In this chapter, we explore some additional topics in consumer theory. We begin with duality theory and investigate more completely the links among utility, indirect utility, and expenditure functions. Then we consider the classic ‘integrability problem’ and ask what conditions a function of prices and income must satisfy in order that it qualify as a demand function for some utility-maximising consumer. The answer to this question will provide a complete characterisation of the restrictions our theory places on observable demand behaviour. We then examine ‘revealed preference’, an alternative approach to demand theory. Finally, we conclude our treatment of the individual consumer by looking at the problem of choice under uncertainty.

2.1 Duality: A Closer Look

As we have seen, the solutions to utility maximisation problems and expenditure minimisation problems are, in a sense, the same. This idea is formally expressed in Theorem 1.9. In this section, we shall explore further the connections among direct utility, indirect utility and expenditure functions. We will show that although our theory of the consumer was developed, quite naturally, beginning with axioms on preferences, an equivalent theory could have been developed beginning with axioms on expenditure behaviour. Indeed, we will show that every function of prices and utility that has all the properties of an expenditure function is in fact an expenditure function, i.e., there is a well-behaved utility function that generates it. Although this result is of some interest in itself, its real significance becomes clear when it is used to characterise completely the observable implications of our theory of the consumer’s demand behaviour. This extraordinary characterisation will follow from the so-called ‘integrability theorem’ taken up in the next section. Given the importance of this result, this section can justifiably be viewed as preparation for the next.

2.1.1 Expenditure and Consumer Preferences

Consider any function of prices and utility, $E(p, u)$, that may or may not be an expenditure function. Now suppose that $E$ satisfies the expenditure function properties 1 to 7 of
Theorem 1.7, so that it is continuous, strictly increasing, and unbounded above in $u$, as well as increasing, homogeneous of degree one, concave, and differentiable in $p$. Thus, $E$ 'looks like' an expenditure function. We shall show that $E$ must then be an expenditure function. Specifically, we shall show that there must exist a utility function on $\mathbb{R}^n_+$ whose expenditure function is precisely $E$. Indeed, we shall give an explicit procedure for constructing this utility function.

To see how the construction works, choose $(p^0, u^0) \in \mathbb{R}^n_+ \times \mathbb{R}_+$, and evaluate $E$ there to obtain the number $E(p^0, u^0)$. Now use this number to construct the (closed) 'half-space' in the consumption set,

$$A(p^0, u^0) \equiv \{ x \in \mathbb{R}^n_+ \mid p^0 \cdot x \geq E(p^0, u^0) \},$$

illustrated in Fig. 2.1(a). Notice that $A(p^0, u^0)$ is a closed convex set containing all points on and above the hyperplane, $p^0 \cdot x = E(p^0, u^0)$. Now choose different prices $p^1$, keep $u^0$ fixed, and construct the closed convex set,

$$A(p^1, u^0) \equiv \{ x \in \mathbb{R}^n_+ \mid p^1 \cdot x \geq E(p^1, u^0) \}.$$

Imagine proceeding like this for all prices $p \gg 0$ and forming the infinite intersection,

$$A(u^0) \equiv \bigcap_{p \gg 0} A(p, u^0) = \{ x \in \mathbb{R}^n_+ \mid p \cdot x \geq E(p, u^0) \text{ for all } p \gg 0 \}. \quad (2.1)$$

The shaded area in Fig. 2.1(b) illustrates the intersection of a finite number of the $A(p, u^0)$, and gives some intuition about what $A(u^0)$ will look like. It is easy to imagine that as more and more prices are considered and more sets are added to the intersection, the shaded area will more closely resemble a superior set for some quasiconcave real-valued function. One might suspect, therefore, that these sets can be used to construct something.

![Figure 2.1](image_url)
very much like a direct utility function representing nice convex, monotonic preferences. This is indeed the case and is demonstrated by the following theorem.

**THEOREM 2.1 Constructing a Utility Function from an Expenditure Function**

Let \( E : \mathbb{R}^n_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) satisfy properties 1 through 7 of an expenditure function given in Theorem 1.7. Let \( A(u) \) be as in (2.1). Then the function \( u : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+ \) given by

\[
u(x) \equiv \max\{u \geq 0 \mid x \in A(u)\}
\]

is increasing, unbounded above, and quasiconcave.

You might be wondering why we have chosen to define \( u(x) \) the way we have. After all, there are many ways one can employ \( E(p, u) \) to assign numbers to each \( x \in \mathbb{R}^n_+ \). To understand why, forget this definition of \( u(x) \) and for the moment suppose that \( E(p, u) \) is in fact the expenditure function generated by some utility function \( u(x) \). How might we recover \( u(x) \) from knowledge of \( E(p, u) \)? Note that by the definition of an expenditure function, \( p \cdot x \geq E(p, u(x)) \) for all \( p \gg 0 \), and, typically, there will be equality for some price. Therefore, because \( E \) is strictly increasing in \( u \), \( u(x) \) is the largest value of \( u \) such that \( p \cdot x \geq E(p, u) \) for all \( p \gg 0 \). That is, \( u(x) \) is the largest value of \( u \) such that \( x \in A(u) \). Consequently, the construction we have given is just right for recovering the utility function that generated \( E(p, u) \) when in fact \( E(p, u) \) is an expenditure function. But the preceding considerations give us a strategy for showing that it is: first, show that \( u(x) \) defined as in the statement of Theorem 2.1 is a utility function satisfying our axioms. (This is the content of Theorem 2.1.) Second, show that \( E \) is in fact the expenditure function generated by \( u(x) \). (This is the content of Theorem 2.2.) We now give the proof of Theorem 2.1.

**Proof:** Note that by the definition of \( A(u) \), we may write \( u(x) \) as

\[
u(x) = \max\{u \geq 0 \mid p \cdot x \geq E(p, u) \forall p \gg 0\}
\]

The first thing that must be established is that \( u(x) \) is well-defined. That is, it must be shown that the set \( \{u \geq 0 \mid p \cdot x \geq E(p, u) \forall p \gg 0\} \) contains a largest element. We shall sketch the argument. First, this set, call it \( B(x) \), must be bounded above because \( E(p, u) \) is unbounded above and increasing in \( u \). Thus, \( B(x) \) possesses an upper bound and hence also a least upper bound, \( \bar{u} \). It must be shown that \( \bar{u} \in B(x) \). But this follows because \( B(x) \) is closed, which we will not show.

Having argued that \( u(x) \) is well-defined, let us consider the claim that it is increasing. Consider \( x^1 \geq x^2 \). Then

\[
p \cdot x^1 \geq p \cdot x^2 \quad \forall p \gg 0,
\]

(P.1)
because all components of $x^1$ are at least as large as the corresponding component of $x^2$. By the definition of $u(x^2)$,

$$p \cdot x^2 \geq E(p, u(x^2)) \quad \forall p \gg 0. \quad (P.2)$$

Together, (P.1) and (P.2) imply that

$$p \cdot x^1 \geq E(p, u(x^2)) \quad \forall p \gg 0. \quad (P.3)$$

Consequently, $u(x^2)$ satisfies the condition: $x^1 \in A(u(x^2))$. But $u(x^1)$ is the largest $u$ satisfying $x^1 \in A(u)$. Hence, $u(x^1) \geq u(x^2)$, which shows that $u(x)$ is increasing.

The unboundedness of $u(\cdot)$ on $\mathbb{R}^n_+$ can be shown by appealing to the increasing, concavity, homogeneity, and differentiability properties of $E(\cdot)$ in $p$, and to the fact that its domain in $u$ is all of $\mathbb{R}^n_+$. We shall not give the proof here (although it can be gleaned from the proof of Theorem 2.2 below).

To show that $u(\cdot)$ is quasiconcave, we must show that for all $x^1, x^2$, and convex combinations $x', u(x') \geq \min[u(x^1), u(x^2)]$. To see this, suppose that $u(x^1) = \min[u(x^1), u(x^2)]$. Because $E$ is strictly increasing in $u$, we know that $E(p, u(x^1)) \leq E(p, u(x^2))$ and that therefore

$$tE(p, u(x^1)) + (1-t)E(p, u(x^2)) \geq E(p, u(x^1)) \quad \forall t \in [0, 1]. \quad (P.4)$$

From the definitions of $u(x^1)$ and $u(x^2)$, we know that

$$p \cdot x^1 \geq E(p, u(x^1)) \quad \forall p \gg 0,$$
$$p \cdot x^2 \geq E(p, u(x^2)) \quad \forall p \gg 0.$$

Multiplying by $t \geq 0$ and $(1-t) \geq 0$, respectively, adding, and using (P.4) gives

$$p \cdot x' \geq E(p, u(x^1)) \quad \forall p \gg 0 \quad \text{and} \quad t \in [0, 1].$$

Consequently, by definition of $u(x')$, $u(x') \geq u(x^1) = \min[u(x^1), u(x^2)]$ as we sought to show.

Theorem 2.1 tells us we can begin with an expenditure function and use it to construct a direct utility function representing some convex, monotonic preferences. We actually know a bit more about those preferences. If we begin with them and derive the associated expenditure function, we end up with the function $E(\cdot)$ we started with!

**Theorem 2.2**

**The Expenditure Function of Derived Utility, $u$, is $E$**

Let $E(p, u)$, defined on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, satisfy properties 1 to 7 of an expenditure function given in Theorem 1.7 and let $u(x)$ be derived from $E$ as in Theorem 2.1. Then for all non-negative
prices and utility,

\[ E(p, u) = \min_x p \cdot x \quad \text{s.t.} \quad u(x) \geq u. \]

That is, \( E(p, u) \) is the expenditure function generated by derived utility \( u(x) \).

**Proof:** Fix \( p^0 > 0 \) and \( u^0 \geq 0 \) and suppose \( x \in \mathbb{R}^n_+ \) satisfies \( u(x) \geq u^0 \). Note that because \( u(\cdot) \) is derived from \( E \) as in Theorem 2.1, we must then have

\[ p \cdot x \geq E(p, u(x)) \quad \forall p \gg 0. \]

Furthermore, because \( E \) is increasing in utility and \( u(x) \geq u^0 \), we must have

\[ p \cdot x \geq E(p, u^0) \quad \forall p \gg 0. \quad (P.1) \]

Consequently, for any given prices \( p^0 \), we have established that

\[ E(p^0, u^0) \leq p^0 \cdot x \quad \forall x \in \mathbb{R}^n_+ \quad \text{s.t.} \quad u(x) \geq u^0. \quad (P.2) \]

But (P.2) then implies that

\[ E(p^0, u^0) \leq \min_{x \in \mathbb{R}^n_+} p^0 \cdot x \quad \text{s.t.} \quad u(x) \geq u^0. \quad (P.3) \]

We would like to show that the first inequality in (P.3) is an equality. To do so, it suffices to find a single \( x^0 \in \mathbb{R}^n_+ \) such that

\[ p^0 \cdot x^0 \leq E(p^0, u^0) \quad \text{and} \quad u(x^0) \geq u^0, \quad (P.4) \]

because this would clearly imply that the minimum on the right-hand side of (P.3) could not be greater than \( E(p^0, u^0) \).

To establish (P.4), note that by Euler’s theorem (Theorem A2.7), because \( E \) is differentiable and homogeneous of degree 1 in \( p \),

\[ E(p, u) = \frac{\partial E(p, u)}{\partial p} \cdot p \quad \forall p \gg 0, \quad (P.5) \]

where we use \( \partial E(p, u)/\partial p \equiv (\partial E(p, u)/\partial p_1, \ldots, \partial E(p, u)/\partial p_n) \) to denote the vector of price-partial derivatives of \( E \). Also, because \( E(p, u) \) is concave in \( p \), Theorem A2.4 implies that for all \( p \gg 0 \),

\[ E(p, u^0) \leq E(p^0, u^0) + \frac{\partial E(p^0, u^0)}{\partial p} \cdot (p - p^0). \quad (P.6) \]
But evaluating (P.5) at \((p^0, u^0)\) and combining this with (P.6) implies that

\[
E(p, u^0) \leq \left( \frac{\partial E(p^0, u^0)}{\partial p} \right) \cdot p \quad \forall \; p \gg 0.
\] (P.7)

Letting \(x^0 = \frac{\partial E(p^0, u^0)}{\partial p}\), note that \(x^0 \in \mathbb{R}^n_+\) because \(E\) is increasing in \(p\). We may rewrite (P.7) now as

\[
p \cdot x^0 \geq E(p, u^0) \quad \forall \; p \gg 0.
\] (P.8)

So, by the definition of \(u(\cdot)\), we must have \(u(x^0) \geq u^0\). Furthermore, evaluating (P.5) at \((p^0, u^0)\) yields \(E(p^0, u^0) = p^0 \cdot x^0\). Thus, we have established (P.4) for this choice of \(x^0\), and therefore we have shown that

\[
E(p^0, u^0) = \min_{x \in \mathbb{R}^n_+} p^0 \cdot x \quad \text{s.t.} \quad u(x) \geq u^0.
\]

Because \(p^0 \gg 0\) and \(u^0 \geq 0\) were arbitrary, we have shown that \(E(p, u)\) coincides with the expenditure function of \(u(x)\) on \(\mathbb{R}^n_+ \times \mathbb{R}_+\).

The last two theorems tell us that any time we can write down a function of prices and utility that satisfies properties 1 to 7 of Theorem 1.7, it will be a legitimate expenditure function for some preferences satisfying many of the usual axioms. We can of course then differentiate this function with respect to product prices to obtain the associated system of Hicksian demands. If the underlying preferences are continuous and strictly increasing, we can invert the function in \(u\), obtain the associated indirect utility function, apply Roy’s identity, and derive the system of Marshallian demands as well. Every time, we are assured that the resulting demand systems possess all properties required by utility maximisation. For theoretical purposes, therefore, a choice can be made. One can start with a direct utility function and proceed by solving the appropriate optimisation problems to derive the Hicksian and Marshallian demands. Or one can begin with an expenditure function and proceed to obtain consumer demand systems by the generally easier route of inversion and simple differentiation.

### 2.1.2 CONVEXITY AND MONOTONICITY

You may recall that after introducing the convexity axiom on preferences, it was stated that 'the predictive content of the theory would be the same with or without it'. This is an opportune time to support that claim and to investigate the import of the monotonicity assumption as well.

For the present discussion, let us suppose only that \(u(x)\) is continuous. Thus, \(u(x)\) need be neither increasing nor quasiconcave.
Let \( e(p, u) \) be the expenditure function generated by \( u(x) \). As we know, the continuity of \( u(x) \) is enough to guarantee that \( e(p, u) \) is well-defined. Moreover, \( e(p, u) \) is continuous.

Going one step further, consider the utility function, call it \( w(x) \), generated by \( e(\cdot) \) in the now familiar way, that is,

\[
w(x) \equiv \max\{u \geq 0 \mid p \cdot x \geq e(p, u) \quad \forall \ p \gg 0\}.
\]

A look at the proof of Theorem 2.1 will convince you that \( w(x) \) is increasing and quasiconcave. Thus, regardless of whether or not \( u(x) \) is quasiconcave or increasing, \( w(x) \) will be both quasiconcave and increasing. Clearly, then, \( u(x) \) and \( w(x) \) need not coincide. How then are they related?

It is easy to see that \( w(x) \geq u(x) \) for all \( x \in \mathbb{R}^n \). This follows because by the definition of \( e(\cdot) \), we have \( e(p, u(x)) \leq p \cdot x \quad \forall \ p \gg 0 \). The desired inequality now follows from the definition of \( w(x) \).

Thus, for any \( u \geq 0 \), the level-\( u \) superior set for \( u(x) \), say \( S(u) \), will be contained in the level-\( u \) superior set for \( w(x) \), say, \( T(u) \). Moreover, because \( w(x) \) is quasiconcave, \( T(u) \) is convex.

Now consider Fig. 2.2. If \( u(x) \) happens to be increasing and quasiconcave, then the boundary of \( S(u) \) yields the negatively sloped, convex indifference curve \( u(x) = u \) in Fig. 2.2(a). Note then that each point on that boundary is the expenditure-minimising bundle to achieve utility \( u \) at some price vector \( p \gg 0 \). Consequently, if \( u(x^0) = u \), then for some \( p^0 \gg 0 \), we have \( e(p^0, u) = p^0 \cdot x^0 \). But because \( e(\cdot) \) is strictly increasing in \( u \), this means that \( w(x^0) \leq u = u(x^0) \). But because \( w(x^0) \geq u(x^0) \) always holds, we must then have \( w(x^0) = u(x^0) \). Because \( u \) was arbitrary, this shows that in this case, \( w(x) = u(x) \) for all \( x \). But this is not much of a surprise in light of Theorems 2.1 and 2.2 and the assumed quasiconcavity and increasing properties of \( u(x) \).

