Preliminary Problem

A special rubber washer is made with two holes cut out as pictured. The area of the smaller of the two holes is \( \frac{1}{7} \) of the whole piece of rubber while the area of the larger hole is \( \frac{1}{4} \) of the whole. If the area of the original piece of rubber was \( 1\frac{3}{8} \text{ in}^2 \), what is the area of the finished washer?

If needed, see Hint before the Chapter Summary.
Integers such as \(-5\) were invented to solve equations like \(x + 5 = 0\). Similarly, a different type of number is needed to solve the equation \(2x = 1\). We need notation for this new number. If multiplication is to work with this new type of number as with whole numbers, then \(2x = x + x = 1\). In other words, the number \(\frac{1}{2}\) (one-half) created to solve the equation must have the property that when added to itself, the result is 1. It is an element of the set of numbers of the form \(\frac{a}{b}\), where \(b \neq 0\) and \(a\) and \(b\) are integers. More generally, numbers of the form \(\frac{a}{b}\) are solutions to equations of the form \(bx = a\). This set \(Q\) of rational numbers is defined as follows:

\[
Q = \left\{ \frac{a}{b} \mid a\text{ and }b\text{ are integers and }b \neq 0 \right\}
\]

Each member of \(Q\) is a fraction. In general, fractions are of the form \(\frac{a}{b}\), where \(b \neq 0\) but \(a\) and \(b\) are not necessarily integers. Each element \(\frac{a}{b}\) of set \(Q\) has \(a\) as the numerator and \(b\) as the denominator.

The English words used for denominators of rational numbers are similar to words to tell “order,” for example, the fourth person in a line, and the glass is three-fourths full. In contrast, \(\frac{3}{4}\) is read “out of four parts, (take) three” in Chinese. The Chinese model enforces the idea of partitioning quantities into equal parts and choosing some number of these parts. The concept of sharing quantities and comparing sizes of shares provides entry points to introduce students to rational numbers.

As early as grade 3 in the Common Core Standards, we find that students should “develop an understanding of fractions, beginning with unit fractions . . . view fractions as being built out of unit fractions . . . use fractions along with visual fraction models to represent parts of a whole.” (p. 21) Additionally by grade 4, students should “understand a fraction as a number on the number line.” (p. 24)

**REMARK** A unit fraction has a numerator of 1.

### 1 The Set of Rational Numbers

The rational number \(\frac{a}{b}\) may also be represented as \(a/b\) or \(a \div b\). The word *fraction* is derived from the Latin word *fractus*, meaning “broken.” The word *numerator* comes from a Latin word meaning “numberer,” and *denominator* comes from a Latin word meaning “namer.” Frequently it is only in the upper grades of middle school that students begin to use integers for the parts of rational numbers, but prospective teachers should know and recognize that rational numbers are negative as well as positive and zero. Some uses of rational numbers that will be considered in this chapter are seen in Table 1.

<table>
<thead>
<tr>
<th>Use</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Division problem or solution to a</td>
<td>The solution to (2x = 3) is (\frac{3}{2}).</td>
</tr>
<tr>
<td>multiplication problem</td>
<td></td>
</tr>
<tr>
<td>Portion, or part, of a whole</td>
<td>Joe received (\frac{1}{2}) of Mary’s salary each month for alimony.</td>
</tr>
<tr>
<td>Ratio</td>
<td>The ratio of Republicans to Democrats on a Senate committee is three to five.</td>
</tr>
<tr>
<td>Probability</td>
<td>When you toss a fair coin, the probability of getting heads is (\frac{1}{2}).</td>
</tr>
</tbody>
</table>
Rational Numbers and Proportional Reasoning

Figure 1 illustrates the use of rational numbers as equal-sized parts of a whole in part (a), a distance on a number line in part (b), and a part of a given set in part (c). The simplest representation is from part (a) where 1 part of 3 equal-sized parts is shaded. The fractional representation for this part is \( \frac{1}{3} \) where the entire bar represents 1 unit and the shaded part is \( \frac{1}{3} \) of the unit whole. [Later the bar model of part (a) will be extended to an area model where the shape may be different than the rectangular bar. Additionally, the bar model is helpful when we consider proportional reasoning later in the chapter.]

An extension of the thinking in the bar model is seen in the remaining parts of Figure 1. For example, part (b) could represent two one-thirds of the unit length, or two-thirds of the unit segment. Part (c) could represent three one-fifths of the whole set, or three-fifths of the whole set.

Early student exposure to rational numbers as fractions usually takes the form of description rather than mathematical notation. They hear phrases such as “one-half of a pizza,” “one-third of a cake,” or “three-fourths of a pie.” They encounter such questions as “If three identical fruit bars are distributed equally among four friends, how much does each receive?” The answer is that each receives \( \frac{3}{4} \) of a bar.

When rational numbers are introduced as fractions that represent a part of a whole, we must pay attention to the whole from which a rational number is derived. For example, if we talk about \( \frac{3}{4} \) of a pizza, then the amount of pizza depends on the size of the pizza, for example, 10” or 12” and the fractional part, \( \frac{3}{4} \).

To understand the meaning of any fraction, \( \frac{a}{b} \), where \( a, b \in W \) and \( b \neq 0 \), using the parts-to-whole model, we must consider each of the following:

1. The whole being considered.
2. The number \( b \) of equal-size parts into which the whole has been divided.
3. The number \( a \) of parts of the whole that are selected.

A fraction \( \frac{a}{b} \) where \( 0 \leq a < b \), is a proper fraction. A proper fraction is less than 1. For example, \( \frac{4}{7} \) is a proper fraction, but \( \frac{7}{4} \) and \( \frac{9}{7} \) are not; \( \frac{7}{4} \) is an improper fraction. In general \( \frac{a}{b} \) is an improper fraction if \( a \geq b > 0 \). An improper fraction is greater than or equal to 1.

**Historical Note**

The early Egyptian numeration system had symbols for fractions with numerators of 1 (unit fractions). Most fractions with other numerators were expressed as a sum of unit fractions, for example, \( \frac{7}{12} = \frac{1}{3} + \frac{1}{4} \).

Fractions with denominator 60 or powers of 60 were seen in Babylon about 2000 BCE, where 12,35 meant \( \frac{12 + \frac{35}{60}}{1} \). This usage was adopted by the Greek astronomer Ptolemy (approximately 125 CE), was used in Islamic and European countries, and is presently used in the measurements of angles.

The modern notation for fractions—a bar between numerator and denominator—is of Hindu origin. It came into general use in Europe in sixteenth-century books.

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Other meanings of fractions can be considered using whole-to-part and part-to-part references. For example whole-to-part might give us an improper fraction and part-to-part allows us to write, for example, the ratio of the number of band students in the school to the number of non-band students in the school.

The Common Core Standards state that grade 3 students should “express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.” (p. 24)

Later students learn that every integer \(n\) can be represented as a rational number because \(n = \frac{n \cdot k}{k}\), where \(k\) is any nonzero integer. In particular, \(0 = \frac{0 \cdot k}{k} = \frac{0}{k}\).

Rational Numbers on a Number Line

In the grade 3 Common Core Standards, we find the following standard:

Represent a fraction \(\frac{a}{b}\) on a number line by marking off \(a\) lengths of \(\frac{1}{b}\) from 0. Recognize that the resulting interval has size \(\frac{a}{b}\) and that its endpoint locates the number \(\frac{a}{b}\) on the number line. (p. 24)

Once the integers 0 and 1 are assigned to points on a line, the unit segment is defined and every other rational number is assigned to a specific point. For example, to represent \(\frac{3}{4}\) on the number line, we divide the segment from 0 to 1 into 4 segments of equal length and mark the line accordingly. Then, starting from 0, we count 3 of these segments and stop at the mark corresponding to the right endpoint of the third segment to obtain the point assigned to the rational number \(\frac{3}{4}\).

The Common Core Standards for grade 3 talk about a lengths of \(\frac{1}{b}\), where \(a\) and \(b\) are both positive (or \(a\) could be 0), but we also use integers as numerators or denominators of rational numbers, though negative integers are not used to talk about lengths. We think of the positive fractions described in the Common Core Standards as marked on a number line on the right side, and as with integers, we can consider the opposites of those fractions reflected over 0 to the left side of the number line as seen in Figure 2. We adopt two conventions for negative fractions, either \(\frac{-a}{b}\) or \(-\frac{a}{b}\).

Figure 2 shows the points that correspond to \(-2, -\frac{5}{4}, -\frac{3}{4}, 0, 3, \frac{5}{4},\) and 2.

Example 1

Describe how to locate the following numbers on the number line of Figure 3: \(\frac{1}{2}, -\frac{1}{2}, \frac{7}{4},\) and \(-\frac{7}{4}\).
Equivalent or Equal Fractions

The grade 4 Common Core Standards state that students should be able to:

- Explain why a fraction \( \frac{a}{b} \) is equivalent to a fraction \( \frac{na}{nb} \) by using visual fraction models with attention to how the number and the size of the parts differ even though the two fractions themselves are the same size. Use the principle to recognize and generate equivalent fractions. (p. 30)

Fractions may be introduced in the classroom through a concrete activity such as paperfolding. In Figure 5(a), 1 of 3 congruent parts, or \( \frac{1}{3} \), is shaded. In this case, the whole is the rectangle. In Figure 5(b), each of the thirds has been folded in half so that now we have 6 sections, and 2 of 6 congruent parts, or \( \frac{2}{6} \), are shaded. Thus, both \( \frac{1}{3} \) and \( \frac{2}{6} \) represent exactly the same shaded portion.

Although the symbols \( \frac{1}{3} \) and \( \frac{2}{6} \) do not look alike, they represent the same rational number and are equivalent fractions, or equal fractions. Equivalent fractions are numbers that represent the same point on a number line. Because they represent equal amounts, we write \( \frac{1}{3} = \frac{2}{6} \) and say that “\( \frac{1}{3} \) equals \( \frac{2}{6} \)”.

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**Rational Numbers and Proportional Reasoning**

**Solution** To decide how to find the point on the number line representing \( \frac{1}{2} \), we consider the unit length 1. We find the point that would be the rightmost endpoint of the segment starting at 0 and ending at the point marking the middle of the unit segment. This is seen in Figure 4.

To find the point on the number line representing \( -\frac{1}{2} \), we find the mirror image of \( \frac{1}{2} \) on the left side of the number line as seen in Figure 4 when it is reflected in 0.

To find the location of \( -\frac{7}{4} \), we first find the image on the right side of 0 by marking the unit length in four parts, duplicating the four parts to mark points between 1 and 2, and then counting 7 of those parts starting at 0. Once \( \frac{7}{4} \) is found on the right side of 0, then its reflection image in 0 gives the point where \( -\frac{7}{4} = -\frac{7}{4} \) should be marked. This is seen in Figure 4.

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**Figure 4**

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**Figure 5**

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Rational Numbers and Proportional Reasoning

Figure 5(c) shows the rectangle with each of the original thirds folded into 4 equal parts with 4 of the 12 parts now shaded. Thus, $\frac{1}{3}$ is equal to $\frac{4}{12}$ because the same portion of the model is shaded. Similarly, we could illustrate that $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}, \ldots$ are all equal.

Fraction strips can be used for generating equivalent fractions, as seen on the student page below. This technique makes use of the Fundamental Law of Fractions, which can be stated as follows: The value of a fraction does not change if its numerator and denominator are multiplied by the same nonzero integer. Under certain assumptions this Law of Fractions can be proved and is stated as a theorem.

**Theorem 1: Fundamental Law of Fractions**

If $\frac{a}{b}$ is a fraction and $n$ a nonzero number, then $\frac{a}{b} = \frac{an}{bn}$.
From the Fundamental Law of Fractions, \( \frac{7}{15} = \frac{-7}{-15} \) because \( \frac{7}{15} = \frac{7(-1)}{15(-1)} = \frac{-7}{-15} \). Similarly, \( \frac{a}{b} = \frac{-a}{-b} \). The form \( \frac{-a}{b} \), where \( b \) is a positive number, is usually preferred.

**Simplifying Fractions**

Theorem 1 implies that if \( d \) is a common factor of \( a \) and \( b \), then \( \frac{a}{b} = \frac{a/d}{b/d} \). Because \( d \) is a divisor of both \( a \) and \( b \), we know that \( a/d = m \) is a nonzero integer. Also \( b/d = n \) is a nonzero integer. Thus, we have the following:

\[
\frac{a}{b} = \frac{a/d}{b/d} = \frac{m}{n}
\]

Hence, we have two equivalent forms of the same rational number. A numerical example is given next.

\[
\frac{60}{210} = \frac{60 \div 10}{210 \div 10} = \frac{6}{21}
\]

This process is referred to as simplifying a fraction. A slightly different look is seen below.

\[
\frac{60}{210} = \frac{6 \cdot 10}{21 \cdot 10} = \frac{6}{21}
\]

Also,

\[
\frac{6}{21} = \frac{2 \cdot 3}{7 \cdot 3} = \frac{2}{7}
\]

We can simplify \( \frac{60}{210} \) because the numerator and denominator have a common factor of 10. We can simplify \( \frac{6}{21} \) because 6 and 21 have a common factor of 3. However, we cannot simplify \( \frac{2}{7} \) because 2 and 7 have no positive common factor other than 1. We could also simplify \( \frac{60}{210} \) in one step:

\[
\frac{60}{210} = \frac{2 \cdot 30}{7 \cdot 30} = \frac{2}{7}
\]

Notice that \( \frac{2}{7} \) is the simplest form of \( \frac{60}{210} \) because both 60 and 210 have been divided by their greatest common divisor, 30.

Suppose we wanted to simplify the rational number \( \frac{-60}{210} \) or its equivalent \( \frac{-60}{210} \). The problem could be addressed by simplifying \( \frac{60}{210} \) as above and then taking the opposite of the result.

This could be thought of as considering \( \frac{-60}{210} = -\left| \frac{60}{210} \right| = -\frac{2}{7} \). Note that the same result could have been found using \( \frac{-60}{210} = -\left| \frac{60}{210} \right| = -\frac{2}{7} \).

**Definition of Simplest Form**

A rational number \( \frac{a}{b} \) is in simplest form or lowest terms if, and only if, \( \text{GCD}(a, b) = 1 \), that is, if \( a \) and \( b \) have no common factor greater than 1.
Scientific/fraction calculators can simplify fractions. For example, to simplify $\frac{6}{12}$, we enter $6 \div 12$ and press SIMP $=$ and $\frac{3}{6}$ appears on the screen. At this point, an indicator tells us that this is not in simplest form, so we press SIMP $=$ again to obtain $\frac{1}{2}$. At any time, we can view the factor that was removed by pressing the $x \div y$ key.

Find a value for $x$ such that $\frac{12}{42} = \frac{x}{210}$.

**Solution** We use the mathematical practice of reasoning to see that because $210 \div 42 = 5$, we can use the Fundamental Law of Fractions to obtain $\frac{12}{42} = \frac{12 \cdot 5}{42 \cdot 5} = \frac{60}{210}$. Hence, $\frac{x}{210} = \frac{60}{210}$, and $x = 60$.

Alternative approach: $\frac{12}{42} = \frac{2 \cdot 6}{7 \cdot 6} = \frac{2}{7} = \frac{2 \cdot 30}{7 \cdot 30} = \frac{60}{210}$. Therefore $x = 60$.

Example 2

Write each of the following fractions in simplest form if they are not already.

a. $\frac{28ab}{42a^2b^2}$
b. $\frac{(a + b)^2}{3a + 3b}$
c. $\frac{x^2 + x}{x + 1}$
d. $\frac{3 + x^2}{3x^2}$
e. $\frac{3 + 3x^2}{3x^2}$
f. $\frac{a^2 - b^2}{a - b}$
g. $\frac{a^2 + b^2}{a + b}$

**Solution**

a. $\frac{28ab}{42a^2b^2} = \frac{2(14ab)}{3a(14ab)} = \frac{2}{3a}$
b. $\frac{(a + b)^2}{3a + 3b} = \frac{(a + b)(a + b)}{3(a + b)} = \frac{a + b}{3}$
c. $\frac{x^2 + x}{x + 1} = \frac{x(x + 1)}{x + 1} = \frac{x}{1} = x$
d. $\frac{3 + x^2}{3x^2}$ cannot be simplified because $3 + x^2$ and $3x^2$ have no factors in common except 1.
e. $\frac{3 + 3x^2}{3x^2} = \frac{3(1 + x^2)}{3x^2} = \frac{1 + x^2}{x^2}$
f. The difference of squares formula: $a^2 - b^2 = (a - b)(a + b)$. Thus,

$$\frac{a^2 - b^2}{a - b} = \frac{(a - b)(a + b)}{(a - b)1} = \frac{a + b}{1} = a + b.$$  

g. The fraction is already in simplest form because $a^2 + b^2$ does not have $(a + b)$ as a factor. Notice that $a^2 + b^2 \neq (a + b)^2$. 

Example 3
When an algebraic expression is written as a fraction, the denominator may not be 0. Thus, when the fraction is simplified, this restriction has to be maintained. For example, in part (c) of Example 3, \( \frac{x^2 + x}{x + 1} = x \) if \( x \neq -1 \), and in part (f) the result holds if \( a - b \neq 0 \), that is, if \( a \neq b \).

Some students think of the Fundamental Law of Fractions as a cancellation property and “simplify” an expression like \( \frac{6 + a^2}{3a} \) by thinking of it as \( \frac{2 \cdot 3 + a \cdot a}{3a} \) and “canceling” equal numbers in the products to obtain \( 2 + a \) as the answer. Emphasizing the factor approach that neither 3 nor \( a \) is a factor of \( 6 + a^2 \) may help to avoid such mistakes.

**Equality of Fractions**

We use three equivalent methods to show that two fractions, such as \( \frac{12}{42} \) and \( \frac{10}{35} \), are equal.

1. Simplify both fractions to simplest form.
   
   \[
   \frac{12}{42} = \frac{2 \cdot 3}{2 \cdot 3 \cdot 7} = \frac{2}{7} \quad \text{and} \quad \frac{10}{35} = \frac{5 \cdot 2}{5 \cdot 7} = \frac{2}{7}
   \]
   
   Thus,
   
   \[
   \frac{12}{42} = \frac{10}{35}.
   \]

2. Rewrite both fractions with the same least common denominator. Since \( \text{LCM}(42, 35) = 210 \), then
   
   \[
   \frac{12}{42} = \frac{60}{210} \quad \text{and} \quad \frac{10}{35} = \frac{60}{210}.
   \]
   
   Thus,
   
   \[
   \frac{12}{42} = \frac{10}{35}.
   \]

3. Rewrite both fractions with a common denominator (not necessarily the least). A common multiple of 42 and 35 may be found by finding the product \( 42 \cdot 35 = 1470 \).
   
   \[
   \frac{12}{42} = \frac{420}{1470} \quad \text{and} \quad \frac{10}{35} = \frac{420}{1470}
   \]
   
   Hence,
   
   \[
   \frac{12}{42} = \frac{10}{35}.
   \]

The third method suggests a general algorithm for determining whether two fractions \( \frac{a}{b} \) and \( \frac{c}{d} \) are equal. Rewrite both fractions with common denominator \( bd \); that is,

\[
\frac{a}{b} = \frac{ad}{bd} \quad \text{and} \quad \frac{c}{d} = \frac{bc}{bd}
\]

Because the denominators are the same, \( \frac{ad}{bd} = \frac{bc}{bd} \) if, and only if, \( ad = bc \). For example, \( \frac{24}{36} = \frac{6}{9} \) because \( 24 \cdot 9 = 216 = 36 \cdot 6 \). In general, the following theorem holds.
Rational Numbers and Proportional Reasoning

Theorem 2: Equality of Fractions

Two fractions \( \frac{a}{b} \) and \( \frac{c}{d} \), with \( b \neq 0 \) and \( d \neq 0 \), are equal if, and only if, \( ad = bc \).

Using a calculator, we determine whether two fractions are equal by using Theorem 2. Since both \( \frac{2}{4} = \frac{1098}{2196} \) and \( \frac{4}{1098} = \frac{1098}{2196} \) yield a display of 4392, we see that \( \frac{2}{4} = \frac{1098}{2196} \).

Ordering Rational Numbers

The grade 4 Common Core Standards state that a student should compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as \( \frac{1}{2} \). Recognize that comparisons are valid only when two fractions refer to the same whole. (p. 30)

Example 4

Jim claims that \( \frac{1}{3} > \frac{1}{2} \) because in Figure 6 the shaded portion for \( \frac{1}{3} \) is larger than the shaded portion for \( \frac{1}{2} \). How would you help him?

Solution

Jim needs to understand as noted in the Common Core Standards that comparisons of two fractions are valid only when they refer to the same whole. In Figure 6, the circle is clearly larger than the square so the two wholes are not the same.

In order to compare two fractions referring to the same whole, it is easiest to compare fractions with like denominators. Children know that \( \frac{7}{8} > \frac{5}{8} \) because if a pizza is divided into 8 parts of equal size, then 7 parts of the pizza is more than 5 parts. Similarly, \( \frac{3}{7} < \frac{4}{7} \). Thus, given two fractions with common positive denominators, the one with the greater numerator is the greater fraction. To make ordering of rational numbers consistent with the ordering of whole numbers and integers we have the following definition.

Definition of Greater Than for Rational Numbers with Like Denominators

If \( a, b, \) and \( c \) are integers and \( b > 0 \), then \( \frac{a}{b} > \frac{c}{b} \) if, and only if, \( a > c \).
To compare fractions with unlike denominators, some students may incorrectly reason that \( \frac{1}{8} > \frac{1}{7} \) because 8 is greater than 7. In another case, they might falsely believe that \( \frac{6}{7} = \frac{7}{8} \) because in both fractions the difference between the numerator and the denominator is 1. Comparing positive fractions with unlike denominators may be aided by using fraction strips to compare the fractions visually. For example, consider the fractions \( \frac{4}{5} \) and \( \frac{11}{12} \) shown in Figure 7.

From Figure 7, students see that each fraction is one piece less than the same-size whole unit. However, they see that the missing piece for \( \frac{11}{12} \) is smaller than the missing piece for \( \frac{4}{5} \), so \( \frac{11}{12} \) must be greater than \( \frac{4}{5} \).

Comparing any fractions with unlike denominators can be accomplished by rewriting the fractions with the same positive common denominator. Using the common denominator \( bd \), we can write the fractions \( \frac{a}{b} \) and \( \frac{c}{d} \) as \( \frac{ad}{bd} \) and \( \frac{bc}{bd} \). Because \( b > 0 \) and \( d > 0 \), then \( bd > 0 \); and we have the following:

\[
\frac{a}{b} > \frac{c}{d} \text{ if, and only if, } \frac{ad}{bd} > \frac{bc}{bd}; \quad \text{and } \frac{ad}{bd} > \frac{bc}{bd} \text{ if, and only if, } ad > bc.
\]

Therefore, we have the following theorem.

**Theorem 3**

If \( a, b, c, \) and \( d \) are integers with \( b > 0 \) and \( d > 0 \), then \( \frac{a}{b} > \frac{c}{d} \text{ if, and only if, } ad > bc. \)

Next consider two fractions with both numerators and denominators positive and with numerators that are the same. For example, consider \( \frac{3}{4} \) and \( \frac{3}{10} \). If the whole is the same for both fractions, this means that we have three \( \frac{1}{4} \)’s and three \( \frac{1}{10} \)’s. Because \( \frac{1}{4} \) is greater than \( \frac{1}{10} \), three of the larger parts is greater than three of the smaller parts. Thus, \( \frac{3}{4} > \frac{3}{10} \).
Denseness of Rational Numbers

The set of rational numbers has a property unlike the set of whole numbers and the set of integers. Consider $\frac{1}{2}$ and $\frac{2}{3}$. To find a rational number between $\frac{1}{2}$ and $\frac{2}{3}$, we first rewrite the fractions with a common denominator, as $\frac{3}{6}$ and $\frac{4}{6}$. Because there is no whole number between the numerators 3 and 4, we next find two fractions equal, respectively, to $\frac{1}{2}$ and $\frac{2}{3}$ with greater denominators. For example, $\frac{1}{2} = \frac{6}{12}$ and $\frac{2}{3} = \frac{8}{12}$, and $\frac{7}{12}$ is between the two fractions $\frac{6}{12}$ and $\frac{8}{12}$. So $\frac{7}{12}$ is between $\frac{1}{2}$ and $\frac{2}{3}$. This property is generalized as follows and stated as a theorem.

Theorem 4: Denseness Property for Rational Numbers

Given any two different rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, there is another rational number between these two numbers.

NOW TRY THIS 1

Explain why there are infinitely many rational numbers between any two rational numbers.

Example 5

a. Find two fractions between $\frac{7}{18}$ and $\frac{1}{2}$.

b. Show that the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$, where $n \in N$, is an increasing sequence; that is, that each term starting from the second term is greater than the preceding term.

Solution

a. Because $\frac{1}{2} = \frac{1 \cdot 9}{2 \cdot 9} = \frac{9}{18}$, we see that $\frac{8}{18}$ or $\frac{4}{9}$ is between $\frac{7}{18}$ and $\frac{9}{18}$. To find another fraction between the given fractions, we find two fractions equal to $\frac{7}{18}$ and $\frac{9}{18}$, respectively, but with greater denominators; for example, $\frac{7}{18} = \frac{14}{36}$ and $\frac{9}{18} = \frac{18}{36}$.

We now see that $\frac{15}{36}, \frac{16}{36}$, and $\frac{17}{36}$ are all between $\frac{14}{36}$ and $\frac{18}{36}$ and thus between $\frac{7}{18}$ and $\frac{1}{2}$.

b. Because the $n$th term of the sequence is $\frac{n}{n+1}$, the next term is $\frac{n+1}{(n+1)+1}$ or $\frac{n+1}{n+2}$. We need to show that for all positive integers $n$, $\frac{n+1}{n+2} > \frac{n}{n+1}$.

