5

Integration

Chapter Preview  We are now at a critical point in the calculus story. Many would argue that this chapter is the cornerstone of calculus because it explains the relationship between the two processes of calculus: differentiation and integration. We begin by explaining why finding the area of regions bounded by the graphs of functions is such an important problem in calculus. Then you will see how antiderivatives lead to definite integrals, which are used to solve the area problem. But there is more to the story. You will also see the remarkable connection between derivatives and integrals, which is expressed in the Fundamental Theorem of Calculus. In this chapter, we develop key properties of definite integrals, investigate a few of their many applications, and present the first of several powerful techniques for evaluating definite integrals.

5.1 Approximating Areas under Curves

The derivative of a function is associated with rates of change and slopes of tangent lines. We also know that antiderivatives (or indefinite integrals) reverse the derivative operation. Figure 5.1 summarizes our current understanding and raises the question: What is the geometric meaning of the integral? The following example reveals a clue.

Figure 5.1

Area under a Velocity Curve
Consider an object moving along a line with a known position function. You learned in previous chapters that the slope of the line tangent to the graph of the position function at a certain time gives the velocity \( v \) at that time. We now turn the situation around. If we know the velocity function of a moving object, what can we learn about its position function?
Recall from Section 3.5 that the displacement of an object moving along a line is the difference between its initial and final position. If the velocity of an object is positive, its displacement equals the distance traveled.

The side lengths of the rectangle in Figure 5.3 have units mi/hr and hr. Therefore, the units of the area are mi/hr \cdot hr = mi, which is a unit of displacement.

Imagine a car traveling at a constant velocity of 60 mi/hr along a straight highway over a two-hour period. The graph of the velocity function \( v = 60 \) on the interval \( 0 \leq t \leq 2 \) is a horizontal line (Figure 5.2). The displacement of the car between \( t = 0 \) and \( t = 2 \) is found by a familiar formula:

\[
\text{displacement} = \text{rate} \cdot \text{time} \\
= 60 \text{ mi/hr} \cdot 2 \text{ hr} = 120 \text{ mi.}
\]

This product is the area of the rectangle formed by the velocity curve and the \( t \)-axis between \( t = 0 \) and \( t = 2 \) (Figure 5.3). In the case of constant positive velocity, we see that the area between the velocity curve and the \( t \)-axis is the displacement of the moving object.

**Figure 5.2**

**Figure 5.3**

**Quick Check 1** What is the displacement of an object that travels at a constant velocity of 10 mi/hr for a half hour, 20 mi/hr for the next half hour, and 30 mi/hr for the next hour?

Because objects do not necessarily move at a constant velocity, we first extend these ideas to positive velocities that change over an interval of time. One strategy is to divide the time interval into many subintervals and approximate the velocity on each subinterval with a constant velocity. Then the displacements on each subinterval are calculated and summed. This strategy produces only an approximation to the displacement; however, this approximation generally improves as the number of subintervals increases.

**Example 1** Approximating the displacement Suppose the velocity in m/s of an object moving along a line is given by the function \( v = r' \), where \( 0 \leq t \leq 8 \). Approximate the displacement of the object by dividing the time interval \([0, 8]\) into \( n \) subintervals of equal length. On each subinterval, approximate the velocity with a constant equal to the value of \( v \) evaluated at the midpoint of the subinterval.

**SOLUTION**

\[\begin{align*}
a. & \quad \text{Begin by dividing } [0, 8] \text{ into } n = 2 \text{ subintervals: } [0, 4] \text{ and } [4, 8]. \\
b. & \quad \text{Divide } [0, 8] \text{ into } n = 4 \text{ subintervals: } [0, 2], [2, 4], [4, 6], \text{ and } [6, 8]. \\
c. & \quad \text{Divide } [0, 8] \text{ into } n = 8 \text{ subintervals of equal length.}
\end{align*}\]

\[\begin{align*}
a. & \quad \text{We divide the interval } [0, 8] \text{ into } n = 2 \text{ subintervals, } [0, 4] \text{ and } [4, 8], \text{ each with length } 4. \text{ The velocity on each subinterval is approximated by evaluating } v \text{ at the midpoint of that subinterval (Figure 5.4a).} \\
& \quad \text{• We approximate the velocity on } [0, 4] \text{ by } v(2) = 2^2 = 4 \text{ m/s. Traveling at } 4 \text{ m/s for } 4 \text{ s results in a displacement of } 4 \text{ m/s} \cdot 4 \text{ s} = 16 \text{ m.} \\
& \quad \text{• We approximate the velocity on } [4, 8] \text{ by } v(6) = 6^2 = 36 \text{ m/s. Traveling at } 36 \text{ m/s for } 4 \text{ s results in a displacement of } 36 \text{ m/s} \cdot 4 \text{ s} = 144 \text{ m.}
\end{align*}\]

Therefore, an approximation to the displacement over the entire interval \([0, 8]\) is

\[
(v(2) \cdot 4 \text{ s}) + (v(6) \cdot 4 \text{ s}) = (4 \text{ m/s} \cdot 4 \text{ s}) + (36 \text{ m/s} \cdot 4 \text{ s}) = 160 \text{ m.}
\]
b. With \( n = 4 \) (Figure 5.4b), each subinterval has length 2. The approximate displacement over the entire interval is
\[
\frac{1}{2} \text{m/s} \cdot 2 \text{s} + \frac{9}{2} \text{m/s} \cdot 2 \text{s} + \frac{25}{2} \text{m/s} \cdot 2 \text{s} + \frac{49}{2} \text{m/s} \cdot 2 \text{s} = 168 \text{ m}.
\]

c. With \( n = 8 \) subintervals (Figure 5.4c), the approximation to the displacement is 170 m. In each case, the approximate displacement is the sum of the areas of the rectangles under the velocity curve.

The midpoint of each subinterval is used to approximate the velocity over that subinterval.

The progression in Example 1 may be continued. Larger values of \( n \) mean more rectangles; in general, more rectangles give a better fit to the region under the curve (Figure 5.5). With the help of a calculator, we can generate the approximations in Table 5.1 using \( n = 1, 2, 4, 8, 16, 32, \) and 64 subintervals. Observe that as \( n \) increases, the approximations appear to approach a limit of approximately 170.7 m. The limit is the exact displacement, which is represented by the area of the region under the velocity curve. This strategy of taking limits of sums is developed fully in Section 5.2.

