11.41

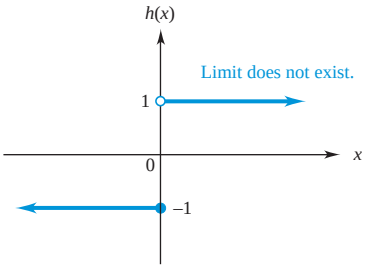


Figure 11.42

(b) $h(x)$ in Figure 11.42 at $x = 0$

Solution The graph in Figure 11.42 shows that $h(0) = -1$. Also, as x approaches 0 from the left, $h(x)$ is -1. However, as x approaches 0 from the right, $h(x)$ is 1. As mentioned in Section 11.1, for a limit to exist at a particular value of x, the values of $h(x)$ must

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approach a single number. Since no single number is approached by the values of $h(x)$ as x approaches 0, $\lim_{x\to 0} h(x)$ does not exist, and part (b) of the definition fails.

(c) $g(x)$ at $x = 4$ in Figure 11.43

Solution In Figure 11.43, the heavy dot above 4 shows that $g(4)$ is defined. In fact, $g(4) = 1$. However, the graph also shows that

$$\lim_{x\to 4} g(x) = -2,$$

so $\lim_{x\to 4} g(x) \neq g(4)$, and part (c) of the definition fails.

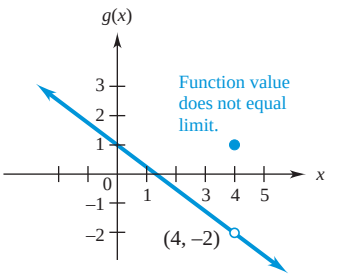


Figure 11.43

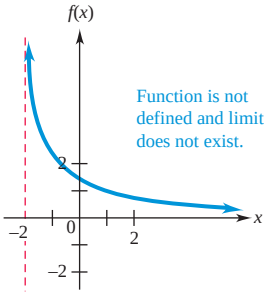


Figure 11.44

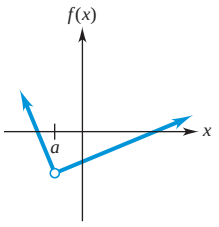
(d) $f(x)$ in Figure 11.44 at $x = -2$

Solution The function f graphed in Figure 11.44 is not defined at -2, and $\lim_{x\to -2} f(x)$ does not exist. Either of these reasons is sufficient to show that f is not continuous at -2. (Function f *is* continuous at any value of x greater than -2, however.) ✓2

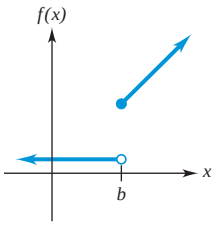
Checkpoint 2

Tell why the given functions are discontinuous at the indicated point.

(a)



(b)



Continuity at Endpoints

One-sided limits (see Section 11.2) can be used to define continuity at an endpoint of a graph.* Consider the endpoint $(1, 0)$ of the graph of the half-circle function $f(x) = \sqrt{1 - x^2}$ in Figure 11.45. Note that f has a left-hand limit at $x = 1$. Furthermore,

$$f(1) = 0 \quad \text{and} \quad \lim_{x\to 1^-} f(x) = 0, \quad \text{so that} \quad \lim_{x\to 1^-} f(x) = f(1).$$

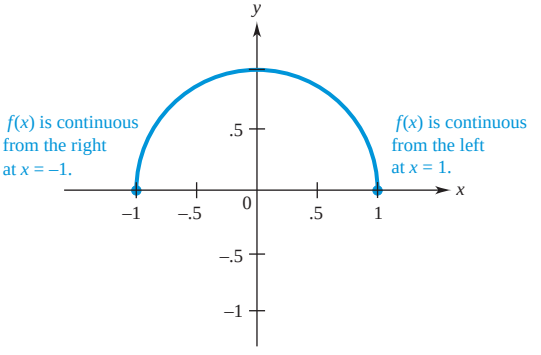


Figure 11.45

We say that $f(x)$ is *continuous from the left* at $x = 1$. A similar situation occurs with the other endpoint $(-1, 0)$: There is a right-hand limit at $x = -1$, and we have $\lim_{x\to -1^+} f(x) = 0 = f(-1)$. We say that $f(x)$ is *continuous from the right* at $x = -1$.

*The preceding definition of continuity does not apply to an endpoint because the two-sided limit $\lim_{x\to c} f(x)$ does not exist when $(c, f(c))$ is an endpoint.

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This situation is an example of the following definition.

Continuity from the Left and from the Right

A function f is **continuous from the left** at $x = c$ if

(a) $f(c)$ is defined;

(b) $\lim_{x \to c^-} f(x)$ exists;

(c) $\lim_{x \to c^-} f(x) = f(c)$.

Similarly, f is **continuous from the right** at $x = c$ if

(a) $f(c)$ is defined;

(b) $\lim_{x \to c^+} f(x)$ exists;

(c) $\lim_{x \to c^+} f(x) = f(c)$.

Note that a function which is continuous at $x = c$ is automatically continuous from the left and from the right at $x = c$. ✓3

✓Checkpoint 3

State whether the given function is continuous from the left, continuous from the right, or neither at the given value of x.

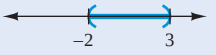
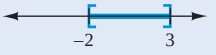
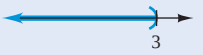
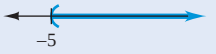
(a) $f(x)$ in Figure 11.40(b) at $x = 2$

(b) $h(x)$ in Figure 11.42 at $x = 0$

(c) $g(x)$ in Figure 11.43 at $x = 4$

Continuity on an Interval

When discussing the continuity of a function, it is often helpful to use interval notation, which was introduced in Chapter 1. The following chart should help you recall how it is used.

Interval	Name	Description	Interval Notation
	Open interval	$-2 < x < 3$	$(-2, 3)$
	Closed interval	$-2 \le x \le 3$	$[-2, 3]$
	Open interval	$x < 3$	$(-\infty, 3)$
	Open interval	$x > -5$	$(-5, \infty)$

Remember, the symbol ∞ does not represent a number; ∞ is used for convenience in interval notation to indicate that the interval extends without bound in the positive direction. Also, $-\infty$ indicates no bound in the negative direction. ✓4

Continuity at a point was defined previously. Continuity on an interval is defined as follows.

✓Checkpoint 4

Write each of the following in interval notation.

(a)

(b)

(c)

Continuity on Open and Closed Intervals

A function f is **continuous on the open interval (a, b)** if f is continuous at every x-value in the interval (a, b).

A function f is **continuous on the closed interval $[a, b]$** if

1. f is continuous on the open interval (a, b);
2. f is continuous from the right at $x = a$;
3. f is continuous from the left at $x = b$.

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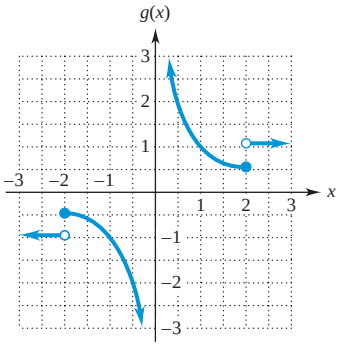


Figure 11.46

Intuitively, a function is continuous on an interval if you can draw the entire graph of f over that interval without lifting your pencil from the paper.

Example 2 Is the function of Figure 11.46 continuous on the given x-intervals?

(a) $(0, 2)$

Solution The function g is discontinuous only at $x = -2$, 0, and 2. Hence, g is continuous at every point of the open interval $(0, 2)$, which does not include 0 or 2.

(b) $[0, 2]$

Solution The function g is not defined at $x = 0$, so it is not continuous from the right there. Therefore, it is not continuous on the closed interval $[0, 2]$.

(c) $(1, 3)$

Solution This interval contains a point of discontinuity at $x = 2$. So g is not continuous on the open interval $(1, 3)$.