The case depicted in Fig. 2.2(b) is more interesting. There, \( u(x) \) is neither increasing nor quasiconcave. Again, the boundary of \( S(u) \) yields the indifference curve \( u(x) = u \).

Note that some bundles on the indifference curve never minimise the expenditure required to obtain utility level \( u \) regardless of the price vector. The thick lines in Fig. 2.2(c) show those bundles that do minimise expenditure at some positive price vector. For those bundles \( x \) on the thick line segments in Fig. 2.2(c), we therefore have as before that \( w(x) = u(x) = u \). But because \( w(x) \) is quasiconcave and increasing, the \( w(x) = u(x) \) indifference curve must be as depicted in Fig. 2.2(d). Thus, \( w(x) \) differs from \( u(x) \) only as much as is required to become strictly increasing and quasiconcave.

Given the relationship between their indifference curves, it is clear that if some bundle maximises \( u(x) \) subject to \( p \cdot x \leq y \), then the same bundle maximises \( w(x) \) subject to \( p \cdot x \leq y \). (Careful, the converse is false.) Consequently, any observable demand behaviour that can be generated by a non-increasing, non-quasiconcave utility function, like \( u(x) \), can also be generated by an increasing, quasiconcave utility function, like \( w(x) \).
Figure 2.2. Duality between expenditure and utility.
It is in this sense that the assumptions of monotonicity and convexity of preferences have no observable implications for our theory of consumer demand.¹

2.1.3 INDIRECT UTILITY AND CONSUMER PREFERENCES

We have seen how duality allows us to work from the expenditure function to the direct utility function. Because the expenditure and indirect utility functions are so closely related (i.e., are inverses of each other), it should come as no surprise that it is also possible to begin with an indirect utility function and work back to the underlying direct utility function. In this section, we outline the duality between direct and indirect utility functions.

Suppose that \( u(x) \) generates the indirect utility function \( v(p, p \cdot x) \). Then by definition, for every \( x \in \mathbb{R}^n_+ \), \( v(p, p \cdot x) \geq u(x) \) holds for every \( p \gg 0 \). In addition, there will typically be some price vector for which the inequality is an equality. Evidently, then we may write

\[
   u(x) = \min_{p \in \mathbb{R}^n_+} v(p, p \cdot x). \tag{2.2}
\]

Thus, (2.2) provides a means for recovering the utility function \( u(x) \) from knowledge of only the indirect utility function it generates. The following theorem gives one version of this result, although the assumptions are not the weakest possible.

**THEOREM 2.3**

**Duality Between Direct and Indirect Utility**

Suppose that \( u(x) \) is quasiconcave and differentiable on \( \mathbb{R}^n_+ \) with strictly positive partial derivatives there. Then for all \( x \in \mathbb{R}^n_+ \), \( v(p, p \cdot x) \), the indirect utility function generated by \( u(x) \), achieves a minimum in \( p \) on \( \mathbb{R}^n_+ \), and

\[
   u(x) = \min_{p \in \mathbb{R}^n_+} v(p, p \cdot x). \tag{T.1}
\]

**Proof:** According to the discussion preceding Theorem 2.3, the left-hand side of (T.1) never exceeds the right-hand side. Therefore, it suffices to show that for each \( x \gg 0 \), there is some \( p \gg 0 \) such that

\[
   u(x) = v(p, p \cdot x). \tag{P.1}
\]

¹Before ending this discussion, we give a cautionary note on the conclusion regarding monotonicity. The fact that the demand behaviour generated by \( u(x) \) in the preceding second case could be captured by the increasing function \( w(x) \) relies on the assumption that the consumer only faces non-negative prices. For example, if with two goods, one of the prices, say, \( p_2 \) were negative, then we may have a situation such as that in Fig. 2.2(e), where \( x^* \) is optimal for the utility function \( u(x) \) but not for the increasing function \( w(x) \). Thus, if prices can be negative, monotonicity is not without observable consequences.
EXAMPLE 2.1 Let us take a particular case and derive the direct utility function. Suppose such that $V$ function that $v$ imises $(P.1)$ holds for $(p^0, x^0)$, but because $x^0$ was arbitrary, we may conclude that for every $x \gg 0$, $(P.1)$ holds for some $p \gg 0$. 

As in the case of expenditure functions, one can show by using (T.1) that if some function $V(p, y)$ has all the properties of an indirect utility function given in Theorem 1.6, then $V(p, y)$ is in fact an indirect utility function. We will not pursue this result here, however. The interested reader may consult Diewert (1974).

Finally, we note that (T.1) can be written in another form, which is sometimes more convenient. Note that because $v(p, y)$ is homogeneous of degree zero in $(p, y)$, we have $v(p, p \cdot x) = v(p/(p \cdot x), 1)$ whenever $p \cdot x > 0$. Consequently, if $x \gg 0$ and $p^* \gg 0$ minimises $v(p, p \cdot x)$ for $p \in \mathbb{R}^n_+$, then $\tilde{p} = p^*/(p^* \cdot x) > 0$ minimises $v(p, 1)$ for $p \in \mathbb{R}^n_+$ such that $p \cdot x = 1$. Moreover, $v(p^*, p^* \cdot x) = v(\tilde{p}, 1)$. Thus, we may rewrite (T.1) as

$$u(x) = \min_{p \in \mathbb{R}^n_+} v(p, 1) \quad \text{s.t. } \quad p \cdot x = 1. \quad (T.1')$$

Whether we use (T.1) or $(T.1')$ to recover $u(x)$ from $v(p, y)$ does not matter. Simply choose that which is more convenient. One disadvantage of (T.1) is that it always possesses multiple solutions because of the homogeneity of $v$ (i.e., if $p^*$ solves (T.1), then so does $q^i p^*$ for all $i > 0$). Consequently, we could not, for example, apply Theorem A2.22 (the Envelope theorem) as we shall have occasion to do in what follows. For purposes such as these, $(T.1')$ is distinctly superior.

EXAMPLE 2.1 Let us take a particular case and derive the direct utility function. Suppose that $v(p, y) = y(p_1' + p_2')^{-1/r}$. From the latter part of Example 1.2, we know this satisfies all necessary properties of an indirect utility function. We will use $(T.1')$ to recover $u(x)$. Setting $y = 1$ yields $v(p, 1) = (p_1' + p_2')^{-1/r}$. The direct utility function therefore will be the minimum-value function,

$$u(x_1, x_2) = \min_{p_1, p_2} (p_1' + p_2')^{-1/r} \quad \text{s.t. } \quad p_1 x_1 + p_2 x_2 = 1.$$
First, solve the minimisation problem and then evaluate the objective function at the solution to form the minimum-value function. The first-order conditions for the Lagrangian require that the optimal \( p_1^* \) and \( p_2^* \) satisfy

\[-((p_1^*)^r + (p_2^*)^r)^{-1}(p_1^*)^{r-1} - \lambda^* x_1 = 0, \tag{E.1}\]
\[-((p_2^*)^r + (p_2^*)^r)^{-1}(p_2^*)^{r-1} - \lambda^* x_2 = 0, \tag{E.2}\]
\[1 - p_1^* x_1 - p_2^* x_2 = 0. \tag{E.3}\]

Eliminating \( \lambda^* \) from (E.1) and (E.2) gives

\[p_1^* = p_2^* \left( \frac{x_1}{x_2} \right)^{1/(r-1)}. \tag{E.4}\]

Substituting from (E.4) into (E.3) and using (E.4) again, after a bit of algebra, gives the solutions

\[p_1^* = \left( \frac{x_1^{r/(r-1)} + x_2^{r/(r-1)}}{x_1^{r/(r-1)} + x_2^{r/(r-1)}} \right)^{1/(r-1)}, \tag{E.5}\]
\[p_2^* = \left( \frac{x_1^{r/(r-1)} + x_2^{r/(r-1)}}{x_1^{r/(r-1)} + x_2^{r/(r-1)}} \right)^{1/(r-1)}. \tag{E.6}\]

Substituting these into the objective function and forming \( u(x_1, x_2) \), we obtain

\[u(x_1, x_2) = \left( \left( x_1^{r/(r-1)} + x_2^{r/(r-1)} \right)^{1/r} \right) \]
\[= \left( \left( x_1^{r/(r-1)} + x_2^{r/(r-1)} \right)^{1-r} \right)^{-1/r} \]
\[= \left( x_1^{r/(r-1)} + x_2^{r/(r-1)} \right)^{(r-1)/r}. \]

Defining \( \rho \equiv r/(r-1) \) yields

\[u(x_1, x_2) = \left( x_1^\rho + x_2^\rho \right)^{1/\rho}. \tag{E.7}\]

This is the CES direct utility function we started with in Example 1.2, as it should be.

The last duality result we take up concerns the consumer’s inverse demand functions. Throughout the chapter, we have concentrated on the ordinary Marshallian demand functions, where quantity demanded is expressed as a function of prices and income.
Occasionally, it is convenient to work with demand functions in inverse form. Here we view the demand price for commodity $i$ as a function of the quantities of good $i$ and of all other goods and write $p_i = p_i(x)$. Duality theory offers a simple way to derive the system of consumer inverse demand functions, as the following theorem shows, where we shall simply assume differentiability as needed.

**THEOREM 2.4 (Hotelling, Wold) Duality and the System of Inverse Demands**

Let $u(x)$ be the consumer’s direct utility function. Then the inverse demand function for good $i$ associated with income $y = 1$ is given by

$$p_i(x) = \frac{\partial u(x)/\partial x_i}{\sum_{j=1}^{n} x_j(\partial u(x)/\partial x_j)}.$$

**Proof:** By the definition of $p(x)$, we have $u(x) = v(p(x), 1)$ and $[p(x)] \cdot x = 1$ for all $x$. Consequently, by the discussion preceding Theorem 2.3 and the normalisation argument,

$$u(x) = v(p(x), 1) = \min_{p \in \mathbb{R}^n_{++}} v(p, 1) \quad \text{s.t.} \quad p \cdot x = 1. \quad (P.1)$$

Consider now the Lagrangian associated with the minimisation problem in (P.1),

$$L(p, \lambda) = v(p, 1) - \lambda(1 - p \cdot x).$$

Applying the Envelope theorem yields

$$\frac{\partial u(x)}{\partial x_i} = \frac{\partial L(p^*, \lambda^*)}{\partial x_i} = \lambda^* p^*_i, \quad i = 1, \ldots, n. \quad (P.2)$$

where $p^* = p(x)$, and $\lambda^*$ is the optimal value of the Lagrange multiplier. Assuming $\partial u(x)/\partial x_i > 0$, we have then that $\lambda^* > 0$.

Multiplying (P.2) by $x_i$ and summing over $i$ gives

$$\sum_{i=1}^{n} x_i \frac{\partial u(x)}{\partial x_i} = \lambda^* \sum_{i=1}^{n} p^*_i x_i$$

$$= \lambda^* \sum_{i=1}^{n} p_i(x) x_i$$

$$= \lambda^*,$$ \hspace{1cm} (P.3)

because $[p(x)] \cdot x = 1$. Combining (P.2) and (P.3) and recalling that $p^*_i = p_i(x)$ yields the desired result.
EXAMPLE 2.2  Let us take the case of the CES utility function once again. If \( u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho} \), then

\[
\frac{\partial u(x)}{\partial x_j} = \left( x_1^\rho + x_2^\rho \right)^{(1/\rho)-1} x_j^{\rho-1}.
\]

Multiplying by \( x_j \), summing over \( j = 1, 2 \), forming the required ratios, and invoking Theorem 2.4 gives the following system of inverse demand functions when income \( y = 1 \):

\[
p_1 = x_1^{\rho-1} \left( x_1^\rho + x_2^\rho \right)^{-1},
\]

\[
p_2 = x_2^{\rho-1} \left( x_1^\rho + x_2^\rho \right)^{-1}.
\]

Notice carefully that these are precisely the solutions (E.5) and (E.6) to the first-order conditions in Example 2.1, after substituting for \( r \equiv \rho/(\rho - 1) \). This is no coincidence. In general, the solutions to the consumer’s utility-maximisation problem give Marshallian demand as a function of price, and the solutions to its dual, the (normalised) indirect utility-minimisation problem, give inverse demands as functions of quantity.

\[\square\]

2.2 INTEGRABILITY

In Chapter 1, we showed that a utility-maximising consumer’s demand function must satisfy homogeneity of degree zero, budget balancedness, symmetry, and negative semidefiniteness, along with Cournot and Engel aggregation. But, really, there is some redundancy in these conditions. In particular, we know from Theorem 1.17 that both aggregation results follow directly from budget balancedness. There is another redundancy as well. Of the remaining four conditions, only budget balancedness, symmetry, and negative semidefinite-ness are truly independent: homogeneity of degree zero is implied by the others. In fact, homogeneity is implied by budget balancedness and symmetry alone, as the following theorem demonstrates.

THEOREM 2.5  **Budget Balancedness and Symmetry Imply Homogeneity**

If \( \mathbf{x}(\mathbf{p}, y) \) satisfies budget balancedness and its Slutsky matrix is symmetric, then it is homogeneous of degree zero in \( \mathbf{p} \) and \( y \).

**Proof:** Recall from the proof of Theorem 1.17 that when budget balancedness holds, we may differentiate the budget equation with respect to prices and income to obtain for, \( i = 1, \ldots, n \),

\[
\sum_{j=1}^{n} p_j \frac{\partial x_j(\mathbf{p}, y)}{\partial p_i} = -x_i(\mathbf{p}, y),
\]

(P.1)
and
\[ \sum_{j=1}^{n} p_j \frac{\partial x_j(p, y)}{\partial y} = 1. \quad (P.2) \]

Fix \( p \) and \( y \), then let \( f_i(t) = x_i(tp, ty) \) for all \( t > 0 \). We must show that \( f_i(t) \) is constant in \( t \) or that \( f_i'(t) = 0 \) for all \( t > 0 \).

Differentiating \( f_i \) with respect to \( t \) gives
\[ f_i'(t) = \sum_{j=1}^{n} \frac{\partial x_i(tp, ty)}{\partial p_j} p_j + \frac{\partial x_i(tp, ty)}{\partial y} y. \quad (P.3) \]

Now by budget balancedness, \( tp \cdot x(tp, ty) = ty \), so that dividing by \( t > 0 \), we may write
\[ y = \sum_{j=1}^{n} p_j x_j(tp, ty). \quad (P.4) \]

Substituting from (P.4) for \( y \) in (P.3) and rearranging yields
\[ f_i'(t) = \sum_{j=1}^{n} p_j \left[ \frac{\partial x_i(tp, ty)}{\partial p_j} + \frac{\partial x_i(tp, ty)}{\partial y} x_j(tp, ty) \right]. \]

But the term in square brackets is the \( ij \)th entry of the Slutsky matrix, which, by assumption, is symmetric. Consequently we may interchange \( i \) and \( j \) within those brackets and maintain equality. Therefore,
\[
\begin{align*}
    f_i'(t) &= \sum_{j=1}^{n} p_j \left[ \frac{\partial x_i(tp, ty)}{\partial p_j} + \frac{\partial x_i(tp, ty)}{\partial y} x_j(tp, ty) \right] \\
    &= \left[ \sum_{j=1}^{n} p_j \frac{\partial x_i(tp, ty)}{\partial p_j} \right] + x_i(tp, ty) \left[ \sum_{j=1}^{n} p_j \frac{\partial x_j(tp, ty)}{\partial y} \right] \\
    &= \frac{1}{t} \left[ \sum_{j=1}^{n} p_j \frac{\partial x_j(tp, ty)}{\partial p_j} \right] + x_i(tp, ty) \frac{1}{t} \left[ \sum_{j=1}^{n} p_j \frac{\partial x_j(tp, ty)}{\partial y} \right] \\
    &= \frac{1}{t} [-x_i(tp, ty)] + x_i(tp, ty) \frac{1}{t} [1] \\
    &= 0,
\end{align*}
\]

where the second-to-last equality follows from (P.1) and (P.2) evaluated at \((tp, ty)\). \( \blacksquare \)
Thus, if \( x(p, y) \) is a utility-maximiser’s system of demand functions, we may (compactly) summarise the implications for observable behaviour we have so far discovered in the following three items alone:

- **Budget Balancedness**: \( \mathbf{p} \cdot x(p, y) = y \).
- **Negative Semidefiniteness**: The associated Slutsky matrix \( s(p, y) \) must be negative semidefinite.
- **Symmetry**: \( s(p, y) \) must be symmetric.

We would like to know whether or not this list is exhaustive. That is, are these the *only* implications for observable behaviour that flow from our utility-maximisation model of consumer behaviour? Are there perhaps other, additional implications that we have so far not discovered? Remarkably, it can be shown that this list is in fact complete – there are no other independent restrictions imposed on demand behaviour by the theory of the utility-maximising consumer.