**Table 5.1** Approximations to the area under the velocity curve \( v = t^2 \) on \([0, 8]\)

<table>
<thead>
<tr>
<th>Number of subintervals</th>
<th>Length of each subinterval</th>
<th>Approximate displacement (area under curve)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8 s</td>
<td>128.0 m</td>
</tr>
<tr>
<td>2</td>
<td>4 s</td>
<td>160.0 m</td>
</tr>
<tr>
<td>4</td>
<td>2 s</td>
<td>168.0 m</td>
</tr>
<tr>
<td>8</td>
<td>1 s</td>
<td>170.0 m</td>
</tr>
<tr>
<td>16</td>
<td>0.5 s</td>
<td>170.5 m</td>
</tr>
<tr>
<td>32</td>
<td>0.25 s</td>
<td>170.625 m</td>
</tr>
<tr>
<td>64</td>
<td>0.125 s</td>
<td>170.65625 m</td>
</tr>
</tbody>
</table>
Chapter 5 • Integration

Approximating Areas by Riemann Sums

We wouldn’t spend much time investigating areas under curves if the idea applied only to computing displacements from velocity curves. However, the problem of finding areas under curves arises frequently and turns out to be immensely important—as you will see in the next two chapters. For this reason, we now develop a systematic method for approximating areas under curves. Consider a function \( f \) that is continuous and nonnegative on an interval \([a, b]\). The goal is to approximate the area of the region \( R \) bounded by the graph of \( f \) and the \( x \)-axis from \( x = a \) to \( x = b \) (Figure 5.6). We begin by dividing the interval \([a, b]\) into \( n \) subintervals of equal length,\n\[ [x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n], \]
where \( a = x_0 \) and \( b = x_n \) (Figure 5.7). The length of each subinterval, denoted \( \Delta x \), is found by dividing the length of the entire interval by \( n \):
\[ \Delta x = \frac{b - a}{n}. \]

DEFINITION Regular Partition
Suppose \([a, b]\) is a closed interval containing \( n \) subintervals
\[ [x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n] \]
of equal length \( \Delta x = \frac{b - a}{n} \) with \( a = x_0 \) and \( b = x_n \). The endpoints \( x_0, x_1, x_2, \ldots, x_{n-1}, x_n \) of the subintervals are called grid points, and they create a regular partition of the interval \([a, b]\). In general, the \( k \)th grid point is
\[ x_k = a + k\Delta x, \text{ for } k = 0, 1, 2, \ldots, n. \]

QUICK CHECK 3 If the interval \([1, 9]\) is partitioned into 4 subintervals of equal length, what is \( \Delta x \)? List the grid points \( x_0, x_1, x_2, x_3, \) and \( x_4 \).

In the \( k \)th subinterval \([x_{k-1}, x_k]\), we choose any point \( x_k^* \) and build a rectangle whose height is \( f(x_k^*) \), the value of \( f \) at \( x_k^* \) (Figure 5.8). The area of the rectangle on the \( k \)th subinterval is
\[ \text{height} \cdot \text{base} = f(x_k^*) \Delta x, \text{ where } k = 1, 2, \ldots, n. \]

Summing the areas of the rectangles in Figure 5.8, we obtain an approximation to the area of \( R \), which is called a Riemann sum:
\[ f(x_1^*) \Delta x + f(x_2^*) \Delta x + \cdots + f(x_n^*) \Delta x. \]

Three notable Riemann sums are the left, right, and midpoint Riemann sums.
For the left Riemann sum,
\[ x_1^* = a + 0 \cdot \Delta x, \quad x_2^* = a + 1 \cdot \Delta x, \]
and in general, \( x_k^* = a + (k - 1) \Delta x, \)
for \( k = 1, \ldots, n. \)

For the right Riemann sum,
\[ x_1^* = a + 1 \cdot \Delta x, \quad x_2^* = a + 2 \cdot \Delta x, \]
and in general \( x_k^* = a + k \Delta x, \)
for \( k = 1, \ldots, n. \)

For the midpoint Riemann sum,
\[ x_1^* = a + \frac{1}{2} \Delta x, \quad x_2^* = a + \frac{3}{2} \Delta x, \]
and in general, \( x_k^* = a + \frac{k}{2} \Delta x = \frac{x_k + x_{k-1}}{2}, \)
for \( k = 1, \ldots, n. \)

**DEFINITION** Riemann Sum

Suppose \( f \) is defined on a closed interval \([a, b] \), which is divided into \( n \) subintervals of equal length \( \Delta x \). If \( x_k^* \) is any point in the \( k \)th subinterval \([x_{k-1}, x_k]\), for \( k = 1, 2, \ldots, n \), then
\[ f(x_1^*) \Delta x + f(x_2^*) \Delta x + \cdots + f(x_n^*) \Delta x \]
is called a Riemann sum for \( f \) on \([a, b] \). This sum is called

- a left Riemann sum if \( x_k^* \) is the left endpoint of \([x_{k-1}, x_k]\) (Figure 5.9);
- a right Riemann sum if \( x_k^* \) is the right endpoint of \([x_{k-1}, x_k]\) (Figure 5.10); and
- a midpoint Riemann sum if \( x_k^* \) is the midpoint of \([x_{k-1}, x_k]\) (Figure 5.11), for
  \( k = 1, 2, \ldots, n. \)

We now use this definition to approximate the area under the curve \( y = \sin x \).

**EXAMPLE 2** Left and right Riemann sums

Let \( R \) be the region bounded by the graph of \( f(x) = \sin x \) and the \( x \)-axis between \( x = 0 \) and \( x = \pi/2 \).

a. Approximate the area of \( R \) using a left Riemann sum with \( n = 6 \) subintervals.

Illustrate the sum with the appropriate rectangles.

b. Approximate the area of \( R \) using a right Riemann sum with \( n = 6 \) subintervals.

Illustrate the sum with the appropriate rectangles.

c. Do the area approximations in parts (a) and (b) underestimate or overestimate the actual area under the curve?
SOLUTION  Dividing the interval \([a, b] = [0, \pi/2]\) into \(n = 6\) subintervals means the length of each subinterval is

\[
\Delta x = \frac{b - a}{n} = \frac{\pi/2 - 0}{6} = \frac{\pi}{12}
\]

a. To find the left Riemann sum, we set \(x_1^*, x_2^*, \ldots, x_6^*\) equal to the left endpoints of the six subintervals. The heights of the rectangles are \(f(x_k^*)\), for \(k = 1, \ldots, 6\).