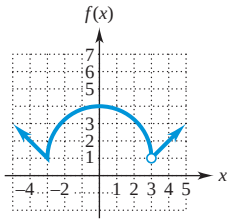
(d) $[-2, -1]$

Solution The function g is continuous on the open interval $(-2, -1)$, continuous from the right at $x = -2$, and continuous from the left at $x = -1$. Hence, g is continuous on the closed interval $[-2, -1]$. ✓5

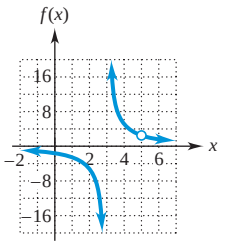
Checkpoint 5

Are the functions whose graphs are shown continuous on the indicated intervals?

(a) $[-4, -1]$; $[2, 5]$



(b) $(1, 3)$; $[4, 6]$



Checkpoint 6

Suppose a smart phone data plan for heavy users charges \$50 for the first 5 GB of data and then \$15 for each additional 2 GB of data (or fraction thereof). If $C(x)$ is the cost (in dollars) of using x GB of data, find any points of discontinuity for C on the interval $(0, 10)$.

Example 3 **Business** The monthly cost $C(x)$ (in dollars) of a popular smart phone data plan for the use of x gigabytes of data is \$30 for the first 2 GB of data plus \$10 for each additional GB of data (or fraction thereof). (Data from: Verizon Wireless, 2012 prices.)

(a) Sketch the graph of $y = C(x)$ on the interval $0 \le x \le 5$.

Solution For any amount of data usage up to and including 2 GB, the charge is \$30. The charge for using more than 2 GB but less than 3 GB of data is \$40. Similar results lead to the graph in Figure 11.47.

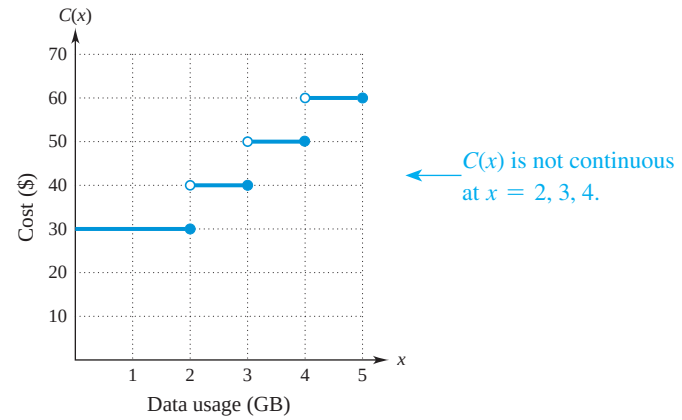


Figure 11.47

(b) Find any points of discontinuity for C on the interval $(0, 5)$.

Solution As the graph suggests, C is discontinuous at $x = 2, 3, 4$ gigabytes. However, C is continuous from the left at each of these x-values. ✓6

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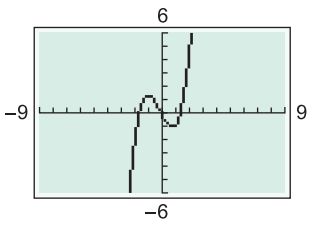

Figure 11.48

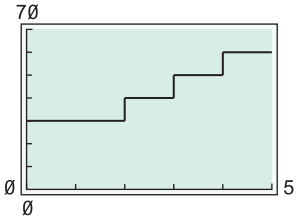

Figure 11.49

Continuity and Technology

Because of the way a graphing calculator or computer graphs functions, the graph of a continuous function may look like disconnected, closely spaced line segments (Figure 11.48). Furthermore, calculators often have trouble accurately portraying graphs at points of discontinuity. For instance, "jump discontinuities" (such as those at $x = 2$, 3, and 4 in Figure 11.47) may appear as "stair steps," with vertical line segments connecting separate pieces of the graph (Figure 11.49). Similarly, a hole in the graph (such as the one at $x = 4$ in Figure 11.43) may not be visible on a calculator screen, depending on what viewing window is used. Nor can a calculator indicate whether an endpoint is included in the graph.

One reason these inaccuracies appear is due to the manner in which a calculator graphs: It plots points and connects them with line segments. In effect, the calculator assumes that the function is continuous unless it actually computes a function value and determines that it is undefined. In that case, it will skip a pixel and not connect points on either side of it.

The moral of this story is that a calculator or computer is only a tool. In order to use it correctly and effectively, you must understand the mathematics involved. When you do, it is usually easy to interpret screen images correctly.

Continuity and Differentiability

As shown earlier in this chapter, a function fails to have a derivative at a point where the function is not defined, where the graph of the function has a "sharp point," or where the graph has a vertical tangent line. (See Figure 11.50.)

The function graphed in Figure 11.50 is continuous on the interval (x_1, x_2) and has a derivative at each point on this interval. On the other hand, the function is also continuous on the interval $(0, x_2)$, but does *not* have a derivative at each point on the interval. (See x_1 on the graph.)

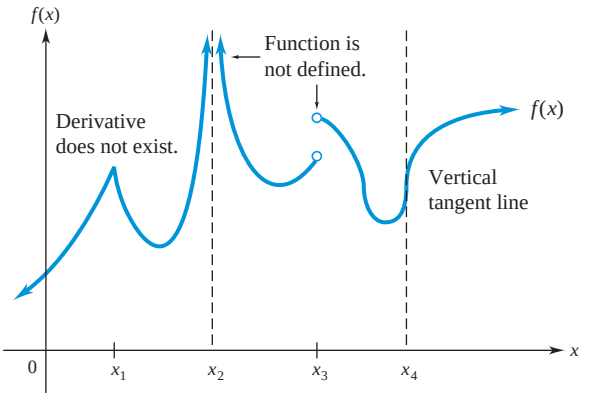

Figure 11.50

A similar situation holds in the general case.

If the derivative of a function exists at a point, then the function is continuous at that point. However, a function may be continuous at a point and not have a derivative there.

Example 4 **Business** The Fair Labor Standards Act requires overtime pay for covered, nonexempt employees at a rate of not less than one and one-half times an employee's regular rate of pay after 40 hours of work in a workweek. A fast food

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restaurant pays an entry-level employee $10 per hour for regular pay and time-and-a-half for overtime pay. Let $P(x)$ represent the employee's gross pay prior to taxes and other withholdings (in dollars) for working x hours in a week. The graph of $y = P(x)$ is shown in Figure 11.51. Find any points where P is discontinuous. Also find any points where P is not differentiable. (Data from: United States Department of Labor.)

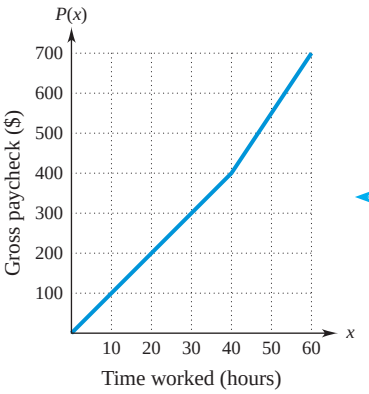


Figure 11.51

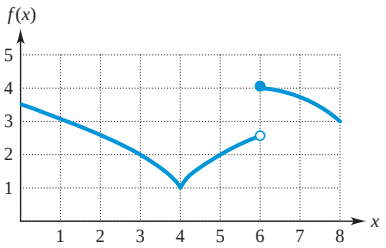
Checkpoint 7

State whether the function whose graph is shown is continuous and/or differentiable at the given values of x.