The terms of the sequence are positive. The inequality will be true if, and only if, $(n+1)(n+1) > n(n+2)$. This inequality is equivalent to

$$n^2 + 2n + 1 > n^2 + 2n$$
$$2n + 1 > 2n$$
$$1 > 0$$

which is true.

Therefore we have an increasing sequence.
Another way to find a number between any two rational numbers involves adding numerators and adding denominators. In Example 5(a), to find a number between \( \frac{7}{18} \) and \( \frac{1}{2} \) we could add the numerators and add the denominators to produce \( \frac{7 + 1}{18 + 2} = \frac{8}{20} \). We see that \( \frac{7}{18} < \frac{8}{20} \) because 140 < 144. Also, \( \frac{8}{20} < \frac{1}{2} \) because 16 < 20. The general property is stated in the following theorem and explored in Now Try This 2.

**Theorem 5**

Let \( \frac{a}{b} \) and \( \frac{c}{d} \) be any rational numbers with positive denominators such that \( \frac{a}{b} < \frac{c}{d} \). Then,

\[
\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}.
\]

**NOW TRY THIS 2**

Prove Theorem 5. (Hint: Prove that \( \frac{a}{b} < \frac{a + c}{b + d} \) and \( \frac{a + c}{b + d} < \frac{c}{d} \).

The proof of Theorem 5 suggested in Now Try This 2 also proves Theorem 4.

**Assessment 1A**

1. Write a sentence that illustrates the use of \( \frac{7}{8} \) in each of the following ways.
   a. As a division problem
   b. As part of a whole
   c. As a ratio

2. For each of the following, write a fraction to approximate the shaded portion as part of the whole.
   a. 
   ![Diagram](image1)
   b. 
   ![Diagram](image2)
   c. 
   ![Diagram](image3)
   d. 
   ![Diagram](image4)

3. If the entire rectangle is a whole, what fraction represents the shaded portion of the figure?
   ![Diagram](image5)

4. For each of the following four squares, write a fraction to describe the shaded portion. What property of fractions does the diagram illustrate?
   a. 
   ![Diagram](image6)
   b. 
   ![Diagram](image7)
   c. 
   ![Diagram](image8)
   d. 
   ![Diagram](image9)

5. Based on your observations, could the shaded portions in the following figures represent the indicated fractions? If not, tell why.
   a. 
   ![Diagram](image10)
   b. 
   ![Diagram](image11)
   c. 
   ![Diagram](image12)
6. In each case, subdivide the whole shown on the right to show the equivalent fraction.
   a. \[ \frac{1}{4} = \frac{2}{8} \]
   b. \[ \frac{1}{3} = \frac{3}{9} \]
   c. \[ \frac{1}{2} = \frac{3}{6} \]

7. Referring to the figure, represent each of the following quantities as a fraction.
   a. The dots in the interior of the circle as a part of all the dots
   b. The dots in the interior of the rectangle as a part of all the dots
   c. The dots in the intersection of the interiors of the rectangle and the circle as a part of all the dots
   d. The dots outside the circular region but inside the rectangular region as part of all the dots

8. Use the Venn diagram pictured with A representing the people in an arts class and B representing the people in a botany class to express the fraction of elements in the indicated sets as a part of the universal set.
   a. \( A \cap B \)
   b. \( A - B \)
   c. \( A \cup B \)
   d. \( U \)

9. For each of the following fractions, write three equal fractions.
   a. \( \frac{2}{9} \)
   b. \( \frac{-2}{5} \)
   c. \( \frac{0}{3} \)
   d. \( \frac{a}{2} \)

10. Find the simplest form for each of the following fractions.
    a. \( \frac{156}{93} \)
    b. \( \frac{27}{45} \)
    c. \( \frac{-65}{91} \)

11. For each of the following fractions, choose the expression in parentheses that equals or describes best the given fraction.
    a. \( \frac{0}{7} \) (1, undefined, 0)
    b. \( \frac{5}{0} \) (undefined, 5, 0)
    c. \( \frac{0}{5} \) (undefined, 5, 0)

12. Find the simplest form for each of the following fractions.
    a. \( \frac{a^2 - b^2}{3a + 3b} \)
    b. \( \frac{14x^2y}{63xy^2} \)

13. Determine whether the following pairs are equal.
    a. \( \frac{5}{8} \) and \( \frac{375}{1000} \)
    b. \( \frac{18}{54} \) and \( \frac{23}{69} \)

14. Determine whether the following pairs are equal by changing both to have the same denominator.
    a. \( \frac{10}{16} \) and \( \frac{12}{18} \)
    b. \( \frac{-21}{86} \) and \( \frac{-51}{215} \)

15. Draw an area model to show that \( \frac{3}{4} = \frac{6}{8} \).

16. If a fraction is equal to \( \frac{3}{4} \) and the sum of the numerator and denominator is 84, what is the fraction?

17. Mr. Gomez filled his car's 16 gal gas tank. He took a trip and used \( \frac{7}{8} \) of the gas.
   a. Draw an arrow in the following figure to show what his gas gauge looked like after the trip:
   b. How many gallons of gas were used?

18. Solve for \( x \) in each of the following.
    a. \( \frac{2}{3} = \frac{x}{16} \)
    b. \( \frac{3}{4} = \frac{-27}{x} \)

19. For each of the following pairs of fractions, replace the comma with the correct symbol \(<, =, >\) to make a true statement.
    a. \( \frac{7}{8}, \frac{5}{6} \)
    b. \( \frac{-7}{8}, \frac{-4}{5} \)

20. Arrange each of the following in decreasing order.
    a. \( \frac{11}{11}, \frac{11}{11}, \frac{11}{11} \)
    b. \( \frac{22}{16}, \frac{13}{17}, \frac{19}{30} \)

21. For each of the following, find two rational numbers between the given fractions.
    a. \( \frac{3}{7} \) and \( \frac{4}{7} \)
    b. \( \frac{-7}{9} \) and \( \frac{-8}{9} \)

22. a. 6 oz is what part of a pound? A ton?
    b. A dime is what fraction of a dollar?
    c. 15 min is what fraction of an hour?
    d. 8 hr is what fraction of a day?
23. Determine whether the following is true: If \( a, b, c \) are integers and \( b < 0 \), then \( \frac{a}{b} > \frac{c}{b} \) if, and only if, \( a > c \).

24. Based on your visual observation write a fraction to represent the shaded portion.

![Shaded Fraction](image)

25. Fill in missing numbers for \( x, y, z, \) and \( w \) to create a sequence with each term being greater than the preceding term.

\[
\begin{align*}
&\text{a. } \frac{1}{3}, \frac{y}{4}, \frac{z}{5}, \frac{6}{6} \\
&\text{b. } \frac{-3}{x}, \frac{-4}{y}, \frac{-5}{z}, \frac{-6}{w} \\
\end{align*}
\]

26. Explain why in the Fundamental Law of Fractions (Theorem 1), \( n \) must be nonzero.

27. Prove that any integer \( n \) can be written as a fraction.

28. A typical English ruler is marked in sixteenths of an inch. Sketch a ruler marking \( \frac{3}{8} \) in.

29. Ten light bulbs were in a chandelier. One-fifth of the bulbs were not shining. How many light bulbs were not shining?

30. At a party, there were 35 guests. Two-fifths of the guests were men. What fraction of the guests were women?

---

**Assessment 1B**

1. Write a sentence that illustrates the use of \( \frac{7}{10} \) in each of the following ways:
   a. As a division problem
   b. As part of a whole
   c. As a ratio

2. For each of the following, write a fraction to approximate the shaded portion of the whole.
   a. [Diagram]
   b. [Diagram]
   c. [Diagram]
   d. [Diagram]
   e. [Diagram]

3. If the entire rectangle is a whole, what fraction represents the shaded portion of the figure?

![Shaded Fraction](image)

4. Complete each of the following figures so that it illustrates \( \frac{3}{5} \).
   a. [Diagram]
   b. [Diagram]
   c. [Diagram]
   d. [Diagram]
   e. [Diagram]

5. Based on your observations, could the shaded portions in the following figures represent the indicated fractions? Tell why.

![Shaded Fractions](image)

6. If each of the following models represents the given fraction, draw a model that represents the whole. Shade your answer.

   a. [Model]
   b. [Model]
   c. [Model]
   d. [Model]

7. Referring to the figure, represent each of the following quantities as a fraction.

   ![Quantities](image)
   a. The dots outside the circular region as a part of all the dots
   b. The dots outside the rectangular region as a part of all the dots
   c. The dots in the union of the rectangular and the circular regions as a part of all the dots
   d. The dots inside the circular region but outside the rectangular region as a part of all the dots.
8. Use the Venn diagram pictured with $A$ representing the people in an algebra class and $B$ representing the people in a biology class to express the fraction of elements in the indicated sets as a part of the universal set.

\[ U = \{13, 4, 17, 9\} \]

a. $A \cup B$

b. $A - B$

c. $\emptyset$

d. $A \cup B$

9. For each of the following, write three fractions equal to the given fractions.

a. $\frac{1}{3}$  

b. $\frac{4}{5}$  

c. $\frac{3}{7}$  

d. $\frac{a}{3}$

10. Find the simplest form for each of the following fractions.

a. $0 \div 68$

b. $\frac{84}{91}$

c. $\frac{-662}{703}$

11. For each of the following, choose the expression in parentheses that equals or describes best the given fraction.

a. $\frac{6 + x}{3x} = \left(\frac{2 + x}{x}, \frac{1}{3}\right)$, cannot be simplified

b. $\frac{2^6 + 2^2}{2^7 + 2^2} = \left(\frac{12}{18}, \frac{2}{3}\right)$, cannot be simplified

c. $\frac{2^{100} + 2^{98}}{2^{100} - 2^{98}} = \left(\frac{2196}{3}, \frac{5}{3}\right)$, too large to simplify

12. Find the simplest form for each of the following fractions.

a. $a^2 + ab$

b. $\frac{a}{3a + ab}$

13. Determine whether the following pairs are equal.

a. $\frac{6}{3} = \frac{3,750}{10,000}$

b. $\frac{17}{27} = \frac{25}{45}$

14. Determine whether the following pairs are equal by changing both to the same denominator.

a. $\frac{5}{12} = \frac{36}{144}$

b. $\frac{-21}{430} = \frac{-51}{215}$

15. Draw an area model to show that $\frac{2}{3} = \frac{6}{9}$.

16. A board is needed that is exactly $\frac{11}{32}$ in. wide to fit a hole. Can a board that is $\frac{3}{8}$ in. be shaved down to fit the hole? If so, how much must be shaved from the board?

17. The following two parking meters are next to each other with the times left as shown. Which meter has more time left on it? How much more?

18. Solve for $x$ in each of the following.

a. $\frac{2}{3} = \frac{x}{18}$

b. $\frac{3}{x} = \frac{3x}{x^2}$

19. For each of the following pairs of fractions, replace the comma with the correct symbol ($<$, $=$, $>$) to make a true statement.

\[
\begin{align*}
a. & \quad \frac{1}{7}, 8 \\
b. & \quad \frac{2}{5}, 10 \\
c. & \quad \frac{0}{7}, 17
\end{align*}
\]

20. For each of the following, find two rational numbers between the given fractions.

\[
\begin{align*}
a. & \quad \frac{-1}{3} \text{ and } \frac{3}{4} \\
b. & \quad \text{ and } \frac{83}{6} \\
c. & \quad \text{ and } \frac{3}{4}
\end{align*}
\]

21. a. 12 oz is what part of a pound?

b. A nickel is what fraction of a dollar?

c. 25 min is what fraction of an hour?

d. 16 hr is what fraction of a 24-hr day?

22. Read each measurement as shown on the following ruler.

23. Fill in missing numbers for $x, y, z,$ and $w$ to create a sequence with each term being less than the preceding term.

\[
\begin{align*}
a. & \quad \frac{5}{x}, \frac{y}{z}, \frac{w}{6} \\
b. & \quad \frac{3}{x}, \frac{4}{y}, \frac{5}{z}
\end{align*}
\]

24. Use the Fundamental Law of Fractions (Theorem 1) to show that 0 could be written in infinitely many ways.

25. Prove that a negative integer $n$ can be written as a fraction with a positive denominator.

26. A typical metric ruler is marked in millimeters (mm) where $1$ mm $= \frac{1}{1000}$ m (where $m$ is the designation of meter). How many millimeters would 5 meters be?

27. Ten light bulbs were in a chandelier. Three-fifth of the bulbs were shining. What fraction of the light bulbs were not shining?

28. At a party, there were 40 guests. One-fifth of the guests were men. What fraction of the guests were women?

29. Answer each of the following.

a. If the area of the entire square is 1 square unit, find the area of each tangram piece.

b. If the area of piece $a$ is 1 square unit, find the area of each tangram piece.
Rational Numbers and Proportional Reasoning

Mathematical Connections 1

Reasoning

1. Explain why 25 cents is one-fourth of a dollar, yet 15 minutes is one-fourth of an hour. Why should these one-fourths not be equal?

2. In each of two different fourth-grade classes, \( \frac{1}{3} \) of the members are girls. Does each class have the same number of girls? Explain your answer.

3. Consider the set of all fractions equal to \( \frac{1}{2} \). If you take any 10 of those fractions, add their numerators to obtain the numerator of a new fraction and add their denominators to obtain the denominator of a new fraction, how does the new fraction relate to \( \frac{1}{2} \)? Generalize what you found and explain.

4. Draw a Venn diagram showing the relationship among natural numbers, whole numbers, integers, and rational numbers. Use subset relations to explain your Venn diagram.

Open-Ended

5. Make three statements about yourself or your environment and use fractions in each. Explain why your statements are true. For example, your parents have three children, two of whom live at home; hence \( \frac{2}{3} \) of their children live at home.

6. Consider the demographics of your class including gender and ethnicity. Write fractions to describe the demographics of the class.

7. Sketch four different windows having different numbers of window panes in each. Shade \( \frac{3}{4} \) of the panes in each window.

Cooperative Learning

8. Assume that the shortest person in your group is 1 unit tall and do the following:
   a. Find rational numbers to approximately represent the heights of other members of the group.
   b. Make a number line and plot the rational number for each person ordered according to height.

9. Assume the tallest person in your group is 1 unit tall and do the following:
   a. Find rational numbers to approximately represent the heights of other members of the group.
   b. Make a number line and plot the rational number for each person ordered according to height.

Connecting Mathematics to the Classroom

10. A student asks if \( \frac{0}{6} \) is in its simplest form. How do you respond?

11. A student writes \( \frac{15}{53} \leq \frac{1}{3} \) because \( 3 \cdot 15 < 53 \cdot 1 \). Another student writes \( \frac{15}{93} = \frac{1}{3} \). Where is the fallacy?

12. A student claims that there are no numbers between \( \frac{999}{1000} \) and 1 because they are so close together. What is your response?

13. A student argued that a pizza cut into 12 pieces was more than a pizza cut into 6 pieces. How would you respond?

14. Ann claims that she cannot show \( \frac{3}{4} \) of the following faces because some are big and some are small. What do you tell her?

15. How would you respond to each of the following students?
   a. Iris claims that if we have two positive rational numbers, the one with the greater numerator is the greater.
   b. Shirley claims that if we have two positive rational numbers, the one with the greater denominator is the lesser.

16. Steve claims that the shaded circles below cannot represent \( \frac{2}{3} \) since there are 10 circles shaded and \( \frac{2}{3} \) is less than 1. How do you respond?

17. Carl says that \( \frac{3}{8} > \frac{2}{3} \) because \( 3 > 2 \) and \( 8 > 3 \). How would you help Carl?

18. Mr. Jimenez and Ms. Cortez gave the same test. In Mr. Jimenez’s class, 20 out of 25 students passed, and in Ms. Cortez’s class, 24 of 30 passed. One of Ms. Cortez’s students claimed that the classes did equally well. How could you explain the student’s reasoning?

National Assessments

National Assessment of Educational Progress (NAEP) Questions

What fraction of the figure is shaded?

NAEP, Grade 4, 2007

In which of the following are the three fractions arranged from least to greatest?

A. \( \frac{2}{7}, \frac{1}{2}, \frac{5}{9} \)  
B. \( \frac{1}{2}, \frac{2}{7}, \frac{5}{9} \)  
C. \( \frac{1}{5}, \frac{2}{9}, \frac{5}{7} \)  
D. \( \frac{1}{9}, \frac{2}{7}, \frac{5}{9} \)  
E. \( \frac{2}{5}, \frac{1}{9}, \frac{2}{7} \)

NAEP, Grade 8, 2007

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Rational Numbers and Proportional Reasoning

Addition and subtraction of rational numbers is very much like addition and subtraction of whole numbers and integers. We first demonstrate the addition of two rational numbers with like denominators, \(\frac{2}{5} + \frac{1}{5}\), using an area model in Figure 8(a) and a number line model in Figure 8(b).

Why does the area model in Figure 8(a) make sense? Suppose that someone gives us \(\frac{2}{5}\) of a pie initially and then gives us another \(\frac{1}{5}\) of the pie. In Figure 8(a), \(\frac{2}{5}\) is represented by 2 pieces when the pie is cut into 5 equal-size pieces, and \(\frac{1}{5}\) is represented by 1 piece of the 5 equal-size pieces. So

2 Objectives

Students will be able to understand and explain

- Addition of rational numbers with like and unlike denominators.
- Rational numbers as mixed numbers.
- Subtraction of rational numbers with like and unlike denominators.
- Properties of addition and subtraction of rational numbers.
- Addition properties of equality.
- Estimation with rational numbers.

After the cuts were made, who had the longest pieces of string?
A. Kim  
B. Les  
C. Mario  
D. Nina

NAEP, Grade 4, 2013

Which fraction has a value closest to \(\frac{1}{2}\)?
A. \(\frac{5}{8}\)  
B. \(\frac{1}{5}\)  
C. \(\frac{2}{7}\)  
D. \(\frac{1}{5}\)

NAEP, Grade 4, 2011
you have \(2 + 1 = 3\) pieces of the 5 equal-size pieces, or \(\frac{3}{5}\) of the total (whole) pie. The number line model in Figure 8(b) works the same as the number line model for whole numbers.

Using a bar model, the addition \(\frac{3}{5} + \frac{4}{5}\) is depicted in Figure 9.

![Figure 9](image)

In Figure 9(a), we have \(\frac{3}{5}\) and \(\frac{4}{5}\) pictured as parts of two equal-sized wholes where each whole consists of five parts. In Figure 9(b), we have \(\frac{3}{5} + \frac{4}{5}\) again pictured as parts of two equal-sized wholes, where each whole consists of five parts. We see that there is one whole and two parts of another whole shaded. Additionally, we see that there are seven of the one-fifth parts shaded. Thus, we could say that \(\frac{3}{5} + \frac{4}{5} = \frac{7}{5}\). Also \(\frac{3}{5} + \frac{4}{5} = 1 + \frac{2}{5}\), which could be written as \(1 \frac{2}{5}\). The latter representation is a mixed number, which is discussed later in the section.

The ideas illustrated in Figures 8 and 9 are summarized in the following definition.

**Definition of Addition of Rational Numbers with Like Denominators**

If \(\frac{a}{b}\) and \(\frac{c}{b}\) are rational numbers, then \(\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}\).

**Problem Solving**

**Adding Rational Numbers Problem**

Determine how to add the rational numbers \(\frac{2}{5}\) and \(\frac{1}{4}\).

**Understanding the Problem** We model \(\frac{2}{5}\) and \(\frac{1}{4}\) as parts of the same-sized whole, as seen in Figure 10, but we need a way to combine the two drawings to find the sum.
Devising a Plan  We use the strategy of solving a related problem: adding rational numbers with the same denominators. We can find the sum using reasoning to write each fraction with a common denominator and then complete the computation.

Carrying Out the Plan  We know that $\frac{2}{3}$ has infinitely many representations, including $\frac{4}{6}$, $\frac{6}{9}$, $\frac{8}{12}$, and so on. Also $\frac{1}{4}$ has infinitely many representations, including $\frac{2}{8}$, $\frac{3}{12}$, $\frac{4}{16}$, and so on. We see that $\frac{8}{12}$ and $\frac{3}{12}$ have the same denominator. One is 8 parts of 12 equal parts, while the other is 3 parts of 12 equal parts. Consequently, the sum is $\frac{2}{3} + \frac{1}{4} = \frac{8}{12} + \frac{3}{12} = \frac{11}{12}$. Figure 11 illustrates the addition.

Looking Back  To add two rational numbers with unlike denominators, we considered equal rational numbers with like denominators. A common denominator for $\frac{2}{3}$ and $\frac{1}{4}$ is 12. This is also the least common denominator, or the LCM, of 3 and 4. To add two fractions with unequal denominators such as $\frac{5}{12}$ and $\frac{7}{18}$, we could find equal fractions with LCM(12, 18) = 36 as the denominator. However, any common denominator will work as well, for example, 72 or even 12·18.

By considering the sum $\frac{2}{3} + \frac{1}{4} = \frac{2 \cdot 4}{3 \cdot 4} + \frac{1 \cdot 3}{4 \cdot 3} = \frac{8}{12} + \frac{3}{12} = \frac{8 + 3}{12} = \frac{11}{12}$, we can generalize to find the sum of two rational numbers with unlike denominators, as in the following.

Alternate Definition of Addition of Rational Numbers with Unlike Denominators

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.
Rational Numbers and Proportional Reasoning

The definition of addition of rational numbers with unlike denominators can be applied to like denominators as seen below.

\[
\frac{a}{b} + \frac{c}{b} = \frac{ab + cb}{b \cdot b} = \frac{(a + c)b}{b \cdot b} = \frac{a + c}{b}
\]

Find each of the following sums.

a. \(\frac{2}{15} + \frac{4}{21}\)

b. \(\frac{2}{3} + \frac{1}{5}\)

c. \(\left(\frac{3}{4} + \frac{1}{5}\right) + \frac{1}{6}\)

d. \(\frac{3}{x} + \frac{4}{y}\)

e. \(\frac{2}{a^2b} + \frac{3}{ab^2}\)

**Solution**

a. LCM\((15, 21) = 3 \cdot 5 \cdot 7; \)
\[
\frac{2}{15} + \frac{4}{21} = \frac{2 \cdot 7}{15 \cdot 7} + \frac{4 \cdot 5}{21 \cdot 5} = \frac{14}{105} + \frac{20}{105} = \frac{34}{105}
\]

b. \(\frac{2}{3} + \frac{1}{5} = \frac{(2)(5) + (-3)(1)}{(3)(5)} = \frac{10 + (-3)}{-15} = \frac{7}{-15} = \frac{7(-1)}{15}\)

c. \(\left(\frac{3}{4} + \frac{1}{5}\right) + \frac{1}{6} = \frac{3 \cdot 5 + 4 \cdot 1}{4 \cdot 5} = \frac{19}{20}\)

d. \(\frac{3}{x} + \frac{4}{y} = \frac{3y + 4x}{xy}\)

e. LCM\((a^2b, ab^2) = a^2b^2; \)
\[
\frac{2}{a^2b} + \frac{3}{ab^2} = \frac{2b}{(a^2b)b} + \frac{3a}{a(ab^2)} = \frac{2b + 3a}{a^2b^2}
\]

**Example 6**

**Mixed Numbers**

In everyday life, we often use **mixed numbers**, that is, numbers that are made up of an integer and a proper fraction. Figure 12 shows a nail that is \(2\frac{3}{4}\) in. long. The mixed number \(2\frac{3}{4}\) means \(2 + \frac{3}{4}\). Students may infer that \(2\frac{3}{4}\) means \(2 \times \frac{3}{4}\), since \(xy\) means \(x \cdot y\), but this is not correct.

Also, the number \(-4\frac{3}{4}\) means \(-4 - \frac{3}{4}\), not \(-4 + \frac{3}{4}\).

![Figure 12](image)

A mixed number is a rational number because it can always be written in the form \(\frac{a}{b}\). For example,

\[
2\frac{3}{4} = 2 + \frac{3}{4} = \frac{2}{1} + \frac{3}{4} = \frac{2 \cdot 4 + 1 \cdot 3}{1 \cdot 4} = \frac{8 + 3}{4} = \frac{11}{4}.
\]
Rational Numbers and Proportional Reasoning

Example 7

Change each of the following mixed numbers to the form $\frac{a}{b}$, where $a$ and $b$ are integers:

a. $4 \frac{1}{3}$  
   Solution: $4 \frac{1}{3} = 4 + \frac{1}{3} = \frac{4 \cdot 3 + 1}{3} = \frac{13}{3}$

b. $-3 \frac{2}{5}$
   Solution: $-3 \frac{2}{5} = -\left(3 + \frac{2}{5}\right) = -\left(\frac{3 \cdot 5 + 1 \cdot 2}{5}\right) = -\left(\frac{17}{5}\right) = -\frac{17}{5}$

Example 8

Change $\frac{39}{5}$ to a mixed number.

Solution: We divide 39 by 5 and use the division algorithm as follows:

$$\frac{39}{5} = 7 \cdot \frac{4}{5} = 7 + \frac{4}{5} = 7 \frac{4}{5}$$

In elementary schools, problems like Example 8 are usually computed using division, as follows:

$$5)\overline{29}$$

Hence, $\frac{29}{5} = 5 + \frac{4}{5} = 5 \frac{4}{5}$. The remainder of 4 in the division actually represents $\frac{4}{5}$ of a unit when put in context.

Scientific/fraction calculators can change improper fractions to mixed numbers. For example, if we enter $2 \div 9 \div 5$ and press $\frac{Ab/c}{d}$, then $5 \div 4/5$ appears, which means $5 \frac{4}{5}$.