But how does one even begin to prove such a result? The solution method is ingenious, and its origins date back to Antonelli (1886). The idea is this: suppose we are given a vector-valued function of prices and income, and that we are then somehow able to construct a utility function that generates precisely this same function as its demand function. Then, clearly, that original function must be consistent with our theory of the utility-maximising consumer because it is in fact the demand function of a consumer with the utility function we constructed. Antonelli’s insight was to realise that if the vector-valued function of prices and income we start with satisfies just the three preceding conditions, then there must indeed exist a utility function that generates it as its demand function. The problem of recovering a consumer’s utility function from his demand function is known as the *integrability problem*.

The implications of this are significant. According to Antonelli’s insight, if a function of prices and income satisfies the three preceding conditions, it is the demand function for some utility-maximising consumer. We already know that *only* if a function of prices and income satisfies those same conditions will it be the demand function for a utility-maximising consumer. Putting these two together, we must conclude that those three conditions – and those three conditions alone – provide a complete and definitive test of our theory of consumer behaviour. That is, demand behaviour is consistent with the theory of utility maximisation if and only if it satisfies budget balancedness, negative semidefiniteness, and symmetry. This impressive result warrants a formal statement.

**THEOREM 2.6 Integrability Theorem**

A continuously differentiable function \( x: \mathbb{R}_{++}^{n+1} \to \mathbb{R}_+^n \) is the demand function generated by some increasing, quasiconcave utility function if (and only if, when utility is continuous, strictly increasing, and strictly quasiconcave) it satisfies budget balancedness, symmetry, and negative semidefiniteness.
We now sketch a proof of Antonelli’s result. However, we shall take the modern approach to this problem as developed by Hurwicz and Uzawa (1971). Their strategy of proof is a beautiful illustration of the power of duality theory.

Proof: (Sketch) Since we have already demonstrated the ‘only if’ part, it suffices to prove the ‘if’ part of the statement. So suppose some function \( x(p, y) \) satisfies budget balancedness, symmetry, and negative semidefiniteness. We must somehow show that there is a utility function that generates \( x(\cdot) \) as its demand function.

Consider an arbitrary expenditure function, \( e(p, u) \), generated by some increasing quasiconcave utility function \( u(x) \), and suppose that \( u(x) \) generates the Marshallian demand function \( x^m(p, y) \). At this stage, there need be no relation between \( x(\cdot) \) and \( e(\cdot) \), \( x(\cdot) \) and \( u(\cdot) \), or \( x(\cdot) \) and \( x^m(\cdot) \).

But just for the sake of argument, suppose that \( x(\cdot) \) and \( e(\cdot) \) happen to be related as follows:

\[
\frac{\partial e(p, u)}{\partial p_i} = x_i(p, e(p, u)), \quad \forall (p, u), \quad i = 1, \ldots, n. \tag{P.1}
\]

Can we then say anything about the relationship between \( x(p, y) \) and the utility function \( u(x) \) from which \( e(p, u) \) was derived? In fact, we can. If (P.1) holds, then \( x(p, y) \) is the demand function generated by the utility function \( u(x) \). That is, \( x(p, y) = x^m(p, y) \).

We now sketch why this is so. Note that if Shephard’s lemma were applicable, the left-hand side of (P.1) would be equal to \( x^h(p, u) \), so that (P.1) would imply

\[
x^h(p, u) = x(p, e(p, u)) \quad \forall (p, u). \tag{P.2}
\]

Moreover, if Theorem 1.9 were applicable, the Hicksian and Marshallian demand functions would be related as

\[
x^h(p, u) = x^m(p, e(p, u)) \quad \forall (p, u). \tag{P.3}
\]

Putting (P.2) and (P.3) together yields

\[
x(p, e(p, u)) = x^m(p, e(p, u)) \quad \forall (p, u). \tag{P.4}
\]

But now recall that, as an expenditure function, for each fixed \( p \), \( e(p, u) \) assumes every non-negative number as \( u \) varies over its domain. Consequently, (P.4) is equivalent to

\[
x(p, y) = x^m(p, y) \quad \forall (p, y)
\]
as claimed. (Despite the fact that perhaps neither Shephard’s lemma nor Theorem 1.9 can be applied, the preceding conclusion can be established.)

Thus, if the function \( x(p, y) \) is related to an expenditure function according to (P.1), then \( x(p, y) \) is the demand function generated by some increasing, quasiconcave utility function (i.e., that which, according to Theorem 2.1, generates the expenditure function).
We therefore have reduced our task to showing that there exists an expenditure function \( e(p, u) \) related to \( x(p, y) \) according to (P.1).

Now, finding an expenditure function so that (P.1) holds is no easy task. Indeed, (P.1) is known in the mathematics literature as a system of partial differential equations. Although such systems are often notoriously difficult to actually solve, there is an important result that tells us precisely when a solution is guaranteed to exist. And, for our purposes, existence is enough.

However, before stating this result, note the following. If (P.1) has a solution \( e(p, u) \), then upon differentiating both sides by \( p_j \), we would get

\[
\frac{\partial^2 e(p, u)}{\partial p_j \partial p_i} = \frac{\partial x_i(p, e(p, u))}{\partial p_j} + \frac{\partial e(p, u)}{\partial p_j} \frac{\partial x_i(p, e(p, u))}{\partial y}.
\]

By Shephard’s lemma, using (P.2), and letting \( y = e(p, u) \), this can be written as

\[
\frac{\partial^2 e(p, u)}{\partial p_j \partial p_i} = \frac{\partial x_i(p, y)}{\partial p_j} + x_j(p, y) \frac{\partial x_i(p, y)}{\partial y}.
\]  

Now note that the left-hand side of (P.5) is symmetric in \( i \) and \( j \) by Young’s theorem. Consequently, (P.5) implies that the right-hand side must be symmetric in \( i \) and \( j \) as well. Therefore, symmetry of the right-hand side in \( i \) and \( j \) is a necessary condition for the existence of a solution to (P.1).

Remarkably, it turns out that this condition is also sufficient for the existence of a solution. According to Frobenius’ theorem, a solution to (P.1) exists if and only if the right-hand side of (P.5) is symmetric in \( i \) and \( j \). Take a close look at the right-hand side of (P.5). It is precisely the \( ij \)th term of the Slutsky matrix associated with \( x(p, y) \). Consequently, because that Slutsky matrix satisfies symmetry, a function \( e(p, u) \) satisfying (P.1) is guaranteed to exist.

But will this function be a true expenditure function? Frobenius’ theorem is silent on this issue. However, by Theorem 2.2, \( e(p, u) \) will be an expenditure function if it has all the properties of an expenditure function listed in Theorem 1.7. We now attempt to verify each of those properties.

First, note that because \( e(p, u) \) satisfies (P.1), and because \( x(p, y) \) is non-negative, \( e(p, u) \) is automatically increasing in \( p \), and Shephard’s lemma is guaranteed by construction. Moreover, one can ensure it is continuous in \( (p, u) \), strictly increasing and unbounded in \( u \in \mathbb{R}_+ \), and that \( e(\cdot, u) = 0 \) when \( u = 0 \). As you are asked to show in Exercise 2.4, because (P.1) and budget balancedness are satisfied, \( e(\cdot, u) \) must be homogeneous of degree 1 in \( p \). Thus, the only remaining property of an expenditure function that must be established is concavity in \( p \).

By Theorem A 2.4, \( e(\cdot) \) will be concave in \( p \) if and only if its Hessian matrix with respect to \( p \) is negative semidefinite. But according to (P.5), this will be the case if and only if the Slutsky matrix associated with \( x(p, y) \) is negative semidefinite, which, by assumption, it is.
Altogether we have established the following: A solution \( e(\cdot) \) to (P.1) exists and is an expenditure function if and only if \( x(p, y) \) satisfies budget balancedness, symmetry, and negative semidefiniteness. This is precisely what we set out to show.

Although we have stressed the importance of this result for the theory itself, there are practical benefits as well. For example, if one wishes to estimate a consumer’s demand function based on a limited amount of data, and one wishes to impose as a restriction that the demand function be utility-generated, one is now free to specify any functional form for demand as long as it satisfies budget balancedness, symmetry, and negative semidefiniteness. As we now know, any such demand function is guaranteed to be utility-generated.

To give you a feel for how one can actually recover an expenditure function from a demand function, we consider an example involving three goods.

**Example 2.3** Suppose there are three goods and that a consumer’s demand behaviour is summarised by the functions

\[
x_i(p_1, p_2, p_3, y) = \frac{\alpha_i y}{p_i}, \quad i = 1, 2, 3,
\]

where \( \alpha_i > 0 \), and \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \).

It is straightforward to check that the vector of demands, \( x(p, y) \), satisfies budget balancedness, symmetry, and negative semidefiniteness. Consequently, by Theorem 2.6, \( x(p, y) \) must be utility-generated.

We shall be content to derive an expenditure function satisfying (P.1) in the previous proof. In Exercise 2.5, you are asked to go one step further and use the construction of Theorem 2.1 to recover a utility function generating the expenditure function obtained here. The utility function you recover then will generate the demand behaviour we began with here.

Our task then is to find \( e(p_1, p_2, p_3, u) \) that solves the following system of partial differential equations

\[
\frac{\partial e(p_1, p_2, p_3, u)}{\partial p_i} = \frac{\alpha_i e(p_1, p_2, p_3, u)}{p_i}, \quad i = 1, 2, 3.
\]

First, note that this can be rewritten as

\[
\frac{\partial \ln(e(p_1, p_2, p_3, u))}{\partial p_i} = \frac{\alpha_i}{p_i}, \quad i = 1, 2, 3. \tag{E.1}
\]

Now, if you were asked to find \( f(x) \) when told that \( f'(x) = \alpha / x \), you would have no trouble deducing that \( f(x) = \alpha \ln(x) + \text{constant} \). But (E.1) says just that, where \( f = \ln(e) \). The only additional element to keep in mind is that when partially differentiating with respect to, say, \( p_1 \), all the other variables – \( p_2, p_3 \), and \( u \) – are treated as constants. With this in mind,
it is easy to see that the three equations (E.1) imply the following three:

\[
\begin{align*}
\ln(e(p, u)) &= \alpha_1 \ln(p_1) + c_1(p_2, p_3, u), \\
\ln(e(p, u)) &= \alpha_2 \ln(p_2) + c_2(p_1, p_3, u), \\
\ln(e(p, u)) &= \alpha_3 \ln(p_3) + c_3(p_1, p_2, u),
\end{align*}
\]  
(E.2)

where the \( c_i(\cdot) \) functions are like the constant added before to \( f(x) \). But we must choose the \( c_i(\cdot) \) functions so that all three of these equalities hold simultaneously. With a little thought, you will convince yourself that (E.2) then implies

\[
\ln(e(p, u)) = \alpha_1 \ln(p_1) + \alpha_2 \ln(p_2) + \alpha_3 \ln(p_3) + c(u),
\]

where \( c(u) \) is some function of \( u \). But this means that

\[
e(p, u) = c(u)p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}.
\]

Because we must ensure that \( e(\cdot) \) is strictly increasing in \( u \), we may choose \( c(u) \) to be any strictly increasing function. It does not matter which, because the implied demand behaviour will be independent of such strictly increasing transformations. For example, we may choose \( c(u) = u \), so that our final solution is

\[
e(p, u) = up_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}.
\]

We leave it to you to check that this function satisfies the original system of partial differential equations and that it has all the properties required of an expenditure function.

### 2.3 Revealed Preference

So far, we have approached demand theory by assuming the consumer has preferences satisfying certain properties (complete, transitive, and strictly monotonic); then we have tried to deduce all of the observable properties of market demand that follow as a consequence (budget balancedness, symmetry, and negative semidefiniteness of the Slutsky matrix). Thus, we have begun by assuming something about things we cannot observe – preferences – to ultimately make predictions about something we can observe – consumer demand behaviour.

In his remarkable *Foundations of Economic Analysis*, Paul Samuelson (1947) suggested an alternative approach. Why not start and finish with observable behaviour? Samuelson showed how virtually every prediction ordinary consumer theory makes for a consumer’s observable market behaviour can also (and instead) be derived from a few simple and sensible assumptions about the consumer’s observable choices themselves, rather than about his unobservable preferences.
The basic idea is simple: if the consumer buys one bundle instead of another affordable bundle, then the first bundle is considered to be revealed preferred to the second. The presumption is that by actually choosing one bundle over another, the consumer conveys important information about his tastes. Instead of laying down axioms on a person’s preferences as we did before, we make assumptions about the consistency of the choices that are made. We make this all a bit more formal in the following.

**DEFINITION 2.1 Weak Axiom of Revealed Preference (WARP)**

A consumer’s choice behaviour satisfies WARP if for every distinct pair of bundles $x^0, x^1$ with $x^0$ chosen at prices $p^0$ and $x^1$ chosen at prices $p^1$,

$$p^0 \cdot x^1 \leq p^0 \cdot x^0 \implies p^1 \cdot x^0 > p^1 \cdot x^1.$$ 

In other words, WARP holds if whenever $x^0$ is revealed preferred to $x^1$, $x^1$ is never revealed preferred to $x^0$.

To better understand the implications of this definition, look at Fig. 2.3. In both parts, the consumer facing $p^0$ chooses $x^0$, and facing $p^1$ chooses $x^1$. In Fig. 2.3(a), the consumer’s choices satisfy WARP. There, $x^0$ is chosen when $x^1$ could have been, but was not, and when $x^1$ is chosen, the consumer could not have afforded $x^0$. By contrast, in Fig. 2.3(b), $x^0$ is again chosen when $x^1$ could have been, yet when $x^1$ is chosen, the consumer could have chosen $x^0$, but did not, violating WARP.

Now, suppose a consumer’s choice behaviour satisfies WARP. Let $x(p, y)$ denote the choice made by this consumer when faced with prices $p$ and income $y$. Note well that this is not a demand function because we have not mentioned utility or utility maximisation – it just denotes the quantities the consumer chooses facing $p$ and $y$. To keep this point clear in our minds, we refer to $x(p, y)$ as a choice function. In addition to WARP, we make one

![Figure 2.3](image-url)  
**Figure 2.3.** The Weak Axiom of Revealed Preference (WARP).
other assumption concerning the consumer’s choice behaviour, namely, that for \( p \gg 0 \), the choice \( x(p, y) \) satisfies budget balancedness, i.e., \( p \cdot x(p, y) = y \). The implications of these two apparently mild requirements on the consumer’s choice behaviour are rather remarkable.

The first consequence of WARP and budget balancedness is that the choice function \( x(p, y) \) must be homogeneous of degree zero in \((p, y)\). To see this, suppose \( x^0 \) is chosen when prices are \( p^0 \) and income is \( y^0 \), and suppose \( x^1 \) is chosen when prices are \( p^1 = tp^0 \) and income is \( y^1 = ty^0 \) for \( t > 0 \). Because \( y^1 = ty^0 \), when all income is spent, we must have \( p^1 \cdot x^1 = tp^0 \cdot x^0 \). First, substitute \( tp^0 \) for \( p^1 \) in this, divide by \( t \), and get

\[
    p^0 \cdot x^1 = p^0 \cdot x^0. \tag{2.3}
\]

Then substitute \( p^1 \) for \( tp^0 \) in the same equation and get

\[
    p^1 \cdot x^1 = p^1 \cdot x^0. \tag{2.4}
\]

If \( x^0 \) and \( x^1 \) are distinct bundles for which (2.3) holds, then WARP implies that the left-hand side in (2.4) must be strictly less than the right-hand side – a contradiction. Thus, these bundles cannot be distinct, and the consumer’s choice function therefore must be homogeneous of degree zero in prices and income.

Thus, the choice function \( x(p, y) \) must display one of the additional properties of a demand function. In fact, as we now show, \( x(p, y) \) must display yet another of those properties as well.

In Exercise 1.45, the notion of Slutsky-compensated demand was introduced. Let us consider the effect here of Slutsky compensation for the consumer’s choice behaviour. In case you missed the exercise, the Slutsky compensation is relative to some pre-specified bundle, say \( x^0 \). The idea is to consider the choices the consumer makes as prices vary arbitrarily while his income is compensated so that he can just afford the bundle \( x^0 \). (See Fig. 2.4.) Consequently, at prices \( p \), his income will be \( p \cdot x^0 \). Under these circumstances, his choice behaviour will be given by \( x(p, p \cdot x^0) \).

Figure 2.4. A Slutsky compensation in income.
Now fix $p_0 \gg 0, y_0 > 0$, and let $x^0 = x(p_0, y_0)$. Then if $p^1$ is any other price vector and $x^1 = x(p^1, p^1 \cdot x^0)$, WARP implies that

$$p^0 \cdot x^0 \leq p^0 \cdot x^1. \quad (2.5)$$

Indeed, if $x^1 = x^0$, then (2.5) holds with equality. And if $x^1 \neq x^0$, then because $x^1$ was chosen when $x^0$ was affordable (i.e., at prices $p^1$ and income $p^1 \cdot x^0$), WARP implies that $x^1$ is not affordable whenever $x^0$ is chosen. Consequently, the inequality in (2.5) would be strict.