(a) $x = 2$

(b) $x = 4$

(c) $x = 6$



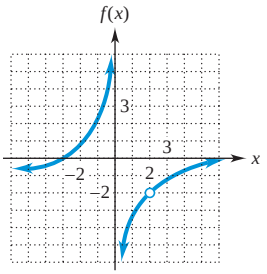
Solution The graph indicates that P is a continuous function since it can be drawn without lifting pencil from paper. However, P is not differentiable at $x = 40$ hours where the graph has a sharp corner. ✓7

TECHNOLOGY TIP In some viewing windows, a calculator graph may appear to have a sharp corner when, in fact, the graph is differentiable at that point. When in doubt, try a different window to see if the corner disappears.

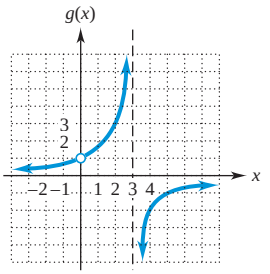
11.9 Exercises

Find all points of discontinuity for the functions whose graphs are shown. (See Example 1.)

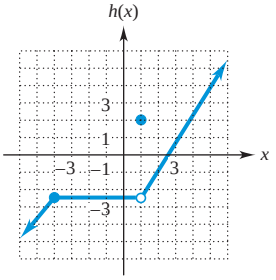
1.



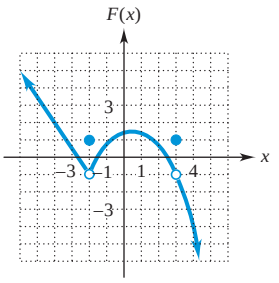
2.



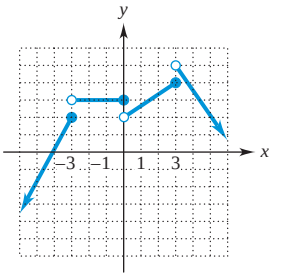
3.



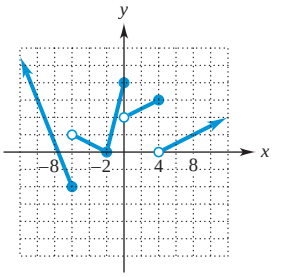
4.



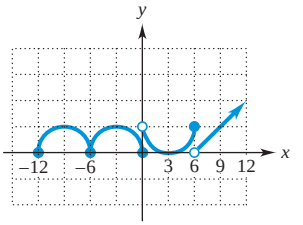
5.



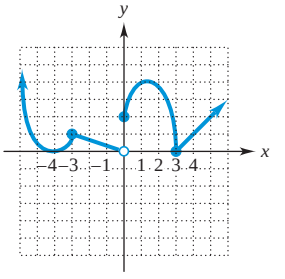
6.



7.



8.



Are the given functions continuous at the given values of x?

9. $f(x) = \dfrac{4}{x - 2}; x = 0, x = 2$

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10. $g(x) = \dfrac{5}{x+5}; x = -5, x = 5$

11. $h(x) = \dfrac{1}{x(x-3)}; x = 0, x = 3, x = 5$

12. $h(x) = \dfrac{-1}{(x-2)(x+3)}; x = 0, x = 2, x = 3$

13. $g(x) = \dfrac{x+2}{x^2-x-2}; x = 1, x = 2, x = -2$

14. $h(x) = \dfrac{3x}{6x^2+15x+6}; x = 0, x = -\dfrac{1}{2}, x = 3$

15. $g(x) = \dfrac{x^2-4}{x-2}; x = 0, x = 2, x = -2$

16. $h(x) = \dfrac{x^2-25}{x+5}; x = 0, x = 5, x = -5$

17. $f(x) = \begin{cases} x-2 & \text{if } x \le 3 \\ 2-x & \text{if } x > 3 \end{cases}; x = 2, x = 3$

18. $g(x) = \begin{cases} e^x & \text{if } x < 0 \\ x+1 & \text{if } 0 \le x \le 3; x = 0, x = 3 \\ 2x-3 & \text{if } x > 3 \end{cases}$

Find all x-values where the function is discontinuous.

19. $g(x) = x^2 - 5x + 12$

20. $f(x) = \dfrac{x^2-9}{x+3}$

21. $h(x) = \dfrac{|x+4|}{x+4}$

22. $f(x) = \dfrac{7+x}{x(x-3)}$

23. $k(x) = e^{\sqrt{x-2}}$

24. $r(x) = \ln\left(\dfrac{x}{x-3}\right)$

In Exercises 25–26, (a) graph the given function, and (b) find all values of x where the function is discontinuous.

25. $f(x) = \begin{cases} 2 & \text{if } x < 1 \\ x+3 & \text{if } 1 \le x < 4 \\ 8 & \text{if } x \ge 4 \end{cases}$

26. $g(x) = \begin{cases} -10 & \text{if } x < -2 \\ x^2+4x-6 & \text{if } -2 \le x < 2 \\ 10 & \text{if } x \ge 2 \end{cases}$

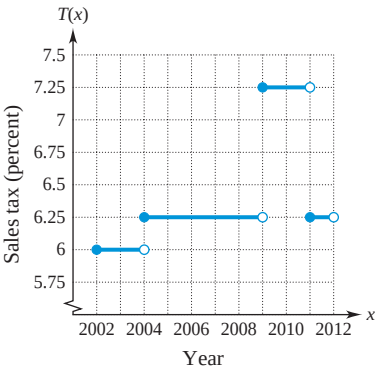
In Exercises 27 and 28, find the constant k that makes the given function continuous at x = 2.

27. $f(x) = \begin{cases} x+k & \text{if } x \le 2 \\ 5-x & \text{if } x > 2 \end{cases}$

28. $g(x) = \begin{cases} x^k & \text{if } x \le 2 \\ 2x+4 & \text{if } x > 2 \end{cases}$

Work these exercises. (See Examples 3 and 4.)

29. **Finance** Officials in California have raised and lowered the state sales tax in various years. The accompanying figure shows the California state sales tax for 2001–2012. Let $T(x)$ represent the sales tax rate (as a percent) in year x. Find the given quantities. (Data from: www.boe.ca.gov/sutax/taxrateshist.htm.)



(a) $\lim\limits_{x \to 2010^-} T(x)$ **(b)** $\lim\limits_{x \to 2010^+} T(x)$

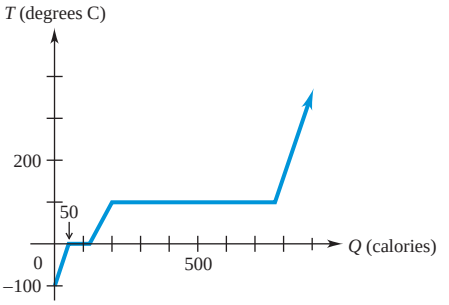
(c) $\lim\limits_{x \to 2010} T(x)$ **(d)** $\lim\limits_{x \to 2011^-} T(x)$

(e) $\lim\limits_{x \to 2011^+} T(x)$ **(f)** $\lim\limits_{x \to 2011} T(x)$

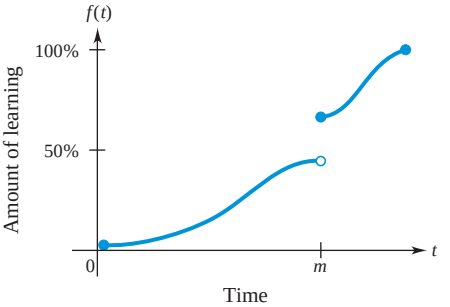
(g) Find all values on the interval (2002, 2012) where T is discontinuous.

(h) Find all values on the interval (2002, 2012) where T is not differentiable.

30. **Natural Science** Suppose a gram of ice is at a temperature of −100°C. The accompanying graph shows the temperature of the ice as an increasing number of calories of heat are applied. Where is this function discontinuous? Where is it differentiable?



31. **Social Science** With certain skills (such as music), learning is rapid at first and then levels off. However, sudden insights may cause learning to speed up sharply. A typical graph of such learning is shown in the accompanying figure. Where is the function discontinuous? Where is it differentiable?