The resulting left Riemann sum (Figure 5.12) is

\[
f(x_1^*) \Delta x + f(x_2^*) \Delta x + \cdots + f(x_6^*) \Delta x \\
= \left( \sin 0 \right) \cdot \frac{\pi}{12} + \left( \sin \frac{\pi}{12} \right) \cdot \frac{\pi}{12} + \left( \sin \frac{\pi}{6} \right) \cdot \frac{\pi}{12} + \left( \sin \frac{\pi}{4} \right) \cdot \frac{\pi}{12} + \left( \sin \frac{5\pi}{12} \right) \cdot \frac{\pi}{12} + \left( \sin \frac{\pi}{2} \right) \cdot \frac{\pi}{12}
\]

\[
\approx 0.863.
\]

b. In a right Riemann sum, the right endpoints are used for \(x_1^*, x_2^*, \ldots, x_6^*\), and the heights of the rectangles are \(f(x_k^*)\), for \(k = 1, \ldots, 6\).

The resulting right Riemann sum (Figure 5.13) is

\[
f(x_1^*) \Delta x + f(x_2^*) \Delta x + \cdots + f(x_6^*) \Delta x \\
= \left( \sin \frac{\pi}{12} \right) \cdot \frac{\pi}{12} + \left( \sin \frac{\pi}{6} \right) \cdot \frac{\pi}{12} + \left( \sin \frac{\pi}{4} \right) \cdot \frac{\pi}{12} + \left( \sin \frac{5\pi}{12} \right) \cdot \frac{\pi}{12} + \left( \sin \frac{\pi}{2} \right) \cdot \frac{\pi}{12}
\]

\[
\approx 1.125.
\]
QUICK CHECK 4 If the function in Example 2 is replaced with \( f(x) = \cos x \), does the left Riemann sum or the right Riemann sum overestimate the area under the curve? \\

Example 3 A midpoint Riemann sum Let \( R \) be the region bounded by the graph of \( f(x) = \sin x \) and the \( x \)-axis between \( x = 0 \) and \( x = \pi/2 \). Approximate the area of \( R \) using a midpoint Riemann sum with \( n = 6 \) subintervals. Illustrate the sum with the appropriate rectangles.

Solution The grid points and the length of the subintervals \( \Delta x = \pi/12 \) are the same as in Example 2. To find the midpoint Riemann sum, we set \( x_1^*, x_2^*, \ldots, x_6^* \) equal to the midpoints of the subintervals. The midpoint of the first subinterval is the average of \( x_0 \) and \( x_1 \), which is

\[
x_1^* = \frac{x_1 + x_0}{2} = \frac{\pi/12 + 0}{2} = \frac{\pi}{24}.
\]

The remaining midpoints are also computed by averaging the two nearest grid points.

![Figure 5.14](image)

The resulting midpoint Riemann sum (Figure 5.14) is

\[
f(x_1^*) \Delta x + f(x_2^*) \Delta x + \cdots + f(x_6^*) \Delta x
\]

\[
= \left( \sin \frac{\pi}{24} \cdot \frac{\pi}{12} + \sin \frac{3\pi}{24} \cdot \frac{\pi}{12} + \sin \frac{5\pi}{24} \cdot \frac{\pi}{12}\right)
\]

\[
+ \left( \sin \frac{7\pi}{24} \cdot \frac{\pi}{12} + \sin \frac{9\pi}{24} \cdot \frac{\pi}{12} + \sin \frac{11\pi}{24} \cdot \frac{\pi}{12}\right)
\]

\[
\approx 1.003.
\]

Comparing the midpoint Riemann sum (Figure 5.14) with the left (Figure 5.12) and right (Figure 5.13) Riemann sums suggests that the midpoint sum is a more accurate estimate of the area under the curve.

Example 4 Riemann sums from tables Estimate the area \( A \) under the graph of \( f \) on the interval \([0, 2]\) using left and right Riemann sums with \( n = 4 \), where \( f \) is continuous but known only at the points in Table 5.2.

Solution With \( n = 4 \) subintervals on the interval \([0, 2]\), \( \Delta x = 2/4 = 0.5 \). Using the left endpoint of each subinterval, the left Riemann sum is

\[
A \approx (f(0) + f(0.5) + f(1.0) + f(1.5)) \Delta x = (1 + 3 + 4.5 + 5.5)0.5 = 7.0.
\]
Using the right endpoint of each subinterval, the right Riemann sum is
\[ A \approx (f(0.5) + f(1.0) + f(1.5) + f(2.0)) \Delta x = (3 + 4.5 + 5.5 + 6.0)0.5 = 9.5. \]
With only five function values, these estimates of the area are necessarily crude. Better estimates are obtained by using more subintervals and more function values.

**Related Exercises 35–38**

### Sigma (Summation) Notation

Working with Riemann sums is cumbersome with large numbers of subintervals. Therefore, we pause for a moment to introduce some notation that simplifies our work.

**Sigma** (or summation notation) is used to express sums in a compact way. For example, the sum \( 1 + 2 + 3 + \cdots + 10 \) is represented in sigma notation as \( \sum_{k=1}^{10} k \). Here is how the notation works. The symbol \( \Sigma \) (sigma, the Greek capital S) stands for sum. The index \( k \) takes on all integer values from the lower limit \( (k = 1) \) to the upper limit \( (k = 10) \). The expression that immediately follows \( \Sigma \) (the summand) is evaluated for each value of \( k \), and the resulting values are summed. Here are some examples:

\[
\begin{align*}
\sum_{k=1}^{99} k &= 1 + 2 + 3 + \cdots + 99 = 4950 \\
\sum_{k=3}^{4} k^2 &= 0^2 + 1^2 + 2^2 + 3^2 = 14 \\
\sum_{k=-2}^{2} (k^2 + k) &= ((-1)^2 + (-1)) + (0^2 + 0) + (1^2 + 1) + (2^2 + 2) = 8.
\end{align*}
\]

The index in a sum is a dummy variable. It is internal to the sum, so it does not matter what symbol you choose as an index. For example,

\[
\sum_{k=1}^{99} k = \sum_{n=1}^{99} n = \sum_{p=1}^{99} p.
\]

Two properties of sums and sigma notation are useful in upcoming work. Suppose that \( \{a_1, a_2, \ldots, a_n\} \) and \( \{b_1, b_2, \ldots, b_n\} \) are two sets of real numbers, and suppose that \( c \) is a real number. Then we can factor multiplicative constants out of a sum:

**Constant Multiple Rule**

\[ \sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k. \]

We can also split a sum into two sums:

**Addition Rule**

\[ \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k. \]

In the coming examples and exercises, the following formulas for sums of powers of integers are essential.