Now, note that by budget balancedness:

$$p_1 \cdot x^0 = p_1 \cdot x(p_1, p_1 \cdot x^0). \quad (2.6)$$

Subtracting (2.5) from (2.6) then implies that for all prices $p^1$,

$$(p^1 - p^0) \cdot x^0 \geq (p^1 - p^0) \cdot x(p^1, p^1 \cdot x^0). \quad (2.7)$$

Because (2.7) holds for all prices $p^1$, let $p^1 = p^0 + tz$, where $t > 0$, and $z \in \mathbb{R}^n$ is arbitrary. Then (2.7) becomes

$$t(z \cdot x^0) \geq t(z \cdot x(p^1, p^1 \cdot x^0)). \quad (2.8)$$

Dividing by $t > 0$ gives

$$z \cdot x^0 \geq z \cdot x(p^0 + tz, (p^0 + tz) \cdot x^0), \quad (2.9)$$

where we have used the fact that $p^1 = p^0 + tz$.

Now for $z$ fixed, we may choose $\bar{t} > 0$ small enough so that $p^0 + rz \gg 0$ for all $t \in [0, \bar{t}]$, because $p_0 \gg 0$. Noting that (2.9) holds with equality when $t = 0$, (2.9) says that the function $f: [0, \bar{t}] \to \mathbb{R}$ defined by the right-hand side of (2.9), i.e.,

$$f(t) = z \cdot x(p^0 + rz, (p^0 + rz) \cdot x^0),$$

is maximised on $[0, \bar{t}]$ at $t = 0$. Thus, we must have $f'(0) \leq 0$. But taking the derivative of $f(t)$ and evaluating at $t = 0$ gives (assuming that $x(\cdot)$ is differentiable):

$$f'(0) = \sum_i \sum_j z_i \left[ \frac{\partial x_i(p^0, y^0)}{\partial p_j} + x_j(p^0, y^0) \frac{\partial x_i(p^0, y^0)}{\partial y} \right] z_j \leq 0. \quad (2.10)$$

Now, because $z \in \mathbb{R}^n$ was arbitrary, (2.10) says that the matrix whose $ij$th entry is

$$\frac{\partial x_i(p^0, y^0)}{\partial p_j} + x_j(p^0, y^0) \frac{\partial x_i(p^0, y^0)}{\partial y} \quad (2.11)$$
must be negative semidefinite. But this matrix is precisely the Slutsky matrix associated with the choice function \( x(p, y) \)!

Thus, we have demonstrated that if a choice function satisfies WARP and budget balancedness, then it must satisfy two other properties implied by utility maximisation, namely, homogeneity of degree zero and negative semidefiniteness of the Slutsky matrix.

If we could show, in addition, that the choice function’s Slutsky matrix was symmetric, then by our integrability result, that choice function would actually be a demand function because we would then be able to construct a utility function generating it.

Before pursuing this last point further, it is worthwhile to point out that if \( x(p, y) \) happens to be a utility-generated demand function then \( x(p, y) \) must satisfy WARP. To see this, suppose a utility-maximising consumer has strictly monotonic and strictly convex preferences. Then we know there will be a unique bundle demanded at every set of prices, and that bundle will always exhaust the consumer’s income. (See Exercise 1.16.) So let \( x^0 \) maximise utility facing prices \( p^0 \), let \( x^1 \) maximise utility facing \( p^1 \), and suppose \( p^0 \cdot x^1 \leq p^0 \cdot x^0 \). Because \( x^1 \), though affordable, is not chosen, it must be because \( u(x^0) > u(x^1) \).

Therefore, when \( x^1 \) is chosen facing prices \( p^1 \), it must be that \( x^0 \) is not available or that \( p^1 \cdot x^0 > p^1 \cdot x^1 \). Thus, \( p^0 \cdot x^1 \leq p^0 \cdot x^0 \) implies \( p^1 \cdot x^0 > p^1 \cdot x^1 \), so WARP is satisfied.

But again what about the other way around? What if a consumer’s choice function always satisfies WARP? Must that behaviour have been generated by utility maximisation? Put another way, must there exist a utility function that would yield the observed choices as the outcome of the utility-maximising process? If the answer is yes, we say the utility function rationalises the observed behaviour.

As it turns out, the answer is yes – and no. If there are only two goods, then WARP implies that there will exist some utility function that rationalises the choices; if, however, there are more than two goods, then even if WARP holds there need not be such a function.

The reason for the two-good exception is related to the symmetry of the Slutsky matrix and to transitivity.

It turns out that in the two-good case, budget balancedness together with homogeneity imply that the Slutsky matrix must be symmetric. (See Exercise 2.9.) Consequently, because WARP and budget balancedness imply homogeneity as well as negative semidefiniteness, then in the case of two goods, they also imply symmetry of the Slutsky matrix. Therefore, for two goods, our integrability theorem tells us that the choice function must be utility-generated.

An apparently distinct, yet ultimately equivalent, explanation for the two-good exception is that with two goods, the pairwise ranking of bundles implied through revealed preference turns out to have no intransitive cycles. (You are, in fact, asked to show this in Exercise 2.9.) And when this is so, there will be a utility representation generating the choice function. Thus, as we mentioned earlier in the text, there is a deep connection between the symmetry of the Slutsky matrix and the transitivity of consumer preferences.

For more than two goods, WARP and budget balancedness imply neither symmetry of the Slutsky matrix nor the absence of intransitive cycles in the revealed preferred to relation. Consequently, for more than two goods, WARP and budget balancedness are not equivalent to the utility-maximisation hypothesis.
This leads naturally to the question: how must we strengthen WARP to obtain a theory of revealed preference that is equivalent to the theory of utility maximisation? The answer lies in the ‘Strong Axiom of Revealed Preference’.

The **Strong Axiom of Revealed Preference (SARP)** is satisfied if, for every sequence of distinct bundles \( \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k \), where \( \mathbf{x}_0 \) is revealed preferred to \( \mathbf{x}_1 \), and \( \mathbf{x}_1 \) is revealed preferred to \( \mathbf{x}_2 \), and \( \mathbf{x}_k \) is revealed preferred to \( \mathbf{x}_0 \), it is not the case that \( \mathbf{x}_k \) is revealed preferred to \( \mathbf{x}_0 \). SARP rules out intransitive revealed preferences and therefore can be used to induce a complete and transitive preference relation, \( \succeq \), for which there will then exist a utility function that rationalises the observed behaviour. We omit the proof of this and instead refer the reader to Houthakker (1950) for the original argument, and to Richter (1966) for an elegant proof.

It is not difficult to show that if a consumer chooses bundles to maximise a strictly quasiconcave and strictly increasing utility function, his demand behaviour must satisfy SARP (see Exercise 2.11). Thus, a theory of demand built only on SARP, a restriction on observable choice, is essentially equivalent to the theory of demand built on utility maximisation. Under both SARP and the utility-maximisation hypothesis, consumer demand will be homogeneous and the Slutsky matrix will be negative semidefinite and symmetric.

In our analysis so far, we have focused on revealed preference axioms and consumer choice functions. In effect, we have been acting as though we had an infinitely large collection of price and quantity data with which to work. To many, the original allure of revealed preference theory was the promise it held of being able to begin with actual data and work from the implied utility functions to predict consumer behaviour. Because real-world data sets will never contain more than a finite number of sample points, more recent work on revealed preference has attempted to grapple directly with some of the problems that arise when the number of observations is finite.

To that end, Afriat (1967) introduced the **Generalised Axiom of Revealed Preference (GARP)**, a slightly weaker requirement than SARP, and proved an analogue of the integrability theorem (Theorem 2.6). According to **Afriat’s theorem**, a finite set of observed price and quantity data satisfy GARP if and only if there exists a continuous, increasing, and concave utility function that rationalises the data. (Exercise 2.12 explores a weaker version of Afriat’s theorem.) However, with only a finite amount of data, the consumer’s preferences are not completely pinned down at bundles ‘out-of-sample’. Thus, there can be many different utility functions that rationalise the (finite) data.

But, in some cases, revealed preference does allow us to make certain ‘out-of-sample’ comparisons. For instance, consider Fig. 2.5. There we suppose we have observed the consumer to choose \( \mathbf{x}_0 \) at prices \( \mathbf{p}_0 \) and \( \mathbf{x}_1 \) at prices \( \mathbf{p}_1 \). It is easy to see that \( \mathbf{x}_0 \) is revealed preferred to \( \mathbf{x}_1 \). Thus, for any utility function that rationalises these data, we must have \( u(\mathbf{x}_0) > u(\mathbf{x}_1) \), by definition. Now suppose we want to compare two bundles such as \( \mathbf{x} \) and \( \mathbf{y} \), which do not appear in our sample. Because \( \mathbf{y} \) costs less than \( \mathbf{x}_1 \) when \( \mathbf{x}_1 \) was chosen, we may deduce that \( u(\mathbf{x}_0) > u(\mathbf{x}_1) > u(\mathbf{y}) \). Also, if more is preferred to less, the utility function must be increasing, so we have \( u(\mathbf{x}) \geq u(\mathbf{x}_0) \). Thus, we have \( u(\mathbf{x}) \geq u(\mathbf{x}_0) > u(\mathbf{x}_1) > u(\mathbf{y}) \) for any increasing utility function that rationalises the observed data, and so we can compare our two out-of-sample bundles directly and
Figure 2.5. Recovering preferences that satisfy GARP.

conclude \( u(x) > u(y) \) for any increasing utility function that could possibly have generated the data we have observed.

But things do not always work out so nicely. To illustrate, say we observe the consumer to buy the single bundle \( x_1 = (1, 1) \) at prices \( p_1 = (2, 1) \). The utility function \( u(x) = x_1^2 x_2 \) rationalises the choice we observe because the indifference curve through \( x_1 \) is tangent there to the budget constraint \( 2x_1 + x_2 = 3 \), as you can easily verify. At the same time, the utility function \( v(x) = x_1 (x_2 + 1) \) will also rationalise the choice of \( x_1 \) at \( p_1 \) as this utility function’s indifference curve through \( x_1 \) will also be tangent at \( x_1 \) to the same budget constraint. This would not be a problem if \( u(x) \) and \( v(x) \) were merely monotonic transforms of one another – but they are not. For when we compare the out-of-sample bundles \( x = (3, 1) \) and \( y = (1, 7) \), in the one case, we get \( u(3, 1) > u(1, 7) \), telling us the consumer prefers \( x \) to \( y \), and in the other, we get \( v(3, 1) < v(1, 7) \), telling us he prefers \( y \) to \( x \).

So for a given bundle \( y \), can we find all bundles \( x \) such that \( u(x) > u(y) \) for every utility function rationalises the data set? A partial solution has been provided by Varian (1982). Varian described a set of bundles such that every \( x \) in the set satisfies \( u(x) > u(y) \) for every \( u(\cdot) \) that rationalises the data. Knoblauch (1992) then showed that Varian’s set is a complete solution – that is, it contains all such bundles.

Unfortunately, consumption data usually contain violations of GARP. Thus, the search is now on for criteria to help decide when those violations of GARP are unimportant enough to ignore and for practical algorithms that will construct appropriate utility functions on data sets with minor violations of GARP.

### 2.4 Uncertainty

Until now, we have assumed that decision makers act in a world of absolute certainty. The consumer knows the prices of all commodities and knows that any feasible consumption bundle can be obtained with certainty. Clearly, economic agents in the real world cannot always operate under such pleasant conditions. Many economic decisions contain some element of uncertainty. When buying a car, for example, the consumer must consider the...
future price of petrol, expenditure on repairs, and the resale value of the car several years
later – none of which is known with certainty at the time of the decision. Decisions like this
involve uncertainty about the outcome of the choice that is made. Whereas the decision
maker may know the probabilities of different possible outcomes, the final result of the
decision cannot be known until it occurs.

At first glance, uncertainty may seem an intractable problem, yet economic theory
has much to contribute. The principal analytical approach to uncertainty is based on the
pathbreaking work of von Neumann and Morgenstern (1944).

2.4.1 PREFERENCES

Earlier in the text, the consumer was assumed to have a preference relation over all con-
sumption bundles \( x \) in a consumption set \( X \). To allow for uncertainty we need only shift
perspective slightly. We will maintain the notion of a preference relation but, instead of
consumption bundles, the individual will be assumed to have a preference relation over
gambles.

To formalise this, let \( A = \{a_1, \ldots, a_n\} \) denote a finite set of outcomes. The \( a_i \)'s
might well be consumption bundles, amounts of money (positive or negative), or anything
at all. The main point is that the \( a_i \)'s themselves involve no uncertainty. On the other hand,
we shall use the set \( A \) as the basis for creating gambles.

For example, let \( A = \{1, -1\} \), where 1 is the outcome ‘win one dollar’, and −1 is
the outcome ‘lose one dollar’. Suppose that you have entered into the following bet with
a friend. If the toss of a fair coin comes up heads, she pays you one dollar, and you pay
her one dollar if it comes up tails. From your point of view, this gamble will result in one
of the two outcomes in \( A \): 1 (win a dollar) or −1 (lose a dollar), and each of these occurs
with a probability of one-half because the coin is fair.

More generally, a simple gamble assigns a probability, \( p_i \), to each of the outcomes
\( a_i \), in \( A \). Of course, because the \( p_i \)'s are probabilities, they must be non-negative, and
because the gamble must result in some outcome in \( A \), the \( p_i \)'s must sum to one. We denote
this simple gamble by \( (p_1 \circ a_1, \ldots, p_n \circ a_n) \). We define the set of simple gambles \( \mathcal{G}_S \) as
follows.

**DEFINITION 2.2 Simple Gambles**

Let \( A = \{a_1, \ldots, a_n\} \) be the set of outcomes. Then \( \mathcal{G}_S \), the set of simple gambles (on \( A \)), is
given by

\[
\mathcal{G}_S = \left\{ (p_1 \circ a_1, \ldots, p_n \circ a_n) \mid p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}.
\]
When one or more of the \( p_i \)'s is zero, we shall drop those components from the expression when it is convenient to do so. For example, the simple gamble \( (\alpha \circ a_1, 0 \circ a_2, \ldots, 0 \circ a_{n-1}, (1 - \alpha) \circ a_n) \) would be written as \( (\alpha \circ a_1, (1 - \alpha) \circ a_n) \). Note that \( G \subseteq \mathbb{S} \) contains \( A \) because for each \( i \), \( (1 \circ a_i) \), the gamble yielding \( a_i \) with probability one, is in \( \mathbb{S} \). To simplify the notation further, we shall write \( a_i \) instead of \( (1 \circ a_i) \) to denote this gamble yielding outcome \( a_i \) with certainty.

Returning to our coin-tossing example where \( A = \{1, -1\} \), each individual, then, was faced with the simple gamble \( (\frac{1}{2} \circ 1, \frac{1}{2} \circ -1) \). Of course, not all gambles are simple. For example, it is quite common for state lotteries to give as prizes tickets for the next state lottery. The compound gambles we shall consider must result in an outcome in \( A \) after finitely many lotteries have been played, some outcome in \( A \) must result. So, if \( g \) is any gamble in \( G \), then \( g = (p_1 \circ g^1, \ldots, p_k \circ g^k) \), for some \( k \geq 1 \) and some gambles \( g^j \in G \), where the \( g^j \)'s might be compound gambles, simple gambles, or outcomes. Of course, the \( p_i \)'s must be non-negative and they must sum to one.\(^{2}\)

\(^{2}\)The objects of choice in decision making under uncertainty are gambles. Analogous to the case of consumer theory, we shall suppose that the decision maker has preferences, \( \succ \), over the set of gambles, \( G \). We shall proceed by positing a number of axioms, called axioms of choice under uncertainty, for the decision maker’s preference relation, \( \succ \). As before, \( \sim \) and \( \succ \) denote the indifference and strict preference relations induced by \( \succ \). The first few axioms will look very familiar and so require no discussion.

**Axiom 1:** Completeness. For any two gambles, \( g \) and \( g' \) in \( G \), either \( g \succ g' \), or \( g' \succ g \).

**Axiom 2:** Transitivity. For any three gambles \( g, g', g'' \) in \( G \), if \( g \succ g' \) and \( g' \succ g'' \), then \( g \succ g'' \).

Because each \( a_i \) in \( A \) is represented in \( G \) as a degenerate gamble, Axioms G1 and G2 imply in particular that the finitely many elements of \( A \) are ordered by \( \succ \). (See
Exercise 2.16.) So let us assume without loss of generality that the elements of \( A \) have been indexed so that \( a_1 \succ a_2 \succ \cdots \succ a_n \).