32. **Business** A pick-your-own strawberry farm charges $10 per gallon of strawberries. The cost of partial gallons is prorated except that there is a minimum charge of $10. Customers who

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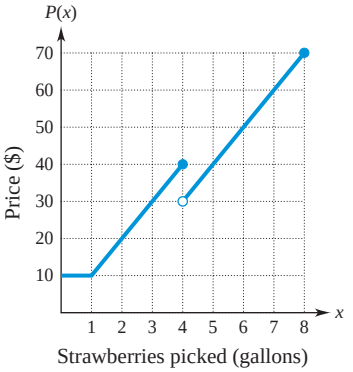
pick more than 4 gallons receive a $10 discount off their total order. The accompanying figure shows the price (in dollars), $P(x)$, of picking x gallons of strawberries. Find the given quantities.

(a) $P(1)$ **(b)** $\lim_{x\to 1} P(x)$

(c) $P(4)$ **(d)** $\lim_{x\to 4} P(x)$

(e) Find all values on the interval (0, 8) where P is discontinuous.

(f) Find all values on the interval (0, 8) where P is not differentiable.



Work the following exercises.

33. Business The accompanying table shows the price (in dollars), $C(x)$, of mailing a first-class letter weighing x ounces domestically in 2012. (Data from: www.usps.gov.)

Weight Not Over . . . (oz)	Price ($)
1	0.45
2	0.65
3	0.85
3.5	1.05

(a) Sketch a graph of $y = C(x)$ on the interval [0, 3.5].

(b) $\lim_{x\to 1^-} C(x)$ **(c)** $\lim_{x\to 1^+} C(x)$

(d) $\lim_{x\to 1} C(x)$ **(e)** $\lim_{x\to 2.5^-} C(x)$

(f) $\lim_{x\to 2.5^+} C(x)$ **(g)** $\lim_{x\to 2.5} C(x)$

(h) Find all values on the interval (0, 3.5) where C is discontinuous.

34. Business A nut supplier offers discounts for buying in bulk. The accompanying table shows the price per pound for jumbo raw Georgia pecans (with the cost of partial pounds being prorated). Let $C(x)$ represent the cost (in dollars) of purchasing x pounds of nuts. Find the cost of buying the given amounts.

(a) ½ pound **(b)** 10 pounds

(c) 11 pounds

(d) Find all values where C is discontinuous.

Pounds Purchased	Price per Pound
$0 < x \le 1$	$16.99
$1 < x \le 10$	$15.99
$10 < x \le 30$	$14.50

35. Business The state of Connecticut has established maximum charges, $C(x)$, (in dollars) for nonconsensual towing of vehicles (up to 10,000 lbs) a distance of x miles. The base charge of $88 includes mileage to the scene, preparing the vehicle, and the first 2 miles of the tow (loaded miles). In addition there is a mileage charge of $4.75 per mile (or fraction of a mile) in excess of 2 loaded miles. Find the given quantities. (Data from: www.ct.gov.)

(a) $C(1)$ **(b)** $\lim_{x\to 1} C(x)$

(c) $C(4)$ **(d)** $\lim_{x\to 4} C(x)$

(e) Find all values where C is discontinuous on the interval (0, 5).

36. Business The cost of hourly parking at Ronald Reagan National Washington Airport is $2 per half hour (or fraction thereof) for the first two hours, and then $4 per hour (or fraction thereof) for the third through eighth hours, with a daily maximum of $36. Let $C(x)$ be the cost (in dollars) of parking for x hours. Find the left-hand, right-hand, and two-sided limits at the given x-values. (Data from: www.metwashairports.com/reagan, 2012 prices.)

(a) $x = 1.5$

(b) $x = 5$

(c) $x = 10$

(d) Find all values where C is discontinuous.

Write each of the given ranges in interval notation.

37.

38.

39.

40.

41. On which of the following intervals is the function in Exercise 6 continuous: (−6, 0); [0, 3]; (4, 8)?

42. On which of the following intervals is the function in Exercise 5 continuous: (−3, 0); [0, 3]; [1, 3]?

43. Health During pregnancy, a woman's weight naturally increases during the course of the event. When she delivers, her weight immediately decreases by the approximate weight of the child. Suppose that a 120-lb woman gains 27 lb during pregnancy, delivers a 7-lb baby, and then, loses the remaining weight during the next 20 weeks.

(a) Graph the weight gain and loss during the pregnancy and the 20 weeks following the birth of the baby. Assume that the pregnancy lasts 40 weeks, that delivery occurs immediately after this time interval, and that the weight gain and loss before and after birth, respectively, are linear.

(b) Is this function continuous? If not, then find the value(s) of t where the function is discontinuous.

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Checkpoint Answers

1. **(a)** $x = -1, 1$ **(b)** $x = -2$ **(c)** $x = 1$
2. **(a)** $f(a)$ does not exist. **(b)** $\lim_{x \to b} f(x)$ does not exist.
3. **(a)** Continuous from the right
   (b) Continuous from the left
   (c) Neither
4. **(a)** $(-5, 3)$ **(b)** $[4, 7]$ **(c)** $(-\infty, -1]$
5. **(a)** Yes; no **(b)** Yes; no
6. $x = 5, 7, 9$ GB
7. **(a)** Continuous, differentiable
   (b) Continuous, not differentiable
   (c) Not continuous, not differentiable

CHAPTER 11 Summary and Review

Key Terms and Symbols

11.1 $\lim_{x \to a} f(x)$ limit of $f(x)$ as x approaches a
limit of a constant function
limit of the identity function
properties of limits
polynomial limits
limit theorem

11.2 $\lim_{x \to a^+} f(x)$ limit of $f(x)$ as x approaches a from the right
$\lim_{x \to a^-} f(x)$ limit of $f(x)$ as x approaches a from the left
$\lim_{x \to \infty} f(x)$ limit of $f(x)$ as x approaches infinity
$\lim_{x \to -\infty} f(x)$ limit of $f(x)$ as x approaches negative infinity
infinite limits

11.3 average rate of change
geometric meaning of average rate of change
velocity
instantaneous rate of change
marginal cost, revenue, profit

11.4 y', derivative of y
$f'(x)$, derivative of $f(x)$
secant line
tangent line
derivative
differentiable

11.5 $\frac{dy}{dx}$, derivative of $y = f(x)$
$D_x[f(x)]$, derivative of $f(x)$
$\frac{d}{dx}[f(x)]$, derivative of $f(x)$
derivative rules
demand function

11.6 $\overline{C}(x)$, average cost per item
product rule
quotient rule
marginal average cost

11.7 $g[f(x)]$, composite function
chain rule
generalized power rule
marginal-revenue product

11.8 derivatives of exponential functions
derivatives of logarithmic functions

11.9 continuous at a point
discontinuous
continuous from the left and from the right
interval notation
continuous on an open or closed interval

Chapter 11 Key Concepts

Limit of a Function

Let f be a function, and let a and L be real numbers. Suppose that as x takes values very close (but not equal) to a (on both sides of a), the corresponding values of $f(x)$ are very close (and possibly equal) to L and that the values of $f(x)$ can be made arbitrarily close to L for all values of x that are close enough to a. Then L is the **limit** of f as x approaches a, written $\lim_{x \to a} f(x) = L$.

One-Sided Limits

The **limit of f as x approaches a from the right**, written $\lim_{x \to a^+} f(x)$, is defined by replacing the phrase "on both sides of a" by "with $x > a$" in the definition of the limit of a function.

The **limit of f as x approaches a from the left**, written $\lim_{x \to a^-} f(x)$, is defined by replacing the phrase "on both sides of a" by "with $x < a$" in the definition of the limit of a function.