We can also use scientific/fraction calculators to add mixed numbers. For example, to add $2 \frac{4}{5} + 3 \frac{5}{6}$, we enter $2 \text{Unit} 4 \sqrt{5} + 3 \text{Unit} 5 \sqrt{6}$, and the display reads $5 \div 49/30$. We then press $\frac{Ab/c}{d}$ to obtain $6 \div 19/30$, which means $6 \frac{19}{30}$.

Adding Mixed Numbers

Because mixed numbers are rational numbers, the method of adding rationals can be used to include mixed numbers. The student page shown on the next page shows a method for computing sums of mixed numbers that uses the commutative and associative properties discussed later in the next section.
Properties of Addition for Rational Numbers

Rational numbers have the following properties for addition: *closure, commutative, associative, additive identity,* and *additive inverse.* To emphasize the additive inverse property of rational numbers, we state it explicitly, as follows.

**Theorem 6: Additive Inverse Property of Rational Numbers**

For any rational number \( \frac{a}{b} \), there exists a unique rational number \( -\frac{a}{b} \), the additive inverse of \( \frac{a}{b} \), such that

\[
\frac{a}{b} + \left( -\frac{a}{b} \right) = 0 = \left( -\frac{a}{b} \right) + \frac{a}{b}
\]
Rational Numbers and Proportional Reasoning

As mentioned earlier, another form of $-\frac{a}{b}$ can be found by considering the sum $\frac{a}{b} + \frac{-a}{b}$. Because $\frac{a}{b} + \frac{-a}{b} = \frac{a + (-a)}{b} = \frac{0}{b} = 0$, it follows that $-\frac{a}{b}$ and $\frac{-a}{b}$ are both additive inverses of $\frac{a}{b}$, so $-\frac{a}{b} = \frac{-a}{b}$.

Find the additive inverses for each of the following:

a. $\frac{3}{5}$

b. $-\frac{5}{11}$

c. $4 \frac{1}{2}$

Solution

a. $-\frac{3}{5}$ or $\frac{3}{5}$

b. $-\left(\frac{-5}{11}\right) = \frac{-(-5)}{11} = \frac{5}{11}$

c. $-4 \frac{1}{2}$, or $\frac{-9}{2}$

Properties of the additive inverse for rational numbers are analogous to those of the additive inverse for integers, as shown in Table 2.

<table>
<thead>
<tr>
<th>Integers</th>
<th>Rational Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $-(a) = a$</td>
<td>1. $\left(\frac{-a}{b}\right) = \frac{a}{b}$</td>
</tr>
<tr>
<td>2. $-(a + b) = -a + -b$</td>
<td>2. $\left(\frac{a}{b} + \frac{c}{d}\right) = \frac{-a}{b} + \frac{-c}{d}$</td>
</tr>
</tbody>
</table>

The set of rational numbers has the addition property of equality, which says that the same number can be added to both sides of an equation.

**Theorem 7: Addition Property of Equality of Rational Numbers**

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers such that $\frac{a}{b} = \frac{c}{d}$ and $\frac{e}{f}$ is any rational number, then

$\frac{a}{b} + \frac{e}{f} = \frac{c}{d} + \frac{e}{f}$

**Subtraction of Rational Numbers**

In elementary school, subtraction of rational numbers is usually introduced by using a take-away model. If we have $\frac{6}{7}$ of a pizza and $\frac{2}{7}$ of the original pizza is taken away, $\frac{4}{7}$ of the pizza remains; that is, $\frac{6}{7} - \frac{2}{7} = \frac{6 - 2}{7} = \frac{4}{7}$. In general, subtraction of rational numbers with like denominators is determined as follows:

$\frac{a}{b} - \frac{c}{d} = a - c$
As with integers, a number line can be used to model subtraction of nonnegative rational numbers. If a line is marked off in units of length $\frac{1}{b}$ and $a \geq c$, then $\frac{a}{b} - \frac{c}{b}$ is equal to $(a - c)$ units of length $\frac{1}{b}$, which implies that $\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$.

When the denominators are not the same, we can perform the subtraction by finding a common denominator. For example,

$$\frac{3}{4} - \frac{2}{3} = \frac{3 \cdot 3}{4 \cdot 3} - \frac{2 \cdot 4}{3 \cdot 4} = \frac{9}{12} - \frac{8}{12} = \frac{9 - 8}{12} = \frac{1}{12}.$$

Subtraction of rational numbers, like subtraction of integers, can be defined in terms of addition as follows.

**Definition of Subtraction of Rational Numbers in Terms of Addition**

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then $\frac{a}{b} - \frac{c}{d}$ is the unique rational number $\frac{e}{f}$ such that $\frac{a}{b} = \frac{c}{d} + \frac{e}{f}$.

As with integers, we can see that subtraction of rational numbers can be performed by adding the additive inverses as stated in the following theorem.

**Theorem 8**

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d}.$$

Now, using Theorem 8, we obtain the following.

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d}$$

$$= \frac{ad + b(-c)}{bd}$$

$$= \frac{ad - bc}{bd}$$

We proved the following theorem, which is sometimes given as a definition of subtraction.

**Theorem 9**

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$
Rational Numbers and Proportional Reasoning

Find each difference.

a. \( \frac{5}{8} - \frac{1}{4} \)  

b. \( 5 \frac{1}{3} - 2 \frac{3}{4} \)

**Solution**

a. One approach is to find the LCM for the denominators. LCM(8, 4) = 8.

\[
\frac{5}{8} - \frac{1}{4} = \frac{5}{8} - \frac{2}{8} = \frac{3}{8}
\]

An alternative approach follows.

\[
\frac{5}{8} - \frac{1}{4} = \frac{5 \cdot 4 - 8 \cdot 1}{8 \cdot 4} = \frac{20 - 8}{32} = \frac{12}{32} = \frac{3}{8}
\]

b. Two methods of solution are given.

\[
\frac{5}{3} - \frac{2}{4} = \frac{16 - 11}{4} = \frac{5}{3} - \frac{2}{4} = \frac{64 - 33}{12} = \frac{31}{12} = \frac{27}{12}
\]

Thus, the sum is \( \frac{1}{3} + \frac{-2}{3} = \frac{-1}{3} \).

The following examples show the use of fractions in algebra.

**Example 10**

Add or subtract, writing your answer in simplest form.

a. \( \frac{x}{2} + \frac{x}{3} \)  

c. \( \frac{2}{a + b} - \frac{2}{a - b} \)

d. \( \frac{1}{x} - \frac{1}{2x^2} \)

**Solution**

a. \( \frac{x}{2} + \frac{x}{3} = \frac{3x}{3 \cdot 2} + \frac{2x}{2 \cdot 3} = \frac{3x + 2x}{6} = \frac{5x}{6} \)

b. We first write each fraction in simplest form.

\[
\frac{2 - x}{6 - 3x} = \frac{2 - x}{3(2 - x)} = \frac{1(2 - x)}{3(2 - x)} = \frac{1}{3}
\]

\[
\frac{4 - 2x}{3x - 6} = \frac{2(x - 2)}{3(x - 2)} = \frac{-2}{3}
\]

Thus, the sum is \( \frac{1}{3} + \frac{-2}{3} = \frac{-1}{3} \).
Rational Numbers and Proportional Reasoning

c. We use Theorem 9.

\[
\frac{2}{a + b} - \frac{2}{a - b} = \frac{2(a - b) - 2(a + b)}{(a + b)(a - b)} \\
= \frac{2a - 2b - 2a - 2b}{(a + b)(a - b)} \\
= -\frac{4b}{(a + b)(a - b)} = -\frac{4b}{a^2 - b^2}
\]

d. \( \frac{1}{x} - \frac{1}{2x^2} = \frac{2x \cdot 1}{2x \cdot x} - \frac{1}{2x^2} \\
= \frac{2x}{2x^2} - \frac{1}{2x^2} \\
= \frac{2x - 1}{2x^2} \)

Estimation with Rational Numbers

Estimation helps us make practical decisions in our everyday lives. For example, suppose we need to double a recipe that calls for \( \frac{3}{4} \) of a cup of flour. Will we need more or less than a cup of flour? Many of the estimation and mental math techniques that we learned to use with whole numbers also work with rational numbers.

The grade 5 Common Core Standards calls for students to “use benchmark fraction and number sense of fractions to estimate mentally and assess the reasonableness of answers.” (p. 36) Estimation plays an important role in judging the reasonableness of computations.

**NOW TRY THIS 3**

A student added \( \frac{3}{4} + \frac{1}{2} \) and obtained \( \frac{4}{6} \). How would you use estimation to show that this answer could not be correct?

Sometimes to obtain an estimate it is desirable to round fractions to a convenient or benchmark fraction, such as \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{3}{4}, \) or 1. For example, if a student had 59 correct answers out of 80 questions, the student answered \( \frac{59}{80} \) of the questions correctly, which is approximately \( \frac{60}{80} \), or \( \frac{3}{4} \). We know \( \frac{60}{80} \) is greater than \( \frac{59}{80} \). The estimate \( \frac{3}{4} \) for \( \frac{59}{80} \) is a high estimate. In a similar way, we can estimate \( \frac{31}{90} \) by \( \frac{30}{90} \) or \( \frac{1}{3} \). In this case, the estimate of \( \frac{1}{3} \) is a low estimate of \( \frac{31}{90} \).

An example of benchmark estimation is seen on the student page shown on the next page.
A sixth-grade class is collecting cans to take to the recycling center. Becky’s group brought the following amounts (in pounds). About how many pounds does her group have all together?

\[
\frac{1}{8}, \frac{3}{4}, \frac{5}{8}, \frac{7}{8}, \frac{10}{16}
\]

**Solution**  
We can estimate the amount by using front-end estimation with the adjustment made by using 0 which is close to \(\frac{1}{8}\), which is close to \(\frac{4}{10}\) and \(\frac{6}{10}\), and 1 which is close to \(\frac{7}{8}\) as benchmark fractions. The front-end estimate is \(1 + 3 + 5 = 9\). The adjustment is \(0 + \frac{1}{2} + 1 + \frac{1}{2}\), or 2. An adjusted estimate would be \(9 + 2 = 11\) lb.

**Example 13**

Estimate each of the following additions.

a. \(\frac{27}{13} + \frac{10}{9}\)  
b. \(3\frac{9}{10} + 2\frac{7}{8} + 1\frac{11}{12}\)  
c. \(3\frac{7}{8} + 1\frac{1}{2} + 2\frac{2}{5} + 5\frac{1}{16}\)

**Solution**

a. Because \(\frac{27}{13}\) is slightly more than 2 and \(\frac{10}{9}\) is slightly more than 1, an estimate might be 3. We know the estimate is low.

b. We first add the whole-number parts to obtain \(3 + 2 = 5\). Because each of the fractions, \(\frac{9}{10}, \frac{7}{8}\), and \(\frac{11}{12}\), is close to but less than 1, their sum is close to but less than 3. The approximate answer is \(5 + 3 = 8\). The estimate is high.

c. Using *grouping to nice numbers*, we group \(3\frac{7}{8} + 5\frac{1}{16}\) and \(11\frac{1}{2} + 2\frac{2}{5}\) to obtain approximately \(9 + 14 = 23\). The estimate is high. (Why?)
1. Perform the following additions or subtractions.
   a. \( \frac{1}{2} + \frac{2}{3} \)
   b. \( \frac{4}{12} - \frac{2}{3} \)
   c. \( \frac{5}{x} + \frac{3}{y} \)
   d. \( \frac{-3}{2x^2y} + \frac{5}{2xy^2} + \frac{7}{x^2} \)
   e. \( \frac{5}{6} + \frac{2}{1} \)
   f. \( \frac{-4}{2} - \frac{3}{6} \)
   g. \( \frac{7}{1} + \frac{3}{12} - \frac{2}{3} \)

2. Change each of the following fractions to a mixed number.
   a. \( \frac{56}{3} \)
   b. \( \frac{-293}{100} \)

3. Change each of the following mixed numbers to a fraction in the form \( \frac{a}{b} \), where \( a \) and \( b \) are integers and \( b \neq 0 \).
   a. \( \frac{6}{4} \)
   b. \( \frac{-3}{5} \)

4. Approximate each of the following situations with a benchmark fraction. Tell whether your estimate is high or low.
   a. Giorgio had 15 base hits out of 46 times at bat.
   b. Ruth made 7 goals out of 41 shots.
   c. Laura answered 62 problems correctly out of 80.

5. Use the information in the table to answer each of the following questions.

<table>
<thead>
<tr>
<th>Team</th>
<th>Games Played</th>
<th>Games Won</th>
</tr>
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<tbody>
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</tr>
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<td>7</td>
</tr>
<tr>
<td>Wildcats</td>
<td>25</td>
<td>6</td>
</tr>
<tr>
<td>Badgers</td>
<td>21</td>
<td>5</td>
</tr>
</tbody>
</table>

   a. Which team won more than \( \frac{1}{2} \) of its games and was closest to winning \( \frac{1}{2} \) of its games?
   b. Which team won less than \( \frac{1}{2} \) of its games and was closest to winning \( \frac{1}{2} \) of its games?
   c. Which team won more than \( \frac{1}{3} \) of its games and was closest to winning \( \frac{1}{3} \) of its games?

6. Sort the following fraction cards into the ovals by estimating in which oval the fraction belongs.

   - Sort these fraction cards.
     - About \( \frac{0}{0} \)
     - About \( \frac{1}{2} \)
     - About \( \frac{1}{1} \)

   a. \( \frac{1}{10} \)
   b. \( \frac{4}{7} \)
   c. \( \frac{8}{12} \)
   d. \( \frac{1}{100} \)
   e. \( \frac{2}{5} \)
   f. \( \frac{1}{18} \)

7. Approximate each of the following fractions by 0, \( \frac{1}{4} \), \( \frac{1}{2} \), \( \frac{3}{4} \) or 1. Tell whether your estimate is high or low.
   a. \( \frac{19}{39} \)
   b. \( \frac{3}{197} \)
   c. \( \frac{150}{201} \)
   d. \( \frac{8}{9} \)

8. Without actually finding the exact answer, state which of the numbers given in parentheses is the best approximation for the given sum.
   a. \( \frac{6}{13} + \frac{7}{15} + \frac{11}{23} + \frac{17}{35} \) (1, 2, 3, 4, 5)
   b. \( \frac{30}{41} + \frac{1}{1000} + \frac{3}{2000} \) (3, 3, 4, 1, 2)

9. Compute each of the following mentally.
   a. \( 1 - \frac{3}{4} \)
   b. \( 3 \frac{3}{8} + 2 \frac{1}{4} - 5 \frac{5}{8} \)

10. The following ruler has regions marked M, A, T, H.

   Use estimation to determine into which region on the ruler each of the following measurements falls. For example, \( \frac{12}{5} \) in.
   a. \( \frac{20}{8} \) in.
   b. \( \frac{36}{8} \) in.
   c. \( \frac{60}{16} \) in.
   d. \( \frac{18}{4} \) in.

11. Use clustering to estimate the following sum.

   \( \frac{3}{1} + \frac{5}{3} + \frac{7}{2} + \frac{7}{9} \)

12. A class consists of \( \frac{2}{5} \) freshmen, \( \frac{1}{4} \) sophomores, and \( \frac{1}{10} \) juniors; the rest are seniors. What fraction of the class is seniors?

13. A clerk sold three pieces of one type of ribbon to different customers. One piece was \( \frac{1}{3} \) yd long, another was \( \frac{3}{4} \) yd long, and the third was \( \frac{1}{2} \) yd long. What was the total length of that type of ribbon sold?
14. Martine bought $\frac{8}{3} \text{ yd}$ of fabric. She wants to make a skirt using $\frac{7}{8} \text{ yd}$, pants using $\frac{3}{8} \text{ yd}$, and a vest using $\frac{2}{3} \text{ yd}$. How much fabric will be left over?

15. Give an example illustrating each of the following properties of rational number addition.
   a. Closure  
   b. Commutative  
   c. Associative

16. Insert five fractions between the numbers 1 and 2 so that the seven numbers (including 1 and 2) constitute part of an arithmetic sequence.

17. a. Check that each of the following statements is true.
   i. $\frac{1}{3} = \frac{1}{4} + \frac{1}{4}$  
   ii. $\frac{1}{4} = \frac{1}{5} + \frac{1}{5}$  
   iii. $\frac{1}{5} = \frac{1}{6} + \frac{1}{5} 

   b. Based on the examples in (a), write $\frac{1}{n}$ as a sum of two unit fractions; that is, as a sum of fractions with numerator 1.

18. Solve for $x$.
   a. $x + \frac{1}{2} = \frac{3}{4}$  
   b. $x - \frac{2}{3} = \frac{5}{6}$

19. Al runs $\frac{5}{8} \text{ mi}$ in 10 min. Bill runs $\frac{7}{8} \text{ mi}$ in 10 min. If both runners continue to run at the same rate, how much farther can Bill run than Al in 20 min?

20. One recipe calls for $\frac{3}{4} \text{ cups}$ of milk and a second recipe calls for $\frac{1}{2} \text{ cups}$ of milk. If you only have 3 cups of milk, can you make both recipes? Why?

21. The table below shows census data from the state of Pennsylvania for 2011.

<table>
<thead>
<tr>
<th>Population</th>
<th>2011</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total 8-yr.</td>
<td>150,155</td>
</tr>
<tr>
<td>Boys</td>
<td>76,432</td>
</tr>
<tr>
<td>Girls</td>
<td>73,723</td>
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<td>Total 9-yr.</td>
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</tr>
</tbody>
</table>

Data from 2011 Census taken from http://www.census.gov/schools/facts/pennsylvania.html

a. About what part of the total population is 8- to 10-year-old boys?
b. About what part of the total population is 8- to 10-year-old girls?
c. If the population of the entire state of Pennsylvania in 2011 was 12,742,886, about what part of the population was boys of ages 8- to 10-year-olds?

22. According to the US Census Bureau, in October 2012, there were about 633,000 3- and 4-year-olds enrolled in schools. Additionally there were about 1,186,000 16- and 17-year-olds enrolled in schools. In the respective ages there were about 1,193,000 and 1,259,000 total. Which age group had the greater fraction in school?

23. There are $360^\circ$ in a circle graph. If $40^\circ$ of the graph represents rent and $5^\circ$ of the graph represents savings, what fractional portion of the whole graph is represented by rent and savings?

24. The energy sources for the US in 2012 included $\frac{37}{100}$ coal, $\frac{3}{10}$ natural gas, $\frac{19}{100}$ nuclear, and $\frac{7}{100}$ hydropower. How much is unaccounted for in this list?

---

Assessment 2B

1. Perform the following additions or subtractions.
   a. $\frac{1}{2} + \frac{2}{3}$
   b. $\frac{5}{12} - \frac{2}{3}$
   c. $\frac{5}{4x} + \frac{3}{2y}$
   d. $\frac{-3}{2x^2y^2} + \frac{5}{2x^2y^2} + \frac{7}{3y}$
   e. $\frac{5}{6} - \frac{2}{8}$
   f. $-\frac{1}{2} + \frac{3}{6}$
   g. $\frac{5}{3} + \frac{5}{6} - \frac{3}{9}$

2. Change each of the following fractions to a mixed number.
   a. $\frac{14}{5}$
   b. $\frac{47}{8}$

3. Change each of the following mixed numbers to a fraction in the form $\frac{a}{b}$, where $a$ and $b$ are integers and $b \neq 0$.
   a. $7 \frac{1}{2}$
   b. $-4 \frac{2}{3}$

4. Place the numbers 2, 5, 6, and 8 in the following boxes to make the equation true.
   $\square + \square = \frac{23}{24}$
5. Use the information in the table to answer each of the following questions.

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</tr>
</tbody>
</table>

a. Which team won less than \( \frac{1}{3} \) of its games and was closest to winning \( \frac{1}{3} \) of its games?

b. Which team won more than \( \frac{1}{4} \) of its games and was closest to winning \( \frac{1}{4} \) of its games?

c. Which teams won less than \( \frac{1}{4} \) of their games?

6. Sort the following fraction cards into the ovals by estimating in which oval the fraction belongs.

7. Approximate each of the following fractions by 0, \( \frac{1}{4} \), \( \frac{1}{2} \), \( \frac{3}{4} \), or 1. Tell whether your estimate is high or low.

8. Without actually finding the exact answer, state which of the numbers given in parentheses is the best approximation for the given sum.

9. Compute each of the following mentally.

10. The following ruler has regions marked M, A, T, H.

Use mental mathematics and estimation to determine into which region on the ruler each of the following measurements falls. For example, \( \frac{12}{5} \) in. falls into region A.

a. \( \frac{9}{8} \) in.

b. \( \frac{18}{8} \) in.

c. \( \frac{50}{16} \) in.

d. \( \frac{17}{4} \) in.

11. A class consists of \( \frac{1}{4} \) freshmen, \( \frac{1}{5} \) sophomores, and \( \frac{1}{10} \) juniors; the rest are seniors. What fraction of the class is seniors?

12. The Naturals Company sells its products in many countries. The following two circle graphs show the fractions of the company's earnings for 2012 and 2014. Based on this information, answer the following questions.

a. In 2012, how much greater was the fraction of sales for Japan than for Canada?

b. In 2014, how much less was the fraction of sales for England than for the United States?

c. How much greater was the fraction of total sales for the United States in 2014 than in 2012?

d. Is it true that the amount of sales in dollars in Australia was less in 2012 than in 2014? Why?
13. A recipe requires $\frac{3}{2}$ cups of milk. Ran put in $\frac{3}{4}$ cups in a bowl. How much more milk does he need?

14. A 15 $\frac{3}{4}$ in. board is cut in a single cut from a 38 $\frac{1}{4}$ in. board. The saw cut takes $\frac{3}{8}$ in. How much of the 38 $\frac{1}{4}$ in. board is left after cutting?

15. Students from Rattlesnake School formed four teams to collect cans for recycling during the months of April and May. A record of their efforts follows.

<table>
<thead>
<tr>
<th>Number of Pounds Collected</th>
<th>Team 1</th>
<th>Team 2</th>
<th>Team 3</th>
<th>Team 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>April</td>
<td>$28 \frac{3}{4}$</td>
<td>$32 \frac{7}{8}$</td>
<td>$28 \frac{1}{2}$</td>
<td>$35 \frac{3}{16}$</td>
</tr>
<tr>
<td>May</td>
<td>$33 \frac{1}{3}$</td>
<td>$28 \frac{5}{12}$</td>
<td>$25 \frac{3}{4}$</td>
<td>$41 \frac{1}{2}$</td>
</tr>
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d. If you continue in this pattern with powers of 2 in the denominator, will the sum ever become greater than 1? Why?

20. The table below shows census data from the state of Pennsylvania for 2011.

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</tbody>
</table>

Data from 2011 Census taken from http://www.census.gov/schools/facts/pennsylvania.html

a. About what fraction more of girls is there in the 8-year-olds are there than in the 10-year-olds?
b. About what part of the total population of 8- to 9-year-olds are girls?
c. If the population of the entire state of Pennsylvania in 2011 was 12,742,886, about what part of the population was girls of ages 8–10?

21. According to the US Census Bureau in October 2012, there were about 633,000 3- and 4-year olds enrolled in schools. Additionally there were about 1,186,000 16- and 17-year-olds enrolled in schools. In the respective ages there were about 1,193,000 and 1,259,000 total. Which of these age groups had the lesser fraction in school?

22. There are 360° in a circle graph. If 50° of the graph represents rent and 7° of the graph represents savings, what fractional portion of the whole graph is not represented by rent and savings?

23. The energy sources for the United States in 2012 included $\frac{37}{100}$ coal, $\frac{3}{10}$ natural gas, $\frac{19}{100}$ nuclear, and $\frac{7}{100}$ hydropower. What fraction more coal than natural gas is used?
3. Explain why we can do the following to convert \(5 \frac{3}{4}\) to a mixed number.
\[
\frac{5 \cdot 4 + 3}{4} = \frac{23}{4}
\]

4. Kara spent \(\frac{1}{2}\) of her allowance on Saturday and \(\frac{1}{3}\) of what she had left on Sunday. Can this situation be modeled as \(\frac{1}{2} - \frac{1}{3}\)? Explain why or why not.

5. Compute \(3 \frac{3}{4} + 5 \frac{1}{3}\) in two different ways and leave your answer as a mixed number. Tell which way you prefer and why.

6. Explain whether each of the following properties holds for subtraction of rational numbers.
   a. Closure
   b. Commutative
   c. Associative
   d. Identity
   e. Inverse

Open-Ended

7. Write a story problem for \(\frac{2}{3} - \frac{1}{4}\).

8. a. Write two fractions whose sum is 1. If one of the fractions is \(\frac{a}{b}\), what is the other?
   b. Write three fractions whose sum is 1.
   c. Write two fractions whose difference is very close to 1 but not exactly 1.

9. a. With the exception of \(\frac{2}{3}\), the Egyptians used only unit fractions (fractions that have numerators of 1). Every unit fraction can be expressed as the sum of two unit fractions in more than one way, for example, \(\frac{1}{2} = \frac{1}{4} + \frac{1}{4}\) and \(\frac{1}{2} = \frac{1}{3} + \frac{1}{6}\). Find at least two different unit fraction representations for each of the following.
   i. \(\frac{1}{3}\)
   ii. \(\frac{1}{7}\)
   b. Show that \(\frac{1}{n} - \frac{1}{n + 1} = \frac{1}{n(n + 1)}\)
   c. Rewrite the equation in part (b) as a sum and then use the sum to answer part (a).
   d. Write \(\frac{1}{17}\) as a sum of two different unit fractions.

Cooperative Learning

10. Interview 10 people and ask them if and when they add and subtract fractions in their lives. Combine their responses with those of the rest of the class to get a view of how “ordinary” people use computation of rational numbers in their daily lives.