---

> Formulas for \( \sum_{k=1}^{n} k^p \), where \( p \) is a positive integer, have been known for centuries. The formulas for \( p = 0, 1, 2, \) and \( 3 \) are relatively simple. The formulas become complicated as \( p \) increases.

---

### Theorem 5.1 Sums of Powers of Integers

Let \( n \) be a positive integer and \( c \) a real number.

\[
\begin{align*}
\sum_{k=1}^{n} c &= cn \\
\sum_{k=1}^{n} k &= \frac{n(n + 1)}{2} \\
\sum_{k=1}^{n} k^2 &= \frac{n(n + 1)(2n + 1)}{6} \\
\sum_{k=1}^{n} k^3 &= \frac{n^2(n + 1)^2}{4}
\end{align*}
\]

**Related Exercises 39–42**

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Riemann Sums Using Sigma Notation

With sigma notation, a Riemann sum has the convenient compact form

\[ f(x_1^*) \Delta x + f(x_2^*) \Delta x + \cdots + f(x_n^*) \Delta x = \sum_{k=1}^{n} f(x_k^*) \Delta x. \]

To express left, right, and midpoint Riemann sums in sigma notation, we must identify the points \( x_k^* \).

- For left Riemann sums, the left endpoints of the subintervals are \( x_k^* = a + (k - 1) \Delta x \), for \( k = 1, \ldots, n \).
- For right Riemann sums, the right endpoints of the subintervals are \( x_k^* = a + k \Delta x \), for \( k = 1, \ldots, n \).
- For midpoint Riemann sums, the midpoints of the subintervals are \( x_k^* = a + (k - \frac{1}{2}) \Delta x \), for \( k = 1, \ldots, n \).

The three Riemann sums are written compactly as follows.

**Definition** Left, Right, and Midpoint Riemann Sums in Sigma Notation

Suppose \( f \) is defined on a closed interval \([a, b]\), which is divided into \( n \) subintervals of equal length \( \Delta x \). If \( x_k^* \) is a point in the \( k \)th subinterval \([x_{k-1}, x_k]\), for \( k = 1, 2, \ldots, n \), then the **Riemann sum** for \( f \) on \([a, b]\) is \( \sum_{k=1}^{n} f(x_k^*) \Delta x \). Three cases arise in practice.

- **Left Riemann sum** if \( x_k^* = a + (k - 1) \Delta x \)
- **Right Riemann sum** if \( x_k^* = a + k \Delta x \)
- **Midpoint Riemann sum** if \( x_k^* = a + (k - \frac{1}{2}) \Delta x \)

**Example 5** Calculating Riemann sums

Evaluate the left, right, and midpoint Riemann sums for \( f(x) = x^3 + 1 \) between \( a = 0 \) and \( b = 2 \) using \( n = 50 \) subintervals. Make a conjecture about the exact area of the region under the curve (Figure 5.15).

**Solution** With \( n = 50 \), the length of each subinterval is

\[ \Delta x = \frac{b - a}{n} = \frac{2 - 0}{50} = \frac{1}{25} = 0.04. \]

The value of \( x_k^* \) for the left Riemann sum is

\[ x_k^* = a + (k - 1) \Delta x = 0 + 0.04(k - 1) = 0.04k - 0.04, \]

for \( k = 1, 2, \ldots, 50 \). Therefore, the left Riemann sum, evaluated with a calculator, is

\[ \sum_{k=1}^{n} f(x_k^*) \Delta x = \sum_{k=1}^{50} f(0.04k - 0.04)0.04 = 5.8416. \]

To evaluate the right Riemann sum, we let \( x_k^* = a + k \Delta x = 0.04k \) and find that

\[ \sum_{k=1}^{n} f(x_k^*) \Delta x = \sum_{k=1}^{50} f(0.04k)0.04 = 6.1616. \]
For the midpoint Riemann sum, we let
\[ x_k^* = a + \left( k - \frac{1}{2} \right) \Delta x = 0 + 0.04 \left( k - \frac{1}{2} \right) = 0.04k - 0.02. \]

The value of the sum is
\[ \sum_{k=1}^{n} f(x_k^*) \Delta x = \sum_{k=1}^{50} f(0.04k - 0.02)0.04 = 5.9992. \]

Because \( f \) is increasing on \([0, 2]\), the left Riemann sum underestimates the area of the shaded region in Figure 5.15 and the right Riemann sum overestimates the area. Therefore, the exact area lies between 5.8416 and 6.1616. The midpoint Riemann sum usually gives the best estimate for increasing or decreasing functions.

Table 5.3 shows the left, right, and midpoint Riemann sum approximations for values of \( n \) up to 200. All three sets of approximations approach a value near 6, which is a reasonable estimate of the area under the curve. In Section 5.2, we show rigorously that the limit of all three Riemann sums as \( n \to \infty \) is 6.

**Table 5.3 Left, right, and midpoint Riemann sum approximations**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( L_n )</th>
<th>( R_n )</th>
<th>( M_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>5.61</td>
<td>6.41</td>
<td>5.995</td>
</tr>
<tr>
<td>40</td>
<td>5.8025</td>
<td>6.2025</td>
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<td>200</td>
<td>5.9601</td>
<td>6.0401</td>
<td>5.99995</td>
</tr>
</tbody>
</table>

Table 5.3 shows the left, right, and midpoint Riemann sum approximations for values of \( n \) up to 200. All three sets of approximations approach a value near 6, which is a reasonable estimate of the area under the curve. In Section 5.2, we show rigorously that the limit of all three Riemann sums as \( n \to \infty \) is 6.