Properties of Limits

Let a, k, A, and B be real numbers, and let f and g be functions such that

$$\lim_{x \to a} f(x) = A \quad \text{and} \quad \lim_{x \to a} g(x) = B.$$

Then

1. $\lim_{x \to a} k = k$ (for any constant k);
2. $\lim_{x \to a} x = a$ (for any real number a);
3. $\lim_{x \to a} [f(x) \pm g(x)] = A \pm B = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$;
4. $\lim_{x \to a} [f(x) \cdot g(x)] = A \cdot B = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$;

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5. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{A}{B} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \qquad (B \neq 0);$

6. for any real number r for which A^r exists,

$$\lim_{x\to a}[f(x)]^r = A^r = [\lim_{x\to a} f(x)]^r.$$

Polynomial Limits

If f is a polynomial function, then $\lim_{x\to a} f(x) = f(a)$.

Limit Theorem

If f and g are functions that have limits as x approaches a, and $f(x) = g(x)$ for all $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

Limits at Infinity

Let f be a function that is defined for all large values of x, and let L be a real number.

Suppose that as x takes larger and larger values, increasing without bound, the corresponding values of $f(x)$ are very close (and possibly equal) to L, and suppose that the values of $f(x)$ can be made arbitrarily close to L by taking large enough values of x. Then L is the **limit of f as x approaches infinity**, written $\lim_{x\to\infty} f(x) = L$.

Let f be a function that is defined for all small negative values of x, and let M be a real number. Suppose that as x takes smaller and smaller negative values, decreasing without bound, the corresponding values of $f(x)$ are very close (and possibly equal) to M, and suppose that the values of $f(x)$ can be made arbitrarily close to M by taking small enough values of x. Then M is the **limit of f as x approaches negative infinity**, written $\lim_{x\to-\infty} f(x) = M$.

Derivatives

The **instantaneous rate of change** of a function f when $x = a$ is

$$\lim_{h\to 0}\frac{f(a+h) - f(a)}{h},$$

provided that this limit exists.

The **tangent line** to the graph of $y = f(x)$ at the point $(a, f(a))$ is the line through this point and having slope $\lim_{h\to 0}\dfrac{f(a+h) - f(a)}{h}$, provided that this limit exists.

The **derivative** of the function f is the function denoted f' whose value at the number x is $f'(x) = \lim_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$, provided that this limit exists.

Rules for Derivatives

(Assume that all indicated derivatives exist.)

Constant Function If $f(x) = k$, where k is any real number, then $\boldsymbol{f'(x) = 0}$.

Power Rule If $f(x) = x^n$, for any real number n, then $\boldsymbol{f'(x) = n \cdot x^{n-1}}$.

Constant Times a Function Let k be a real number. Then the derivative of $y = k \cdot f(x)$ is $\boldsymbol{y' = k \cdot f'(x)}$.

Sum-or-Difference Rule If $f(x) = u(x) \pm v(x)$, then $\boldsymbol{f'(x) = u'(x) \pm v'(x)}$.

Product Rule If $f(x) = u(x) \cdot v(x)$, then $\boldsymbol{f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)}$.

Quotient Rule If $f(x) = \dfrac{u(x)}{v(x)}$, and $v(x) \neq 0$, then $\boldsymbol{f'(x) = \dfrac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}}$.

Chain Rule Let $y = f[g(x)]$. Then $\boldsymbol{y' = f'[g(x)] \cdot g'(x)}$.

Chain Rule (alternative form) If y is a function of u, say, $y = f(u)$, and if u is a function of x, say, $u = g(x)$, then $y = f(u) = f[g(x)]$, and

$$\boldsymbol{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}.$$

Generalized Power Rule Let u be a function of x, and let $y = u^n$ for any real number n. Then

$$\boldsymbol{y' = n \cdot u^{n-1} \cdot u'}.$$

Exponential Function If $y = e^{g(x)}$, then $\boldsymbol{y' = g'(x) \cdot e^{g(x)}}$.

Natural-Logarithm Function If $y = \ln|x|$, then $\boldsymbol{y' = \dfrac{1}{x}}$.

If $y = \ln[g(x)]$, then $\boldsymbol{y' = \dfrac{g'(x)}{g(x)}}$.

Continuity

A function f is **continuous** at $x = c$ if $f(c)$ is defined, $\lim_{x\to c} f(x)$ exists, and $\lim_{x\to c} f(x) = f(c)$.

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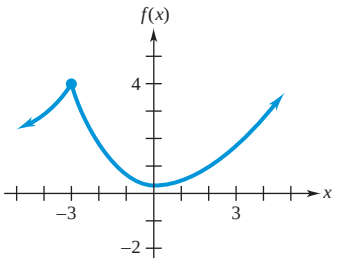
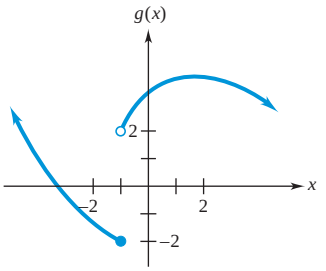
Chapter 11 Review Exercises

In Exercises 1–4, determine graphically or numerically whether the limit exists. If the limit exists, find its (approximate) value.

1. $\lim_{x\to -3} f(x)$; $\lim_{x\to -3^-} f(x)$; $\lim_{x\to -3^+} f(x)$



2. $\lim_{x\to -1} g(x)$; $\lim_{x\to -1^-} g(x)$; $\lim_{x\to -1^+} g(x)$



3. $\lim_{x\to 0} \dfrac{\sqrt{2-x}-\sqrt{2}}{x}$

4. $\lim_{x\to -1} \dfrac{10^x - .1}{x+1}$

Find the given limit if it exists.

5. $\lim_{x\to 5} \dfrac{5x-4}{x-2}$

6. $\lim_{x\to -1} \dfrac{7-3x}{x-4}$

7. $\lim_{x\to 3} \dfrac{x^2-9}{x+3}$

8. $\lim_{x\to 7} \dfrac{49-x^2}{x+7}$

9. $\lim_{x\to 4} \dfrac{x^2-16}{x-4}$

10. $\lim_{x\to -2} \dfrac{x^2-x-6}{x+2}$

11. $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$

12. $\lim_{x\to 16} \dfrac{\sqrt{x}-4}{x-16}$

Find the given limit if it exists.

13. $\lim_{x\to 4^+} \sqrt{x^2-x-12}$

14. $\lim_{x\to 3^-} (5x+\sqrt{9-x^2})$

15. $\lim_{x\to 7} \dfrac{|x-7|}{x-7}$

16. $\lim_{x\to -9} \dfrac{|x+9|}{x+9}$

17. $\lim_{x\to 6^-} \dfrac{x^2+1}{x-6}$

18. $\lim_{x\to -5} \dfrac{8x}{(x+5)^2}$

19. $\lim_{x\to\infty} \dfrac{9x+1}{3x-4x^2}$

20. $\lim_{x\to\infty} \dfrac{5x^2-8x+3}{5-7x^3}$

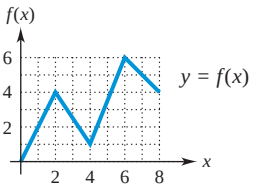
21. $\lim_{x\to\infty} \dfrac{6x^2-x+50}{3-2x^2}$

22. $\lim_{x\to\infty} \dfrac{12x^3+3x^2-7x+1}{6-3x-4x^3}$

Use the accompanying graph to find the average rate of change of f on the given intervals.

23. $x = 0$ to $x = 4$

24. $x = 2$ to $x = 8$



Find the average rate of change for each of the given functions.

25. $f(x) = 3x^2 - 5$, from $x = 1$ to $x = 5$

26. $g(x) = -x^3 + 2x^2 + 1$, from $x = -2$ to $x = 3$

27. $h(x) = \dfrac{6-x}{2x+3}$, from $x = 0$ to $x = 6$

28. $f(x) = e^{2x} + 5\ln x$, from $x = 1$ to $x = 7$

Use the definition of the derivative to find the derivative of each of the given functions.