It seems plausible then that no gamble is better than that giving \( a_1 \) with certainty, and no gamble is worse than that giving \( a_n \) with certainty (although we are not directly assuming this). That is, for any gamble \( g \), it seems plausible that \((\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim g\), when \( \alpha = 1 \), and \( g \succsim (\alpha \circ a_1, (1-\alpha) \circ a_n) \) when \( \alpha = 0 \). The next axiom says that if indifference does not hold at either extreme, then it must hold for some intermediate value of \( \alpha \).

**AXIOM 3:** Continuity. *For any gamble \( g \) in \( G \), there is some probability, \( \alpha \in [0, 1] \), such that \( g \sim (\alpha \circ a_1, (1-\alpha) \circ a_n) \).*

Axiom G3 has implications that at first glance might appear unreasonable. For example, suppose that \( A = \{ $1000, $10, 'death' \} \). For most of us, these outcomes are strictly ordered as follows: \( $1000 \succ $10 \succ 'death' \). Now consider the simple gamble giving \( $10 \) with certainty. According to G3, there must be some probability \( \alpha \) rendering the gamble \((\alpha \circ $1000, (1-\alpha) \circ 'death')\) equally attractive as \( $10 \). Thus, if there is no probability \( \alpha \) at which you would find \( $10 \) with certainty and the gamble \((\alpha \circ $1000, (1-\alpha) \circ 'death')\) equally attractive, then your preferences over gambles do not satisfy G3.

Is, then, Axiom G3 an unduly strong restriction to impose on preferences? Do not be too hasty in reaching a conclusion. If you would drive across town to collect \( $1000 \) – an action involving some positive, if tiny, probability of death – rather than accept a \( $10 \) payment to stay at home, you would be declaring your preference for the gamble over the small sum with certainty. Presumably, we could increase the probability of a fatal traffic accident until you were just indifferent between the two choices. When that is the case, we will have found the indifference probability whose existence G3 assumes.

The next axiom expresses the idea that if two simple gambles each potentially yield only the best and worst outcomes, then that which yields the best outcome with the higher probability is preferred.

**AXIOM 4:** Monotonicity. *For all probabilities \( \alpha, \beta \in [0, 1] \),

\[(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n)\]

if and only if \( \alpha \geq \beta \).

Note that monotonicity implies \( a_1 \succ a_n \), and so the case in which the decision maker is indifferent among all the outcomes in \( A \) is ruled out.

Although most people will usually prefer gambles that give better outcomes higher probability, as monotonicity requires, it need not always be so. For example, to a safari hunter, death may be the worst outcome of an outing, yet the possibility of death adds to the excitement of the venture. An outing with a small probability of death would then be preferred to one with zero probability, a clear violation of monotonicity.

The next axiom states that the decision maker is indifferent between one gamble and another if he is indifferent between their realised, and their realisations occur with the same probabilities.
AXIOM 5: Substitution. If \( g = (p_1 \circ g^1, \ldots, p_k \circ g^k) \) and \( h = (p_1 \circ h^1, \ldots, p_k \circ h^k) \) are in \( G \), and if \( h^i \sim g^i \) for every \( i \), then \( h \sim g \).

Together with G1, Axiom G5 implies that when the agent is indifferent between two gambles he must be indifferent between all convex combinations of them. That is, if \( g \sim h \), then because by G1 \( g \sim g \), Axiom G5 implies \((\alpha \circ g, (1 - \alpha) \circ h) \sim (\alpha \circ g, (1 - \alpha) \circ g) = g\).

Our next, and final, axiom states that when considering a particular gamble, the decision maker cares only about the effective probabilities that gamble assigns to each outcome in \( A \). This warrants a bit of discussion.

For example, suppose that \( A = \{a_1, a_2\} \). Consider the compound gamble yielding outcome \( a_1 \) with probability \( \alpha \), and yielding a lottery ticket with probability \( 1 - \alpha \), where the lottery ticket itself is a simple gamble. It yields the outcome \( a_1 \) with probability \( \beta \) and the outcome \( a_2 \) with probability \( 1 - \beta \).

Now, taken all together, what is the effective probability that the outcome in fact will be \( a_1 \)? Well, \( a_1 \) can result in two mutually exclusive ways, namely, as an immediate result of the compound gamble, or as a result of the lottery ticket. The probability of the first is clearly \( \alpha \). The probability of the second is \((1 - \alpha)\beta\), because to obtain \( a_1 \) via the lottery ticket, \( a_1 \) must not have been the immediate result of the compound gamble and it must have been the result of the lottery ticket. So, all together, the probability that the outcome is \( a_1 \) is the sum, namely, \( \alpha + (1 - \alpha)\beta \), because the two different ways that \( a_1 \) can arise are mutually exclusive. Similarly, the effective probability that the outcome is \( a_2 \), is \((1 - \alpha)(1 - \beta)\).

To say that the decision maker cares only about the effective probabilities on the \( a_i \)'s when considering the preceding compound gamble is to say that the decision maker is indifferent between the compound gamble and the simple gamble \((\alpha + (1 - \alpha)\beta \circ a_1, (1 - \alpha)(1 - \beta) \circ a_2)\) that it induces.

Clearly, one can derive the (unique) effective probabilities on the \( a_i \)'s induced by any compound gamble in a similar way. We shall not spell out the procedure explicitly here, as it is, at least conceptually, straightforward.

For any gamble \( g \in G \), if \( p_i \) denotes the effective probability assigned to \( a_i \) by \( g \), then we say that \( g \) induces the simple gamble \((p_1 \circ a_1, \ldots, p_n \circ a_n) \in G_S \). We emphasise that every \( g \in G \) induces a unique simple gamble. Our final axiom is then as follows.3

AXIOM 6: Reduction to Simple Gambles. For any gamble \( g \in G \), if \((p_1 \circ a_1, \ldots, p_n \circ a_n)\) is the simple gamble induced by \( g \), then \((p_1 \circ a_1, \ldots, p_n \circ a_n) \sim g \).

Note that by G6 (and transitivity G2), an individual’s preferences over all gambles – compound or otherwise – are completely determined by his preferences over simple gambles.

As plausible as G6 may seem, it does restrict the domain of our analysis. In particular, this would not be an appropriate assumption to maintain if one wished to model the behaviour of vacationers in Las Vegas. They would probably not be indifferent between

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3In some treatments, Axioms G5 and G6 are combined into a single ‘independence’ axiom. (See Exercise 2.20.)
playing the slot machines many times during their stay and taking the single once and for all gamble defined by the effective probabilities over winnings and losses. On the other hand, many decisions under uncertainty are undertaken outside of Las Vegas, and for many of these, Axiom G6 is reasonable.

2.4.2 VON NEUMANN-MORGENSTERN UTILITY

Now that we have characterised the axioms preferences over gambles must obey, we once again ask whether we can represent such preferences with a continuous, real-valued function. The answer to that question is yes, which should come as no surprise. We know from our study of preferences under certainty that, here, Axioms G1, G2, and some kind of continuity assumption should be sufficient to ensure the existence of a continuous function representing $\succeq$. On the other hand, we have made assumptions in addition to G1, G2, and continuity. One might then expect to derive a utility representation that is more than just continuous. Indeed, we shall show that not only can we obtain a continuous utility function representing $\succeq$ on $\mathcal{G}$, we can obtain one that is linear in the effective probabilities on the outcomes.

To be more precise, suppose that $u: \mathcal{G} \rightarrow \mathbb{R}$ is a utility function representing $\succeq$ on $\mathcal{G}$. So, for every $g \in \mathcal{G}$, $u(g)$ denotes the utility number assigned to the gamble $g$. In particular, for every $i$, $u$ assigns the number $u(a_i)$ to the degenerate gamble $(1 \circ a_i)$, in which the outcome $a_i$ occurs with certainty. We will often refer to $u(a_i)$ as simply the utility of the outcome $a_i$. We are now prepared to describe the linearity property mentioned above.

**DEFINITION 2.3  Expected Utility Property**

The utility function $u: \mathcal{G} \rightarrow \mathbb{R}$ has the expected utility property if, for every $g \in \mathcal{G}$,

$$u(g) = \sum_{i=1}^{n} p_i u(a_i),$$

where $(p_1 \circ a_1, \ldots, p_n \circ a_n)$ is the simple gamble induced by $g$.

Thus, to say that $u$ has the expected utility property is to say that it assigns to each gamble the expected value of the utilities that might result, where each utility that might result is assigned its effective probability. Of course, the effective probability that $g$ yields utility $u(a_i)$ is simply the effective probability that it yields outcome $a_i$, namely, $p_i$.

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4The function $u(\cdot)$ represents $\succeq$ whenever $g \succeq g'$ if and only if $u(g) \geq u(g')$. See Definition 1.5.

5The expected value of a function $x$ taking on the values $x_1, \ldots, x_n$ with probabilities $p_1, \ldots, p_n$, respectively, is defined to be equal to $\sum_{i=1}^{n} p_i x_i$. Here, the $u(a_i)$’s play the role of the $x_i$’s, so that we are considering the expected value of utility.
Note that if $u$ has the expected utility property, and if $g_s = (p_1 \circ a_1, \ldots, p_n \circ a_n)$ is a simple gamble, then because the simple gamble induced by $g_s$ is $g_s$ itself, we must have

$$u(p_1 \circ a_1, \ldots, p_n \circ a_n) = \sum_{i=1}^{n} p_i u(a_i), \quad \forall \text{ probability vectors } (p_1, \ldots, p_n).$$

Consequently, the function $u$ is completely determined on all of $\mathcal{G}$ by the values it assumes on the finite set of outcomes, $A$.

If an individual’s preferences are represented by a utility function with the expected utility property, and if that person always chooses his most preferred alternative available, then that individual will choose one gamble over another if and only if the expected utility of the one exceeds that of the other. Consequently, such an individual is an **expected utility maximiser**.

Any such function as this will have some obvious analytical advantages because the utility of any gamble will be expressible as a linear sum involving only the utility of outcomes and their associated probabilities. Yet this is clearly a great deal to require of the function representing $\succsim$, and it is unlike anything we required of ordinary utility functions under certainty before. To help keep in mind the important distinctions between the two, we refer to utility functions possessing the expected utility property as **von Neumann-Morgenstern (VNM) utility functions**.

We now present a fundamental theorem in the theory of choice under uncertainty.

**THEOREM 2.7**

**Existence of a VNM Utility Function on $\mathcal{G}$**

Let preferences $\succsim$ over gambles in $\mathcal{G}$ satisfy axioms G1 to G6. Then there exists a utility function $u: \mathcal{G} \to \mathbb{R}$ representing $\succsim$ on $\mathcal{G}$, such that $u$ has the expected utility property.

**Proof:** As in our proof of the existence of a utility function representing the consumer’s preferences in Chapter 1, the proof here will be constructive.

So, consider an arbitrary gamble, $g$, from $\mathcal{G}$. Define $u(g)$ to be the number satisfying

$$g \sim (u(g) \circ a_1, (1 - u(g)) \circ a_n).$$

By G3, such a number must exist, and you are asked to show in Exercise 2.19 that by G4 this number is unique. This then defines a real-valued function, $u$, on $\mathcal{G}$. (Incidentally, by definition, $u(g) \in [0, 1]$ for all $g$.)

It remains to show that $u$ represents $\succsim$, and that it has the expected utility property. We shall begin with the first of these.

So let $g, g' \in \mathcal{G}$ be arbitrary gambles. We claim that the following equivalences hold

$$g \succsim g'$$

(P.1)
if and only if

\[(u(g) \circ a_1, (1-u(g)) \circ a_n) \succsim (u(g') \circ a_1, (1-u(g')) \circ a_n) \tag{P.2}\]

if and only if

\[u(g) \geq u(g'). \tag{P.3}\]

To see this, note that (P.1) iff (P.2) because \(\succsim\) is transitive, and \(g \sim (u(g) \circ a_1, (1-u(g)) \circ a_n)\), and \(g' \sim (u(g') \circ a_1, (1-u(g')) \circ a_n)\), both by the definition of \(u\). Also, (P.2) iff (P.3) follows directly from monotonicity (Axiom G4).

Consequently, \(g \succsim g'\) if and only if \(u(g) \geq u(g')\), so that \(u\) represents \(\succsim\) on \(G\).

To complete the proof, we must show that \(u\) has the expected utility property. So let \(g \in G\) be an arbitrary gamble, and let \(g_s \equiv (p_1 \circ a_1, \ldots, p_n \circ a_n) \in G_s\) be the simple gamble it induces. We must show that

\[u(g) = \sum_{i=1}^{n} p_i u(a_i). \tag{P.4}\]

Because by G6 \(g \sim g_s\), and because \(u\) represents \(\succsim\), we must have \(u(g) = u(g_s)\). It therefore suffices to show that

\[u(g_s) = \sum_{i=1}^{n} p_i u(a_i). \tag{P.4}\]

Now, for each \(i = 1, \ldots, n\), by definition, \(u(a_i)\) satisfies

\[a_i \sim (u(a_i) \circ a_1, (1-u(a_i)) \circ a_n). \tag{P.5}\]

Let \(q^i\) denote the simple gamble on the right in (P.5). That is, \(q^i \equiv (u(a_i) \circ a_1, (1-u(a_i)) \circ a_n)\) for every \(i = 1, \ldots, n\). Consequently, \(q^i \sim a_i\) for every \(i\), so that by the substitution axiom, G5,

\[g' \equiv (p_1 \circ q^1, \ldots, p_n \circ q^n) \sim (p_1 \circ a_1, \ldots, p_n \circ a_n) = g_s. \tag{P.6}\]

We now wish to derive the simple gamble induced by the compound gamble \(g'\). Note that because each \(q^i\) can result only in one of the two outcomes \(a_1\) or \(a_n\), \(g'\) must result only in one of those two outcomes as well. What is the effective probability that \(g'\) assigns to \(a_1\)? Well, \(a_1\) results if for any \(i\), \(q^i\) occurs (probability \(p_i\)) and \(a_1\) is the result of gamble \(q^i\) (probability \(u(a_i)\)). Thus, for each \(i\), there is a probability of \(p_i u(a_i)\) that \(a_1\) will result. Because the occurrences of the \(q^i\)'s are mutually exclusive, the effective probability that \(a_1\) results is the sum \(\sum_{i=1}^{n} p_i u(a_i)\). Similarly, the effective probability of
an is $\sum_{i=1}^{n} p_{i}(1 - u(a_{i}))$, which is equal to $1 - \sum_{i=1}^{n} p_{i}u(a_{i})$, because the $p_{i}$'s sum to one. Therefore, the simple gamble induced by $g'$ is

$$g'_{s} \equiv \left( \left( \sum_{i=1}^{n} p_{i}u(a_{i}) \right) \circ a_{1}, \left( 1 - \sum_{i=1}^{n} p_{i}u(a_{i}) \right) \circ a_{n} \right).$$

By the reduction axiom, G6, it must be the case that $g' \sim g'_{s}$. But the transitivity of $\sim$ together with (P.6) then imply that

$$g_{s} \sim \left( \left( \sum_{i=1}^{n} p_{i}u(a_{i}) \right) \circ a_{1}, \left( 1 - \sum_{i=1}^{n} p_{i}u(a_{i}) \right) \circ a_{n} \right). \quad (P.7)$$

However, by definition (and Exercise 2.19), $u(g_{s})$ is the unique number satisfying

$$g_{s} \sim (u(g_{s}) \circ a_{1}, (1 - u(g_{s})) \circ a_{n}). \quad (P.8)$$

Therefore, comparing (P.7) with (P.8) we conclude that

$$u(g_{s}) = \sum_{i=1}^{n} p_{i}u(a_{i}),$$

as desired.

The careful reader might have noticed that Axiom G1 was not invoked in the process of proving Theorem 2.7. Indeed, it is redundant given the other axioms. In Exercise 2.22, you are asked to show that G2, G3, and G4 together imply G1. Consequently, we could have proceeded without explicitly mentioning completeness at all. On the other hand, assuming transitivity and not completeness would surely have raised unnecessary questions in the reader’s mind. To spare you that kind of stress, we opted for the approach presented here.

The upshot of Theorem 2.7 is this: if an individual’s preferences over gambles satisfy Axioms G1 through G6, then there are utility numbers that can be assigned to the outcomes in $A$ so that the individual prefers one gamble over another if and only if the one has a higher expected utility than the other.

The proof of Theorem 2.7 not only establishes the existence of a utility function with the expected utility property, but it also shows us the steps we might take in constructing such a function in practice. To determine the utility of any outcome $a_{i}$, we need only ask the individual for the probability of the best outcome that would make him indifferent between a best–worst gamble of the form $(\alpha \circ a_{1}, (1 - \alpha) \circ a_{n})$ and the outcome $a_{i}$ with certainty. By repeating this process for every $a_{i} \in A$, we then could calculate the utility associated with any gamble $g \in G$ as simply the expected utility it generates. And if the individual’s preferences satisfy G1 through G6, Theorem 2.7 guarantees that the utility function we obtain in this way represents her preferences.
EXAMPLE 2.4 Suppose $A = \{10, 4, -2\}$, where each of these represent thousands of dollars. We can reasonably suppose that the best outcome is $10$ and the worst is $-2$.