**ALTERNATIVE SOLUTION** It is worth examining another approach to Example 5. Consider the right Riemann sum given previously:
\[ \sum_{k=1}^{n} f(x_k^*) \Delta x = \sum_{k=1}^{50} f(0.04k)0.04. \]

Rather than evaluating this sum with a calculator, we note that
\[ f(0.04k) = (0.04k)^3 + 1 \]
and use the properties of sums:
\[
\sum_{k=1}^{50} f(x_k^*) \Delta x = \sum_{k=1}^{50} ((0.04k)^3 + 1)0.04 \\
= \sum_{k=1}^{50} (0.04k)^3 0.04 + \sum_{k=1}^{50} 1 \cdot 0.04 \\
= (0.04)^4 \sum_{k=1}^{50} k^3 + 0.04 \sum_{k=1}^{50} 1 \\
= (0.04)^4 \sum_{k=1}^{50} k^3 + 0.04 \sum_{k=1}^{50} 1 \\
= \sum_{k=1}^{50} k^3 = \frac{50^2 \cdot 51^2}{4} \\
\]

Using the summation formulas for powers of integers in Theorem 5.1, we find that
\[ \sum_{k=1}^{50} k = 50 \quad \text{and} \quad \sum_{k=1}^{50} k^3 = \frac{50^2 \cdot 51^2}{4} \]

Substituting the values of these sums into the right Riemann sum, its value is
\[ \sum_{k=1}^{50} f(x_k^*) \Delta x = \frac{3851}{625} = 6.1616, \]
confirming the result given by a calculator. The idea of evaluating Riemann sums for arbitrary values of \( n \) is used in Section 5.2, where we evaluate the limit of the Riemann sum as \( n \to \infty \).

Related Exercises 43–52

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SECTION 5.1 EXERCISES

Review Questions
1. Suppose an object moves along a line at 15 m/s, for \( 0 \leq t < 2 \), and at 25 m/s, for \( 2 \leq t \leq 5 \), where \( t \) is measured in seconds. Sketch the graph of the velocity function and find the displacement of the object for \( 0 \leq t \leq 5 \).

2. Given the graph of the positive velocity of an object moving along a line, what is the geometrical representation of its displacement over a time interval \([a, b]\)?

3. Suppose you want to approximate the area of the region bounded by the graph of \( f(x) = \cos x \) and the \( x \)-axis between \( x = 0 \) and \( x = \pi/2 \). Explain a possible strategy.

4. Explain how Riemann sum approximations to the area of a region under a curve change as the number of subintervals increases.

5. Suppose the interval \([1, 3]\) is partitioned into \( n = 4 \) subintervals. What is the subinterval length \( \Delta x \)? List the grid points \( x_0, x_1, x_2, x_3, \) and \( x_4 \). Which points are used for the left, right, and midpoint Riemann sums?

6. Suppose the interval \([2, 6]\) is partitioned into \( n = 4 \) subintervals with grid points \( x_0 = 2, x_1 = 3, x_2 = 4, x_3 = 5, \) and \( x_4 = 6 \). Write, but do not evaluate, the \( a \), \( b \), and \( \text{midpoint} \) Riemann sums for \( f(x) = x^2 \).

7. Does a right Riemann sum underestimate or overestimate the area of the region under the graph of a function that is positive and decreasing on an interval \([a, b]\)? Explain.

8. Does a left Riemann sum underestimate or overestimate the area of the region under the graph of a function that is positive and increasing on an interval \([a, b]\)? Explain.

Basic Skills

9. Approximating displacement The velocity in ft/s of an object moving along a line is given by \( v = 3t^2 + 1 \) on the interval \( 0 \leq t \leq 4 \).

   a. Divide the interval \([0, 4]\) into \( n = 4 \) subintervals, \([0, 1]\), \([1, 2]\), \([2, 3]\), and \([3, 4]\). On each subinterval, assume the object moves at a constant velocity equal to \( v \) evaluated at the midpoint of the subinterval and use these approximations to estimate the displacement of the object on \([0, 4]\) (see part (a) of the figure).

   b. Repeat part (a) for \( n = 8 \) subintervals (see part (b) of the figure).

10. Approximating displacement The velocity in ft/s of an object moving along a line is given by \( v = \sqrt{t} \) on the interval \( 1 \leq t \leq 7 \).

   a. Divide the time interval \([1, 7]\) into \( n = 3 \) subintervals, \([1, 3]\), \([3, 5]\), and \([5, 7]\). On each subinterval, assume the object moves at a constant velocity equal to \( v \) evaluated at the midpoint of the subinterval and use these approximations to estimate the displacement of the object on \([1, 7]\) (see part (a) of the figure).

   b. Repeat part (a) for \( n = 6 \) subintervals (see part (b) of the figure).

11–16. Approximating displacement The velocity of an object is given by the following functions on a specified interval. Approximate the displacement of the object on this interval by subdividing the interval into \( n \) subintervals. Use the left endpoint of each subinterval to compute the height of the rectangles.

   11. \( v = 2t + 1 \) (m/s), for \( 0 \leq t \leq 8 \); \( n = 2 \)

   12. \( v = e^t \) (m/s), for \( 0 \leq t \leq 3 \); \( n = 3 \)

   13. \( v = \frac{1}{2t + 1} \) (m/s), for \( 0 \leq t \leq 8 \); \( n = 4 \)

   14. \( v = t^2 + 4 \) (ft/s), for \( 0 \leq t \leq 12 \); \( n = 6 \)

   15. \( v = 4\sqrt{t} + 1 \) (mi/hr), for \( 0 \leq t \leq 15 \); \( n = 5 \)

   16. \( v = \frac{t + 3}{6} \) (m/s), for \( 0 \leq t \leq 4 \); \( n = 4 \)

17–18. Left and right Riemann sums Use the figures to calculate the left and right Riemann sums for \( f \) on the given interval and for the given value of \( n \).

   17. \( f(x) = x + 1 \) on \([1, 6]\); \( n = 5 \)

   18. \( f(x) = x + 1 \) on \([1, 6]\); \( n = 10 \)
18. \( f(x) = \frac{1}{x} \) on \([1, 5]\); \( n = 4 \)

19–26. Left and right Riemann sums Complete the following steps for the given function, interval, and value of \( n \).

a. Sketch the graph of the function on the given interval.

b. Calculate \( \Delta x \) and the grid points \( x_0, x_1, \ldots, x_n \).

c. Illustrate the left and right Riemann sums. Then determine which Riemann sum underestimates and which sum overestimates the area under the curve.

d. Calculate the left and right Riemann sums.