29. $y = 2x + 3$

30. $y = 4 - 3x$

31. $y = 2x^2 - x - 1$

32. $y = x^2 + 2x$

Find the slope of the tangent line to the given curve at the given value of x. Find the equation of each tangent line.

33. $y = x^2 - 6x$; at $x = 2$

34. $y = 8 - x^2$; at $x = 1$

35. $y = \dfrac{-2}{x+5}$; at $x = -2$

36. $y = \sqrt{6x-2}$; at $x = 3$

37. **Business** Suppose hardware store customers are willing to buy $T(p)$ boxes of nails at p dollars per box, where

$$T(p) = .06p^4 - 1.25p^3 + 6.5p^2 - 18p + 200 \quad (0 < p \le 11).$$

(a) Find the average rate of change in demand for a change in price from \$5 to \$8.

(b) Find the instantaneous rate of change in demand when the price is \$5.

(c) Find the instantaneous rate of change in demand when the price is \$8.

38. Suppose the average rate of change of a function $f(x)$ from $x = 1$ to $x = 4$ is 0. Does this mean that f is constant between $x = 1$ and $x = 4$? Explain.

Find the derivative of each of the given functions.

39. $y = 6x^2 - 7x + 3$

40. $y = x^3 - 5x^2 + 1$

41. $y = 4x^{9/2}$

42. $y = -4x^{-5}$

43. $f(x) = x^{-6} + \sqrt{x}$

44. $f(x) = 8x^{-3} - 5\sqrt{x}$

45. $y = (4t^2+9)(t^3-5)$

46. $y = (-5t+3)(t^3-4t)$

47. $y = 8x^{3/4}(5x+1)$

48. $y = 40x^{-1/4}(x^2+1)$

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49. $f(x) = \dfrac{4x}{x^2 - 8}$

50. $g(x) = \dfrac{-5x^2}{3x + 1}$

51. $y = \dfrac{\sqrt{x} - 6}{3x + 7}$

52. $y = \dfrac{\sqrt{x} + 9}{x - 4}$

53. $y = \dfrac{x^2 - x + 1}{x - 1}$

54. $y = \dfrac{2x^3 - 5x^2}{x + 2}$

55. $f(x) = (4x - 2)^4$

56. $k(x) = (5x - 1)^6$

57. $y = \sqrt{2t - 5}$

58. $y = -3\sqrt{8t - 1}$

59. $y = 2x(3x - 4)^3$

60. $y = 5x^2(2x + 3)^5$

61. $f(u) = \dfrac{3u^2 - 4u}{(2u + 3)^3}$

62. $g(t) = \dfrac{t^3 + t - 2}{(2t - 1)^5}$

63. $y = -6e^{2x}$

64. $y = 8e^{-5x}$

65. $y = e^{-2x^3}$

66. $y = -5e^{x^2}$

67. $y = 5x \cdot e^{2x}$

68. $y = -7x^2 \cdot e^{-3x}$

69. $y = \ln(x^2 + 4x - 1)$

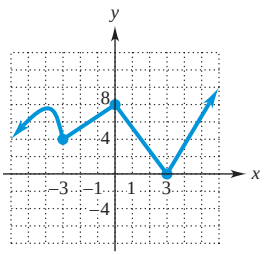
70. $y = \ln(4x^3 + 2x)$

71. $y = \dfrac{\ln 6x}{x^2 - 1}$

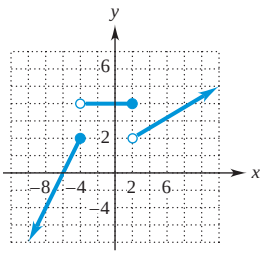
72. $y = \dfrac{\ln(3x + 5)}{x^2 + 5x}$

Find all points of discontinuity for the given graphs.

73.



74.



Are the given functions continuous at the given points?

75. $f(x) = \dfrac{x - 6}{x + 5}; x = 6, x = -5, x = 0$

76. $f(x) = \dfrac{x^2 - 9}{x + 3}; x = 3, x = -3, x = 0$

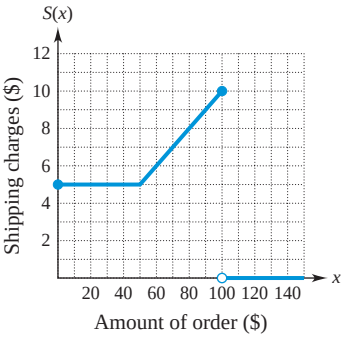
77. $h(x) = \dfrac{2 - 3x}{2 - x - x^2}; x = -2, x = \dfrac{2}{3}, x = 1$

78. $f(x) = \dfrac{x^2 - 4}{x^2 - x - 6}; x = 2, x = 3, x = 4$

Work these problems.

79. **Business** Shipping charges at an online pet supply store are 10% of the total purchase price except that there is a minimum charge of \$5 and orders over \$100 receive free shipping. The accompanying figure shows the charge, $S(x)$, for shipping an order in the amount of x dollars. Find the given limits.

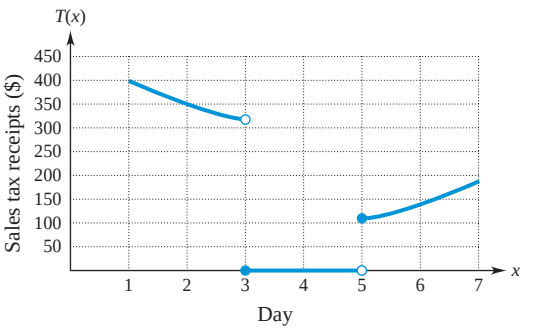
(a) $\lim\limits_{x \to 50} S(x)$ **(b)** $\lim\limits_{x \to 100} S(x)$



80. **Business** The Iowa Department of Revenue held a back-to-school sales tax holiday (during which clothing and footwear were exempt from state sales tax) on August 3–4, 2012. Suppose that the accompanying figure shows the sales tax receipts $T(x)$ (in dollars) from a certain apparel store at time x, where $x = 1$ corresponds to August 1. Find the given limits.

(a) $\lim\limits_{x \to 2} T(x)$ **(b)** $\lim\limits_{x \to 3} T(x)$

(c) $\lim\limits_{x \to 4} T(x)$ **(d)** $\lim\limits_{x \to 5} T(x)$



81. **Business** An airport car rental agency charges \$65 per day (or fraction of a day) to rent a full-size car, plus a one-time airport surcharge of \$5. Let $C(x)$ be the cost of renting a full-size car from this agency for x days. Find the given quantities.

(a) $C(2)$ **(b)** $\lim\limits_{x \to 2^-} C(x)$

(c) $\lim\limits_{x \to 2^+} C(x)$ **(d)** $\lim\limits_{x \to 2} C(x)$

82. **Business** The average cost (in dollars per hundred brochures) of printing x hundred glossy brochures (one page, two-sided, tri-fold) can be approximated by $\overline{C}(x) = \dfrac{450 + 115x}{x}$. Find $\lim\limits_{x \to \infty} \overline{C}(x)$.

83. **Social Science** Throughout the world, the percentage of seats in national parliaments (in a single chamber or lower chamber) held by women is shown in the figure on the next page. Find the average rate of change of the percent of women in national parliaments between the years 2000 and 2004. (Data from: www.worldbank.org.)