To construct the VNM utility function used in the proof of Theorem 2.7, we first have to come up with indifference probabilities associated with each of the three outcomes. We accomplish this by composing best–worst gambles that offer $10$ and $-2$ with as yet unknown probabilities summing to 1. Finally, we ask the individual the following question for each of the three outcomes: ‘What probability for the best outcome will make you indifferent between the best–worst gamble we have composed and the outcome $a_i$ with certainty?’ The answers we get will be the utility numbers we assign to each of the three ultimate outcomes. Suppose we find that

$$10 \sim (1 \circ 10, 0 \circ -2), \quad \text{so} \quad u(10) \equiv 1, \quad (E.1)$$

$$4 \sim (.6 \circ 10, .4 \circ -2), \quad \text{so} \quad u(4) \equiv .6, \quad (E.2)$$

$$-2 \sim (0 \circ 10, 1 \circ -2), \quad \text{so} \quad u(-2) \equiv 0. \quad (E.3)$$

Note carefully that under this mapping, the utility of the best outcome must always be 1 and that of the worst outcome must always be zero. However, the utility assigned to intermediate outcomes, such as $4$ in this example, will depend on the individual’s attitude towards taking risks.

Having obtained the utility numbers for each of the three possible outcomes, we now have every bit of information we need to rank all gambles involving them. Consider, for instance,

$$g_1 \equiv (.2 \circ 4, .8 \circ 10), \quad (E.4)$$

$$g_2 \equiv (.07 \circ -2, .03 \circ 4, .9 \circ 10). \quad (E.5)$$

Which of these will the individual prefer? Assuming that his preferences over gambles satisfy G1 through G6, we may appeal to Theorem 2.7. It tells us that we need only calculate the expected utility of each gamble, using the utility numbers generated in (E.1) through (E.3), to find out. Doing that, we find

$$u(g_1) = .2u(4) + .8u(10) = .92,$$

$$u(g_2) = .07u(-2) + .03u(4) + .9u(10) = .918.$$

Because $g_1$ has the greater expected utility, it must be the preferred gamble! In similar fashion, using only the utility numbers generated in (E.1) through (E.3), we can rank any of the infinite number of gambles that could be constructed from the three outcomes in $A$.

Just think some more about the information we have uncovered in this example. Look again at the answer given when asked to compare $4$ with certainty to the best–worst gamble in (E.2). The best–worst gamble $g$ offered there has an expected value of $E(g) = (.6)(10) + (.4)(-2) = 5.2$. This exceeds the expected value $4$ he obtains under the simple gamble offering $4$ with certainty, yet the individual is indifferent between these
two gambles. Because we assume that his preferences are monotonic, we can immediately conclude that he would strictly prefer the same $4 with certainty to every best–worst gamble offering the best outcome with probability less than .6. This of course includes the one offering $10 and $2 with equal probabilities of .5, even though that gamble and $4 with certainty have the same expected value of $4. Thus, in some sense, this individual prefers to avoid risk. This same tendency is reflected in his ranking of $1 and $10, even though that gamble and $4 with certainty have the same expected value of $4. Thus, in some sense, this individual prefers to avoid risk. This same tendency is reflected in his ranking of $1 and $10, even though that gamble and $4 with certainty have the same expected value of $4.

Let us step back a moment to consider what this VNM utility function really does and how it relates to the ordinary utility function under certainty. In the standard case, if the individual is indifferent between two commodity bundles, both receive the same utility number, whereas if one bundle is strictly preferred to another, its utility number must be larger. This is true, too, of the VNM utility function $u(g)$, although we must substitute the word ‘gamble’ for ‘commodity bundle’.

However, in the consumer theory case, the utility numbers themselves have only ordinal meaning. Any strictly monotonic transformation of one utility representation yields another one. On the other hand, the utility numbers associated with a VNM utility representation of preferences over gambles have content beyond ordinality. To see this, suppose that $A = \{a, b, c\}$, where $a \succ b \succ c$, and that $\succeq$ satisfies G1 through G6. By G3 and G4, there is an $\alpha \in (0, 1)$ satisfying

$$b \sim (\alpha \circ a, (1 - \alpha) \circ c).$$

Note well that the probability number $\alpha$ is determined by, and is a reflection of, the decision maker’s preferences. It is a meaningful number. One cannot double it, add a constant to it, or transform it in any way without also changing the preferences with which it is associated.

Now, let $u$ be some VNM utility representation of $\succeq$. Then the preceding indifference relation implies that

$$u(b) = u(\alpha \circ a, (1 - \alpha) \circ c) = \alpha u(a) + (1 - \alpha)u(c),$$

where the second equality follows from the expected utility property of $u$. But this equality can be rearranged to yield

$$\frac{u(a) - u(b)}{u(b) - u(c)} = \frac{1 - \alpha}{\alpha}.$$
determined by the decision maker’s preferences, so, too, then is the preceding ratio of utility differences.

We conclude that the ratio of utility differences has inherent meaning regarding the individual’s preferences and they must take on the same value for every VNM utility representation of \( \succeq \). Therefore, VNM utility representations provide distinctly more than ordinal information about the decision maker’s preferences, for otherwise, through suitable monotone transformations, such ratios could assume many different values.

Clearly, then, a strictly increasing transformation of a VNM utility representation might not yield another VNM utility representation. (Of course, it still yields a utility representation, but that representation need not have the expected utility property.) This then raises the following question: what is the class of VNM utility representations of a given preference ordering? From the earlier considerations, these must preserve the ratios of utility differences. As the next result shows, this property provides a complete characterisation.

**THEOREM 2.8**

*VNM Utility Functions are Unique up to Positive Affine Transformations*

Suppose that the VNM utility function \( u(\cdot) \) represents \( \succeq \). Then the VNM utility function, \( v(\cdot) \), represents those same preferences if and only if for some scalar \( \alpha \) and some scalar \( \beta > 0 \),

\[
v(g) = \alpha + \beta u(g),
\]

for all gambles \( g \).

**Proof:** Sufficiency is obvious (but do convince yourself), so we only prove necessity here. Moreover, we shall suppose that \( g \) is a simple gamble. You are asked to show that if \( u \) and \( v \) are linearly related for all simple gambles, then they are linearly related for all gambles. As before, we let

\[
A = \{a_1, \ldots, a_n\} \quad \text{and} \quad g \equiv (p_1 \circ a_1, p_2 \circ a_2, \ldots, p_n \circ a_n),
\]

where \( a_1 \succeq \cdots \succeq a_n \), and \( a_1 > a_n \).

Because \( u(\cdot) \) represents \( \succeq \), we have \( u(a_1) \geq \cdots \geq u(a_i) \geq \cdots \geq u(a_n) \), and \( u(a_1) > u(a_n) \). So, for every \( i = 1, \ldots, n \), there is a unique \( \alpha_i \in [0, 1] \) such that

\[
u(a_i) = \alpha_i u(a_1) + (1 - \alpha_i) u(a_n). \tag{P.1}
\]

Note that \( \alpha_i > 0 \) if and only if \( a_i > a_n \).

Now, because \( u(\cdot) \) has the expected utility property, (P.1) implies that

\[
u(a_i) = u(a_i \circ a_1, (1 - \alpha_i) \circ a_n),
\]
which, because \( u(\cdot) \) represents \( \geq \), means that
\[
a_i \sim (\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n).
\] (P.2)

So, because \( v(\cdot) \) also represents \( \geq \), we must have
\[
v(a_i) = v(a_i \circ a_1, (1 - \alpha_i) \circ a_n).
\]
And, because \( v(\cdot) \) has the expected utility property, this implies that
\[
v(a_i) = \alpha_i v(a_1) + (1 - \alpha_i) v(a_n).
\] (P.3)

Together, (P.1) and (P.3) imply that
\[
\begin{align*}
&u(a_1) - u(a_i) = \frac{1 - \alpha_i}{\alpha_i} = \frac{v(a_1) - v(a_i)}{v(a_i) - v(a_n)} \\
&\text{for every } i = 1, \ldots, n \text{ such that } a_i \succ a_n \text{ (i.e., such that } \alpha_i > 0). \\
\end{align*}
\] (P.4)

From (P.4) we may conclude that
\[
(u(a_1) - u(a_i))(v(a_i) - v(a_n)) = (v(a_1) - v(a_i))(u(a_i) - u(a_n)) \\
\text{whenever } a_i > a_n. \text{ However, (P.5) holds even when } a_i \sim a_n \text{ because in this case } u(a_i) = u(a_n) \text{ and } v(a_i) = v(a_n). \text{ Hence, (P.5) holds for all } i = 1, \ldots, n.
\]

Rearranging, (P.5) can be expressed in the form
\[
v(a_i) = \alpha + \beta u(a_i), \quad \text{for all } i = 1, \ldots, n,
\] (P.6)

where
\[
\begin{align*}
\alpha &\equiv \frac{u(a_1)v(a_n) - v(a_1)u(a_n)}{u(a_1) - u(a_n)} \quad \text{and} \quad \beta \equiv \frac{v(a_1) - v(a_n)}{u(a_1) - u(a_n)}.
\end{align*}
\]

Notice that both \( \alpha \) and \( \beta \) are constants (i.e., independent of \( i \)), and that \( \beta \) is strictly positive.

So, for any gamble \( g \), if \( (p_1 \circ a_1, \ldots, p_n \circ a_n) \) is the simple gamble induced by \( g \), then
\[
v(g) = \sum_{i=1}^{n} p_i v(a_i) = \sum_{i=1}^{n} p_i (\alpha + \beta u(a_i)) = \alpha + \beta \sum_{i=1}^{n} p_i u(a_i) = \alpha + \beta u(g),
\]
where the first and last equalities follow because $v(\cdot)$ and $u(\cdot)$ have the expected utility property and the second equality follows from (P.6).

Just before the statement of Theorem 2.8, we stated that the class of VNM utility representations of a single preference relation is characterised by the constancy of ratios of utility differences. This in fact follows from Theorem 2.8, as you are asked to show in an exercise.

Theorem 2.8 tells us that VNM utility functions are not completely unique, nor are they entirely ordinal. We can still find an infinite number of them that will rank gambles in precisely the same order and also possess the expected utility property. But unlike ordinary utility functions from which we demand only an order-preserving numerical scaling, here we must limit ourselves to transformations that multiply by a positive number and/or add a constant term if we are to preserve the expected utility property as well. Yet the less than complete ordinality of the VNM utility function must not tempt us into attaching undue significance to the absolute level of a gamble’s utility, or to the difference in utility between one gamble and another. With what little we have required of the agent’s binary comparisons between gambles in the underlying preference relation, we still cannot use VNM utility functions for interpersonal comparisons of well-being, nor can we measure the ‘intensity’ with which one gamble is preferred to another.

2.4.3 RISK AVERSION

In Example 2.4 we argued that the VNM utility function we created there reflected some desire to avoid risk. Now we are prepared to define and describe risk aversion more formally. For that, we shall confine our attention to gambles whose outcomes consist of different amounts of wealth. In addition, it will be helpful to take as the set of outcomes, $A$, all non-negative wealth levels. Thus, $A = \mathbb{R}_+$. Even though the set of outcomes now contains infinitely many elements, we continue to consider gambles giving only finitely many outcomes a strictly positive effective probability. In particular, a simple gamble takes the form $(p_1 \circ w_1, \ldots, p_n \circ w_n)$, where $n$ is some positive integer, the $w_i$’s are non-negative wealth levels, and the non-negative probabilities, $p_1, \ldots, p_n$, sum to 1. Finally, we shall assume that the individual’s VNM utility function, $u(\cdot)$, is differentiable with $u'(w) > 0$ for all wealth levels $w$.

We now investigate the relationship between a VNM utility function and the agent’s attitude towards risk. The expected value of the simple gamble $g$ offering $w_i$ with probability $p_i$ is given by $E(g) = \sum_{i=1}^{n} p_i w_i$. Now suppose the agent is given a choice between accepting the gamble $g$ on the one hand or receiving with certainty the expected value of $g$ on the other. If $u(\cdot)$ is the agent’s VNM utility function, we can evaluate these two

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6With this framework, it is possible to prove an expected utility theorem along the lines of Theorem 2.7 by suitably modifying the axioms to take care of the fact that $A$ is no longer a finite set.
alternatives as follows:

\[ u(g) = \sum_{i=1}^{n} p_i u(w_i), \]

\[ u(E(g)) = u\left(\sum_{i=1}^{n} p_i w_i\right). \]

The first of these is the VNM utility of the gamble, and the second is the VNM utility of the gamble’s expected value. If preferences satisfy Axioms G1 to G6, we know the agent prefers the alternative with the higher expected utility. When someone would rather receive the expected value of a gamble with certainty than face the risk inherent in the gamble itself, we say they are risk averse. Of course, people may exhibit a complete disregard of risk, or even an attraction to risk, and still be consistent with Axioms G1 through G6. We catalogue these various possibilities, and define terms precisely, in what follows.

As remarked after Definition 2.3, a VNM utility function on \( G \) is completely determined by the values it assumes on the set of outcomes, \( A \). Consequently, the characteristics of an individual’s VNM utility function over the set of simple gambles alone provides a complete description of the individual’s preferences over all gambles. Because of this, it is enough to focus on the behaviour of \( u \) on \( G_S \) to capture an individual’s attitudes towards risk. This, and the preceding discussion, motivate the following definition.

**DEFINITION 2.4 Risk Aversion, Risk Neutrality, and Risk Loving**

Let \( u(\cdot) \) be an individual’s VNM utility function for gambles over non-negative levels of wealth. Then for the simple gamble \( g = (p_1 \circ w_1, \ldots, p_n \circ w_n) \), the individual is said to be

1. risk averse at \( g \) if \( u(E(g)) > u(g) \),
2. risk neutral at \( g \) if \( u(E(g)) = u(g) \),
3. risk loving at \( g \) if \( u(E(g)) < u(g) \).

If for every non-degenerate\(^7\) simple gamble, \( g \), the individual is, for example, risk averse at \( g \), then the individual is said simply to be risk averse (or risk averse on \( G \) for emphasis). Similarly, an individual can be defined to be risk neutral and risk loving (on \( G \)).

Each of these attitudes toward risk is equivalent to a particular property of the VNM utility function. In the exercises, you are asked to show that the agent is risk averse, risk neutral, or risk loving over some subset of gambles if and only if his VNM utility function is strictly concave, linear, or strictly convex, respectively, over the appropriate domain of wealth.

---

\(^7\)A simple gamble is non-degenerate if it assigns strictly positive probability to at least two distinct wealth levels.
To help see the first of these claims, let us consider a simple gamble involving two outcomes:

\[ g \equiv (p \circ w_1, (1 - p) \circ w_2). \]

Now suppose the individual is offered a choice between receiving wealth equal to 
\[ E(g) = pw_1 + (1 - p)w_2 \] with certainty or receiving the gamble \( g \) itself. We can assess the alternatives as follows:

\[
\begin{align*}
    u(g) &= pu(w_1) + (1 - p)u(w_2), \\
    u(E(g)) &= u(pw_1 + (1 - p)w_2).
\end{align*}
\]

Now look at Fig. 2.6. There we have drawn a chord between the two points 
\[ R = (w_1, u(w_1)) \] and \[ S = (w_2, u(w_2)) \], and located their convex combination, 
\[ T = pR + (1 - p)S. \] The abscissa of \( T \) must be \( E(g) \) and its ordinate must be \( u(g) \). (Convince yourself of this.) We can then locate \( u(E(g)) \) on the vertical axis using the graph of the function \( u(w) \) as indicated. The VNM utility function in Fig. 2.6 has been drawn strictly concave in wealth over the relevant region. As you can see, \( u(E(g)) > u(g) \), so the individual is risk averse.

In Fig. 2.6, the individual prefers \( E(g) \) with certainty to the gamble \( g \) itself. But there will be some amount of wealth we could offer with certainty that would make him indifferent between accepting that wealth with certainty and facing the gamble \( g \). We call this amount of wealth the certainty equivalent of the gamble \( g \). When a person is risk averse and strictly prefers more money to less, it is easy to show that the certainty equivalent is less than the expected value of the gamble, and you are asked to do this in the exercises. In effect, a risk-averse person will ‘pay’ some positive amount of wealth to avoid the gamble’s inherent risk. This willingness to pay to avoid risk is measured by the risk premium.

\[ \text{Figure 2.6. Risk aversion and strict concavity of the VNM utility function.} \]
The certainty equivalent and the risk premium, both illustrated in Fig. 2.6, are defined in what follows.