20. \( f(x) = 9 - x \) on \([3, 8]\); \( n = 5 \)

21. \( f(x) = \cos x \) on \([0, \pi/2]\); \( n = 4 \)

22. \( f(x) = \sin^{-1}(x/3) \) on \([0, 3]\); \( n = 6 \)

23. \( f(x) = x^2 - 1 \) on \([2, 4]\); \( n = 4 \)

24. \( f(x) = 2x^2 \) on \([1, 6]\); \( n = 5 \)

25. \( f(x) = e^{x/2} \) on \([1, 4]\); \( n = 6 \)

26. \( f(x) = \ln 4x \) on \([1, 3]\); \( n = 5 \)

27. A midpoint Riemann sum Approximate the area of the region bounded by the graph of \( f(x) = 100 - x^2 \) and the \( x \)-axis on \([0, 10]\) with \( n = 5 \) subintervals. Use the midpoint of each subinterval to determine the height of each rectangle (see figure).

28. A midpoint Riemann sum Approximate the area of the region bounded by the graph of \( f(t) = \cos(t/2) \) and the \( t \)-axis on \([0, \pi] \) with \( n = 4 \) subintervals. Use the midpoint of each subinterval to determine the height of each rectangle (see figure).

29–34. Midpoint Riemann sums Complete the following steps for the given function, interval, and value of \( n \).

a. Sketch the graph of the function on the given interval.

b. Calculate \( \Delta x \) and the grid points \( x_0, x_1, \ldots, x_n \).

c. Illustrate the midpoint Riemann sum by sketching the appropriate rectangles.

d. Calculate the midpoint Riemann sum.

29. \( f(x) = 2x + 1 \) on \([0, 4]\); \( n = 4 \)

30. \( f(x) = 2\cos x \) on \([0, 1]\); \( n = 5 \)

31. \( f(x) = \sqrt{x} \) on \([1, 3]\); \( n = 4 \)

32. \( f(x) = x^2 \) on \([0, 4]\); \( n = 4 \)

33. \( f(x) = 1/x \) on \([1, 6]\); \( n = 5 \)

34. \( f(x) = 4 - x \) on \([-1, 4]\); \( n = 5 \)

35–36. Riemann sums from tables Evaluate the left and right Riemann sums for \( f \) over the given interval for the given value of \( n \).

35. \( n = 4; [0, 2] \)

\[
\begin{array}{cccc}
 x & 0 & 0.5 & 1 \\
f(x) & 5 & 3 & 2 \\
\end{array}
\]

36. \( n = 8; [1, 5] \)

\[
\begin{array}{cccc}
x & 1 & 1.5 & 2 \\
f(x) & 0 & 2 & 3 \\
\end{array}
\]

37. Displacement from a table of velocities The velocities (in \( \text{mi/hr} \)) of an automobile moving along a straight highway over a two-hour period are given in the following table.

<table>
<thead>
<tr>
<th>( t ) (hr)</th>
<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
<th>1.25</th>
<th>1.5</th>
<th>1.75</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v ) (mi/hr)</td>
<td>50</td>
<td>50</td>
<td>60</td>
<td>55</td>
<td>65</td>
<td>50</td>
<td>60</td>
<td>70</td>
<td></td>
</tr>
</tbody>
</table>

a. Sketch a smooth curve passing through the data points.

b. Find the midpoint Riemann sum approximation to the displacement on \([0, 2]\) with \( n = 2 \) and \( n = 4 \).

38. Displacement from a table of velocities The velocities (in \( \text{m/s} \)) of an automobile moving along a straight freeway over a four-hour period are given in the following table.

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v ) (m/s)</td>
<td>20</td>
<td>25</td>
<td>30</td>
<td>35</td>
<td>30</td>
<td>35</td>
<td>40</td>
<td>40</td>
<td></td>
</tr>
</tbody>
</table>

a. Sketch a smooth curve passing through the data points.

b. Find the midpoint Riemann sum approximation to the displacement on \([0, 4]\) with \( n = 2 \) and \( n = 4 \) subintervals.

39. Sigma notation Express the following sums using sigma notation. (Answers are not unique.)

a. \( 1 + 2 + 3 + 4 + 5 \)

b. \( 4 + 5 + 6 + 7 + 8 + 9 \)

c. \( 1^2 + 2^2 + 3^2 + 4^2 \)

d. \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{50} \)

40. Sigma notation Express the following sums using sigma notation. (Answers are not unique.)

a. \( 1 + 3 + 5 + 7 + \cdots + 99 \)

b. \( 4 + 9 + 14 + \cdots + 44 \)

c. \( 3 + 8 + 13 + \cdots + 63 \)

d. \( \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \cdots + \frac{1}{49\cdot50} \)
41. Sigma notation Evaluate the following expressions.
   a. \( \sum_{k=1}^{10} k \)  
   b. \( \sum_{k=1}^{10} (2k + 1) \)  
   c. \( \sum_{k=1}^{4} k^2 \)  
   d. \( \sum_{n=1}^{10} (1 + n^2) \)  
   e. \( \sum_{m=1}^{4} \frac{2m + 2}{3} \)  
   f. \( \sum_{j=1}^{3} (3j - 4) \)  
   g. \( \sum_{p=1}^{5} (2p + p^2) \)  
   h. \( \sum_{n=0}^{n} \frac{n!}{2} \)

42. Evaluating sums Evaluate the following expressions by two methods.
   (i) Use Theorem 5.1.  
   (ii) Use a calculator.
   a. \( \sum_{k=1}^{45} k \)  
   b. \( \sum_{k=1}^{45} (5k - 1) \)  
   c. \( \sum_{k=1}^{45} 2k^2 \)  
   d. \( \sum_{n=1}^{50} (1 + n^2) \)  
   e. \( \sum_{n=1}^{75} \frac{2m + 2}{3} \)  
   f. \( \sum_{j=1}^{75} (3j - 4) \)  
   g. \( \sum_{p=1}^{50} (2p + p^2) \)  
   h. \( \sum_{n=0}^{n} (n^2 + 3n - 1) \)

43–46. Riemann sums for larger values of \( n \) Complete the following steps for the given function \( f \) and interval.
   a. For the given value of \( n \), use sigma notation to write the left, right, and midpoint Riemann sums. Then evaluate each sum using a calculator.
   b. Based on the approximations found in part (a), estimate the area of the region bounded by the graph of \( f \) and the \( x \)-axis on the interval.
   43. \( f(x) = \sqrt{x} \) on \([0, 4]\); \( n = 40 \)
   44. \( f(x) = x^2 + 1 \) on \([-1, 1]\); \( n = 50 \)
   45. \( f(x) = x^3 - 1 \) on \([2, 7]\); \( n = 75 \)
   46. \( f(x) = \cos 2x \) on \([0, \pi/4]\); \( n = 60 \)