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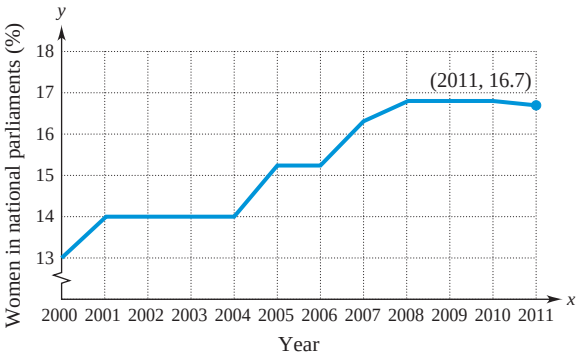

84. **Social Science** Use the figure in the preceding exercise to find the average rate of change of the percent of women in national parliaments between the years 2004 and 2011.

85. **Business** The amount of revenue (in thousands of dollars) generated by Google, Inc. can be approximated by $f(x) = 306x^2 + 133x - 1139$, where $x = 2$ corresponds to the year 2002. Find the instantaneous rate of change of revenue with respect to time in the year 2012. (Data from: www.morningstar.com.)

86. **Natural Science** Suppose that $D(x)$ represents the depth of a certain lake (in feet) at a distance x feet from the bank (in the direction toward the center of the lake). Explain what each of the given equations means in practical terms for a person standing in the lake.

(a) $D(20) = 5$ **(b)** $D'(20) = .3$

87. **Social Science** The youth age dependency ratio in the United States, the proportion of youth dependents (people younger than 15) per 100 working-age population (people ages 15–64), can be approximated by $f(x) = .024x^2 - .47x + 32.33$, where $x = 0$ corresponds to the year 2000. Find the youth age dependency ratio and the rate of change of this number in the given year. (Data from: www.worldbank.org.)

(a) 2007 **(b)** 2014

88. **Finance** Total gross domestic savings (in billions of dollars) can be approximated by $S(x) = .836x^4 - 20.3x^3 + 152x^2 - 333x + 1697$, where $x = 0$ corresponds to the year 2000. How is the total gross domestic savings changing in the year 2010? (Data from: www.worldbank.org.)

89. **Business** The number of cell phone subscriptions (in millions) in India can be approximated by

$$f(t) = 4.8e^{.5t},$$

where $t = 0$ corresponds to the year 2000. Find the number of cell phone subscriptions in India and the rate of change of this number in the year 2011. (Data from: www.worldbank.org.)

90. **Health** Based on historical data and future projections for the period 1970–2020, life expectancy (in years) for a person born in the United States in year x can be approximated by

$$L(x) = 15.08 \ln x + 7.3,$$

where $x = 70$ corresponds to the year 1970. (Data from: United States Census Bureau.)

(a) Find the projected life expectancy for a person born in 2020.

(b) If another person is born one year later, approximately how would her or his life expectancy differ from the person born in 2020?

91. **Business** A popular food truck that drives to various locations to sell lunch items has determined that the cost (in hundreds of dollars) of making x hundred sandwiches can be approximated by

$$C(x) = \frac{250x + 45}{2x + 40}.$$

(a) Find the marginal-cost function.

(b) If the food truck operators are currently making 1000 sandwiches per week, find the approximate cost of making an additional 100 sandwiches.

92. **Business** Consider the cost function given in the previous exercise.

(a) Find the average-cost function.

(b) Find the marginal average-cost function.

(c) If the food truck operators are currently making 1000 sandwiches per week, approximately how will the average cost change if they make an additional 100 sandwiches?

93. **Business** The daily demand function for coffee at the campus coffee cart can be approximated by $p = \sqrt{25 - .1x}$, where p is the price (in dollars) and x is the number of cups of coffee demanded.

(a) Find the revenue function.

(b) Find the marginal revenue function.

(c) If the demand is currently 90 cups of coffee, find the revenue.

(d) If the demand is currently 90 cups of coffee, find the approximate revenue from selling one additional cup of coffee.

94. **Business** The cost function for the coffee cart described in the previous exercise can be approximated by $C(x) = 120 + .75x$, where $C(x)$ is the cost (in dollars) of producing x cups of coffee.

(a) Find the profit function.

(b) Find the marginal profit function.

(c) If the demand is currently 120 cups of coffee, find the approximate profit from selling one additional cup of coffee.

95. **Business** Consider the demand equation given in Exercise 93.

(a) Solve the demand equation for x in terms of p.

(b) Find the rate of change of demand with respect to price.

(c) If the price is currently \$3 per cup, approximately what effect would raising the price by \$1 have on the demand?

96. **Business** Sales at a fireworks outlet (in thousands of dollars) on day x can be approximated by $S(x) = 1 + 5xe^{-.3x}$, where $x = 1$ corresponds to July 1. Find the amount of sales and rate of change of sales on the given day.

(a) July 2 **(b)** July 4

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97. **Business** Use the graph of $y = S(x)$ shown in Exercise 79 to answer the given questions.

(a) At what x-values is the function S not continuous?

(b) At what x-values is the function S not differentiable?

98. **Business** The monthly cost of a popular cell phone plan is \$50 for the first 450 minutes of airtime plus \$.50 per minute (or fraction of a minute) for usage over 450 minutes. Let $C(x)$ be the cost of using x minutes of airtime. Find the given quantities.

(a) $C(400)$

(b) $\lim_{x \to 400} C(x)$

(c) Is $C(x)$ continuous at $x = 400$?

(d) $C(460)$

(e) $\lim_{x \to 460} C(x)$

(f) Is $C(x)$ continuous at $x = 460$?

99. **Natural Science** Under certain conditions, the length of the monkeyface prickleback, a west-coast game fish, can be approximated by

$$L = 71.5(1 - e^{-.1t})$$

and its weight by

$$W = .01289 \cdot L^{2.9},$$

where L is the length in cm, t is the age in years, and W is the weight in grams.*

(a) Find the approximate length of a five-year-old monkeyface.

(b) Find how fast the length of a five-year-old monkeyface is increasing.

(c) Find the approximate weight of a five-year-old monkeyface. (*Hint:* Use your answer from part (a).)

(d) Find the rate of change of the weight with respect to length for a five-year-old monkeyface.

(e) Using the chain rule and your answers to parts (b) and (d), find how fast the weight of a five-year-old monkeyface is increasing.

*William H. Marshall and Tina Wyllie Echeverria, "Characteristics of the Monkeyface Prickleback," *California Fish and Game* 78, no. 2 (spring 1992).

Case Study 11 Price Elasticity of Demand

Any retailer who sells a product or a service is concerned with how a change in price affects demand for the article. The sensitivity of demand to price changes varies with different items. For smaller items, such as soft drinks, food staples, and lightbulbs, small percentage changes in price will not affect the demand for the item much. However, sometimes a small percentage change in price on big-ticket items, such as cars, homes, and furniture, can have significant effects on demand.

One way to measure the sensitivity of changes in price to demand is by the ratio of percent change in demand to percent change in price. If q represents the quantity demanded and p the price of the item, this ratio can be written as

$$\frac{\Delta q/q}{\Delta p/p},$$

where Δq represents the change in q and Δp represents the change in p. The ratio is usually negative, because q and p are positive, while Δq and Δp typically have opposite signs. (An *increase* in price causes a *decrease* in demand.) If the absolute value of this quantity is large, it shows that a small increase in price can cause a relatively large decrease in demand.