Connecting Mathematics to the Classroom

11. Kendra showed that \(\frac{1}{3} + \frac{3}{4} = \frac{4}{7}\) by using the following figure. How would you help her?
\[
\begin{array}{c}
\text{\ } \\
\text{\ } + \\
\text{\ } \\
\end{array} 
\begin{array}{c}
\text{\ } \\
\text{\ } = \\
\text{\ } \\
\end{array} 
\begin{array}{c}
\text{\ } \\
\text{\ } \\
\text{\ } \\
\end{array}
\]

12. To show \(2 \frac{3}{4} = \frac{11}{4}\), the teacher drew the following picture.

Ken said this shows a picture of \(\frac{11}{12}\), not \(\frac{11}{4}\). What is Ken thinking and how should the teacher respond?

13. Jill claims that for positive fractions, \(\frac{a}{b} + \frac{a}{c} = \frac{a}{b + c}\) because the fractions have a common numerator. How do you respond?

14. Explain the error pattern on Jon’s test.
   a. \(\frac{13}{35} = \frac{1}{5}\)
   b. \(\frac{27}{73} = \frac{2}{3}\)
   c. \(\frac{16}{64} = \frac{1}{4}\)
   d. \(\frac{5}{8} + \frac{2}{5} = \frac{6}{8} + \frac{3}{4} = \frac{9}{8} + \frac{1}{3} = \frac{8}{11}\)
   e. \(\frac{8}{5} - \frac{6}{4} = \frac{2}{4} - \frac{3}{8} - \frac{2}{3} = \frac{1}{5}\)
   f. \(\frac{2}{7} - \frac{1}{3} = \frac{1}{4}\)

Review Problems

15. Simplify each rational number if possible.
   a. \(\frac{14}{21}\)
   b. \(\frac{117}{153}\)
   c. \(\frac{5}{7}\)
   d. \(\frac{a^2 + a}{1 + a}\)
   e. \(\frac{a^2 + 1}{a + 1}\)
   f. \(\frac{a^2 - b^2}{a - b}\)

16. Determine whether the fractions in each of the following pairs are equal.
   a. \(\frac{a^2}{b} \quad \text{and} \quad \frac{a^2b}{b^2}\)
   b. \(\frac{377}{400} \quad \text{and} \quad \frac{378}{401}\)
   c. \(\frac{0}{10} \quad \text{and} \quad \frac{0}{10}\)
   d. \(\frac{a}{b} \quad \text{and} \quad \frac{a + 1}{b + 1}\) where \(a \neq b\)

17. There are 206 bones in the body. Can the fraction \(\frac{27}{103}\) represent the number of bones in both hands as part of the total number of bones in the body? Explain your answer.

18. Explain why there are infinitely many fractions equivalent to \(\frac{3}{5}\).
19. Mary ate \( \frac{3}{5} \) of the cookies and left the rest for Suzanne. What fraction of the cookies are left for Suzanne?

20. On a number line, explain why \(-\frac{1}{100}\) is greater than \(-\frac{1}{10}\).

**National Assessments**

**National Assessment of Educational Progress (NAEP) Questions**

\[
\frac{1}{20}, \frac{4}{20}, \frac{7}{20}, \frac{10}{20}, \frac{13}{20}, \ldots
\]

If the pattern shown continues, what is the first fraction in the pattern that will be greater than 1?

A. \(\frac{20}{20}\)  
B. \(\frac{21}{20}\)  
C. \(\frac{22}{20}\)  
D. \(\frac{25}{20}\)

**NAEP, Grade 4, 2013**

3 Multiplication, Division, and Estimation with Rational Numbers

**Multiplication of Rational Numbers**

To motivate the definition of multiplication of rational numbers, we use the interpretation of multiplication as repeated addition. Using repeated addition, we interpret \(3 \left( \frac{3}{4} \right)\) as follows:

\[
3 \left( \frac{3}{4} \right) = \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = \frac{9}{4} = 2 \frac{1}{4}
\]

The area model in Figure 13 is another way to calculate this product.

We next consider \(\left( \frac{3}{4} \right)^3\). How should this product be interpreted? If the commutative property of multiplication of rational numbers is to hold, then \(\left( \frac{3}{4} \right)^3 = 3 \left( \frac{3}{4} \right) = \frac{9}{4}\).
Rational Numbers and Proportional Reasoning

Next, we consider another interpretation of multiplication. What is $\frac{3}{4}$ of 3? In Figure 14(a) consider the 1-unit bar broken into fourths. Thus, there are four equal parts of the 1 unit, each of length $\frac{1}{4}$ unit. In Figure 14(b) suppose the length of the bar was 3 units. We want the same type of action to occur so it is divided into three-fourths using the same strategy. But analogously, each part would be 3 times the length of the bars in Figure 13(a). If this strategy is used then $\frac{3}{4}$ of 3 is seen as the shaded portion and $\frac{3}{4} \cdot 3 = \frac{9}{4}$ of 1 or simply $\frac{9}{4}$ is $\frac{3}{4}$ of 3.

![Figure 14](image)

If forests once covered about $\frac{3}{5}$ of Earth’s land and only about $\frac{1}{2}$ of these forests remain, what fraction of Earth is covered with forests today? We need to find $\frac{1}{2}$ of $\frac{3}{5}$, and can use an area model to find the answer.

Figure 15(a) shows a rectangle representing the whole separated into fifths, with $\frac{3}{5}$ shaded. To find $\frac{1}{2}$ of $\frac{3}{5}$, we divide the shaded portion of the rectangle in Figure 15(a) into two congruent parts and take one of those parts. The result would be the green portion of Figure 15(b). However, the green portion represents 3 parts out of 10, or $\frac{3}{10}$ of the whole. Thus,

$$\frac{1}{2} \text{ of } \frac{3}{5} = \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10} = \frac{1 \cdot 3}{2 \cdot 5}.$$  

An area model like the one in Figure 14 is used on the student page shown on the next page.
If $\frac{5}{6}$ of the population of a certain city are college graduates and $\frac{7}{11}$ of the city’s college graduates are female, what fraction of the population of that city is female college graduates?

**Solution**  The fraction should be $\frac{7}{11}$ of $\frac{5}{6}$, or $\frac{7 \cdot 5}{11 \cdot 6} = \frac{35}{66}$.

The fraction of the population who are female college graduates is $\frac{35}{66}$. 
Properties of Multiplication of Rational Numbers

Multiplication of rational numbers has properties analogous to the properties of multiplication of integers. These include the following properties for multiplication: closure, commutative, associative, and multiplicative identity. When we expand from the set of integers to the set of rationals, we pick up an additional property; that is, the multiplicative inverse property. For emphasis, we list the last two properties.

**Theorem 10: Identity and Inverse Properties of Multiplication of Rational Numbers**

**Multiplicative Identity Property of Rational Numbers**

The rational number 1 is the unique number such that for every rational number \( \frac{a}{b} \),

\[
1 \cdot \frac{a}{b} = \frac{a}{b} = \frac{a}{b} \cdot 1.
\]

**Multiplicative Inverse Property of Rational Numbers**

For any nonzero rational number \( \frac{a}{b} \) the multiplicative inverse (reciprocal) is the unique rational number \( \frac{b}{a} \) such that

\[
\frac{a}{b} \cdot \frac{b}{a} = 1 = \frac{b}{a} \cdot \frac{a}{b}.
\]

**Example 15**

Find the multiplicative inverse, if possible, for each of the following rational numbers.

a. \( \frac{2}{3} \)  
   b. \( -\frac{2}{5} \)  
   c. 4  
   d. 0  
   e. \( 6\frac{1}{2} \)

**Solution**

a. \( \frac{3}{2} \)

b. \( \frac{5}{2} \), or \( -\frac{5}{2} \)

c. Because \( 4 = \frac{4}{1} \), the multiplicative inverse of 4 is \( \frac{1}{4} \).

d. Even though \( 0 = \frac{0}{1} \) is undefined; there is no multiplicative inverse of 0.

e. Because \( 6\frac{1}{2} = \frac{13}{2} \), the multiplicative inverse of \( 6\frac{1}{2} \) is \( \frac{2}{13} \).
Theorem 11: Properties of Rational Number Operations

Distributive Properties of Multiplication Over Addition and Subtraction for Rational Numbers

Let \( \frac{a}{b}, \frac{c}{d}, \) and \( \frac{e}{f} \) be any rational numbers. Then

\[
\frac{a}{b} \left( \frac{c}{d} + \frac{e}{f} \right) = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}; \quad \text{and} \quad \frac{a}{b} \left( \frac{c}{d} - \frac{e}{f} \right) = \frac{a}{b} \cdot \frac{c}{d} - \frac{a}{b} \cdot \frac{e}{f}.
\]

Multiplication Property of Equality for Rational Numbers

Let \( \frac{a}{b}, \frac{c}{d}, \) and \( \frac{e}{f} \) be any rational numbers such that \( \frac{a}{b} = \frac{c}{d} \); then \( \frac{a}{b} \cdot \frac{e}{f} = \frac{c}{d} \cdot \frac{e}{f} \).

Multiplication Properties of Inequality for Rational Numbers

Let \( \frac{a}{b}, \frac{c}{d}, \) and \( \frac{e}{f} \) be any rational numbers. Then

a. \( \frac{a}{b} > \frac{c}{d} \) and \( \frac{e}{f} > 0 \), then \( \frac{a}{b} \cdot \frac{e}{f} > \frac{c}{d} \cdot \frac{e}{f} \).

b. \( \frac{a}{b} > \frac{c}{d} \) and \( \frac{e}{f} < 0 \), then \( \frac{a}{b} \cdot \frac{e}{f} < \frac{c}{d} \cdot \frac{e}{f} \).

Multiplication Property of Zero for Rational Numbers

Let \( \frac{a}{b} \) be any rational number; then \( \frac{a}{b} \cdot 0 = 0 \cdot \frac{a}{b} = 0 \).

Example 16

A bicycle is on sale at \( \frac{3}{4} \) of its original price. If the sale price is $330, what was the original price?

Solution

Let \( x \) be the original price. Then \( \frac{3}{4} \) of the original price is \( \frac{3}{4} x \). Because the sale price is $330, we have \( \frac{3}{4} x = 330 \). Solving for \( x \) gives

\[
\frac{4}{3} \cdot \frac{3}{4} x = \frac{4}{3} \cdot 330
\]

\[
1 \cdot x = 440
\]

\[
x = 440.
\]

Thus, the original price was $440.

An alternative approach follows. Suppose the bar in Figure 16 represents the original price. We know that \( \frac{3}{4} \) of the original price is the sale price $330 as shown. We have 3 of the \( \frac{1}{4} \) parts of the original price is $330, and one part, \( \frac{1}{4} \) of the original price, must be $110. Thus, the original price is $330 + $110 = $440.

![Figure 16](image)
Multiplication with Mixed Numbers

In Figure 17, Johnny just figured out how to multiply mixed numbers while his colleagues seem to be struggling. How might we help them?

In Figure 17, Johnny may have decided one way to multiply $2 \frac{1}{2} \cdot 2 \frac{1}{2}$ is to change the mixed numbers to improper fractions and use the definition of multiplication as shown.

$$2 \frac{1}{2} \cdot 2 \frac{1}{2} = \frac{5}{2} \cdot \frac{5}{2} = \frac{25}{4}$$

We could then change $\frac{25}{4}$ to the mixed number $6 \frac{1}{4}$.

Another way to multiply mixed numbers uses the distributive property of multiplication over addition, as seen below.

$$2 \frac{1}{2} \cdot 2 \frac{1}{2} = \left(2 + \frac{1}{2}\right)\left(2 + \frac{1}{2}\right)$$

$$= \left(2 + \frac{1}{2}\right)^2 + \left(2 + \frac{1}{2}\right)\frac{1}{2}$$

$$= 2 \cdot 2 + \frac{1}{2} \cdot 2 + 2 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$$

$$= 4 + 1 + 1 + \frac{1}{4}$$

$$= 6 + \frac{1}{4}$$

$$= 6 \frac{1}{4}$$

Multiplication of fractions enables us to obtain equivalent fractions, to perform addition and subtraction of fractions, as well as to solve equations in a different way, as shown in the following example.
Use the definition of multiplication of fractions and its properties to justify the following.

a. The Fundamental Law of Fractions: \( \frac{a}{b} \cdot \frac{m}{n} = \frac{am}{bn} \) if \( b \neq 0, n \neq 0 \).

b. Addition of fractions using a common denominator.

**Solution**

a. \( \frac{a}{b} = \frac{a \cdot 1}{b} = \frac{a}{b} \cdot \frac{n}{n} = \frac{an}{bn} \)

b. \( \frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot b}{bd} \)

We apply and extend division of whole numbers by recalling that \( 6 \div 3 \) means “How many 3s are there in 6?” We found that \( 6 \div 3 = 2 \) because \( 3 \cdot 2 = 6 \) and, in general, when \( a, b, c \in \mathbb{W} \), \( a \div b = c \) if, and only if, \( c \) is the unique whole number such that \( bc = a \). Consider \( 3 \div \left( \frac{1}{2} \right) \), which is equivalent to finding how many halves there are in 3. We see from the area model in Figure 18 that there are 6 half pieces in the 3 whole pieces. We record this as \( 3 \div \left( \frac{1}{2} \right) = 6 \). This is true because \( \left( \frac{1}{2} \right)6 = 3 \).

Another way to show that \( 3 \div \left( \frac{1}{2} \right) = 6 \) is on a ruler. In Figure 19 we see that there are six \( \frac{1}{2} \)s in 3.
With whole numbers, one way to think about division is in terms of repeated subtraction. We found that \(6 \div 2 = 3\) because 2 could be subtracted from 6 three times; that is, \(6 - 3 \cdot 2 = 0\). Similarly, with \(3 \div \frac{1}{2}\), we want to know how many halves can be subtracted from 3. Because \(3 - 6 \left( \frac{1}{2} \right) = 0\), we know that \(3 \div \frac{1}{2} = 6\).

Next, consider \(\frac{3}{4} \div \frac{1}{8}\). This means “How many \(\frac{1}{8}\)s are in \(\frac{3}{4}\)?” Figure 20 shows that there are six \(\frac{1}{8}\)s in the shaded portion, which represents \(\frac{3}{4}\) of the whole. Therefore, \(\frac{3}{4} \div \frac{1}{8} = 6\). This is true because \(\left( \frac{1}{8} \right) 6 = \frac{3}{4}\).

The measurement, or number-line, model may be used to understand division of fractions. For example, consider \(\frac{7}{8} \div \frac{3}{4}\). First we draw a number line divided into eighths, as shown in Figure 21. Next we want to know how many \(\frac{3}{4}\)s there are in \(\frac{7}{8}\). The bar of length \(\frac{3}{4}\) is made up of 6 equal-size pieces of length \(\frac{1}{8}\). We see that there is at least one length of \(\frac{3}{4}\) in \(\frac{7}{8}\). If we put another bar of length \(\frac{3}{4}\) on the number line, we see there is 1 more of the 6 equal-length segments needed to make \(\frac{7}{8}\). Therefore, the answer is \(1 \frac{1}{6}\), or \(\frac{7}{6}\).

Additionally we know that \(\frac{3}{4} \cdot \frac{7}{6} = \frac{21}{24} = \frac{7}{8}\), so the answer is correct.

In the previous examples, we saw a relationship between division and multiplication of rational numbers. We can define division for rational numbers formally in terms of multiplication in the same way that we defined division for whole numbers.

**Definition of Division of Rational Numbers**

Let \(\frac{a}{b}\) and \(\frac{c}{d}\) be rational numbers such that \(\frac{c}{d} \neq 0\). Then \(\frac{a}{b} \div \frac{c}{d} = \frac{e}{f}\) if, and only if, \(\frac{e}{f}\) is the unique rational number such that \(\frac{c}{d} \cdot \frac{e}{f} = \frac{a}{b}\).
Rational Numbers and Proportional Reasoning

In the above definition of division, \( \frac{c}{d} \neq 0 \) because division by 0 is not defined.

NOW TRY THIS 4

Students often confuse finding half of a number with dividing by one-half. Notice that

\[
a \div 2 = \frac{a}{2} = \frac{1}{2}a, \text{ but } a \div \frac{1}{2} = x \text{ if, and only if, } \frac{1}{2}x = a \text{ which implies } 2\left(\frac{1}{2}x\right) = 2a \text{ and } x = 2a.
\]

Write a real-life story that will help students see the difference between finding half of a number and division by \( \frac{1}{2} \).

The partial student page below illustrates a method for dividing called “invert and multiply.”

Algorithm for Division of Rational Numbers

Does the invert-and-multiply method make sense based on what we know about rational numbers? Consider what such a division might mean. For example, using the definition of division of rational numbers,

\[
\frac{2}{3} \div \frac{5}{7} = x \text{ if, and only if, } \frac{2}{3} = \frac{5}{7}x.
\]

To solve for \( x \), we multiply both sides of the equation by \( \frac{7}{5} \), the reciprocal of \( \frac{5}{7} \). Thus,

\[
\frac{7}{5} \cdot \frac{2}{3} = \frac{7}{5} \left(\frac{5}{7}x\right) = \left(\frac{7}{5} \cdot \frac{5}{7}\right)x = 1 \cdot x = x.
\]

Therefore, \( \frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} \). This illustrates the “invert-and-multiply” method, or the “use the reciprocal and multiply” method.
A traditional justification of this rule follows. The algorithm for division of fractions is usually justified in the middle grades by using the Fundamental Law of Fractions, $\frac{a}{b} = \frac{ac}{bc}$, where $a$, $b$, and $c$ are fractions, or equivalently, the identity property of multiplication. For example,

$$\frac{2}{3} \div \frac{5}{7} = \frac{\frac{2}{3}}{\frac{5}{7}} = \frac{2}{3} \cdot \frac{7}{5} = \frac{2 \cdot 7}{3 \cdot 5}.$$ 

**Theorem 12: Algorithm for Division of Fractions**

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers with $\frac{c}{d} \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$ 

**Alternative Algorithm for Division of Rational Numbers**

An alternative algorithm for division of fractions can be found by first dividing fractions that have equal denominators. For example, $\frac{9}{10} \div \frac{3}{10} = \frac{9}{10} \div \frac{3}{10} = \frac{9}{3}$ and $\frac{15}{23} \div \frac{5}{23} = \frac{15}{23} \div \frac{5}{23} = \frac{15}{5}$. These examples suggest that when two fractions with the same denominator are divided, the result can be obtained by dividing the numerator of the first fraction by the numerator of the second; that is, $\frac{a}{b} \div \frac{c}{b} = \frac{a}{c}$. To divide fractions with different denominators, we rename the fractions so that the denominators are equal. Thus,

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{bc}{bd} = \frac{ad}{bd} \cdot \frac{bd}{bc} = \frac{ad}{bc}.$$ 

**NOW TRY THIS 5**

Show that $\frac{a}{b} \div \frac{c}{d}$ and $\frac{a}{b} \div \frac{c}{d}$ are equal.

The next three examples illustrate the use of division of rational numbers.

**Example 18**

A radio station provides 36 min for public service announcements for every 24 hr of broadcasting.

a. What part of the 24-hr broadcasting day is allotted to public service announcements?

b. How many $\frac{3}{4}$-min public service announcements can be broadcast in the 36 min?

**Solution**

a. There are 60 min in an hour and 60 $\cdot$ 24 min in the broadcasting day. Thus, $36 \div (60 \cdot 24)$, or $\frac{1}{40}$, of the day is allotted for the announcements.

b. $36 \div \left(\frac{3}{4}\right) = 36 \left(\frac{4}{3}\right) = 48$ announcements are broadcast.
We have $35 \frac{1}{2}$ yd of material available to make towels. Each towel requires $\frac{3}{8}$ yd of material.

**a.** How many towels can be made?

**b.** How much material will remain?

**Solution**

**a.** We need to find the integer part of the answer to $35 \frac{1}{2} \div \frac{3}{8}$.

$$35 \frac{1}{2} \div \frac{3}{8} = \frac{71}{2} \cdot \frac{8}{3} = \frac{284}{3} = 94 \frac{2}{3}$$

Thus, we can make 94 towels.

**b.** Because the division in part (a) was by $\frac{3}{8}$, the amount of material remaining is $\frac{2}{3}$ of $\frac{3}{8}$, or $\frac{2 \cdot 3}{8} = \frac{1}{4}$ yd.

---

**Example 20**

A bookstore has a shelf that is $37 \frac{1}{2}$ in. long. Each book that is to be placed on the shelf is $1 \frac{1}{4}$ in. thick. How many books can be placed on the shelf?

**Solution** We need to find how many $1 \frac{1}{4}$’s there are in $37 \frac{1}{2}$.

$$\frac{37 \frac{1}{2}}{1 \frac{1}{4}} = \frac{75}{2} \div \frac{5}{4}$$

$$= \frac{75 \cdot 4}{2 \cdot 5}$$

$$= \frac{300}{10}$$

$$= 30$$

Note that we could compute $\frac{75 \cdot 4}{5}$ by first eliminating common factors; that is, $\frac{75 \cdot 4}{5} = \frac{15 \cdot 2}{1 \cdot 1} = \frac{30}{1} = 30$.

Therefore, 30 books can be placed on the shelf.

---

**Mental Math and Estimation with Rational Numbers**

Mental math strategies developed with whole numbers can also be used with rational numbers.
Example 21

Use rational number properties to mentally compute the following products.

a. \((12 \cdot 25)\frac{1}{4}\)  
b. \(\left(\frac{5}{1} - \frac{1}{6}\right)12\)  
c. \(\frac{4}{5}(20)\)

Solution
Each computation shown is a possible approach.

a. \((12 \cdot 25)\frac{1}{4} = 25\left(\frac{12}{4}\right) = 25 \cdot 3 = 75\)

b. \(\left(\frac{5}{1} - \frac{1}{6}\right)12 = \left(\frac{5}{6}\right)12 = 5 \cdot 12 + \frac{1}{6} \cdot 12 = 60 + 2 = 62\)

c. \(\frac{4}{5}(20) = 4\left(\frac{1}{5} \cdot 20\right) = 4 \cdot 4 = 16\)

Similarly, estimation strategies developed with whole number can be used with rational numbers.

Example 22

Estimate each of the following.

a. \(\frac{3}{4} \cdot \frac{1}{8}\)  
b. \(\frac{24}{7} \div \frac{4}{8}\)

Solution

a. Using rounding, the product will be close to \(3 \cdot 8 = 24\). If we use the range strategy, we can say the product must be between \(3 \cdot 7 = 21\) and \(4 \cdot 8 = 32\).

b. We can use compatible numbers and think of the estimate as \(24 \div 4 = 6\) or \(25 \div 5 = 5\).

Extending the Notion of Exponents

Recall that \(a^m\) was defined for any whole number \(a\) and any natural number \(m\) as the product of \(m\) \(a\)'s. We define \(a^m\) for any rational number \(a\) in a similar way as follows.

**Definition of \(a\) to the \(m\)th Power**

\[a^m = a \cdot a \cdot a \ldots \cdot a, \text{ where } a \text{ is any rational number and } m \text{ is any natural number.}\]

From the definition, \(a^3 \cdot a^2 = (a \cdot a \cdot a)(a \cdot a) = a^{3+2} = a^5\). In a similar way, it follows that

**Statement 1:** If \(a\) is a rational number and \(m\) and \(n\) are any natural numbers, \(a^m \cdot a^n = a^{m+n}\).

If Statement 1 is to be true for all whole numbers \(m\) and \(n\), then because \(a^1 \cdot a^0 = a^{1+0} = a^1\), we must have \(a^0 = 1\). Hence, it is useful to give meaning to \(a^0\) when \(a \neq 0\) as follows.

**Statement 2:** For any nonzero rational number \(a\), \(a^0 = 1\).

If \(a^m \cdot a^n = a^{m+n}\) is extended to all integer powers of \(a\), then how should \(a^{-1}\) be defined? If Statement 1 is to be true for all integers \(m\) and \(n\), then \(a^{-1} \cdot a^3 = a^{-1+3} = a^2 = 1\). Therefore, \(a^{-1} = \frac{1}{a^1}\). This is true in general and we have the following.

**Statement 3:** Definition: For any nonzero rational number \(a\) and any natural number \(n\), \(a^{-n} = \frac{1}{a^n}\).
Rational Numbers and Proportional Reasoning

In elementary grades the definition of $a^{-n}$ is typically motivated by looking at patterns. Notice that as the following exponents decrease by 1, the numbers on the right are divided by 10. Thus, the pattern might be continued, as shown.

$$
\begin{align*}
10^3 &= 10 \cdot 10 \cdot 10 \\
10^2 &= 10 \cdot 10 \\
10^1 &= 10 \\
10^0 &= 1 \\
10^{-1} &= \frac{1}{10} = \frac{1}{10^1} \\
10^{-2} &= \frac{1}{10} \cdot \frac{1}{10} = \frac{1}{10^2} \\
10^{-3} &= \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} = \frac{1}{10^3}
\end{align*}
$$

If the pattern is extended in this way, then we would predict that $10^{-n} = \frac{1}{10^n}$. Notice that this is inductive reasoning and hence is not a mathematical justification.

Consider whether the property $a^m \cdot a^n = a^{m+n}$ can be extended to include all powers of $a$, where the exponents are integers. For example, is it true that $2^4 \cdot 2^{-3} = 2^{4-3} = 2^1$? The definitions of $2^3$ and the properties of nonnegative exponents ensure this is true, as shown next.

$$2^4 \cdot 2^{-3} = 2^4 \cdot \frac{1}{2^3} = \frac{2^4}{2^3} = \frac{2^1 \cdot 2^3}{2^3} = 2^1$$

Similarly, $2^4 \cdot 2^{-3} = 2^{4-3} = 2^7$ is true because

$$2^4 \cdot 2^{-3} = \frac{1}{2^3} \cdot \frac{1}{2^3} = \frac{1 \cdot 1}{2^4 \cdot 2^3} = \frac{1}{2^4 \cdot 3} = \frac{1}{2^7} = 2^7.$$

In general, with integer exponents, the following theorem holds.