47–52. Approximating areas with a calculator Use a calculator and right Riemann sums to approximate the area of the given region. Present your calculations in a table showing the approximations for \( n = 10, 30, 60, \) and 80 subintervals. Comment on whether your approximations appear to approach a limit.
   47. The region bounded by the graph of \( f(x) = 4 - x^2 \) and the \( x \)-axis on the interval \([-2, 2]\)
   48. The region bounded by the graph of \( f(x) = x^2 + 1 \) and the \( x \)-axis on the interval \([0, 2]\)
   49. The region bounded by the graph of \( f(x) = 2 - 2 \sin x \) and the \( x \)-axis on the interval \([-\pi/2, \pi/2]\)
   50. The region bounded by the graph of \( f(x) = 2 + \sin x \) and the \( x \)-axis on the interval \([1, 2]\)
   51. The region bounded by the graph of \( f(x) = \ln x \) and the \( x \)-axis on the interval \([1, e]\)
   52. The region bounded by the graph of \( f(x) = \sqrt{x} + 1 \) and the \( x \)-axis on the interval \([0, 3]\)

Further Explorations

53. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.
   a. Consider the linear function \( f(x) = 2x + 5 \) and the region bounded by its graph and the \( x \)-axis on the interval \([3, 6]\).
      Suppose the area of this region is approximated using midpoint Riemann sums. Then the approximations give the exact area of the region for any number of subintervals.
   b. A left Riemann sum always overestimates the area of a region bounded by a positive increasing function and the \( x \)-axis on an interval \([a, b]\).
   c. For an increasing or decreasing nonconstant function on an interval \([a, b]\) and a given value of \( n \), the value of the midpoint Riemann sum always lies between the values of the left and right Riemann sums.

54. Riemann sums for a semicircle Let \( f(x) = \sqrt{1 - x^2} \).
   a. Show that the graph of \( f \) is the upper half of a circle of radius 1 centered at the origin.
   b. Estimate the area between the graph of \( f \) and the \( x \)-axis on the interval \([-1, 1]\) using a midpoint Riemann sum with \( n = 25 \).
   c. Repeat part (b) using \( n = 75 \) rectangles.
   d. What happens to the midpoint Riemann sums on \([-1, 1]\) as \( n \to \infty \)?

55–58. Sigma notation for Riemann sums Use sigma notation to write the following Riemann sums. Then evaluate each Riemann sum using Theorem 5.1 or a calculator.
   55. The right Riemann sum for \( f(x) = x + 1 \) on \([0, 4]\) with \( n = 50 \)
   56. The left Riemann sum for \( f(x) = e^x \) on \([0, \ln 2]\) with \( n = 40 \)
   57. The midpoint Riemann sum for \( f(x) = x^3 \) on \([3, 11]\) with \( n = 32 \)
   58. The midpoint Riemann sum for \( f(x) = 1 + \cos \pi x \) on \([0, 2]\) with \( n = 50 \)

59–62. Identifying Riemann sums Fill in the blanks with right or midpoint, an interval, and a value of \( n \). In some cases, more than one answer may work.
   59. \( \sum_{k=1}^{4} f(1 + k) \cdot 1 \) is a ______ Riemann sum for \( f \) on the interval \([-__, __] \) with \( n = __\).
   60. \( \sum_{k=1}^{4} f(2k + 1) \cdot 1 \) is a ______ Riemann sum for \( f \) on the interval \([-__, __] \) with \( n = __\).
   61. \( \sum_{k=1}^{4} f(1.5 + k) \cdot 1 \) is a ______ Riemann sum for \( f \) on the interval \([-__, __] \) with \( n = __\).
   62. \( \sum_{k=1}^{8} f\left(1.5 + \frac{k}{2}\right) \cdot \frac{1}{2} \) is a ______ Riemann sum for \( f \) on the interval \([-__, __] \) with \( n = __\).
63. Approximating areas Estimate the area of the region bounded by the graph of \( f(x) = x^2 + 2 \) and the \( x \)-axis on \([0, 2]\) in the following ways.
   a. Divide \([0, 2]\) into \( n = 4 \) subintervals and approximate the area of the region using a left Riemann sum. Illustrate the solution geometrically.
   b. Divide \([0, 2]\) into \( n = 4 \) subintervals and approximate the area of the region using a midpoint Riemann sum. Illustrate the solution geometrically.
   c. Divide \([0, 2]\) into \( n = 4 \) subintervals and approximate the area of the region using a right Riemann sum. Illustrate the solution geometrically.

64. Approximating area from a graph Approximate the area of the region bounded by the graph (see figure) and the \( x \)-axis by dividing the interval \([0, 6]\) into \( n = 3 \) subintervals. Use a left and right Riemann sum to obtain two different approximations.

65. Approximating area from a graph Approximate the area of the region bounded by the graph (see figure) and the \( x \)-axis by dividing the interval \([1, 7]\) into \( n = 6 \) subintervals. Use a left and right Riemann sum to obtain two different approximations.

Applications

66. Displacement from a velocity graph Consider the velocity function for an object moving along a line (see figure).
   a. Describe the motion of the object over the interval \([0, 6]\).
   b. Use geometry to find the displacement of the object between \( t = 0 \) and \( t = 2 \).
   c. Use geometry to find the displacement of the object between \( t = 2 \) and \( t = 5 \).
   d. Assuming that the velocity remains 10 m/s, for \( t \geq 5 \), find the function that gives the displacement between \( t = 0 \) and any time \( t \geq 5 \).

67. Displacement from a velocity graph Consider the velocity function for an object moving along a line (see figure).
   a. Describe the motion of the object over the interval \([0, 6]\).
   b. Use geometry to find the displacement of the object between \( t = 0 \) and \( t = 2 \).
   c. Use geometry to find the displacement of the object between \( t = 2 \) and \( t = 5 \).
   d. Assuming that the velocity remains 10 m/s, for \( t \geq 5 \), find the function that gives the displacement between \( t = 0 \) and any time \( t \geq 5 \).