**DEFINITION 2.5  Certainty Equivalent and Risk Premium**

The certainty equivalent of any simple gamble \( g \) over wealth levels is an amount of wealth, \( CE \), offered with certainty, such that

\[
 u(CE) \equiv u(CE) = u \left( \frac{1}{2} \left( w_0 + h \right) \right) + u \left( \frac{1}{2} \left( w_0 - h \right) \right).
\]

The risk premium is an amount of wealth, \( P \), such that

\[
 u(g) \equiv u \left( E(g) - P \right).
\]

**EXAMPLE 2.5** Suppose \( u(w) \equiv \ln(w) \). Because this is strictly concave in wealth, the individual is risk averse. Let \( g \) offer 50–50 odds of winning or losing some amount of wealth, \( h \), so that if the individual’s initial wealth is \( w_0 \),

\[
g \equiv ((1/2) \circ (w_0 + h), (1/2) \circ (w_0 - h)),
\]

where we note that \( E(g) = w_0 \). The certainty equivalent for \( g \) must satisfy

\[
 \ln(CE) = (1/2) \ln(w_0 + h) + (1/2) \ln(w_0 - h) = \ln(w_0^2 - h^2)^{1/2}.
\]

Thus, \( CE = \left( w_0^2 - h^2 \right)^{1/2} < E(g) \) and \( P = w_0 - \left( w_0^2 - h^2 \right)^{1/2} > 0 \).

Many times, we not only want to know whether someone is risk averse, but also how risk averse they are. Ideally, we would like a summary measure that allows us both to compare the degree of risk aversion across individuals and to gauge how the degree of risk aversion for a single individual might vary with the level of their wealth. Because risk aversion and concavity of the VNM utility function in wealth are equivalent, the seemingly most natural candidate for such a measure would be the second derivative, \( u''(w) \), a basic measure of a function’s ‘curvature’. We might think that the greater the absolute value of this derivative, the greater the degree of risk aversion.

But this will not do. Although the sign of the second derivative does tell us whether the individual is risk averse, risk loving, or risk neutral, its size is entirely arbitrary. Theorem 2.8 showed that VNM utility functions are unique up to affine transformations. This means that for any given preferences, we can obtain virtually any size second derivative we wish through multiplication of \( u(\cdot) \) by a properly chosen positive constant. With this and other considerations in mind, Arrow (1970) and Pratt (1964) have proposed the following measure of risk aversion.

**DEFINITION 2.6  The Arrow-Pratt Measure of Absolute Risk Aversion**

The Arrow-Pratt measure of absolute risk aversion is given by

\[
 Ra(w) \equiv -\frac{u''(w)}{u'(w)}.
\]
Note that the sign of this measure immediately tells us the basic attitude towards risk: \( R_a(w) \) is positive, negative, or zero as the agent is risk averse, risk loving, or risk neutral, respectively. In addition, any positive affine transformation of utility will leave the measure unchanged: adding a constant affects neither the numerator nor the denominator; multiplication by a positive constant affects both numerator and denominator but leaves their ratio unchanged.

To demonstrate the effectiveness of the Arrow-Pratt measure of risk aversion, we now show that consumers with larger Arrow-Pratt measures are indeed more risk averse in a behaviourally significant respect: they have lower certainty equivalents and are willing to accept fewer gambles.

To see this, suppose there are two consumers, 1 and 2, and that consumer 1 has VNM utility function \( u(w) \), and consumer 2’s VNM utility function is \( v(w) \). Wealth, \( w \), can take on any non-negative number. Let us now suppose that at every wealth level, \( w \), consumer 1’s Arrow-Pratt measure of risk aversion is larger than consumer 2’s. That is,

\[
R^1_a(w) = -\frac{u''(w)}{u'(w)} > -\frac{v''(w)}{v'(w)} = R^2_a(w) \quad \text{for all } w \geq 0, \tag{2.12}
\]

where we are assuming that both \( u' \) and \( v' \) are always strictly positive.

For simplicity, assume that \( v(w) \) takes on all values in \([0, \infty)\). Consequently, we may define \( h: [0, \infty) \rightarrow \mathbb{R} \) as follows:

\[
h(x) = u(v^{-1}(x)) \quad \text{for all } x \geq 0. \tag{2.13}
\]

Therefore, \( h \) inherits twice differentiability from \( u \) and \( v \) with

\[
h'(x) = \frac{u'(v^{-1}(x))}{v'(v^{-1}(x))} > 0, \quad \text{and}
\]

\[
h''(x) = \frac{u'(v^{-1}(x))(u''(v^{-1}(x))/u'(v^{-1}(x)) - v''(v^{-1}(x))/v'(v^{-1}(x)))}{[v'(v^{-1}(x))]^2} < 0
\]

for all \( x > 0 \), where the first inequality follows because \( u', v' \) > 0, and the second follows from (2.12). Therefore, \( h \) is a strictly increasing, strictly concave function.

Consider now a gamble \( (p_1 \circ w_1, \ldots, p_n \circ w_n) \) over wealth levels. We can use (2.13) and the fact that \( h \) is strictly concave to show that consumer 1’s certainty equivalent for this gamble is lower than consumer 2’s.

To see this, let \( \hat{w}_i \) denote consumer \( i \)'s certainty equivalent for the gamble. That is,

\[
\sum_{i=1}^{n} p_i u(w_i) = u(\hat{w}_1), \tag{2.14}
\]

\[
\sum_{i=1}^{n} p_i v(w_i) = v(\hat{w}_2). \tag{2.15}
\]
We wish to show that \( \hat{w}_1 < \hat{w}_2 \).

Putting \( x = v(w) \) in (2.13) and using (2.14) gives

\[
\begin{align*}
u(\hat{w}_1) &= \sum_{i=1}^{n} p_i h(v(w_i)) \\
&< h \left( \sum_{i=1}^{n} p_i v(w_i) \right) \\
&= h(v(\hat{w}_2)) \\
&= u(\hat{w}_2),
\end{align*}
\]

where the inequality, called Jensen’s inequality, follows because \( h \) is strictly concave, and the final two equalities follow from (2.15) and (2.13), respectively. Consequently, \( u(\hat{w}_1) < u(\hat{w}_2) \), so that because \( u \) is strictly increasing, \( \hat{w}_1 < \hat{w}_2 \) as desired.

We may conclude that consumer 1’s certainty equivalent for any given gamble is lower than 2’s. And from this it easily follows that if consumers 1 and 2 have the same initial wealth, then consumer 2 (the one with the globally lower Arrow-Pratt measure) will accept any gamble that consumer 1 will accept. (Convince yourself of this.) That is, consumer 1 is willing to accept fewer gambles than consumer 2.

Finally, note that in passing, we have also shown that (2.12) implies that consumer 1’s VNM utility function is more concave than consumer 2’s in the sense that (once again putting \( x = v(w) \) in (2.13))

\[
u(w) = h(v(w)) \quad \text{for all } w \geq 0, \tag{2.16}\]

where, as you recall, \( h \) is a strictly concave function. Thus, according to (2.16), \( u \) is a ‘concavification’ of \( v \). This is yet another (equivalent) expression of the idea that consumer 1 is more risk averse than consumer 2.

\( R_a(w) \) is only a local measure of risk aversion, so it need not be the same at every level of wealth. Indeed, one expects that attitudes toward risk, and so the Arrow-Pratt measure, will ordinarily vary with wealth, and vary in ‘sensible’ ways. Arrow has proposed a simple classification of VNM utility functions (or utility function segments) according to how \( R_a(w) \) varies with wealth. Quite straightforwardly, we say that a VNM utility function displays constant, decreasing, or increasing absolute risk aversion over some domain of wealth if, over that interval, \( R_a(w) \) remains constant, decreases, or increases with an increase in wealth, respectively.

Decreasing absolute risk aversion (DARA) is generally a sensible restriction to impose. Under constant absolute risk aversion, there would be no greater willingness to accept a small gamble at higher levels of wealth, and under increasing absolute risk aversion, we have rather perverse behaviour: the greater the wealth, the more averse one becomes to accepting the same small gamble. DARA imposes the more plausible restriction that the individual be less averse to taking small risks at higher levels of wealth.
EXAMPLE 2.6 Consider an investor who must decide how much of his initial wealth \( w \) to put into a risky asset. The risky asset can have any of the positive or negative rates of return \( r_i \) with probabilities \( p_i, i = 1, \ldots, n \). If \( \beta \) is the amount of wealth to be put into the risky asset, final wealth under outcome \( i \) will be \( (w - \beta) + (1 + r_i)\beta = w + \beta r_i \). The investor’s problem is to choose \( \beta \) to maximise the expected utility of wealth. We can write this formally as the single-variable optimisation problem

\[
\max_{\beta} \sum_{i=1}^{n} p_i u(w + \beta r_i) \quad \text{s. t.} \quad 0 \leq \beta \leq w. \tag{E.1}
\]

We first determine under what conditions a risk-averse investor will decide to put no wealth into the risky asset. In this case, we would have a corner solution where the objective function in (E.1) reaches a maximum at \( \beta^* = 0 \), so its first derivative must be non-increasing there. Differentiating expected utility in (E.1) with respect to \( \beta \), then evaluating at \( \beta^* = 0 \), we therefore must have

\[
\sum_{i=1}^{n} p_i u'(w + \beta^* r_i) r_i = u'(w) \sum_{i=1}^{n} p_i r_i \leq 0.
\]

The sum on the right-hand side is just the expected return on the risky asset. Because \( u'(w) \) must be positive, the expected return must be non-positive. Because you can easily verify that the concavity of \( u \) in wealth is sufficient to ensure the concavity of (E.1) in \( \beta \), we conclude that a risk-averse individual will abstain completely from the risky asset if and only if that asset has a non-positive expected return. Alternatively, we can say that a risk-averse investor will always prefer to place some wealth into a risky asset with a strictly positive expected return.

Now assume that the risky asset has a positive expected return. As we have seen, this means we can rule out \( \beta^* = 0 \). Let us also suppose that \( \beta^* < w \). The first- and second-order conditions for an interior maximum of (E.1) tell us that

\[
\sum_{i=1}^{n} p_i u'(w + \beta^* r_i) r_i = 0 \tag{E.2}
\]

and

\[
\sum_{i=1}^{n} p_i u''(w + \beta^* r_i) r_i^2 < 0, \tag{E.3}
\]

respectively, where (E.3) is strict because of risk aversion.

Now we ask what happens to the amount of wealth devoted to the risky asset as wealth increases. Casual empiricism suggests that as wealth increases, a greater absolute amount of wealth is placed into risky assets, i.e., that risky assets are ‘normal’ rather than
‘inferior’ goods. We will show that this is so under DARA. Viewing $\beta^*$ as a function of $w$, differen- 
tiating (E.2) with respect to $w$, we find that

$$
\frac{d\beta^*}{dw} = \frac{-\sum_{i=1}^{n} p_i u''(w + \beta^* r_i) r_i}{\sum_{i=1}^{n} p_i u'(w + \beta^* r_i) r_i^2}.
$$

(E.4)

Risk aversion ensures that the denominator in (E.4) will be negative, so risky assets will be ‘normal’ only when the numerator is also negative. DARA is sufficient to ensure this.

To see this, note that the definition of $R_a(w + \beta^* r_i) r_i$ implies

$$
-u''(w + \beta^* r_i) r_i \equiv R_a(w + \beta^* r_i) r_i u'(w + \beta^* r_i), \quad i = 1, \ldots, n.
$$

(E.5)

Under DARA, $R_a(w) > R_a(w + \beta^* r_i)$ whenever $r_i > 0$, and $R_a(w) < R_a(w + \beta^* r_i)$ whenever $r_i < 0$. Multiplying both sides of these inequalities by $r_i$, we obtain in both cases,

$$
R_a(w) r_i > R_a(w + \beta^* r_i) r_i, \quad i = 1, \ldots, n.
$$

(E.6)

Substituting $R_a(w)$ for $R_a(w + \beta^* r_i)$ in (E.5) and using (E.6), we obtain

$$
-u''(w + \beta^* r_i) r_i < R_a(w) r_i u'(w + \beta^* r_i), \quad i = 1, \ldots, n.
$$

Finally, taking expectations of both sides gives

$$
-\sum_{i=1}^{n} p_i u''(w + \beta^* r_i) r_i < R_a(w) \sum_{i=1}^{n} p_i r_i u'(w + \beta^* r_i) = 0,
$$

(E.7)

where the last equality follows from (E.2).

Thus, when behaviour displays DARA, (E.4) is positive and more wealth will be put into the risky asset as wealth increases.

**EXAMPLE 2.7** A risk-averse individual with initial wealth $w_0$ and VNM utility function $u(\cdot)$ must decide whether and for how much to insure his car. The probability that he will have an accident and incur a dollar loss of $L$ in damages is $\alpha \in (0, 1)$. How much insurance, $x$, should he purchase?

Of course, the answer depends on the price at which insurance is available. Let us suppose that insurance is available at an actuarially fair price, i.e., one that yields insurance companies zero expected profits. Now, if $\rho$ denotes the rate at which each dollar of insurance can be purchased, the insurance company’s expected profits per dollar of insurance sold (assuming zero costs) will be $\alpha(\rho - 1) + (1 - \alpha)\rho$. Setting this equal to zero implies that $\rho = \alpha$. 
So, with the price per dollar of insurance equal to $\alpha$, how much insurance should our risk-averse individual purchase? Because he is an expected utility maximiser, he will choose that amount of insurance, $x$, to maximise his expected utility,

$$\alpha u(w_0 - \alpha x - L + x) + (1 - \alpha) u(w_0 - \alpha x). \quad (E.1)$$

Differentiating (E.1) with respect to $x$ and setting the result to zero yields

$$(1 - \alpha)\alpha u'(w_0 - \alpha x - L + x) - \alpha(1 - \alpha) u'(w_0 - \alpha x) = 0,$$

which, on dividing by $(1 - \alpha)\alpha$, yields

$$u'(w_0 - \alpha x - L + x) = u'(w_0 - \alpha x).$$

But because the individual is risk averse, $u'' < 0$, so that the marginal utility of wealth is strictly decreasing in wealth. Consequently, equality of the preceding marginal utilities of wealth implies equality of the wealth levels themselves, i.e.,

$$w_0 - \alpha x - L + x = w_0 - \alpha x,$$

which implies that

$$x = L.$$

Consequently, with the availability of actuarially fair insurance, a risk-averse individual fully insures against all risk. Note that at the optimum, the individual’s wealth is constant and equal to $w_0 - \alpha L$ whether or not he has an accident.

### 2.5 Exercises

2.1 Show that budget balancedness and homogeneity of $x(p, y)$ are unrelated conditions in the sense that neither implies the other.

2.2 Suppose that $x(p, y) \in \mathbb{R}_+^n$ satisfies budget balancedness and homogeneity on $\mathbb{R}_+^{n+1}$. Show that for all $(p, y) \in \mathbb{R}_+^{n+1}$, $s(p, y) \cdot p = 0$, where $s(p, y)$ denotes the Slutsky matrix associated with $x(p, y)$.

2.3 Derive the consumer’s direct utility function if his indirect utility function has the form $v(p, y) = yp_1^{\alpha}p_2^{\beta}$ for negative $\alpha$ and $\beta$.

2.4 Suppose that the function $e(p, u) \in \mathbb{R}_+$, not necessarily an expenditure function, and $x(p, y) \in \mathbb{R}_+^n$, not necessarily a demand function, satisfy the system of partial differential equations given in Section 2.2. Show the following:

(a) If $x(p, y)$ satisfies budget balancedness, then $e(p, u)$ must be homogeneous of degree one in $p$.

(b) If $e(p, u)$ is homogeneous of degree one in $p$ and for each $p$, it assumes every non-negative value as $u$ varies, then $x(p, y)$ must be homogeneous of degree zero in $(p, y)$. 


2.5 Consider the solution, \( e(p, u) = up_1^\alpha p_2^\beta / p_3^\gamma \) at the end of Example 2.3.

(a) Derive the indirect utility function through the relation \( e(p, v(p, y)) = y \) and verify Roy’s identity.

(b) Use the construction given in the proof of Theorem 2.1 to recover a utility function generating \( e(p, u) \). Show that the utility function you derive generates the demand functions given in Example 2.3.

2.6 A consumer has expenditure function \( e(p_1, p_2, u) = up_1 p_2 / (p_1 + p_2) \). Find a direct utility function, \( u(x_1, x_2) \), that rationalises this person’s demand behaviour.

2.7 Derive the consumer’s inverse demand functions, \( p_1(x_1, x_2) \) and \( p_2(x_1, x_2) \), when the utility function is of the Cobb-Douglas form, \( u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha} \) for \( 0 < \alpha < 1 \).

2.8 The consumer buys bundle \( x^i \) at prices \( p^i \), \( i = 0, 1 \). Separately for parts (a) to (d), state whether these indicated choices satisfy WARP:

(a) \( p^0 = (1, 3), x^0 = (4, 2); p^1 = (3, 5), x^1 = (3, 1) \).