**Theorem 13**

For any nonzero rational number $a$ and any integers $m$ and $n$, $a^m \cdot a^n = a^{m+n}$.

If $a = 0$ and either $m$ or $n$ is negative, then $a^m \cdot a^n$ is undefined.

Other properties of exponents can be developed by using the properties of rational numbers. For example,

$$\frac{2^5}{2^3} = \frac{2^5 \cdot 2^2}{2^3} = 2^2 = 2^{5-3} \quad \text{and} \quad \frac{2^5}{2^8} = \frac{2^5}{2^5 \cdot 2^3} = \frac{1}{2^3} = 2^{-3} = 2^{5-8}.$$

With integer exponents, the following theorem holds.

**Theorem 14**

For any nonzero rational number $a$ and any integers $m$ and $n$, $\frac{a^m}{a^n} = a^{m-n}$.
Rational Numbers and Proportional Reasoning

At times, we may encounter an expression like \( (2^4)^3 \). This expression can be written as a single power of 2 as follows:

\[
(2^4)^3 = 2^4 \cdot 2^4 \cdot 2^4 = 2^{4+4+4} = 2^{12}
\]

In general, if \( a \) is any rational number and \( m \) and \( n \) are positive integers, then

\[
(a^m)^n = a^{mn}.
\]

Does this theorem hold for negative-integer exponents? For example, does \( (2^3)^{-4} = 2^{(3)(-4)} = 2^{-12} \)? The answer is yes because \( (2^3)^{-4} = \frac{1}{(2^3)^4} = \frac{1}{2^{12}} = 2^{-12} \). Also, \( (2^{-3})^4 = \left(\frac{1}{2^3}\right)^4 = \frac{1}{2^{12}} = 2^{-12} \).

Theorem 15
For any nonzero rational number \( a \) and any integers \( m \) and \( n \),

\[
(a^m)^n = a^{mn}.
\]

Using the definitions and theorems developed, we derive additional properties. For example:

\[
\left(\frac{2}{3}\right)^4 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{2 \cdot 2 \cdot 2 \cdot 2}{3 \cdot 3 \cdot 3 \cdot 3} = \frac{2^4}{3^4}
\]

This property is generalized as follows.

Theorem 16
For any nonzero rational number \( \frac{a}{b} \) and any integer \( m \),

\[
\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.
\]

From the definition of negative exponents, the preceding theorem, and division of fractions, we have

\[
\left(\frac{a}{b}\right)^m = \frac{1}{\left(\frac{a}{b}\right)^m} = \frac{1}{\frac{a^m}{b^m}} = \frac{b^m}{a^m} = \left(\frac{b}{a}\right)^m.
\]

Theorem 17
For any nonzero rational number \( \frac{a}{b} \) and any integer \( m \),

\[
\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m.
\]
Rational Numbers and Proportional Reasoning

A property similar to the one in Theorem 16 holds for multiplication. For example,

\[(2 \cdot 3)^{-3} = \frac{1}{(2 \cdot 3)^3} = \frac{1}{2^3 \cdot 3^3} = \left(\frac{1}{2^3}\right) \cdot \left(\frac{1}{3^3}\right) = 2^{-3} \cdot 3^{-3}.\]

Theorem 18 considers the distributive property of exponentiation over multiplication.

**Theorem 18**

For any nonzero rational numbers \(a\) and \(b\) and any integer \(m\),

\[(a \cdot b)^m = a^m \cdot b^m.\]

Theorem 18 is also true when \(a\) or \(b\) = 0 and \(m > 0\).

The properties of exponents are summarized below.

**Theorem 19: Properties of Exponents**

For any nonzero rational numbers \(a\) and \(b\) and integers \(m\) and \(n\), the following are true.

<table>
<thead>
<tr>
<th>a. (a^0 = 1)</th>
<th>e. ((a^m)^n = a^{mn})</th>
</tr>
</thead>
<tbody>
<tr>
<td>b. (a^{-m} = \frac{1}{a^m})</td>
<td>f. ((\frac{a}{b})^m = \frac{a^m}{b^m})</td>
</tr>
<tr>
<td>c. (a^m \cdot a^n = a^{m+n})</td>
<td>g. ((\frac{a}{b})^{-m} = \left(\frac{b}{a}\right)^m)</td>
</tr>
<tr>
<td>d. (\frac{a^m}{a^n} = a^{m-n})</td>
<td>h. ((ab)^m = a^m b^m)</td>
</tr>
</tbody>
</table>

Notice that property (h) is for multiplication. Analogous properties do not hold for addition and subtraction. For example, in general, \((a + b)^1 \neq a^{-1} + b^{-1}\). To see why, a numerical example is sufficient, but it is instructive to write each side with positive exponents:

\[(a + b)^1 = \frac{1}{a + b} \]

\[a^{-1} + b^{-1} = \frac{1}{a} + \frac{1}{b}\]

We know from addition of fractions, \(\frac{1}{a + b} \neq \frac{1}{a} + \frac{1}{b}\). Therefore, \((a + b)^1 \neq a^{-1} + b^{-1}\).

**Example 23**

In each of the following statements, show each equality or inequality is true in general for non-zero rational numbers \(x\), \(a\), and \(b\).

<table>
<thead>
<tr>
<th>a. ((-x)^2 \neq -x^2)</th>
<th>e. ((a^2 + b^2)^1 \neq a^2 + b^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>b. ((-x)^3 = -x^3)</td>
<td></td>
</tr>
<tr>
<td>c. (ab^{-1} \neq (ab)^{-1}) unless (a = 1) or (a = -1)</td>
<td></td>
</tr>
<tr>
<td>d. ((a^2 b^2)^1 = a^2 b^2)</td>
<td></td>
</tr>
</tbody>
</table>

Solution

a. \((-x)^2 = \frac{1}{(x)^2} = \frac{1}{x^2}\)

\(-x^2 = -(x^2) = -\left( \frac{1}{x^2} \right) = \frac{-1}{x^2}\)

Hence, \((-x)^2 \neq -x^2\).

b. \((-x)^3 = \frac{1}{(-x)^3} = \frac{1}{x^3}\)

\(-x^3 = -(x^3) = -\left( \frac{1}{x^3} \right)\)

Hence, \((-x)^3 = -x^3\).

c. \(ab^{-1} = a\left( \frac{1}{b} \right) = \frac{a}{b}\) but \((ab)^{-1} = \frac{1}{ab}\). Hence, \(ab^{-1} \neq (ab)^{-1}\).

d. \((a^{-2}b^{-2})^{-1} = (a^{-2})^{-1}(b^{-2})^{-1} = a^{2}\left( \frac{1}{b^{2}} \right) = a^2b^2\)

e. \((a^2 + b^2)^{-1} = \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^{-1} = \left( \frac{a^2 + b^2}{a^2 \cdot b^2} \right)^{-1} = \frac{a^2 b^2}{a^2 + b^2} \neq a^2 + b^2\)

Observe that all the properties of exponents refer to powers with either the same base or the same exponent. To evaluate expressions using exponents where different bases and powers are used, perform all the computations or rewrite the expressions in either the same base or the same exponent, if possible. For example, \(\frac{27^4}{81^4}\) can be rewritten \(\frac{27^4}{81^4} = \left( \frac{3}{3} \right)^4 = \frac{3^{12}}{3^{12}} = 1\).

Example 24

Perform the following computations and express your answers without negative exponents.

a. \(16^2 \cdot 8^{-3}\)

b. \(20^2 \div 2^4\)

c. \((10^1 + 5 \cdot 10^{-2} + 3 \cdot 10^{-3})10^4\)

d. \((x^3y^{-2})^4\)

Solution

a. \(16^2 \cdot 8^{-3} = (2^4)^2 \cdot (2^3)^{-3} = 2^{8} \cdot 2^{-9} = 2^{8+(-9)} = 2^{-1} = \frac{1}{2}\)

b. \(\frac{20^2}{2^4} = \frac{(2^2)^2 \cdot 5^2}{2^4} = \frac{2^4 \cdot 5^2}{2^4} = 5^2\) or 25

c. \((10^1 + 5 \cdot 10^{-2} + 3 \cdot 10^{-3})10^4 = 10^1 \cdot 10^4 + 5 \cdot 10^2 \cdot 10^4 + 3 \cdot 10^{-3} \cdot 10^4\)
\(= 10^{1+4} + 5 \cdot 10^{2+4} + 3 \cdot 10^{3+4}\)
\(= 10^5 + 5 \cdot 10^6 + 3 \cdot 10^9\)
\(= 153\)

d. \((x^3y^{-2})^4 = (x^3)^4 \cdot (y^{-2})^4 = x^{3(4)} \cdot y^{-2(4)} = x^{12}y^{-8} = \left( \frac{1}{x^{12}} \right)y^8 = \frac{y^8}{x^{12}}\) if \(x, y \neq 0\)
Assessment 3A

1. In the following figures, a unit rectangle is used to illustrate the product of two fractions. Name the fractions and their products.
   a.  
   b.  

2. Use a rectangular region to illustrate each of the following products.
   a. \( \frac{1}{3} \times \frac{2}{4} \)
   b. \( \frac{2}{3} \times \frac{1}{4} \)

3. Find each of the following products. Write your answers in simplest form.
   a. \( \frac{49}{65} \times \frac{26}{98} \)
   b. \( \frac{a}{b} \times \frac{b}{d} \)
   c. \( \frac{xy}{z} \times \frac{z^2}{x^3 y^2} \)

4. Use the distributive property of multiplication over addition to find each product.
   a. \( 4 \frac{1}{2} \times 2 \frac{1}{3} \)
   b. \( 3 \frac{1}{2} \times 2 \frac{1}{2} \)

5. Find the multiplicative inverse of each of the following.
   a. \( -\frac{1}{3} \)
   b. \( 3 \frac{1}{5} \)
   c. \( \frac{x}{y}, \text{if } x \neq 0 \)
   d. \( -7 \)

6. Solve for \( x \).
   a. \( \frac{2}{3} x = \frac{7}{6} \)
   b. \( \frac{3}{4} + x = \frac{1}{2} \)
   c. \( \frac{5}{6} + \frac{2}{3} x = \frac{3}{4} \)
   d. \( \frac{2x}{3} - \frac{1}{4} = x + \frac{1}{6} + \frac{1}{2} \)

7. Show that the following properties do not hold for the division of rational numbers.
   a. Commutative
   b. Associative

8. Compute the following mentally. Find the exact answers.
   a. \( 3 \frac{1}{4} \times 8 \)
   b. \( 7 \frac{1}{4} \times 4 \)

9. Choose from among the numbers in parentheses the number that best approximates each of the following.
   a. \( 3 \frac{11}{12} \times 5 \frac{3}{100} \) (8, 20, 15, 16)
   b. \( 2 \frac{1}{10} \times 7 \frac{7}{8} \) (16, 14, 4, 3)
   c. \( \frac{1}{101} \div \frac{1}{103} \) (0, 1, \( \frac{2}{1} \), \( \frac{3}{4} \))

10. Without actually doing the computations, choose the phrase in parentheses that correctly describes each.
    a. \( \frac{13}{17} \) (greater than 1, less than 1)
    b. \( \frac{3}{7} \div \frac{5}{9} \) (greater than 1, less than 1)
    c. \( \frac{4}{3} \div \frac{2}{3} \) (greater than 2, less than 2)

11. A sewing project requires \( 6 \frac{1}{8} \) yd of material that sells for $4 per yard and \( 3 \frac{1}{4} \) yd of material that sells for $3 per yard. Choose the best estimate for the cost of the project:
    a. Between $30 and $40
    b. Between $20 and $34
    c. Between $36 and $40
    d. Between $33 and $40

12. Five-eighths of the students at Salem State College live in dormitories. If 6000 students at the college live in dormitories, how many students are there in the college?

13. Alberto owns \( \frac{5}{9} \) of the stock in the N.W. Tofu Company. His sister Renatta owns half as much stock as Alberto. What part of the stock is owned by neither Alberto nor Renatta?

14. A suit is on sale for $180.00. What was the original price of the suit if the discount was \( \frac{1}{4} \) of the original price?

15. John took all his money out of his savings account. He spent $50.00 on a radio and \( \frac{3}{5} \) of what remained on presents. Half of what was left he put in his checking account, and the remaining $35.00 he donated to charity. How much money did John originally have in his savings account?
16. Al gives \( \frac{1}{2} \) of his marbles to Bev. Bev gives \( \frac{1}{2} \) of these to Carl.
Carl gives \( \frac{1}{2} \) of these to Dani. If Dani was given four marbles, how many did Al have originally?

17. Write each of the following in simplest form using positive exponents in the final answer.
   a. \( 3^2 \cdot 3^6 \)
   b. \( 3^2 \cdot 3^6 \)
   c. \( 5^{15} \cdot 5^4 \)
   d. \( 5^{15} \cdot 5^4 \)
   e. \( (\cdot 5)^2 \)
   f. \( \frac{a^4}{a^3} \)

18. Write each of the following in simplest form using positive exponents in the final answer.
   a. \( \left( \frac{1}{2} \right)^3 \cdot \left( \frac{1}{2} \right)^7 \)
   b. \( \left( \frac{1}{2} \right)^9 + \left( \frac{1}{2} \right)^6 \)
   c. \( \left( \frac{2}{3} \right)^5 \cdot \left( \frac{4}{5} \right)^2 \)
   d. \( \left( \frac{3}{5} \right)^7 + \left( \frac{3}{5} \right)^7 \)

19. If \( a \) and \( b \) are rational numbers, with \( a \neq 0 \) and \( b \neq 0 \), and if \( m \) and \( n \) are integers, which of the following statements are always true? Justify your answers.
   a. \( a^m \cdot b^n = (a^m \cdot b^n) \)
   b. \( a^m \cdot b^n = (a^m)^n \)
   c. \( a^m \cdot b^n = (a)^{m+n} \)
   d. \( (ab)^0 = 1 \)
   e. \( (a + b)^m = a^m + b^m \)
   f. \( (a + b)^{-m} = \frac{1}{a^m} + \frac{1}{b^m} \)

20. Solve for the integer \( n \) in each of the following.
   a. \( 2^n = 32 \)
   b. \( n^2 = 36 \)
   c. \( 2^n \cdot 2^7 = 2^9 \)
   d. \( 2^5 \cdot 2^7 = 8 \)

21. Solve each of the following inequalities for \( x \), where \( x \) is an integer.
   a. \( 3^x \leq 9 \)
   b. \( 25^x < 125 \)
   c. \( 3^{2x} > 27 \)
   d. \( 4^x > 1 \)

22. Determine which fraction in each of the following pairs is greater.
   a. \( \left( \frac{1}{2} \right)^3 \text{ or } \left( \frac{1}{2} \right)^4 \)
   b. \( \left( \frac{3}{4} \right)^{10} \text{ or } \left( \frac{3}{4} \right)^{10} \)
   c. \( \left( \frac{4}{3} \right)^{10} \text{ or } \left( \frac{4}{3} \right)^{10} \)
   d. \( \left( \frac{3}{4} \right)^{10} \text{ or } \left( \frac{4}{3} \right)^{10} \)
   e. \( 32^{10} \text{ or } 4^{10} \)
   f. \( (\cdot 27)^{15} \text{ or } (\cdot 3)^{75} \)

23. Show that the arithmetic mean of two rational numbers is between the two numbers; that is, prove if \( \frac{a}{b} < \frac{c}{d} \) then
   \[ \frac{a}{b} < \frac{1}{2} \left( \frac{a}{b} + \frac{c}{d} \right) < \frac{c}{d} \]

24. In the Corcoran School of Design in 2014, \( \frac{17}{25} \) of the students were male.
   a. What fraction were female?
   b. Does this imply 17 students are male?

25. The reported tax revenue in dollars for Washington DC in 2011 is shown below.

<table>
<thead>
<tr>
<th>Revenue Category</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corporate Income</td>
<td>$36,802,396</td>
</tr>
<tr>
<td>Personal Income</td>
<td>32,020,924</td>
</tr>
<tr>
<td>Social Security &amp; Other Taxes</td>
<td>620,501</td>
</tr>
<tr>
<td>Hotel Tax</td>
<td>212,565,755</td>
</tr>
<tr>
<td>Property Taxes</td>
<td>183,005,144</td>
</tr>
<tr>
<td>Excise &amp; Fees</td>
<td>21,723,515</td>
</tr>
<tr>
<td>Sales Tax</td>
<td>187,656,061</td>
</tr>
<tr>
<td>Restaurant</td>
<td>119,487,765</td>
</tr>
<tr>
<td>Retail</td>
<td>58,122,531</td>
</tr>
<tr>
<td>Airport</td>
<td>7,155,614</td>
</tr>
<tr>
<td>Car Rental</td>
<td>2,890,151</td>
</tr>
<tr>
<td>Total</td>
<td>674,394,296</td>
</tr>
</tbody>
</table>

26. According to the Washington DC City Government, the following lists the number of homicides in the city since 2004.

<table>
<thead>
<tr>
<th>Year</th>
<th>Homicides</th>
</tr>
</thead>
<tbody>
<tr>
<td>2014</td>
<td>76</td>
</tr>
<tr>
<td>2013</td>
<td>49</td>
</tr>
<tr>
<td>2012</td>
<td>88</td>
</tr>
<tr>
<td>2011</td>
<td>108</td>
</tr>
<tr>
<td>2010</td>
<td>132</td>
</tr>
<tr>
<td>2009</td>
<td>144</td>
</tr>
<tr>
<td>2008</td>
<td>186</td>
</tr>
<tr>
<td>2007</td>
<td>181</td>
</tr>
<tr>
<td>2006</td>
<td>169</td>
</tr>
<tr>
<td>2005</td>
<td>196</td>
</tr>
<tr>
<td>2004</td>
<td>198</td>
</tr>
</tbody>
</table>

a. What year showed the greatest fractional increase from the previous year?
b. What year showed the greatest fractional decrease from the previous year?

27. If \( \frac{1}{33} \) of all deer in the United States are in Mississippi and \( \frac{2}{7} \) of the deer are in Montana, what total fractional part of the deer population is in those two states?

28. If \( \frac{1}{4} \) of an estate is to be distributed equally to 8 cousins, what fractional part of the entire estate does each cousin receive?
1. In the following figure a unit rectangle is used to illustrate the product of two fractions. Name the fractions and their products.
   a. b. 

2. Use a rectangular region to illustrate each of the following products.
   a. \( \frac{2}{5} \cdot \frac{1}{3} \)  
   b. \( \frac{2}{7} \cdot \frac{2}{3} \)

3. Find each of the following products of rational numbers. Write your answers in simplest form.
   a. \( \frac{2}{3} \cdot \frac{3}{4} \)  
   b. \( \frac{22}{7} \cdot \frac{2}{3} \)  
   c. \( \frac{-5}{2} \cdot \frac{1}{2} \)  
   d. \( \frac{3}{4} \cdot \frac{2}{1} \)  
   e. \( \frac{a^2 \cdot b^2}{b^3} \)  
   f. \( \frac{x^3 \cdot y}{z} \cdot \frac{z}{x^3 y} \)

4. Use the distributive property to find each product of rational numbers.
   a. \( \frac{2}{5} \cdot \frac{3}{5} \)  
   b. \( \left( \frac{x}{y} + 1 \right) \left( \frac{y}{x} - 1 \right) \)
   c. \( 248 \cdot \frac{2}{5} \cdot 100 \cdot \frac{1}{8} \)

5. Find the multiplicative inverse of each of the following.
   a. \( \frac{6}{7} \)  
   b. \( 8 \)  
   c. \( 4 \cdot \frac{1}{5} \)  
   d. \( -1 \cdot \frac{1}{2} \)

6. Solve for \( x \).
   a. \( \frac{2}{3} = \frac{11}{6} \)  
   b. \( \frac{3}{4} + x = \frac{1}{3} \)  
   c. \( \frac{5}{6} - \frac{2}{3} = \frac{3}{4} \)  
   d. \( \frac{2x}{3} + \frac{1}{4} = \frac{x}{6} - \frac{1}{2} \)

7. Find a fraction such that if you add the denominator to the numerator and place the sum over the original denominator, the new fraction has triple the value of the original fraction.

8. Compute the following mentally; find the exact answers.
   a. \( 3 \cdot \frac{1}{2} \)  
   b. \( 7 \cdot \frac{3}{4} \)  
   c. \( 9 \cdot \frac{1}{5} \)  
   d. \( 8 \cdot \frac{2}{3} \)  
   e. \( 3 \div \frac{1}{2} \)  
   f. \( 3 \div \frac{1}{2} \)  
   g. \( 3 \div \frac{1}{5} \)  
   h. \( \frac{1}{2} \div 2 \)

9. Choose from among the numbers in parentheses the number that best approximates each of the following.
   a. \( 20 \frac{2}{3} \cdot \frac{9}{7} \)  
   b. \( 21, 24, \frac{1}{2}, 10 \)
   c. \( \frac{1}{10} \cdot \frac{1}{100} \)  
   d. \( 1, 1001, 0 \)

10. Without actually doing the computations, choose the phrase in parentheses that correctly describes each.
   a. \( 4 \frac{1}{3} \div 2 \frac{13}{100} \) (greater than 2, less than 2)
   b. \( 16 \div 4 \frac{3}{18} \) (greater than 4, less than 4)
   c. \( 16 \div 3 \frac{8}{9} \) (greater than 4, less than 4)

11. When you multiply a certain number by 3 and then subtract \( \frac{7}{18} \), you get the same result as when you multiply the number by 2 and add \( \frac{5}{12} \). What is the number?

12. Di Paloma University had a faculty reduction and lost \( \frac{1}{5} \) of its faculty. If 320 faculty members were left after the reduction, how many members were there originally?

13. A person has 29 \( \frac{1}{2} \) yd of material available to make doll outfits. Each outfit requires \( \frac{3}{4} \) yd of material.
   a. How many outfits can be made?
   b. How much material will be left over?

14. Every employee’s salary at the Sunrise Software Company increases each year by \( \frac{1}{10} \) of that person’s salary the previous year.
   a. If Martha’s present annual salary is $100,000, what will her salary be in 2 yr?
   b. If Aaron’s present salary is $99,000, what was his salary 1 yr ago?
   c. If Juanita’s present salary is $363,000, what was her salary 2 yr ago?

15. Jasmine is reading a book. She has finished \( \frac{3}{5} \) of the book and has 82 pages left to read. How many pages has she read?

16. Peter, Paul, and Mary start at the same time walking around a circular track in the same direction. Peter takes \( \frac{1}{2} \) hr to walk around the track. Paul takes \( \frac{5}{12} \) hr, and Mary takes \( \frac{1}{3} \) hr.
   a. How many minutes does it take each person to walk around the track?
   b. How many times will each person go around the track before all three meet again at the starting line?
17. Write each of the following rational numbers in simplest form using positive exponents in the final answer.
   a. \( \left( \frac{1}{3} \right)^{-1} \)
   b. \( \frac{a^3}{a} \)
   c. \( \frac{(a^4)^3}{a^3} \)
   d. \( \frac{a}{a^3} \)
   e. \( \frac{a^3}{a^2} \)

18. Write each of the following in simplest form using positive exponents in the final answer.
   a. \( \left( \frac{1}{2} \right)^{10} + \left( \frac{3}{2} \right)^{-1} \)
   b. \( \frac{2}{3} \cdot \left( \frac{4}{5} \right)^2 \)
   c. \( \left( \frac{3}{5} \right)^7 + \left( \frac{5}{3} \right)^4 \)
   d. \( \left[ \left( \frac{5}{6} \right)^3 \right]^{-1} \)

19. If \( a \) and \( b \) are rational numbers, with \( a \neq 0 \) and \( b \neq 0 \), and if \( m \) and \( n \) are integers, which of the following statements are always true? Justify your answers.
   a. \( \frac{a^m}{b^n} = \left( \frac{a}{b} \right)^{m-n} \)
   b. \( (ab)^m = \frac{1}{a^m} \cdot \frac{1}{b^m} \)
   c. \( \left( \frac{2}{a^1 + b^1} \right)^{-1} = \frac{1}{2} \cdot \frac{1}{a + b} \)
   d. \( 2(a^{-1} + b^{-1})^{-1} = \frac{2ab}{a + b} \)
   e. \( a^{mn} = a^m \cdot a^n \)
   f. \( \left( \frac{a}{b} \right)^{-1} = \frac{b}{a} \)

20. Solve, if possible, for \( n \) where \( n \) is an integer in each of the following.
   a. \( 2^n = 32 \)
   b. \( n^3 = \frac{-1}{27} \)
   c. \( 2^n \cdot 2^7 = 1024 \)
   d. \( 2^n \cdot 7^2 = 64 \)
   e. \( (2 + n)^3 = 2^4 + n^2 \)
   f. \( 3^n = 27 \)

21. Solve each of the following inequalities for \( x \), where \( x \) is an integer.
   a. \( 3^x \geq 81 \)
   b. \( 4^x \geq 8 \)
   c. \( 3^{2x} \leq 27 \)
   d. \( 2^x < 1 \)

22. Determine which fraction in each of the following pairs is greater.
   a. \( \left( \frac{4}{3} \right)^{10} \) or \( \left( \frac{4}{3} \right)^{8} \)
   b. \( \left( \frac{3}{4} \right)^{10} \) or \( \left( \frac{4}{5} \right)^{10} \)
   c. \( \left( \frac{4}{3} \right)^{10} \) or \( \left( \frac{5}{4} \right)^{10} \)
   d. \( \left( \frac{3}{4} \right)^{100} \) or \( \left( \frac{3}{4} \right)^9 \)

23. In the following, determine which number is greater.
   a. \( 3^{2^{100}} \) or \( 4^{2^{100}} \)
   b. \( (-27)^{15} \) or \( (-3)^{50} \)

24. Brandy bought a horse for $270 and immediately started paying for his keep. She sold the horse for $540. Considering the cost of his keep she found that she had lost an amount equal to half of what she paid for the horse plus one-fourth of the cost of his keep. How much did Brandy lose on the horse?