Applying some algebra, we can rewrite this ratio as

$$\frac{\Delta q/q}{\Delta p/p} = \frac{\Delta q}{q} \cdot \frac{p}{\Delta p} = \frac{p}{q} \cdot \frac{\Delta q}{\Delta p}.$$

Suppose $q = f(p)$. (Note that this is the inverse of the way our demand functions have been expressed so far; previously, we had $p = D(q)$.) Then $\Delta q = f(p + \Delta p) - f(p)$. It follows that

$$\frac{\Delta q}{\Delta p} = \frac{f(p + \Delta p) - f(p)}{\Delta p}.$$

As $\Delta p \to 0$, this quotient becomes

$$\lim_{\Delta p \to 0} \frac{\Delta q}{\Delta p} = \lim_{\Delta p \to 0} \frac{f(p + \Delta p) - f(p)}{\Delta p} = \frac{dq}{dp},$$

and

$$\lim_{\Delta p \to 0} \frac{p}{q} \cdot \frac{\Delta q}{\Delta p} = \frac{p}{q} \cdot \frac{dq}{dp}.$$

The quantity

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}$$

is positive because dq/dp is negative. E is called **elasticity of demand** and measures the instantaneous responsiveness of demand to price. For example, E may be .6 for physician services (expenses that have considerable price increases each year, but still have a high demand), but may be 2.3 for restaurant meals (highly enjoyable, but high-cost items and not necessities). These numbers indicate that the demand for physician services is much less responsive to price changes than the demand for restaurant meals. Another factor that impacts elasticity of demand is the availability of substitute products. For example, if one airline increases prices on a particular route but other airlines serving the route do not, then rather than paying the higher prices, consumers could fly with a different airline (the substitute product).

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If $E < 1$, the relative change in demand is less than the relative change in price, and the demand is called **inelastic.** If $E > 1$, the relative change in demand is greater than the relative change in price, and the demand is called **elastic.** When $E = 1$, the percentage changes in price and demand are relatively equal, and the demand is said to have **unit elasticity.**

Sometimes elasticity is counterintuitive. The addiction to illicit drugs is an excellent example. The quantity of the drug demanded by addicts, if anything, increases, no matter what the cost. Thus, illegal drugs are an inelastic commodity.

Example 1 Suppose that the demand for flat screen televisions is expressed by the equation

$$q = -.025p + 20.45,$$

where q is the annual demand (in millions of televisions) and p is the price of the product (in dollars).

(a) Calculate and interpret the elasticity of demand when $p = \$200$ and when $p = \$500$.

Solution Since $q = -.025p + 20.45$, we have $dq/dp = -.025$, so that

$$\begin{aligned} E &= -\frac{p}{q} \cdot \frac{dq}{dp} \\ &= -\frac{p}{-.025p + 20.45} \cdot (-.025) \\ &= \frac{.025p}{-.025p + 20.45}. \end{aligned}$$

Let $p = 200$ to get

$$E = \frac{.025(200)}{-.025(200) + 20.45} \approx .324.$$

Since $.324 < 1$, the demand was inelastic, and a percentage change in price resulted in a smaller percentage change in demand. For example, a 10% increase in price will cause a 3.24% decrease in demand.

If $p = 500$, then

$$E = \frac{.025(500)}{-.025(500) + 20.45} \approx 1.57.$$

Since $1.57 > 1$, the price is elastic. At this point, a percentage increase in price resulted in a greater percentage decrease in demand. A 10% increase in price resulted in a 15.7% decrease in demand.

(b) Determine the price at which demand had unit elasticity ($E = 1$). What is the significance of this price?

Solution Demand had unit elasticity at the price p that made $E = 1$, so we must solve the equation

$$\begin{aligned} \frac{.025p}{-.025p + 20.45} &= 1 \\ .025p &= -.025p + 20.45 \\ .05p &= 20.45 \\ p &= 409. \end{aligned}$$

Demand had unit elasticity at a price of \$409 per flat screen television. Unit elasticity indicates that the changes in price and demand are about the same.

The definitions from the preceding discussion can be expressed in the manner that follows.

Elasticity of Demand

Let $q = f(p)$, where q is the demand at a price p. The **elasticity of demand** is as follows:

Demand is **inelastic** if $E < 1$.

Demand is **elastic** if $E > 1$.

Demand has **unit elasticity** if $E = 1$.

Exercises

1. The monthly demand for beef in a given region can be expressed by the equation $q = -3.003p + 675.23$, where q is the monthly demand in tons and p is the price in dollars per 100 pounds. Determine the elasticity of demand when the price is \$70.*

2. Acme Stationery sells designer-brand pens. The demand equation for annual sales of these pens is $q = -1000p + 70{,}000$, where p is the price per pen. Normally, the pens sell for \$30 each. They are very popular, and Acme has been thinking of raising the price by one third.†
   (a) Find the elasticity of demand if $p = \$30$.
   (b) Find the elasticity of demand if the price is raised by one third. Is raising the price a good idea?

3. The monthly demand for lodging in a certain city is given by $q = -2481.52p + 472{,}191.2$, where p is the nightly rate.§
   (a) Find E when $p = \$100$ and when $p = \$75$.
   (b) At what price is there unit elasticity?

4. The following multiple-choice question is a released sample question from the Major Field Test in Economics.** The demand for a good is given by $q = 100 - 4p^2$, where $q =$ quantity demanded per unit of time and $p =$ the price per unit. At a price of \$4, the absolute value of the price elasticity of demand is approximately equal to which of the following?
   (a) ∞ **(b)** 36.0 **(c)** 3.6
   (d) 1.0 **(e)** 0.28

5. What must be true about demand if $E = 0$ everywhere?

*Adapted from *How Demand and Supply Determine Price*, Agricultural Marketing Manual, Alberta, Canada, February 1999.

†Taken from R. Horn, *Economics 331: Warm-up Problems: Supply, Demand, Elasticity.*

§Extracted from Bjorn Hanson, *Price Elasticity of Lodging Demand* (Pricewaterhouse-Coopers, 2000).

**The Major Field Tests are developed by ETS (http://www.ets.org/Media/Tests/MFT/pdf/2008/MFT_Economics_Sample_Items_08.pdf).

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Extended Project

Sensitivity to higher prices was evident in the tobacco market following increases in taxes that included large disparities in tax rates for cigarettes and pipe tobacco versus roll-your-own tobacco. Data for prices and quantities demanded are summarized in the following table for the year prior to and following the implementation of the tax increases.* In this project, the price data includes only the cost of the tax rather than the total cost of the product plus tax since the increases in taxes were the significant aspect of the changes in price.

	Cigarettes		Roll-Your-Own Tobacco		Pipe Tobacco	
Year	**Price (dollars per thousand sticks)**	**Quantity (billions of sticks)**	**Price (dollars per pound)**	**Quantity (thousands of pounds)**	**Price (dollars per pound)**	**Quantity (thousands of pounds)**
2008	19.50	337.64	1.10	19,673	1.10	3150
2010	50.33	296.23	24.78	6152	2.83	20,817

(a) The data show that the percent increase in the tax on cigarettes was $\dfrac{50.33 - 19.50}{19.50} = 158\%$. Find the percent increase for pipe tobacco and roll-your-own tobacco. Were the increases similar for all products or were there large disparities?

(b) For cigarettes, use a linear model (the equation of a line) to express the quantity demanded as a function of price.

(c) Repeat part (b) for roll-your-own tobacco.

(d) For cigarettes, determine the elasticity of demand $E = -\dfrac{p}{q}\dfrac{dq}{dp}$ at the lower price prior to the tax increases and also at the higher price following the tax increases. Was the elasticity of demand for cigarettes inelastic ($E < 1$) or elastic ($E > 1$) at the lower price? At the higher price?

(e) Repeat part (d) for roll-your-own tobacco.

(f) Factors that impact elasticity of demand include price, the necessity of the product, and availability of substitute products. What is a possible explanation for the large difference in elasticity of demand for cigarettes and roll-your-own tobacco at the higher price?

(g) You work for a tobacco store that sells a wide variety of products including those listed in the table above. Your employer has asked you to write a report that summarizes the elasticity of demand information you have calculated, provides a possible explanation for the large differences in elasticity of demand for cigarettes and roll-your-own tobacco after the price increases, and comments on the implications for the tobacco store.

*Tobacco Taxes: Large Disparities in Rates for Smoking Products Trigger Significant Market Shifts to Avoid Higher Taxes, United States Government Accountability Office Report to Congressional Committees, April 2012.

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