(b) \( p^0 = (1, 6), x^0 = (10, 5); p^1 = (3, 5), x^1 = (8, 4) \).

(c) \( p^0 = (1, 2), x^0 = (3, 1); p^1 = (2, 2), x^1 = (1, 2) \).

(d) \( p^0 = (2, 6), x^0 = (20, 10); p^1 = (3, 5), x^1 = (18, 4) \).

2.9 Suppose there are only two goods and that a consumer’s choice function \( x(p, y) \) satisfies budget balancedness, \( p \cdot x(p, y) = y \forall (p, y) \). Show the following:

(a) If \( x(p, y) \) is homogeneous of degree zero in \((p, y)\), then the Slutsky matrix associated with \( x(p, y) \) is symmetric.

(b) If \( x(p, y) \) satisfies WARP, then the ‘revealed preferred to’ relation, \( R \), has no intransitive cycles.

(By definition, \( x^i R x^j \) if and only if \( x^i \) is revealed preferred to \( x^j \).)

2.10 Hicks (1956) offered the following example to demonstrate how WARP can fail to result in transitive revealed preferences when there are more than two goods. The consumer chooses bundle \( x^i \) at prices \( p^i \), \( i = 0, 1, 2 \), where

\[
\begin{align*}
  p^0 &= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} & x^0 &= \begin{pmatrix} 5 \\ 19 \\ 9 \end{pmatrix} \\
  p^1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} & x^1 &= \begin{pmatrix} 12 \\ 12 \\ 12 \end{pmatrix} \\
  p^2 &= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} & x^2 &= \begin{pmatrix} 27 \\ 11 \\ 1 \end{pmatrix}.
\end{align*}
\]

(a) Show that these data satisfy WARP. Do it by considering all possible pairwise comparisons of the bundles and showing that in each case, one bundle in the pair is revealed preferred to the other.

(b) Find the intransitivity in the revealed preferences.
2.11 Show that if a consumer chooses bundles to maximise a strictly quasiconcave and strictly increasing utility function, his demand behaviour satisfies SARP.

2.12 This exercise guides you through a proof of a simplified version of Afriat’s Theorem. Suppose that a consumer is observed to demand bundle $x^1$ when the price vector is $p^1$, and bundle $x^2$ when the price vector is $p^2$. This produces the finite data set $D = \{(x^1, p^1), (x^2, p^2), \ldots, (x^K, p^K)\}$. We say that the consumer’s choice behaviour satisfies GARP on the finite data set $D$ if for every finite sequence, $(x^{k_1}, p^{k_1}), (x^{k_2}, p^{k_2}), \ldots, (x^{k_m}, p^{k_m})$, of points in $D$, if $p^{k_1} \cdot x^{k_1} \geq p^{k_2} \cdot x^{k_2}, \ldots, p^{k_{m-1}} \cdot x^{k_{m-1}} \geq p^{k_m} \cdot x^{k_m}$, then $p^{k_1} \cdot x^{k_1} \leq p^{k_m} \cdot x^{k_m}$.

In other words, GARP holds if whenever $x^{k_1}$ is revealed preferred to $x^{k_2}$, and $x^{k_2}$ is revealed preferred to $x^{k_3}$, and so on, and $x^{k_{m-1}}$ is revealed preferred to $x^{k_m}$, then $x^{k_1}$ is at least as expensive as $x^{k_m}$ when $x^{k_m}$ is chosen. (Note that SARP is stronger, requiring that $x^{k_1}$ be strictly more expensive than $x^{k_m}$.)

Assume throughout this question that the consumer’s choice behaviour satisfies GARP on the data set $D = \{(x^1, p^1), (x^2, p^2), \ldots, (x^K, p^K)\}$ and that $p^k \in \mathbb{R}^n_+$ for every $k = 1, \ldots, K$.

For each $k = 1, 2, \ldots, n$, define
\[
\phi(x^k) = \min_{k_1, \ldots, k_m} p^{k_1} \cdot (x^{k_2} - x^{k_1}) + p^{k_2} \cdot (x^{k_3} - x^{k_2}) + \ldots + p^{k_m} \cdot (x^k - x^{k_m}),
\]
where the minimum is taken over all sequences $k_1, \ldots, k_m$ of distinct elements of $\{1, 2, \ldots, K\}$ such that $p^{k_j} \cdot (x^{k_{j+1}} - x^{k_j}) \leq 0$ for every $j = 1, 2, \ldots, m - 1$, and such that $p^{k_m} \cdot (x^k - x^{k_m}) \leq 0$.

Note that at least one such sequence always exists, namely the ‘sequence’ consisting of one number, $k_1 = k$. Note also that there are only finitely many such sequences because their elements are distinct. Hence, the minimum above always exists.

(a) Prove that for all $k, j \in \{1, 2, \ldots, K\}$, $\phi(x^k) \leq \phi(x^j) + p^j \cdot (x^j - x^k)$ whenever $p^j \cdot x^j \leq p^j \cdot x^k$.

We next use the non-positive function $\phi(\cdot)$ to define a utility function $u: \mathbb{R}^n_+ \rightarrow \mathbb{R}$.

For every $x \in \mathbb{R}^n_+$ such that $p^k \cdot (x - x^k) \leq 0$ for at least one $k \in \{1, \ldots, K\}$, define $u(x) \leq 0$ as follows:
\[
u(x) = \min_{k} (\phi(x^k) + p^k \cdot (x - x^k)),
\]
where the minimum is over all $k \in \{1, \ldots, K\}$ such that $p^k \cdot (x - x^k) \leq 0$.

For every $x \in \mathbb{R}^n_+$ such that $p^k \cdot (x - x^k) > 0$ for every $k \in \{1, \ldots, K\}$, define $u(x) \geq 0$ as follows:
\[
u(x) = x_1 + \ldots + x_n.
\]

(b) Prove that for every $k \in \{1, \ldots, K\}$, $u(x^k) = \phi(x^k)$.

(c) Prove that $u(\cdot)$ is strongly increasing i.e., $u(x') > u(x)$ whenever every coordinate of $x'$ is at least as large as the corresponding coordinate of $x$ and at least one coordinate is strictly larger.

(d) Prove that for every $k \in \{1, \ldots, K\}$ and every $x \in \mathbb{R}^n$, if $p^k \cdot x \leq p^k \cdot x^k$, then $u(x) \leq u(x^k)$ and therefore, by (c), the second inequality is strict if the first is strict.

(e) Prove that $u(\cdot)$ is quasiconcave.

Altogether, (a)–(e) prove the following: If a finite data set satisfies GARP, then there is a strongly increasing quasiconcave utility function that rationalises the data in the sense that each chosen bundle
maximises the consumer’s utility among all bundles that are no more expensive than the chosen bundle at the prices at which it was chosen. (Afriat’s Theorem proves that utility function can, in addition, be chosen to be continuous and concave.)

(f) Prove a converse. That is, suppose that a strictly increasing utility function rationalises a finite data set. Prove that the consumer’s behaviour satisfies GARP on that data set.

2.13 Answer the following.

(a) Suppose that a choice function \( x(p, y) \in \mathbb{R}_+^n \) is homogeneous of degree zero in \((p, y)\). Show that WARP is satisfied \( \forall (p, y) \) iff it is satisfied on \( \{ (p, 1) \mid p \in \mathbb{R}_+^n \} \).

(b) Suppose that a choice function \( x(p, y) \in \mathbb{R}_+^n \) satisfies homogeneity and budget balancedness. Suppose further that whenever \( p_1 \) is not proportional to \( p_0 \), we have \( (p_1) \sim s(p_0, y)p_1 < 0 \). Show that \( x(p, y) \) satisfies WARP.

2.14 Consider the problem of insuring an asset against theft. The value of the asset is $D, the insurance cost is $I per year, and the probability of theft is \( p \). List the four outcomes in the set \( A \) associated with this risky situation. Characterise the choice between insurance and no insurance as a choice between two gambles, each involving all four outcomes in \( A \), where the gambles differ only in the probabilities assigned to each outcome.

2.15 We have assumed that an outcome set \( A \) has a finite number of elements, \( n \). Show that as long as \( n \geq 2 \), the space \( \mathcal{G} \) will always contain an infinite number of gambles.

2.16 Using Axioms G1 and G2, prove that at least one best and at least one worst outcome must exist in any finite set of outcomes, \( A = \{a_1, \ldots, a_n\} \) whenever \( n \geq 1 \).

2.17 Let \( A = \{a_1, a_2, a_3\} \), where \( a_1 > a_2 > a_3 \). The gamble \( g \) offers \( a_2 \) with certainty. Prove that if \( g \sim (\alpha \circ a_1, (1 - \alpha) \circ a_3) \), the \( \alpha \) must be strictly between zero and 1.

2.18 In the text, it was asserted that, to a safari hunter, death may be the worst outcome of an outing, yet an outing with the possibility of death is preferred to one where death is impossible. Characterise the outcome set associated with a hunter’s choice of outings, and prove this behaviour violates the combined implications of Axioms G3 and G4.

2.19 Axiom G3 asserts the existence of an indifference probability for any gamble in \( \mathcal{G} \). For a given gamble \( g \in \mathcal{G} \), prove that the indifference probability is unique using G4.

2.20 Consider the following ‘Independence Axiom’ on a consumer’s preferences, \( \succsim \), over gambles: If \( (p_1 \circ a_1, \ldots, p_n \circ a_n) \sim (q_1 \circ a_1, \ldots, q_n \circ a_n) \), then for every \( \alpha \in [0, 1] \), and every simple gamble \((r_1 \circ a_1, \ldots, r_n \circ a_n)\),

\[
((\alpha p_1 + (1 - \alpha)r_1) \circ a_1, \ldots, (\alpha p_n + (1 - \alpha)r_n) \circ a_n)
\sim
((\alpha q_1 + (1 - \alpha)r_1) \circ a_1, \ldots, (\alpha q_n + (1 - \alpha)r_n) \circ a_n).
\]

(Note this axiom says that when we combine each of two gambles with a third in the same way, the individual’s ranking of the two new gambles is independent of which third gamble we used.) Show that this axiom follows from Axioms G5 and G6.
2.21 Using the definition of risk aversion given in the text, prove that an individual is risk averse over gambles involving non-negative wealth levels if and only if his VNM utility function is strictly concave on \( \mathbb{R}_+ \).

2.22 Suppose that \( \succcurlyeq \) is a binary relation over gambles in \( G \) satisfying Axioms G2, G3, and G4. Show that \( \succcurlyeq \) satisfies G1 as well.

2.23 Let \( u \) and \( v \) be utility functions (not necessarily VNM) representing \( \succcurlyeq \) on \( G \). Show that \( v \) is a positive affine transformation of \( u \) if and only if for all gambles \( g^1, g^2, g^3 \in G \), with no two indifferent, we have

\[
\frac{u(g^1) - u(g^2)}{u(g^2) - u(g^3)} = \frac{v(g^1) - v(g^2)}{v(g^2) - v(g^3)}.
\]

2.24 Reconsider Example 2.7 and show that the individual will less than fully insure if the price per unit of insurance, \( \rho \), exceeds the probability of incurring an accident, \( \alpha \).

2.25 Consider the quadratic VNM utility function \( U(w) = a + bw + cw^2 \).

(a) What restrictions if any must be placed on parameters \( a, b, \) and \( c \) for this function to display risk aversion?

(b) Over what domain of wealth can a quadratic VNM utility function be defined?

(c) Given the gamble

\[ g = ((1/2) \circ (w + h), (1/2) \circ (w - h)), \]

show that \( CE < E(g) \) and that \( P > 0 \).

(d) Show that this function, satisfying the restrictions in part (a), cannot represent preferences that display decreasing absolute risk aversion.

2.26 Let \( u(w) = -(b - w)^c \). What restrictions on \( w, b, \) and \( c \) are required to ensure that \( u(w) \) is strictly increasing and strictly concave? Show that under those restrictions, \( u(w) \) displays increasing absolute risk aversion.

2.27 Show that for \( \beta > 0 \), the VNM utility function \( u(w) = \alpha + \beta \ln(w) \) displays decreasing absolute risk aversion.

2.28 Let \( u(x_1, x_2) = \ln(x_1) + 2 \ln(x_2) \). If \( p_1 = p_2 = 1 \), will this person be risk loving, risk neutral, or risk averse when offered gambles over different amounts of income?

2.29 Using the definitions of risk aversion, certainty equivalent, and risk premium, prove that \( CE < E(g) \) (or \( P > 0 \)) for all \( g \in G \) is necessary and sufficient for risk aversion.

2.30 Prove that an individual is risk neutral if and only if each of the following is satisfied:

(a) The VNM utility function is linear in wealth.

(b) \( C = E(g) \) for all \( g \in G \).

(c) \( P = 0 \) for all \( g \in G \).

What are the three equivalent necessary and sufficient conditions for risk loving?
2.31 Prove that for any VNM utility function, the condition \( u'''(w) > 0 \) is necessary but not sufficient for DARA.

2.32 If a VNM utility function displays constant absolute risk aversion, so that \( R_a(w) = \alpha \) for all \( w \), what functional form must it have?

2.33 Suppose a consumer’s preferences over wealth gambles can be represented by a twice differentiable VNM utility function. Show that the consumer’s preferences over gambles are independent of his initial wealth if and only if his utility function displays constant absolute risk aversion.

2.34 Another measure of risk aversion offered by Arrow and Pratt is their relative risk aversion measure, \( R_r(w) \equiv R_a(w)w \). In what sense is \( R_r(w) \) an ‘elasticity’? If \( u(w) \) displays constant relative risk aversion, what functional form must it have?

2.35 An investor must decide how much of initial wealth \( w \) to allocate to a risky asset with unknown rate of return \( r \), where each outcome \( r_i \) occurs with probability \( p_i, i = 1, \ldots, n \). Using the framework of Example 2.6, prove that if the investor’s preferences display increasing absolute risk aversion, the risky asset must be an ‘inferior’ good.

2.36 Let \( S_i \) be the set of all probabilities of winning such that individual \( i \) will accept a gamble of winning or losing a small amount of wealth, \( h \). Show that for any two individuals \( i \) and \( j \), where \( R_a(w) > R_a(w) \), it must be that \( S_i \subset S_j \). Conclude that the more risk averse the individual, the smaller the set of gambles he will accept.

2.37 An infinitely lived agent must choose his lifetime consumption plan. Let \( x_t \) denote consumption spending in period \( t \), \( y_t \) denote income expected in period \( t \), and \( r > 0 \), the market rate of interest at which the agent can freely borrow or lend. The agent’s intertemporal utility function takes the additively separable form

\[
\begin{align*}
  u^*(x_0, x_1, x_2, \ldots) &= \sum_{t=0}^{\infty} \beta^t u(x_t),
\end{align*}
\]

where \( u(x) \) is increasing and strictly concave, and \( 0 < \beta < 1 \). The intertemporal budget constraint requires that the present value of expenditures not exceed the present value of income:

\[
\begin{align*}
  \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t x_t &\leq \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t y_t.
\end{align*}
\]

(a) What interpretation can you give to parameter \( \beta \)?

(b) Write down the first-order conditions for optimal choice of consumption in period \( t \).

(c) Assuming that consumption in all other periods remains constant, sketch an indifference curve showing the intertemporal trade-off between \( x_t \) and \( x_{t+1} \) alone. Carefully justify the slope and curvature you have depicted.

(d) How does consumption in period \( t \) vary with the market interest rate?

(e) Show that lifetime utility will always increase with an income increase in any period.

(f) If \( \beta = 1/(1 + r) \), what is the consumption plan of the agent?

(g) Describe the agent’s consumption plan if \( \beta > 1/(1 + r) \) and if \( \beta < 1/(1 + r) \).
2.38 Consider a two-period version of the preceding exercise where

\[ u(x_t) = -(1/2)(x_t - 2)^2 \quad \text{and} \quad t = 0, 1. \]

(a) If \( y_0 = 1, \ y_1 = 1, \) and \( \beta = 1/(1 + r), \) solve for optimal consumption in each period and calculate the level of lifetime utility the agent achieves.

Suppose, now, that the agent again knows that income in the initial period will be \( y_0 = 1. \) However, there is uncertainty about what next period’s income will be. It could be high, \( y_H^1 = 3/2; \) or it could be low, \( y_L^1 = 1/2. \) He knows it will be high with probability \( 1/2. \) His problem now is to choose the initial period consumption, \( x_0; \) the future consumption if income is high, \( x_H^1; \) and the future consumption if income is low, \( x_L^1, \) to maximise (inter-temporal) expected utility.

(b) Again, assuming that \( \beta = 1/(1 + r), \) formulate the agent’s optimisation problem and solve for the optimal consumption plan and the level of lifetime utility.

(c) How do you account for any difference or similarity in your answers to parts (a) and (b)?