25. In 2014 in the Corcoran School of Design, \( \frac{8}{25} \) of the students were female. What fraction was male?

26. The reported tax revenue in dollars for Washington DC in 2011 is shown below.

<table>
<thead>
<tr>
<th>Tax Type</th>
<th>Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corporate Income</td>
<td>$36,802,396</td>
</tr>
<tr>
<td>Personal Income</td>
<td>32,020,924</td>
</tr>
<tr>
<td>Social Security &amp; Other Taxes</td>
<td>620,501</td>
</tr>
<tr>
<td>Hotel Tax</td>
<td>212,565,755</td>
</tr>
<tr>
<td>Property Taxes</td>
<td>183,005,144</td>
</tr>
<tr>
<td>Excise &amp; Fees</td>
<td>21,723,515</td>
</tr>
<tr>
<td>Sales Tax</td>
<td>187,656,061</td>
</tr>
<tr>
<td>Restaurant</td>
<td>119,487,765</td>
</tr>
<tr>
<td>Retail</td>
<td>58,122,531</td>
</tr>
<tr>
<td>Airport</td>
<td>7,155,614</td>
</tr>
<tr>
<td>Car Rental</td>
<td>2,890,151</td>
</tr>
</tbody>
</table>

   **Total:** 674,394,296

   a. Approximately what is the fractional part of the total tax is the difference in hotel tax and property tax?
   b. If the sales tax was increased \( \frac{11}{11} \) approximately what fraction of the total tax package would be gained?

27. According to the Washington DC City Government, the following lists the number of homicides in the city since 2004.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2004</td>
<td>123</td>
<td>105</td>
<td>124</td>
<td>132</td>
<td>144</td>
<td>156</td>
<td>169</td>
<td>196</td>
<td>198</td>
</tr>
<tr>
<td>2011</td>
<td>132</td>
<td>124</td>
<td>132</td>
<td>144</td>
<td>156</td>
<td>169</td>
<td>196</td>
<td>198</td>
<td>198</td>
</tr>
</tbody>
</table>

a. What is the fractional decrease in homicides from 2005 to 2012?
b. What is the fractional increase in homicides from 2012 to 2013?

28. If \( \frac{1}{3} \) of all deer in the United States are in Mississippi, and \( \frac{2}{7} \) of the deer are in Montana, what fractional part of the deer population is not in those two states?

29. If \( \frac{1}{4} \) of an estate consists of \( \frac{2}{3} \) stocks and \( \frac{3}{5} \) bonds, what part of the estate is each of the stocks and bonds?
Mathematical Connections 3

Reasoning
1. Suppose you divide a natural number $n$ by a positive rational number less than 1. Will the answer always be less than $n$, sometimes less than $n$, or never less than $n$? Why?
2. If the fractions represented by points $C$ and $D$ on the following number line are multiplied, what point best represents the product? Explain why.

3. If the product of two numbers is 1 and one of the numbers is greater than 1, what do you know about the other number? Explain your answer.

Open-Ended
4. Write a story or model for $\frac{3}{4} + \frac{1}{2}$.
5. Complete a survey of your class asking questions such as their favorite color, favorite type of shoe, and favorite math concept. Use fractions to summarize your information and find any differences in male and female preference.
6. Consider the demographics of students in each undergraduate class (freshman, sophomore, junior, and senior) at your university. Describe the fractional differences in males and females in each class.

Cooperative Learning
7. Choose a brick building on your campus. Measure the height of one brick and the thickness of mortar between bricks. Estimate the height of the building and then calculate the height of the building. Were rational numbers used in your computations?
8. Have each student in your group choose a state or city and find the demographics on taxes as in Exercise 26 in Assessment 3B. Compare fractions representing sales taxes for each chosen location. Which has the highest? Lowest?
9. In each group of a class, describe your group using fractions in the explanation. Use at least three sets of fractions in your description. Pass the descriptions to the teacher who will then distribute the descriptions to a different group. The goal is to describe your group well enough that others can identify the groups.

Connecting Mathematics to the Classroom
10. Bente says to do the problem $12 \frac{1}{4} \div 3 \frac{3}{4}$ you just find $12 \div 3 = 4$ and $\frac{1}{4} + \frac{3}{4} = \frac{1}{2} = 4 \frac{1}{2}$. How do you respond?
11. Amy says that dividing a number by $\frac{1}{2}$ is the same as taking half of a number. How do you respond?
12. Dani says that if we have $\frac{3}{5} \div \frac{2}{5}$, we could just multiply $\frac{3}{5} \cdot \frac{5}{2} = \frac{3 \cdot 1}{5} \cdot \frac{2}{10}$ Is she correct? Explain why.

13. Noah says that dividing a number by 2 is the same as multiplying it by $\frac{1}{2}$. He wants to know if he is right, and if so, why. How do you respond?
14. Jim is not sure when to use multiplication by a fraction and when to use division. He has the following list of problems. How would you help him solve these problems in a way that would enable him to solve similar problems on his own?
   a. If $\frac{3}{4}$ of a package of sugar fills $\frac{1}{2}$ c. How many cups of sugar are in a full package of sugar?
   b. How many packages of sugar are needed to fill 2 c?
   c. If $\frac{1}{3}$ c sugar is required to make two loaves of challah, how many cups of sugar are needed for three loaves?
   d. If $\frac{3}{4}$ c sugar is required for 1 gal of punch, how many gallons can be made with 2 c of sugar?
   e. If you have $22 \frac{3}{8}$ in. of ribbon, and need $1 \frac{1}{4}$ in. to decorate one doll, how many dolls can be decorated, and how much ribbon will be left over?
15. A student claims that division always makes things smaller so $5 \div \left(\frac{1}{2}\right)$ cannot be 10 because 10 is greater than the number 5 she started with. How do you respond?
16. A student simplified the fraction $\frac{m + n}{p + n} \div \frac{m}{p}$. How would you help this student?
17. Jillian says she learned that 17 divided by 5 can be written as $17 \div 5 = 3 \frac{2}{3}$, but she thinks that writing $17 \div 5 = \frac{17}{5} = 3 \frac{2}{5}$ is much better. How do you respond?

Review Problems
18. Perform each of the following computations. Leave your answers in simplest form or as mixed numbers.
   a. $\frac{-3}{16} + \frac{7}{4}$
   b. $\frac{1}{6} + \frac{4}{9} + \frac{5}{3}$
   c. $\frac{5}{2^3} \cdot \frac{5}{3^2}$
   d. $\frac{3}{5} + \frac{4}{6}$
   e. $\frac{1}{6} - \frac{5}{8}$
   f. $\frac{1}{3} - \frac{5}{12}$
19. Each student at Sussex Elementary School takes one foreign language. Two-thirds of the students take Spanish, $\frac{1}{9}$ take French, $\frac{1}{18}$ take German, and the rest take some other foreign language. If there are 720 students in the school, how many do not take Spanish, French, or German?
20. Find each sum or difference; simplify if possible.
   a. $\frac{3x}{xy^2} + \frac{y}{x^2}$
   b. $\frac{a}{x^2} - \frac{b}{x^2}$
   c. $\frac{a^2}{a^2 - b^2} - \frac{a - b}{a + b}$
21. Determine which of the following is always correct.
   a. \( \frac{ab + c}{b} = a + c \)
   b. \( \frac{a + b}{a + c} = \frac{b}{c} \)
   c. \( \frac{ab + ac}{ac} = \frac{b + c}{c} \)

National Assessments

National Assessment of Educational Progress (NAEP) Questions

Both figures below show the same scale. The marks on the scale have no labels except the zero point.

The weight of the cheese is \( \frac{1}{2} \) pound. What is the total weight of the two apples?

NAEP, Grade 8, 2007

Jim has \( \frac{3}{4} \) of a yard of string which he wishes to divide into pieces each \( \frac{1}{8} \) of a yard long. How many pieces will he have?

A. 3  
B. 4  
C. 6  
D. 8

NAEP, Grade 8, 2003

Nick has a whole pizza.

Nick says he will eat \( \frac{1}{2} \) of the pizza.

He says he will give \( \frac{3}{8} \) of the pizza to Sam and \( \frac{3}{8} \) of the pizza to Joe.

Can Nick do what he says?

☐ Yes  ☐ No

NAEP, Grade 4, 2013

Proportional Reasoning

Proportional reasoning is an extremely important concept taught in grades K–8. Proportionality has connections to most, if not all, of the other foundational middle-school topics and can provide a context to study these topics.

For grade 7, the Common Core Standards state that students should “analyze proportional relationships and use them to solve real-world and mathematical problems.” Additionally we find that students should “decide whether two quantities are in a proportional relationship,” “identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams and verbal descriptions,” represent proportional relationships in equations,” and “use proportional relationships to solve multistep ratio and percent problems.” (p. 48)

Ratios are encountered in everyday life. For example, there may be a 2-to-3 ratio of Democrats to Republicans on a certain legislative committee, a friend may be given a speeding ticket for driving 69 miles per hour, or eggs may cost $2.40 a dozen. Each of these illustrates a ratio.

Definition of Ratio

A ratio, denoted as \( \frac{a}{b} \), \( a:b \), or \( a \div b \), where \( a \) and \( b \) are rational numbers, is a comparison of two quantities.
Rational Numbers and Proportional Reasoning

A ratio of 1:3 for boys to girls in a class means that the number of boys is $\frac{1}{3}$ that of girls; that is, there is 1 boy for every 3 girls. We could also say that the ratio of girls to boys is 3:1, or that there are 3 times as many girls as boys. Ratios can represent part-to-whole or whole-to-part comparisons. For example, if the ratio of boys to girls in a class is 1:3, then the ratio of boys (part) to children (whole) is 1:4. If there are $b$ boys and $g$ girls, then $\frac{b}{g} = \frac{1}{3}$ and $g = 3b$. Also, the ratio of boys to the entire class is $\frac{b}{b + g} = \frac{b}{b + 3b} = \frac{b}{4b} = \frac{1}{4}$. We could also say that the ratio of all children (whole) to boys (part) is 4:1. Some ratios give part-to-part comparisons, such as the ratio of the number of boys to girls or the number of students to one teacher. For example, a school might say that the average ratio of students to teachers cannot exceed 24:1.

The ratio of 1:3 for boys to girls in a class does not tell us how many boys and how many girls there are in the class. It only tells us the relative size of the groups. There could be 2 boys and 6 girls, or 3 boys and 9 girls, or 4 boys and 12 girls, or some other numbers that give a ratio equal to $\frac{1}{3}$.

There were 7 males and 12 females in the Dew Drop Inn on Monday evening. In the game room next door were 14 males and 24 females.

a. Express the number of males to females at the inn as a ratio (part-to-part).

b. Express the number of males to females at the game room as a ratio (part-to-part).

c. Express the number of males in the game room to the number of people in the game room as a ratio (part-to-whole).

Solution

a. The ratio is $\frac{7}{12}$.

b. The ratio is $\frac{14}{24} = \frac{7}{12}$.

c. The ratio is $\frac{14}{38} = \frac{7}{19}$.

Example 25

Proportions

In a study, children were shown a picture of a carton of orange juice and were told that the orange juice was made from orange concentrate and water. Then they were shown two glasses—a large glass and a small glass—and they were told that both glasses were filled with orange juice from the carton. They were then asked if the orange juice from each of the two glasses would taste equally “orangey” or if one would taste more “orangey.” About half of the students said they were not equally orangey. Of those about half the students said the larger glass would be more “orangey” and about half said the smaller glass would be more “orangey.” These students may have been thinking of only one quantity—the water alone or the orange concentrate alone.

Suppose Recipe A for an orange drink calls for 2 cans of orange concentrate for every 3 cans of water. We could say that the ratio of cans of orange concentrate to cans of water is 2:3.
Rational Numbers and Proportional Reasoning

We represent this pictorially in Figure 22(a), where O represents a can of orange concentrate and W represents a can of water. In Figure 22(b) and (c), we continue the process of adding 2 cans of orange concentrate for every 3 cans of water.

\[
\begin{array}{ccccccc}
\text{Recipe A} \\
O & O & O & O & O & O & O \\
W & W & W & W & W & W & W \\
(a) & (b) & (c)
\end{array}
\]

Figure 22

From Figure 22 we could develop and continue the ratio table, as shown in Table 3.

<table>
<thead>
<tr>
<th>Cans of Orange Concentrate</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cans of Water</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In Table 3, the ratios \(\frac{2}{3}\) and \(\frac{4}{6}\) are equal. The equation \(\frac{2}{3} = \frac{4}{6}\) is a proportion. In general, we have the following definition.

**Definition of a Proportion**

A proportion is a statement that two given ratios are equal.

If Recipe B calls for 4 cans of orange concentrate for every 8 cans of water, then the ratio of cans of orange concentrate to cans of water for this recipe is 4:8. We picture this in Figure 23(a).

\[
\begin{array}{ccccccc}
\text{Recipe B} \\
O & O & O & O & O & O & O \\
W & W & W & W & W & W & W \\
W & W & W & W & W & W & W \\
(a) & (b)
\end{array}
\]

Figure 23

Which of the two recipes produces a drink that tastes more “orangey”? In Figure 22(a), we see that in Recipe A there are 2 cans of orange concentrate for every 3 cans of water. In Figure 23(a), we see that in Recipe B there are 4 cans of orange concentrate for every 8 cans of water.

To compare the two recipes, we need either the same number of cans of orange concentrate or the same number of cans of water. Either is possible. Figure 22(b) shows that for Recipe A there are 4 cans of orange concentrate for every 6 cans of water. In Recipe B, for 4 cans of orange concentrate there are 8 cans of water. Recipe B calls for more water per 4 cans of orange concentrate.
concentrate, so it is less “orangey.” An alternative is to observe that in Figure 23(b), Recipe B shows that there are 2 cans of orange concentrate for every 4 cans of water. We compare this with Figure 22(a), showing 2 cans of orange concentrate for every 3 cans of water, and reach the same conclusion.

From our work in Section 1, we know that \( \frac{2}{3} = \frac{4}{6} \) because \( 2 \cdot 6 = 3 \cdot 4 \). Hence \( \frac{2}{3} = \frac{4}{6} \) is a proportion. Also \( \frac{2}{3} \neq \frac{4}{8} \) because \( 2 \cdot 8 \neq 3 \cdot 4 \); this is not a proportion. In general, we have the following theorem that follows from Theorem 2 developed in Section 1.

**Theorem 20**

If \( a, b, c, \) and \( d \) are rational numbers and \( b \neq 0 \) and \( d \neq 0 \), then

\[
\frac{a}{b} = \frac{c}{d}
\]

is a proportion if, and only if, \( ad = bc \).

The proportion \( \frac{a}{b} = \frac{c}{d} \) may be read as “\( a \) is to \( b \) as \( c \) is to \( d \).”

Students in the lower grades typically experience problems that are *additive*. Consider the problem below.

**Allie and Bente type at the same speed. Allie started typing first. When Allie had typed 8 pages, Bente had typed 4 pages. When Bente has typed 10 pages, how many has Allie typed?**

This is an example of an *additive* relationship. Students should reason that since the two people type at the same speed, when Bente has typed an additional 6 pages, Allie should have also typed an additional 6 pages, so she should have typed \( 8 + 6 \), or 14, pages.

Next consider the following problem:

**Carl can type 8 pages for every 4 pages that Dan can type. If Dan has typed 12 pages, how many pages has Carl typed?**

If students try an *additive* approach, they will conclude that since Dan has typed 8 more pages than in the original relationship, then Carl should have typed an additional 8 pages for a total of 16 pages. However, the correct reasoning is that since Carl types twice as fast as Dan he will type twice as many pages as Dan. Therefore, when Dan has typed 12 pages, Carl has typed 24 pages. The relationship between the ratios is *multiplicative*. Another way to solve this problem is to set up the proportion \( \frac{8}{4} = \frac{x}{12} \), where \( x \) is the number of pages that Carl will type, and solve for \( x \).

Because \( \frac{8}{4} = \frac{8 \cdot 3}{4 \cdot 3} = \frac{24}{12} \), then \( x = 24 \) pages.

In the problem above, one term in the proportion is missing:

\[
\frac{8}{4} = \frac{x}{12}
\]

One way to solve the equation is to multiply both sides by 12, as follows:

\[
\frac{8}{4} \cdot 12 = \frac{x}{12} \cdot 12
\]

\[
8 \cdot 3 = x
\]

\[
24 = x
\]
Another method of solution uses Theorem 20. This is often called the *cross-multiplication method*. The equation \( \frac{8}{4} = \frac{x}{12} \) is a proportion if, and only if,

\[
8 \cdot 12 = 4x \\
96 = 4x \\
24 = x.
\]

If there are 3 cars for every 8 students at a high school, how many cars are there for 1200 students?

**Solution**  We use the strategy of setting up a table, as shown in Table 4.

<table>
<thead>
<tr>
<th>Number of cars</th>
<th>3</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of students</td>
<td>8</td>
<td>1200</td>
</tr>
</tbody>
</table>

The ratio of cars to students is always the same:

\[
\frac{\text{Cars}}{\text{Students}} \rightarrow \frac{3}{8} = \frac{x}{1200} \\
3 \cdot 1200 = 8x \\
3600 = 8x \\
450 = x
\]

Thus, there are 450 cars.

Next consider two car rental companies where the rates for 1–4 days are given in Table 5.

<table>
<thead>
<tr>
<th></th>
<th>Ace Car Rental</th>
<th>Better Car Rental</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Days</strong></td>
<td><strong>Cost</strong></td>
<td><strong>Days</strong></td>
</tr>
<tr>
<td>1</td>
<td>$20</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$40</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>$60</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>$80</td>
<td>4</td>
</tr>
</tbody>
</table>

The first two days for Ace Car Rental rates can be used to write a proportion because \( \frac{1 \text{ day}}{\$20} = \frac{2 \text{ days}}{\$40} \). In a proportion, the units of measure must be in the same relative positions. In this case, the numbers of days are in the numerators and the costs are in the denominators.

For the Better Car Rental we see that \( \frac{1 \text{ day}}{\$20} \neq \frac{2 \text{ days}}{\$35} \), so a proportion is not formed.
Consider Table 6, which is a ratio table built from the values for Ace Car Rental.

<table>
<thead>
<tr>
<th>Days (d)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost (c)</td>
<td>20</td>
<td>40</td>
<td>60</td>
<td>80</td>
</tr>
</tbody>
</table>

The ratios \( \frac{d}{c} \) are all equal, that is, \( \frac{1}{20} = \frac{2}{40} = \frac{3}{60} = \frac{4}{80} \). Thus, each pair of ratios forms a proportion. In this case, \( \frac{d}{c} = \frac{1}{20} \) for all values of \( c \) and \( d \). This is expressed by saying that \( d \) is proportional to \( c \) or \( d \) varies proportionally to \( c \) or \( d \) varies directly with \( c \). In this case, \( d = \frac{1}{20}c \) for every \( c \) and \( d \). The number \( \frac{1}{20} \) is the constant of proportionality. We can say that gas used by a car is proportional to the miles traveled or lottery profits vary directly with the number of tickets sold.

**Definition of Constant of Proportionality**

If the variables \( x \) and \( y \) are related by the equality \( y = kx \) (or \( k = \frac{y}{x} \)), then \( y \) is proportional to \( x \) and \( k \) is the constant of proportionality between \( y \) and \( x \).

A central idea in proportional reasoning is that a relationship between two quantities is such that the ratio of one quantity to the other remains unchanged as the numerical values of both quantities change.

It is important to remember that in the ratio \( a:b \) or \( \frac{a}{b} \), \( a \) and \( b \) do not have to be whole numbers. For example, if in Eugene, Oregon, \( \frac{7}{10} \) of the population exercise regularly, then \( \frac{3}{10} \) of the population do not exercise regularly, and the ratio of those who do to those who do not is \( \frac{7}{10} : \frac{3}{10} \). This ratio can be written 7:3.

It is important to pay special attention to units of measure when working with proportions. For example, if a turtle travels 5 in. every 10 sec, how many feet does it travel in 50 sec? If units of measure are ignored, we might set up the following proportion:

\[
\frac{5}{10} = \frac{x}{50}
\]

In this proportion the units of measure are not listed. A more informative proportion that often prevents errors is the following:

\[
\frac{5 \text{ in.}}{10 \text{ sec}} = \frac{x \text{ in.}}{50 \text{ sec}}
\]

This implies that \( x = 25 \). Consequently, since 12 in. = 1 ft, the turtle travels \( \frac{25}{12} \) ft, or \( 2 \frac{1}{12} \) ft, or 2 ft 1 in. in 50 sec.
Another approach for solving proportions uses the **scaling strategy**. Suppose we are asked whether it is better to buy 12 tickets for $15.00 or 20 tickets for $23.00. One way to approach the problem is to find the cost of a common number of tickets from each scenario.

Because \( \text{LCM}(12, 20) = 60 \), we could choose to find the cost of 60 tickets under each plan.

In the first plan, since 12 tickets cost $15.00, then 60 tickets cost $75.00.

In the second plan, since 20 tickets cost $23.00, then 60 tickets cost $69.00.

Therefore, the second plan is a better buy.

The **unit-rate strategy** for solving this problem involves finding the cost of one ticket under each plan and then comparing unit costs.

In the first plan, since 12 tickets cost $15.00, then 1 ticket costs $1.25.

In the second plan, since 20 tickets cost $23.00, then 1 ticket costs $1.15.

The grade 6 Common Core Standards state the following:

Solve unit rate problems including those involving unit pricing and constant speed. For example, if it took 7 hours to mow 4 lawns, then at that rate how many lawns could be mowed in 35 hours? (p. 42)

**NOW TRY THIS 6**

Work the problem posed in the grade 6 Common Core Standards.

---

**Example 27**

Kai, Paulus, and Judy made $2520 for painting a house. Kai worked 30 hr, Paulus worked 50 hr, and Judy worked 60 hr. They divided the money in proportion to the number of hours worked. If they all earn the same rate of pay, how much did each earn?

**Solution**  Let \( x \) be the unit rate or the rate of pay per hour. Then \( 30x \) denotes the amount of money that Kai received; Paulus received \( 50x \) and Judy received \( 60x \). Because the total amount of money received is \( 30x + 50x + 60x \), we have

\[
30x + 50x + 60x = 2520
\]

\[
140x = 2520
\]

\[
x = 18 \text{ (dollars per hour)}.
\]

Hence,

Kai received \( 30x = 30 \cdot 18 \), or $540.

Paulus received \( 50x = 50 \cdot 18 \), or $900.

Judy received \( 60x = 60 \cdot 18 \), or $1080.

Dividing each of the amounts by 18 shows that the proportion is as required.
Rational Numbers and Proportional Reasoning

Consider the proportion \( \frac{15}{30} = \frac{3}{6} \). Because the ratios in the proportion are equal and because equal nonzero fractions have equal reciprocals, it follows that \( \frac{30}{15} = \frac{6}{3} \). Also notice that the proportions are true because each results in \( 15 \cdot 6 = 30 \cdot 3 \). In general, we have the following theorem.

**Theorem 21**

For any rational numbers \( \frac{a}{b} \) and \( \frac{c}{d} \), with \( a \neq 0 \) and \( c \neq 0 \), \( \frac{a}{b} = \frac{c}{d} \) if, and only if, \( \frac{b}{a} = \frac{d}{c} \).

Consider \( \frac{15}{30} = \frac{3}{6} \) again. Notice that \( \frac{15}{3} = \frac{30}{6} \); that is, the ratio of the numerators is equal to the ratio of the corresponding denominators. In general, we have the following theorem.

**Theorem 22**

For any rational numbers \( \frac{a}{b} \) and \( \frac{c}{d} \), with \( c \neq 0 \), \( \frac{a}{b} = \frac{c}{d} \) if, and only if, \( \frac{a}{c} = \frac{b}{d} \).

**Scale Drawings**

Ratios and proportions are used in scale drawings. For example, if the scale is 1:300, then the length of 1 cm in such a drawing represents 300 cm, or 3 m in true size. The scale is the ratio of the size of the drawing to the size of the object. The following example shows the use of scale drawings.

The floor plan of the main floor of a house in Figure 24 is drawn in the scale of 1:300. Find the dimensions in meters of the living room.

![Figure 24](image)

**Solution**  If the dimensions of the living room are approximately 3 cm by 2 cm. Because the scale is 1:300, 1 cm in the drawing represents 300 cm, or 3 m in true size. Hence, 3 cm represents \( 3 \cdot 3 = 9 \) m, and 2 cm represents \( 2 \cdot 3 = 6 \) m. Hence, the dimensions of the living room are approximately 9 m by 6 m.
The ratio of men to women at a party was 5:2 before 14 more women appeared. At that point, the ratio was 4:3. How many men and how many women were at the party?

**Solution**  We find the solution using a bar method. In Figure 25 each unit of men and women contain the same number of people giving the ratio 5:2.

![Figure 25](image)

When 14 women joined the group, we have the situation in Figure 26(a). Additionally the ratio is now 4:3, and the number of people in each original unit has changed. The number of men has not changed so 5 of the original units equal 4 of the new ones.

![Figure 26](image)

Thus, we have the following equations.

For the men: 5 original units = 4 new units.
For the women: 2 original units + 14 = 3 new units.

Adding the two together we get

7 original units + 14 = 7 new units, or
1 original unit + 2 = 1 new unit.

Equivalently, 4 original units + 8 = 4 new units.
Substituting we have, 4 old units + 8 = 5 old units, or 8 = 1 original unit.

In the Figure 25, there are 5 units of men at 8 men per unit = 40 men, and 2 units of women at 8 women per unit for 16 women. Thus, there were 56 people at the party originally. When 14 more women joined, there were 70 total.

**Assessment 4A**

1. Answer the following regarding the English alphabet.
   a. Determine the ratio of vowels to consonants.
   b. What is the ratio of consonants to vowels?
   c. What is the ratio of consonants to letters?
   d. Write a word that has a ratio of 2:3 of vowels to consonants.

2. Solve for \( x \) in each of the following proportions.
   a. \( \frac{12}{x} = \frac{18}{45} \)
   b. \( \frac{x}{7} = \frac{-10}{21} \)
   c. \( \frac{5}{7} = \frac{3x}{98} \)
   d. \( \frac{1}{2} \) is to 5 as \( x \) is to 15.
3. a. If the ratio of boys to girls in a class is 2:3, what is the ratio of boys to all the students in the class? Why?
b. If the ratio of boys to girls in a class is \( m:n \), what is the ratio of boys to all the students in the class? 
c. If \( \frac{3}{5} \) of the class are girls, what is the ratio of boys to girls?
4. There are approximately 2 lb of muscle for every 5 lb of body weight. For a 90-lb person, approximately how much of the weight is muscle?
5. Which is a better buy—4 grapefruits for $80c or 12 grapefruits for $2?
6. On a map, \( \frac{1}{3} \) in. represents 5 mi. If New York and Alussim are 18 in. apart on the map, what is the actual distance between them?
7. David reads 40 pages of a book in 50 min. How many pages should he be able to read in 80 min if he reads at a constant rate?
8. Two numbers are in the ratio 3:4. Find the numbers if a. their sum is 98. 
b. their product is 768.
9. Gary, Bill, and Carmella invested in a corporation in the ratio of 2:3:5, respectively. If they divide the profit of $82,000 proportionally to their investment, approximately how much will each receive?
10. Sheila and Dora worked 3 \( \frac{1}{2} \) hr and 4 \( \frac{1}{2} \) hr, respectively, on a programming project. They were paid $176 for the project. How much did each earn if they are both paid at the same rate?
11. Vonna scored 75 goals in her soccer kicking practice. If her success-to-failure rate is \( \frac{V}{a} \), then \( \frac{a}{V} \). Express each of the following ratios in the form \( \frac{a}{b} \).
   a. If the ratio of boys to girls in a class is 2:3, what is the ratio of boys to all the students in the class? Why?
   b. If the ratio of boys to girls in a class is \( m:n \), what is the ratio of boys to all the students in the class?
   c. If \( \frac{3}{5} \) of the class are girls, what is the ratio of boys to girls?
12. Express each of the following ratios in the form \( \frac{a}{b} \), where \( a \) and \( b \) are natural numbers.
   a. \( \frac{1}{2} : 1 \)
   b. \( \frac{1}{3} : \frac{1}{3} \)
   c. \( \frac{1}{5} : \frac{2}{7} \)
13. Use Theorems 21 and 22 to write three other proportions that follow from the following proportion.
   \( \frac{12c}{56 oz} = \frac{16c}{48 oz} \)
14. The rise and span for a house roof are identified as shown on the drawing. The pitch of a roof is the ratio of the rise to the half-span.
   a. If the rise is 10 ft and the span is 28 ft, what is the pitch?
   b. If the span is 16 ft and the pitch is \( \frac{3}{4} \), what is the rise?
15. Gear ratios are used in industry. A gear ratio is the comparison of the number of teeth on two gears. When two gears are meshed, the revolutions per minute (rpm) are inversely proportional to the number of teeth; that is,
   \[ \text{rpm of large gear} = \frac{\text{Number of teeth on small gear}}{\text{Number of teeth on large gear}} \]
   a. The rpm ratio of the large gear to the small gear is 4:6. If the small gear has 18 teeth, how many teeth does the large gear have?
   b. The large gear revolves at 200 rpm and has 60 teeth. How many teeth are there on the small gear, which has an rpm of 600?
16. A Boeing 747 jet is approximately 230 ft long and has a wingspan of 195 ft. If a scale model of the plane is about 40 cm long, approximately what is the model's wingspan?
17. A recipe calls for 1 tsp of mustard seeds, 3 c of tomato sauce, \( \frac{1}{2} \) c of chopped scallions, and \( \frac{1}{4} \) c of beans. If one ingredient is altered as specified, how much must the other ingredients be changed to keep the proportions the same?
   a. 2 c of tomato sauce
   b. 1 c of chopped scallions
   c. \( \frac{3}{4} \) c of beans
18. The electrical resistance of a wire, measured in ohms (\( \Omega \)), is proportional to the length of the wire. If the electrical resistance of a 5-ft wire is 4 \( \Omega \), what is the resistance of 20 ft of the same wire?
19. In a photograph of a father and his daughter, the daughter's height is 2 cm and the father's height is 6 cm. If the father is actually 183 cm tall, how tall is the daughter?
20. The amount of gold in jewelry and other products is measured in karats (K), where 24K represents pure gold. The mark 14K on a chain indicates that the ratio between the mass of the gold in the chain and the mass of the chain is 14:24. If a gold ring is marked 18K and it weighs 4 oz, what is the value of the gold in the ring if pure gold is valued at $1800 per oz?
21. If Amber is paid $8 per hour for typing, the table shows how much she earns.

<table>
<thead>
<tr>
<th>Hours (h)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wages (w)</td>
<td>$8</td>
<td>$16</td>
<td>$24</td>
<td>$32</td>
<td>$40</td>
</tr>
</tbody>
</table>

   a. How much would Amber make for a 40-hr work week?
   b. What is the constant of proportionality?
22. a. In Room A there are 1 man and 2 women; in Room B there are 2 men and 4 women; and in Room C there are 5 men and 10 women. If all the people in Rooms B and C go to Room A, what will be the ratio of men to women in Room A?
   b. Prove the following generalization of the proportions used in part (a).
   \[ \frac{a}{b} = \frac{c}{d} = \frac{e}{f}, \text{then} \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{a + c + e}{b + d + f} \]
23. Use the bar method of Example 29 to solve the following exercise. One-half of the length of stick A is \( \frac{2}{3} \) of the length of stick B. Stick B is 18 cm shorter than stick A. What is the length of both sticks?

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24. A car travels about 26 miles on 1 gallon of gas while a truck travels about 250 miles on 14 gallons of gas. Which gets the better gas mileage?

25. Susan bikes 20 miles in 2 hours while Nick bikes 32 miles in 3 hours. Who travels faster?

26. There are 40 students in a classroom, and the desired ratio of students to computers is 3:1. How many computers are needed for the classroom to achieve the desired ratio?

27. The ratio of oblong tables to round tables at a conference is 5:1. The total number of tables at the conference is 102. How many of each type are there?

28. A recipe uses \( \frac{1}{2} \) c flour to make 2 dozen cookies. Is 4 c flour enough to make 6 dozen cookies with this recipe? Explain your answer.

---

### Assessment 4B

1. Answer the following regarding the letters in the word *Mississippi*.
   - a. Determine the ratio of vowels to consonants.
   - b. What is the ratio of consonants to vowels?
   - c. What is the ratio of consonants to letters?

2. Solve for \( x \) in each of the following proportions.
   - a. \( \frac{5}{x} = \frac{30}{42} \)
   - b. \( \frac{x}{8} = \frac{-12}{32} \)
   - c. \( \frac{7}{8} = \frac{3x}{48} \)
   - d. \( \frac{x}{2} \) is to 8 as \( x \) is to 24

3. There are 5 adult drivers to each teenage driver in Aluossim. If there are 12,345 adult drivers in Aluossim, how many teenage drivers are there?

4. A candle is 30 in. long. After burning for 12 min, the candle is 25 in. long. If it continues to burn at the same rate, how long will it take for the whole candle to burn?

5. A rectangular yard has a width-to-length ratio of 5:9. If the distance around the yard is 2800 ft, what are the dimensions of the yard?

6. A grasshopper can jump 20 times its length. If jumping ability in humans (height) were proportional to a grasshopper’s (length), how far could a 6-ft-tall person jump?

7. Jim found out that after working for 9 months he had earned 6 days of vacation time. How many days per year does he earn at this rate?

8. At Rattlesnake School the teacher–student ratio is 1:30. If the school has 1200 students, how many additional teachers must be hired to change the ratio to 1:20?

9. At a particular time, the ratio of the height of an object that is perpendicular to the ground to the length of its shadow is the same for all objects. If a 30-ft tree casts a shadow of 12 ft, how tall is a tree that casts a shadow of 14 ft?

10. The following table shows several possible widths \( W \) and corresponding lengths \( L \) of a rectangle whose area is 10 ft\(^2\).

<table>
<thead>
<tr>
<th>Width (( W )) (Feet)</th>
<th>Length (( L )) (Feet)</th>
<th>Area (Square Feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} )</td>
<td>20</td>
<td>( \frac{1}{2} \cdot 20 = 10 )</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>( 1 \cdot 10 = 10 )</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>( 2 \cdot 5 = 10 )</td>
</tr>
<tr>
<td>( 2\frac{1}{2} )</td>
<td>4</td>
<td>( 2\frac{1}{2} \cdot 4 = 10 )</td>
</tr>
<tr>
<td>4</td>
<td>( 2\frac{1}{2} )</td>
<td>( 4 \cdot 2\frac{1}{2} = 10 )</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>( 5 \cdot 2 = 10 )</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>( 10 \cdot 1 = 10 )</td>
</tr>
<tr>
<td>20</td>
<td>( \frac{1}{2} )</td>
<td>( 20 \cdot \frac{1}{2} = 10 )</td>
</tr>
</tbody>
</table>

\[
\frac{L}{\text{Area} = 10 \text{ ft}^2} = \frac{1}{W}
\]

11. Find three sets of \( x \)- and \( y \)-values for the following proportions.
   - \( \frac{4 \text{ tickets}}{\$20} = \frac{x \text{ tickets}}{\$y} \)

12. If rent is $850 for each 2 weeks, how much is the rent for 7 weeks?

13. Leonardo da Vinci in his drawing *Vitruvian Man* showed that the man’s armspan was equal to the man’s height. Some other ratios are listed below.

   - a. Length of hand = \( \frac{7}{9} \)
   - b. Distance from elbow to end of hand = \( \frac{8}{5} \)
   - c. Length of hand = \( \frac{14}{3} \)
   - d. Distance from shoulder to elbow = \( \frac{5}{3} \)
   - e. Length of big toe = \( \frac{3}{2} \)
   - f. Length of foot = \( \frac{7}{9} \)

   Using the ratios above, answer the following questions.
   - a. If the length of a big toe is 6 cm, how long should the hand be?
   - b. If a hand is 21 cm, how long is the foot?

14. On a city map, a rectangular park has a length of 4 in. If the actual length and width of the park are 300 ft and 200 ft, respectively, how wide is the park on the map?

15. Jim’s car will travel 240 mi on 15 gal of gas. How far can he expect to go on 3 gal of gas?

16. Some model railroads use an O scale in replicas of actual trains. The O scale uses the ratio 1 in./48 in. How many feet long is the actual locomotive if an O scale replica is 18 in. long?

17. On an American flag, what is the ratio of stars to stripes?

18. On an American flag, the ratio of the length of the flag to its width must be 19:10.

   - a. If a flag is to be \( \frac{9}{2} \) ft long, how wide should it be?
   - b. The flag that was placed on the Moon measured 5 ft by 3 ft. Does this ratio form a proportion with the official length-to-width ratio? Why?

19. If \( \frac{x}{y} = \frac{a}{b}, a \neq 0, x \neq 0 \), is true, what other proportions do you know are true?
20. If a certain recipe takes \(\frac{1}{2}\) c flour and 4 c milk, how much milk should be used if the cook only has 1 c flour?

21. To estimate the number of fish in a lake, scientists use a tagging and recapturing technique. A number of fish are captured, tagged, and then released back into the lake. After a while, some fish are captured and the number of tagged fish is counted.

Let \(T\) be the total number of fish captured, tagged, and released into the lake, \(n\) the number of fish in a recaptured sample, and \(t\) the number of fish found tagged in that sample. Finally, let \(x\) be the number of fish in the lake. The assumption is that the ratio between tagged fish and the total number of fish in any sample is approximately the same, and hence scientists assume \(\frac{t}{n} = \frac{T}{x}\). Suppose 173 fish were captured, tagged, and released. Then 68 fish were recaptured among them 21 were found to be tagged. Estimate the number of fish in the lake.

Reasoning
1. Iris has found some dinosaur bones and a fossil footprint.

   The length of the footprint is 40 cm, the length of the thigh bone is 100 cm, and the length of the body is 700 cm.

   a. Iris found a new track that she believes was made by the same species of dinosaur. If the footprint was 30 cm long and if the same ratio of foot length to body length holds, how long is the dinosaur?

   b. In the same area, Iris also found a 50-cm thigh bone. Do you think this thigh bone belonged to the same dinosaur that made the 30-cm footprint that Iris found? Why or why not?

2. Suppose a 10-in. circular pizza costs $4.00. To find the price, \(x\), of a 14-in. circular pizza, is it correct to set up the proportion \(\frac{x}{4} = \frac{14}{10}\)? Why or why not?

3. Prove that if \(\frac{a}{b} = \frac{c}{d}\) and \(a \neq b\), then the following are true.

   a. \(\frac{a + b}{b} = \frac{c + d}{d}\) (Hint: \(\frac{a}{b} + 1 = \frac{c}{d} + 1\))

   b. \(\frac{a}{a + b} = \frac{c}{c + d}\)

   c. \(\frac{a - b}{a + b} = \frac{c - d}{c + d}\)

   d. \(\frac{a}{c - d} = \frac{b}{a - b}\)

4. Nell said she can tell just by looking at the ratios 15:7 and 15:8 that these do not form a proportion. Is she correct? Why?

5. Sol had photographs that were 4 in. by 6 in., 5 in. by 7 in., and 8 in. by 10 in. Do the dimensions vary proportionately? Explain why.

6. Can \(\frac{a}{b}\) and \(\frac{a + b}{b}\) ever form a proportion? Why?

7. In a condo complex, \(\frac{2}{3}\) of the men were married to \(\frac{3}{4}\) of the women. What is the ratio of married people to the total adult population of the condo complex? Explain how you can obtain this ratio without knowing the actual number of men or women.

Open-Ended
8. List three real-world situations that involve ratio and proportion.

9. Find the ratio of Democrats to Republicans in the U.S. House of Representatives and the U.S. Senate. Determine how many each party would need to hold a majority and how many would be needed to stop a filibuster under the existing rules.

10. Most fertilizers used for gardening and crops are listed with a ratio of 1:2:1. Research the meaning of this ratio and find an acceptable mixture for flowers and foliage. If a fertilizer package showed a ratio of 10:20:10, what would be the meaning of this?

11. Research the golden ratio that the Greeks may have used in the design of the Parthenon. Write a report on this ratio and include a drawing of a golden rectangle.

Cooperative Learning
12. In *Gulliver's Travels* by Jonathan Swift we find the following:

   The seamstresses took my measure as I lay on the ground, one a standing at my neck and another at mid-leg, with a strong cord extended, that each held by the end, while the third measured the length of the cord with a rule of an inch long. Then they measured my right thumb and desired no more; for by a mathematical computation, that twice around the thumb is once around the wrist, and so on to the neck and the waist; and with the help of my old shirt, which I displayed on the ground before them for a pattern, they fitted me exactly.

   a. Explore the measurements of those in your group to see if you believe the ratios mentioned for Gulliver.

   b. Suppose the distance around a person's thumb is 9 cm. What is the distance around the person's neck?
Rational Numbers and Proportional Reasoning

c. What ratio could be used to compare a person's height to armspan?
d. Do you think there is a ratio between foot length and height? If so, what might it be?
e. Estimate other body ratios and then see how close you are to actual measurements.

Connecting Mathematics to the Classroom

13. Mary is working with measurements and writes the following proportion.
   \[
   \frac{12 \text{ in.}}{1 \text{ ft}} = \frac{5 \text{ ft}}{60 \text{ in.}}
   \]
   How would you help her?

14. Nora said she can use division to decide whether two ratios form a proportion; for example, \(32:8\) and \(40:10\) form a proportion because \(32 ÷ 8 = 4\) and \(40 ÷ 10 = 4\). Is she correct? Why?

15. Al is 5 ft tall and has a shadow that is 18 in. long. At the same time, a tree has a shadow that is 15 ft long. Al sets up and solves the proportion as follows:
   \[
   \frac{5 \text{ ft}}{15 \text{ ft}} = \frac{18 \text{ in.}}{x}, \text{ so } x = 54 \text{ in.}
   \]
   How would you help him?

16. Mandy read that the arm of the Statue of Liberty is 42 ft long. She would like to know how long the Statue of Liberty's nose is. How would you advise her to proceed?

17. One student in the class says that her sister is in a school using the Singapore math materials and much of the work with fractions is done with bars. The student continues, saying, "I don't understand how bars can help me understand it with an exercise like that found in exercise 13 above." Can you help her?

Review Problems

18. If the numerator of a rational number is 6 times the denominator and the numerator is also 5 more than the denominator, what are the numerator and denominator?

19. Explain whether \(\frac{3}{4}\) is a proper fraction or an improper fraction and why.

20. Explain why any integer is a rational number.

21. A student says that \(\frac{20}{30}\) can be simplified by crossing out the 0s and that in general, this procedure works for simplifying fractions. Explain whether or not the statement is true using the rational number \(\frac{25}{35}\).

22. In an old Sam Loyd puzzle, a watch is described as having stopped when the minute and hour hands formed a straight line and the second hand was not on 12. At what time can this happen?

National Assessments

National Assessment of Educational Progress (NAEP) Questions

Sarah has a part-time job at Better Burgers restaurant and is paid $5.50 for each hour she works. She has made the chart below to reflect her earnings but needs your help to complete it.

A. Fill in the missing entries in the chart.

<table>
<thead>
<tr>
<th>Hours Worked</th>
<th>Money Earned (in dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$5.50</td>
</tr>
<tr>
<td>4</td>
<td>$38.50</td>
</tr>
<tr>
<td>7 (\frac{3}{4})</td>
<td>$42.63</td>
</tr>
</tbody>
</table>

B. If Sarah works \(b\) hours, then in terms of \(b\), how much will she earn?

NAEP, Grade 8, 2007

The length of a photograph is 5 inches and its width is 3 inches. The photograph is enlarged proportionally. The length of the enlarged photograph is 10 inches. What is the width of the enlarged photograph?

A. 6 inches
B. 7 inches
C. 9 inches
D. 15 inches
E. 16 \(\frac{2}{3}\) inches

NAEP, Grade 8, 2013

Hint for Solving the Preliminary Problem

Find the fraction of the whole washer that the two holes represent and then find the portion not represented by the holes. This should aid in finding the area of the washer.
## Chapter Summary

### Key Concepts

<table>
<thead>
<tr>
<th>Section 1</th>
<th>Definitions, Descriptions, and Theorems</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rational number</strong></td>
<td>A number in the form ( \frac{a}{b} ) or ( a/b ), where ( a ) and ( b ) are integers and ( b \neq 0 ). In the rational number ( \frac{a}{b} ), ( a ) is the numerator and ( b ) is the denominator.</td>
</tr>
</tbody>
</table>
| **Uses of rational numbers** | - Division problem  
- Portion, or part, of a whole  
- Ratio  
- Probability |
| **Types of fractions** |  
- **Proper fraction**—any fraction \( \frac{a}{b} \), where \( 0 \leq a < b \); a proper fraction is always less than 1.  
- **Improper fraction**—any fraction \( \frac{a}{b} \), where \( a \geq b > 0 \); an improper fraction is always greater than or equal to 1.  
- **Equivalent, or equal, fractions**—numbers that represent the same point on a number line. |
| **Fundamental Law of Fractions** | *Theorem:* If \( \frac{a}{b} \) is a fraction and \( n \) is a nonzero rational number, then \( \frac{a}{b} = \frac{an}{bn} \). |
| **Simplest form** | A rational number \( \frac{a}{b} \) is in *simplest form*, or *lowest terms* if, and only if, \( \text{GCD}(a, b) = 1 \); that is, if \( a \) and \( b \) have no common factor greater than 1. |
| **Equality of fractions** | Two fractions \( \frac{a}{b} \) and \( \frac{c}{d} \), where \( b \neq 0 \) and \( d \neq 0 \), are equal (or equivalent) if, and only if, \( ad = bc \). |
| **Ordering of rational numbers** |  
- **Like denominators**—If \( a, b, \) and \( c \) are any integers and \( b > 0 \), then \( \frac{a}{b} > \frac{c}{b} \) if, and only if, \( a > c \).  
- **Unlike denominators**—If \( a, b, c, \) and \( d \) are any integers such that \( b > 0 \) and \( d > 0 \), then \( \frac{a}{b} > \frac{c}{d} \) if, and only if \( ad > bc \). |
| **Denseness property for rational numbers** | Given any two different rational numbers \( \frac{a}{b} \) and \( \frac{c}{d} \), there is another rational numbers between these two numbers.  
*Theorem:* Let \( \frac{a}{b} \) and \( \frac{c}{d} \) be any rational numbers with positive denominators, where \( \frac{a}{b} < \frac{c}{d} \). Then \( \frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d} \). |

### Section 2

| Addition of rational numbers | Like denominators—If \( \frac{a}{b} \) and \( \frac{c}{b} \) are rational numbers, then \( \frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} \).  
Unlike denominators—If \( \frac{a}{b} \) and \( \frac{c}{d} \) are rational numbers, then \( \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \). |
**Section 2**

Addition properties for rational numbers

- **Additive inverse**—for any rational number \( \frac{a}{b} \), there exists a unique rational number \(-\frac{a}{b}\) such that \( \frac{a}{b} + \left(-\frac{a}{b}\right) = 0 = \left(-\frac{a}{b}\right) + \frac{a}{b} \).

- **Property of equality**—if \( \frac{a}{b} \) and \( \frac{c}{d} \) are any rational numbers such that \( \frac{a}{b} = \frac{c}{d} \) and \( \frac{e}{f} \) is a rational number, then \( \frac{a}{b} + \frac{e}{f} = \frac{c}{d} + \frac{e}{f} \).

Subtraction of rational numbers

If \( \frac{a}{b} \) and \( \frac{c}{d} \) are any rational numbers, then \( \frac{a}{b} - \frac{c}{d} \) is the unique rational number \( \frac{e}{f} \) such that \( \frac{a}{b} = \frac{c}{d} + \frac{e}{f} \). If \( \frac{a}{b} \) and \( \frac{c}{d} \) are any rational numbers, then \( \frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \left(-\frac{c}{d}\right) \).

Estimation with rational numbers

Round fractions to convenient, or benchmark numbers, such as \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{3}, \frac{3}{4} \), or 1.

**Section 3**

Multiplication of rational numbers

If \( \frac{a}{b} \) and \( \frac{c}{d} \) are any rational numbers, then \( \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \).

Multiplication properties for rational numbers

- **Multiplicative identity**—the rational number 1 is the unique number such that for every rational number \( \frac{a}{b} \), \( 1 \cdot \frac{a}{b} = \frac{a}{b} = \frac{a}{b} \cdot 1 \).

- **Multiplicative inverse**—for any nonzero rational number \( \frac{a}{b} \), the multiplicative inverse (reciprocal) is the unique rational number \( \frac{b}{a} \) such that \( \frac{a}{b} \cdot \frac{b}{a} = 1 = \frac{b}{a} \cdot \frac{a}{b} \).

- **Property of 0**—let \( \frac{a}{b} \) be any rational number. Then \( \frac{a}{b} \cdot 0 = 0 = 0 \cdot \frac{a}{b} \).

- **Property of equality**—let \( \frac{a}{b}, \frac{c}{d}, \) and \( \frac{e}{f} \) be any rational numbers such that \( \frac{a}{b} = \frac{c}{d} \). Then \( \frac{a}{b} \cdot \frac{c}{d} = \frac{e}{f} \).

- **Properties of inequality**—let \( \frac{a}{b}, \frac{c}{d}, \) and \( \frac{e}{f} \) be any rational numbers:
  - If \( \frac{a}{b} > \frac{c}{d} \) and \( \frac{e}{f} > 0 \), then \( \frac{a}{b} \cdot \frac{e}{f} > \frac{c}{d} \cdot \frac{e}{f} \).
  - If \( \frac{a}{b} > \frac{c}{d} \) and \( \frac{e}{f} < 0 \), then \( \frac{a}{b} \cdot \frac{e}{f} < \frac{c}{d} \cdot \frac{e}{f} \).

Distributive property of multiplication over addition or subtraction

Let \( \frac{a}{b}, \frac{c}{d}, \) and \( \frac{e}{f} \) be any rational numbers. Then

\[
\frac{a}{b} \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a \cdot c}{b \cdot d} + \frac{a \cdot e}{b \cdot f} \quad \text{and} \quad \frac{a}{b} \left(\frac{c}{d} - \frac{e}{f}\right) = \frac{a \cdot c}{b \cdot d} - \frac{a \cdot e}{b \cdot f}.
\]

Division of rational numbers

If \( \frac{a}{b} \) and \( \frac{c}{d} \) are any rational numbers, with \( \frac{c}{d} \neq 0 \), then \( \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \). if, and only if, \( \frac{c}{d} \) is the unique rational number such that \( \frac{c \cdot e}{d} = \frac{a}{b} \).

Algorithm for Division of Fractions

If \( \frac{a}{b} \) and \( \frac{c}{d} \) are any rational numbers, with \( \frac{c}{d} \neq 0 \), then \( \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \).
# Rational Numbers and Proportional Reasoning

## Section 3

**Exponents of rational numbers**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^m$</td>
<td>If $a$ is any rational number and $m$ is any natural number, then $a^m = a \cdot a \cdot \ldots \cdot a$ (m factors)</td>
</tr>
<tr>
<td>$a^{-n}$</td>
<td>For any nonzero rational number $a$ and any natural number $n$, $a^{-n} = \frac{1}{a^n}$</td>
</tr>
</tbody>
</table>

**Theorems**

1. For any nonzero rational number $a$ and any integers $m$ and $n$, $a^m \cdot a^n = a^{m+n}$.
2. For any nonzero rational number $a$ and any integers $m$ and $n$, $\frac{a^m}{a^n} = a^{m-n}$.
3. For any nonzero rational number $a$ and any integers $m$ and $n$, $(a^m)^n = a^{mn}$.
4. For any nonzero rational number $\frac{a}{b}$ and any integer $m$, $(\frac{a}{b})^{m} = a^{m}b^{-m}$.
5. For any nonzero rational number $\frac{a}{b}$ and any integer $m$, $a^{m}b^{m} = (a \cdot b)^{m}$.

## Section 4

**Ratio**

A comparison of two quantities $a$ and $b$, where $a$ and $b$ are rational numbers, denoted as $\frac{a}{b}$ or $a:b$ and read as “$a$ to $b$”.

**Proportion**

A statement that two given ratios are equal.

Theorem: If $a$, $b$, $c$, and $d$ are rational numbers and $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if, and only if, $ad = bc$.

**Constant of Proportionality**

If the variables $x$ and $y$ are related by the equality $y = kx$, or $k = \frac{y}{x}$, then $y$ is proportional to $x$ and $k$ is the constant of proportionality between $y$ and $x$.

**Proportionality theorems**

Theorem: For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, with $a \neq 0$ and $c \neq 0$, $\frac{a}{b} = \frac{c}{d}$ if, and only if, $\frac{b}{a} = \frac{d}{c}$.

Theorem: For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, with $c \neq 0$, $\frac{a}{b} = \frac{c}{d}$ if, and only if, $\frac{a}{c} = \frac{b}{d}$.

## Chapter Review

1. **For each of the following, draw a diagram illustrating the fraction.**
   - a. $\frac{3}{4}$
   - b. $\frac{2}{3}$
   - c. $\frac{3}{4}$, $\frac{2}{3}$

2. **Write three rational numbers equal to $\frac{5}{6}$.**

3. **Write each of the following rational numbers in simplest form.**
   - a. $\frac{24}{28}$
   - b. $\frac{ax^2}{bx}$
   - c. $\frac{0}{17}$
   - d. $\frac{45}{81}$
   - e. $\frac{b^2 + bx}{b + x}$
   - f. $\frac{16}{216}$
   - g. $\frac{x + a}{x - a}$
   - h. $\frac{xa}{x + a}$
4. In each of the following pairs, replace the comma with $>$, $<$, or $=$ to make a true statement.
   a. $\frac{6}{10}$, $\frac{120}{200}$
   b. $\frac{-3}{4}$, $\frac{-5}{6}$
   c. $\left(\frac{4}{5}\right)^{10}$, $\left(\frac{4}{5}\right)^{20}$
   d. $\left(1 + \frac{1}{5}\right)^{2}$, $\left(1 + \frac{1}{5}\right)^{4}$

5. Find the additive and multiplicative inverses for each of the following.
   a. 3
   b. $3 \frac{1}{7}$
   c. $\frac{5}{6}$
   d. $-\frac{3}{4}$

6. Order the following numbers from least to greatest.
   $\frac{-7}{8}$, 0, $-2 \frac{1}{3}$, $\frac{69}{140}$, $\frac{71}{140}$, $\frac{74}{140}$, $\frac{74}{300}$, $\frac{71}{140}$, $\frac{69}{140}$, $\frac{74}{140}$, $\frac{74}{300}$

7. Can $\frac{4}{5}$, $\frac{7}{8}$, $\frac{5}{14}$ be written as $\frac{4}{8}$, $\frac{7}{14}$, $\frac{5}{14}$ to obtain the same answer? Why or why not?

8. Use mental math to compute the following. Explain your method.
   a. $\frac{1}{3} \cdot (8 \cdot 9)$
   b. $36 \cdot 1 \frac{5}{6}$

9. John has $54 \frac{1}{4}$ yd of material.
   a. If he needs to cut the cloth into pieces that are $3 \frac{1}{12}$ yd long, how many pieces can he cut?
   b. How much material will be left over?

10. Without actually performing the given operations, choose the most appropriate estimate (among the numbers in parentheses) for the following expressions.
    a. $30 \frac{3}{8}$, $8 \frac{1}{3}$ (15, 20, 8)
    b. $\left(\frac{3}{800} + \frac{4}{5000} + \frac{15}{6}\right)6$ (15, 0, 132)
    c. $\frac{1}{404} + \frac{1}{1609}$ (1, 4, 0)

11. Write a story problem that models $4 \frac{5}{8} + \frac{1}{2}$. Solve the problem by drawing appropriate diagrams.

12. Find two rational numbers between $\frac{3}{4}$ and $\frac{4}{5}$.

13. Suppose the $\Box$ button on your calculator is broken, but the $\sqrt{ }$ button works. Explain how you could compute $\sqrt{504792}$.

14. Jim is starting a diet. When he arrived home, he ate $\frac{1}{3}$ of the half of a pizza that was left from the previous night. The whole pizza contains approximately 2000 calories. How many calories did Jim consume?

15. If a person got heads on a flip of a fair coin one-half the time and obtained 376 heads, how many times was the coin flipped?

16. If a person obtained 240 heads when flipping a coin 1000 times, what fraction of the time did the person obtain heads? Put the answer in simplest form.

17. If the University of New Mexico won $\frac{3}{4}$ of its women’s basketball games and $\frac{5}{8}$ of its men’s basketball games, explain whether it is reasonable to say that the university won $\frac{3}{4} + \frac{5}{8}$ of its basketball games.

18. The carvings of the faces at Mount Rushmore in South Dakota measure 60 ft from chin to forehead. If the distance from chin to forehead is typically 9 in., and the distance between the pupils of the eyes is typically $2 \frac{1}{2}$ in., what is the approximate distance between the pupils on the carving of George Washington’s head?

19. A student argues that the following fraction is not a rational number because it is not the quotient of two integers:
    $\frac{2}{3}$, $\frac{3}{4}$
    How would you respond?

20. Molly wants to fertilize 12 acres of park land. If it takes $9 \frac{1}{3}$ bags for each acre, how many bags does she need?

21. If $\frac{2}{3}$ of all students in the academy are female and $\frac{2}{5}$ of those are blondes, what fraction describes the number of blond females in the academy?

22. Explain which is greater: $-\frac{11}{9}$ or $-\frac{12}{10}$.

23. Solve for $x$ in each of the following.
    a. $7x = 343$  
    b. $2^{3x} = \frac{1}{512}$
    c. $2x - \frac{5}{3} = \frac{5}{6}$  
    d. $x + 2\frac{1}{2} = 5\frac{2}{3}$
    e. $\frac{20 + x}{x} = \frac{4}{5}$  
    f. $2x + 4 = 3x - \frac{1}{3}$

24. Write each of the following in simplest form. Leave all answers with positive exponents.
    a. $\frac{(x^3a^{-1})^2}{xa^{-1}}$  
    b. $\frac{x^2y^2}{x^2y^2}$

25. Find each sum or difference.
    a. $\frac{3a}{xy^2} + \frac{b}{x^2y^2}$  
    b. $\frac{5}{xy^2} - \frac{2}{3x}$
    c. $\frac{a}{x^2y^2z} - \frac{b}{xyz}$
    d. $\frac{7}{2x^2y^2} + \frac{5}{2x^2y^2}$
26. Mike drew the following picture to find out how many pieces of ribbon \( \frac{1}{2} \) yd long could be cut from a strip of ribbon \( \frac{3}{4} \) yd long.

<table>
<thead>
<tr>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>left over</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{3}{4} )</td>
</tr>
</tbody>
</table>

From the picture he concluded that \( 1 \frac{3}{4} + \frac{1}{2} \) is 3 pieces with \( \frac{1}{4} \) yd left over, so the answer is 3 \( \frac{1}{4} \) pieces. He checked this using the algorithm \( \frac{7}{4} \div \frac{2}{1} = \frac{14}{4} = 3 \frac{1}{2} \) and is confused why he has two different answers. How would you help him?

27. Tom tossed a coin 30 times and got 17 heads.
   a. What is the ratio of heads to coin tosses?
   b. What is the ratio of heads to tails?
   c. What is the ratio of tails to heads?

28. Which bottle of juice is a better buy (cost per fluid ounce):
    - 48 fl oz for $3 or 64 fl oz for $4?

29. Eighteen-karat gold contains 18 parts (grams) gold and 6 parts (grams) other metals. Amy’s new ring contains 12 parts gold and 3 parts other metals. Is the ring 18-karat gold? Why?

30. A recipe for fruit salad serves 4 people. It calls for 3 oranges and 16 grapes. How many oranges and grapes do you need to serve 11 people?

31. If the scale on a drawing of a house is 1 cm to 2 meters, what is the length of the house if it measures 3 cm on the scale drawing?

32. In water \( (H_2O) \), the ratio of the weight of oxygen to the weight of hydrogen is approximately 8:1. How many ounces of hydrogen are in 1 lb of water?

33. A manufacturer produces the same kind of computer chip in two plants. In the first plant, the ratio of defective chips to good chips is 15:100 and in the second plant, that ratio is 12:100. A buyer of a large number of chips is aware that some come from the first plant and some from the second. However, she is not aware of how many come from each. The buyer would like to know the ratio of defective chips to good chips in any given order. Can she determine that ratio? If so, explain how. If not, explain why not.

34. Suppose the ratio of the lengths of the sides in two squares is 1:x. What is the ratio of their areas? \( A = x^2 \)

35. The Grizzlies won 18 games and lost 7.
   a. What is the ratio of games won to games lost?
   b. What is the ratio of games won to games played?

36. Express each of the following as a ratio \( \frac{a}{b} \) where \( a \) and \( b \) are whole numbers.
   a. \( \frac{1}{5} : 1 \)
   b. \( \frac{2}{5} : \frac{3}{4} \)

37. The ratio of boys to girls in Mr. Good’s class is 3 to 5, the ratio of boys to girls in Ms. Garcia’s is the same, and you know that there are 15 girls in Ms. Garcia’s class. How many boys are in Ms. Garcia’s class?

38. If the ratio of the number of states of the United States using the Common Core Standards to those not using it is 9:10, how many states are not using these Standards?

39. From about 1978 to 1985, there was a trend towards conservatism among university professors with about \( \frac{1}{20} \) of professors identifying themselves as strongly left-wing, about a third identifying themselves as liberals, about \( \frac{1}{4} \) identifying themselves as moderates, \( \frac{1}{4} \) as conservative, and \( \frac{1}{20} \) as strongly conservative. What fraction identified themselves with none of the labels?

40. Since 1985, the fraction of liberal professors has grown steadily, with research finding somewhere between 7 to 9 liberals for each professor of another political persuasion. What statement with ratios could you make about professors of other political persuasions?

41. In an equilateral triangle, all sides have the same length and the perimeter is the sum of those lengths. What is the ratio of the perimeters of two equilateral (all sides of equal length) triangles whose sides each have lengths 6 cm and 10 cm respectively?

42. If a cup of coffee with 1 oz of cream is compared to another cup with \( \frac{9}{10} \) oz cream, which has more cream?

43. In a package of tulip bulbs, the seller guarantees that at least \( \frac{45}{99} \) of the bulbs will bloom. If you planted a package of 121 bulbs, at least how many would be expected not to bloom?

44. A woman’s will decreed that her cats be shared among her three daughters as follows: \( \frac{1}{2} \) of the cats to the eldest daughter, \( \frac{1}{3} \) of the cats to the middle daughter, and \( \frac{1}{9} \) of the cats to the youngest daughter. Since the woman had 17 cats, the daughters decided that they could not carry out their mother’s wishes. The judge who held the will agreed to lend the daughters a cat so that they could share the cats as their mother wished. Now, \( \frac{1}{2} \) of 18 is 9, \( \frac{1}{3} \) of 18 is 6; and \( \frac{1}{9} \) of 18 is 2. Since 9 + 6 + 2 = 17, the daughters were able to divide the 17 cats and return the borrowed cat. They obviously did not need the extra cat to carry out their mother’s bequest, but they could not divide 17 into halves, thirds, and ninths. Has the woman’s will really been followed?

45. Prince Juan was allowed to take a number of bags of gold as he went into exile. However, a guard at the first bridge he crossed demanded half the bags of gold plus one more bag. Juan met this demand and proceeded to the next bridge. Guards at the second, third, and fourth bridges made identical demands, all of which the prince met. When Juan finally crossed all the bridges, a single bag of gold was left. With how many bags did Juan start?
Answers to Problems


Assessment 1A

1. Answers vary. a. The solution to \( 8x = 7 \) is \( \frac{7}{8} \) b. Jane ate \( \frac{7}{8} \) of the pizza. c. The ratio of boys to girls is 7 to 8. 2. a. \( \frac{1}{6} \) b. \( \frac{1}{4} \) c. \( \frac{2}{6} = \frac{1}{3} \) d. \( \frac{7}{12} \) e. \( \frac{5}{8} \) f. \( \frac{3}{8} = \frac{2}{4} \) c. \( \frac{2}{3} \) e. \( \frac{7}{8} \) d. \( \frac{8}{12} = \frac{2}{3} \) The diagrams illustrate the Fundamental Law of Fractions. 5. a. No, the parts do not have equal areas. The shaded part could be \( \frac{1}{4} \) of the circle, but we can’t tell from the figure. b. Yes c. Yes 6. Answers vary. a. 11. a. 12. a. \( \frac{3}{1} \) b. \( \frac{54}{8} \) c. \( \frac{16}{23} \) d. \( \frac{92}{44} \) e. \( \frac{1}{2} \) f. \( \frac{1}{3} \) g. \( \frac{8}{12} = \frac{1}{3} \) h. \( \frac{7}{2} \) i. \( \frac{23}{3} = \frac{7}{2} \) j. Answers vary. 36. \( \frac{22}{4} = \frac{1}{2} \) 37. \( \frac{29}{4} \) 4. Answers vary. a. \( \frac{1}{3} \) high b. \( \frac{1}{6} \) low c. \( \frac{3}{4} \) low d. \( \frac{1}{2} \) high 5. a. Beavers b. Ducks c. Bears 6. Answers vary. About 3A 7. a. \( \frac{1}{2} \) high b. 0, low c. \( \frac{3}{4} \) high d. 1, high 8. a. \( \frac{3}{4} \) b. \( \frac{3}{4} \) 9. a. \( \frac{1}{4} \) b. 0 10. a. A b. H c. T d. H 11. Answers vary. Approximately 4 \( \times \) 3 = 12 12. a. \( \frac{1}{4} \) b. \( \frac{3}{4} \) 13. 6 \( \frac{7}{12} \) yd 14. 2 \( \frac{5}{6} \) yd 15. Answers vary. a. \( \frac{1}{4} + \frac{3}{4} \) b. \( \frac{1}{2} + \frac{3}{4} = \frac{3}{4} + \frac{1}{2} \) c. \( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) \) 16. \( \frac{1}{7} \) \( \frac{8}{9} \) \( \frac{30}{6} \) \( \frac{1}{2} \) 17. a. \( \frac{1}{7} \) b. \( \frac{1}{4} \) c. \( \frac{1}{3} \) d. \( \frac{1}{12} \) e. \( \frac{1}{16} \) f. \( \frac{1}{48} \) g. \( \frac{1}{3} \) h. \( \frac{1}{3} \) 18. \( \frac{5}{6} \) 19. \( \frac{3}{4} \) 20. No, you need \( \frac{1}{4} \) cup more milk. 21. Answers vary. a. About \( \frac{23}{45} \) b. About \( \frac{22}{45} \) c. About \( \frac{2}{127} \) 22. The 16- and 17-year-olds 23. \( \frac{1}{8} \) 24. \( \frac{7}{100} \)

Assessment 3A

1. Answers vary. a. \( \frac{1}{4} \) b. \( \frac{1}{3} \) c. \( \frac{1}{2} \) d. \( \frac{1}{3} \) 2. Answers vary. a. b. \( \frac{2}{3} \) 3. Answers vary. a. \( \frac{1}{5} \) b. \( \frac{1}{a} \) c. \( \frac{a}{b} \) d. \( \frac{1}{c} \) e. \( \frac{y}{x} \) f. \( \frac{1}{d} \) g. \( \frac{1}{e} \) h. \( \frac{3}{2} \) i. \( \frac{3}{2} \) 4. a. \( \frac{10}{2} \) b. \( \frac{3}{2} \) 5. a. \( \frac{3}{3} \) b. \( \frac{3}{10} \) 6. a. \( \frac{23}{12} \) b. \( \frac{3}{2} \) c. \( \frac{1}{8} \) d. \( \frac{3}{2} \) 7. Answers vary. For example, a. \( 6 + 2 \neq 2 + 6 \) b. \( 8 + 4 \neq 8 + 4 + 4 + 2 \) a. \( \frac{23}{12} \) b. \( \frac{3}{2} \) c. \( \frac{3}{2} \) d. \( \frac{3}{2} \)
c. Greater than 2 11. d 12. 9600 students 13. $\frac{1}{6}$
14. $240$ 15. $225$ 16. 32 marbles 17. a. $\frac{1}{31}$ b. $\frac{3}{11}$
18. c. $\left(\frac{2}{3}\right)^{\prime}$ d. 1 19. a., b., c., e., and f. are false; counterexamples
20. a. 5
21. a. $x \leq 2$ b. $x < 2$ c. $x \geq 2$
22. a. $\left(\frac{1}{2}\right)^{3}$ b. $\left(\frac{3}{4}\right)^{8}$ c. $\left(\frac{4}{3}\right)^{10}$ d. $\left(\frac{5}{4}\right)^{10}$
23. Answers vary.
24. a. $\frac{1}{2}$ b. About $\frac{1}{11}$ b. About $\frac{1}{6}$
25. Answers vary.
26. a. 2012 to 2013 b. 2008 to 2009 27. $\frac{73}{231}$ 28. $\frac{1}{32}$

Assessment 4A
1. a. 5:21 b. 21:5 c. 21:26 d. Answers vary. For example, minor.
2. a. 30 b. $\frac{3}{2}$ c. 23 $\frac{1}{3}$ d. 10 $\frac{1}{2}$
3. a. 2:5. Because the ratio is 2:3, there are 2x boys and 3x girls; hence, the ratio of boys to all students is $\frac{2x}{2x + 3x} = \frac{2}{5}$. b. $m:(m + n)$ c. 3:2. 4. 36 lb 5. 12 grapefruits for $2$ 6. 270 mi 7. 64 pages 8. a. 42, 56 b. 24 and 32, or $-24$ and $-32$
9. $16,400$; $24,000$; $41,000$ 10. $77$ and $99$ 11. 135 12. a. $\frac{1}{6}$ b. $\frac{1}{12}$ c. $\frac{7}{12}$
13. Answers vary. 14. a. $\frac{5}{9}$ b. 6 ft 15. a. 27 b. 20 16. Approximately 34 cm
17. a. $\frac{2}{3}$ tsp mustard seeds, 1 c scallions, $\frac{1}{6}$ c beans
b. $\frac{2}{3}$ tsp mustard seeds, 2 c tomato sauce, $\frac{1}{6}$ c beans
c. $\frac{7}{13}$ tsp mustard seeds, $\frac{8}{13}$ c tomato sauce, $\frac{21}{26}$ c scallions
18. 16 ohms 19. 61 cm 20. The ratio between the mass of the gold in the ring and the mass of the ring is $\frac{18}{24}$.
If $x$ is the number of ounces of pure gold in the ring which weighs 4 oz, we have $\frac{18}{24} = \frac{x}{4}$. Hence, $x = \frac{(18 \cdot 4)}{24}$, or 3 oz.
Consequently, the price of the gold in the ring is $3 \cdot $1800 or $5400. 21. a. $\frac{320}{8}$ b. 8 22. a. 1:2
b. Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$. Then, $a = br$, $c = dr$, $e = fr$. So, $a + c + e = br + dr + fr$
$c + e = r(b + d + f)$
$c + e = r$.
23. As seen in the drawing, $\frac{1}{2}$ of stick A is $\frac{2}{3}$ of stick B, and stick B is 18 cm shorter than stick A.
We have the following: Stick B is 3 · 18 unit sections while stick A is 4 · 18 unit sections. Thus, the lengths of the sticks are A, 72 cm, and B, 54 cm. 24. The car 25. Nick 26. 14 computers 27. 17 round ones and 85 oblong ones 28. No, 6 dozen cookies would take $\frac{1}{2}$ c flour.

Chapter Review
1. Answers vary. a. b. c.
2. Answers vary; for example, $\frac{10}{12} = \frac{5}{6}$. 3. a. $\frac{6}{7}$
b. $\frac{ax}{b}$ c. 0 d. $\frac{5}{9}$ e. b. f. $\frac{2}{27}$ g. Cannot be simplified
h. Cannot be simplified 4. a. $= b. > c. > d. <$
5. a. $-3$, $\frac{1}{3}$ b. $-3$, $\frac{17}{22}$ c. $-\frac{5}{6}$, $\frac{4}{3}$ d. $-\frac{3}{4}$
6. $-2$, $\frac{3}{4}$ 0, $\frac{71}{140}$, $\frac{69}{140}$, $\frac{71}{140}$, $\frac{74}{75}$
7. Yes. By the definition of multiplication and the commutative and associative laws of multiplication, we can do the following:
8. Answers vary. a. 24, because $\frac{1}{3}(8 \cdot 9)$ is equal to $\left(\frac{1}{3} \cdot 9\right) \cdot 8 = 3 \cdot 8 = 24$. b. 66, because $36 \cdot \frac{5}{6}$ is equal to $36 \cdot \frac{11}{6} = 6 \cdot 11 = 66$. 9. a. 17 pieces b. $\frac{5}{6}$
10. a. 15 b. 15 c. 4 11. Answers vary.
12. Answers vary. For example, $\frac{76}{100}$ and $\frac{78}{100}$
13. $\frac{5}{0} \div \frac{47}{9} \div \frac{2}{x} \div \frac{1}{x} = $
14. Approximately 333 calories
15. 752 times 16. $\frac{240}{100} = \frac{6}{25}$

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17. It is not reasonable to say that the university won \( \frac{3}{4} + \frac{5}{8} \), or \( \frac{11}{8} \), of its basketball games. The correct fraction cannot be determined without additional information but it is between \( \frac{5}{8} \) and \( \frac{3}{4} \). 18. 16 \( \frac{2}{3} \) ft 19. You should show him that the given fraction could be written as an integer over an integer. In this case, the result is \( \frac{8}{9} \). 20. 112 bags 21. \( \frac{4}{15} \) 22. \( \frac{12}{10} \) is greater than \( -\frac{11}{9} \) because \(-12 \cdot 9 > -11 \cdot 10\). Alternatively, 
\[
\frac{-12}{10} - \frac{-11}{9} = \frac{-108}{90} - \frac{-110}{90} = \frac{2}{90}, \quad \text{which is positive; therefore} \\
\frac{-12}{10} > \frac{-11}{9}.
\]
23. a. 3 b. 3 c. \( \frac{5}{4} \) or \( \frac{1}{4} \) d. \( \frac{19}{6} \) or \( \frac{3}{6} \)

Rational Numbers and Proportional Reasoning

Answers to Now Try This

1. Consider two rational numbers \( \frac{a}{b} \) and \( \frac{c}{d} \) where \( \frac{a}{b} < \frac{c}{d} \). By the denseness property of rational numbers we can find a rational number \( x_1 \) between the two fractions. Since \( \frac{a}{b} < x_1 \), there is a rational number \( x_2 \) between \( \frac{a}{b} \) and \( x_1 \). We next can find a rational number \( x_3 \) between \( \frac{a}{b} \) and \( x_2 \) so on. This process can be repeated indefinitely, and hence we obtain infinitely many rational numbers \( x_1, x_2, x_3, \ldots \) between \( \frac{a}{b} \) and \( \frac{c}{d} \). 2. Because \( \frac{a}{b} < \frac{c}{d} \) with \( b > 0 \) and \( d > 0 \), Theorem 3 implies \( ad < bc \).

Adding \( ab \) to each side of the inequality, we now have \( ab + ad < ab + bc \). Thus, by factoring \( a(b + d) < b(a + c) \).

3. \( \frac{3}{4} \) is greater than \( \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} = 1 \), so \( \frac{3}{4} + \frac{1}{2} > 1 \). \( \frac{4}{6} \) is less than 1, so it cannot be the correct answer for \( \frac{3}{4} + \frac{1}{2} \). 4. Answers vary. For example, consider the following. If Caleb has $10.00, how many chocolate bars can he buy if a. the price of one bar is $2.00? b. the price of one bar is \( \frac{1}{2} \). For (a) the answer is \( 10 \div 2 \) or 5. For (b) the answer is \( 10 \div \frac{1}{2} \), which is the same as finding the number of \( \frac{1}{2} \)s in 10. Since there are two halves in 1, in 10 there are 20. Hence Caleb can buy 20 bars.

5. \( \frac{a + c}{b + d} = \frac{a}{c} \cdot \frac{d}{b} = \frac{a}{c} \cdot \frac{1}{b} = \frac{a}{b} \cdot \frac{c}{d} = \frac{ad}{bc} = \frac{a}{b} + \frac{c}{d} \)

6. 20 lawns

Answer to Preliminary Problem

The area of the holes is \( \frac{1}{4} + \frac{1}{7} = \frac{11}{28} \) of the whole washer. Thus the area of the rubber in the finished product is 
\( 1 - \frac{11}{28} = \frac{17}{28} \) of the whole washer. If the area of the original piece of rubber is \( \frac{3}{8} \), then the area of the finished washer is 
\( \left( \frac{17}{28} \right) \left( \frac{3}{8} \right) = \frac{17}{224} \) in.\(^2 \) or about \( \frac{5}{6} \) in.\(^2 \).
Credits

Credits are listed in the order of appearance.

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