68. Flow rates Suppose a gauge at the outflow of a reservoir measures the flow rate of water in units of \( \text{ft}^3/\text{hr} \). In Chapter 6, we show that the total amount of water that flows out of the reservoir is the area under the flow rate curve. Consider the flow-rate function shown in the figure.
   a. Find the amount of water (in units of \( \text{ft}^3 \)) that flows out of the reservoir over the interval \([0, 4]\).
   b. Find the amount of water that flows out of the reservoir over the interval \([8, 10]\).
   c. Does more water flow out of the reservoir over the interval \([0, 4]\) or \([4, 6]\)?
   d. Show that the units of your answer are consistent with the units of the variables on the axes.

69. Mass from density A thin 10-cm rod is made of an alloy whose density varies along its length according to the function shown in the figure. Assume density is measured in units of g/cm.
In Chapter 6, we show that the mass of the rod is the area under the density curve.

a. Find the mass of the left half of the rod \((0 \leq x \leq 5)\).

b. Find the mass of the right half of the rod \((5 \leq x \leq 10)\).

c. Find the mass of the entire rod \((0 \leq x \leq 10)\).

d. Find the point along the rod at which it will balance (called the center of mass).

**70–71. Displacement from velocity** The following functions describe the velocity of a car (in mi/hr) moving along a straight highway for a 3-hr interval. In each case, find the function that gives the displacement of the car over the interval \([0, t]\), where \(0 \leq t \leq 3\).

70. \(v(t) = \begin{cases} 40 & \text{if } 0 \leq t \leq 1.5 \\ 50 & \text{if } 1.5 < t \leq 3 \end{cases}\)

71. \(v(t) = \begin{cases} 30 & \text{if } 0 \leq t \leq 2 \\ 50 & \text{if } 2 < t \leq 2.5 \\ 44 & \text{if } 2.5 < t \leq 3 \end{cases}\)

72–75. Functions with absolute value Use a calculator and the method of your choice to approximate the area of the following regions. Present your calculations in a table, showing approximations using \(n = 16, 32, 64\) subintervals. Comment on whether your approximations appear to approach a limit.

72. The region bounded by the graph of \(f(x) = |25 - x^2|\) and the \(x\)-axis on the interval \([0, 10]\)

73. The region bounded by the graph of \(f(x) = |x(x^2 - 1)|\) and the \(x\)-axis on the interval \([-1, 1]\)

74. The region bounded by the graph of \(f(x) = |\cos 2x|\) and the \(x\)-axis on the interval \([0, \pi]\)

75. The region bounded by the graph of \(f(x) = |1 - x^3|\) and the \(x\)-axis on the interval \([-1, 2]\)

### Additional Exercises

76. Riemann sums for constant functions Let \(f(x) = c\), where \(c > 0\), be a constant function on \([a, b]\). Prove that any Riemann sum for any value of \(n\) gives the exact area of the region between the graph of \(f\) and the \(x\)-axis on \([a, b]\).

77. Riemann sums for linear functions Assume that the linear function \(f(x) = mx + c\) is positive on the interval \([a, b]\). Prove that the midpoint Riemann sum with any value of \(n\) gives the exact area of the region between the graph of \(f\) and the \(x\)-axis on \([a, b]\).

78. Shape of the graph for left Riemann sums Suppose a left Riemann sum is used to approximate the area of the region bounded by the graph of a positive function and the \(x\)-axis on the interval \([a, b]\). Fill in the following table to indicate whether the resulting approximation underestimates or overestimates the exact area in the four cases shown. Use a sketch to explain your reasoning in each case.

<table>
<thead>
<tr>
<th>Function</th>
<th>Concave up on ([a, b])</th>
<th>Concave down on ([a, b])</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x) = x^2)</td>
<td>Increasing on ([a, b])</td>
<td>Increasing on ([a, b])</td>
</tr>
<tr>
<td>(f(x) = -x^2)</td>
<td>Decreasing on ([a, b])</td>
<td>Decreasing on ([a, b])</td>
</tr>
</tbody>
</table>

79. Shape of the graph for right Riemann sums Suppose a right Riemann sum is used to approximate the area of the region bounded by the graph of a positive function and the \(x\)-axis on the interval \([a, b]\). Fill in the following table to indicate whether the resulting approximation underestimates or overestimates the exact area in the four cases shown. Use a sketch to explain your reasoning in each case.

<table>
<thead>
<tr>
<th>Function</th>
<th>Concave up on ([a, b])</th>
<th>Concave down on ([a, b])</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x) = x^2)</td>
<td>Increasing on ([a, b])</td>
<td>Increasing on ([a, b])</td>
</tr>
<tr>
<td>(f(x) = -x^2)</td>
<td>Decreasing on ([a, b])</td>
<td>Decreasing on ([a, b])</td>
</tr>
</tbody>
</table>

### QUICK CHECK ANSWERS

1. 45 mi
2. 0.25, 0.125, 7.875
3. \(\Delta x = 2\); \{1, 3, 5, 7, 9\}
4. The left sum overestimates the area.
5.2 Definite Integrals

We introduced Riemann sums in Section 5.1 as a way to approximate the area of a region bounded by a curve \( y = f(x) \) and the \( x \)-axis on an interval \([a, b]\). In that discussion, we assumed \( f \) to be nonnegative on the interval. Our next task is to discover the geometric meaning of Riemann sums when \( f \) is negative on some or all of \([a, b]\). Once this matter is settled, we proceed to the main event of this section, which is to define the definite integral. With definite integrals, the approximations given by Riemann sums become exact.

Net Area

How do we interpret Riemann sums when \( f \) is negative at some or all points of \([a, b]\)? The answer follows directly from the Riemann sum definition.

**EXAMPLE 1 Interpreting Riemann sums** Evaluate and interpret the following Riemann sums for \( f(x) = 1 - x^2 \) on the interval \([a, b]\) with \( n \) equally spaced subintervals.

a. A midpoint Riemann sum with \([a, b] = [1, 3] \) and \( n = 4 \)

b. A left Riemann sum with \([a, b] = [0, 3] \) and \( n = 6 \)

**SOLUTION**

a. The length of each subinterval is \( \Delta x = \frac{b - a}{n} = \frac{3 - 1}{4} = 0.5 \). So the grid points are

\[
x_0 = 1, \quad x_1 = 1.5, \quad x_2 = 2, \quad \text{and} \quad x_3 = 2.5, \quad \text{and} \quad x_4 = 3.
\]

To compute the midpoint Riemann sum, we evaluate \( f \) at the midpoints of the subintervals, which are

\[
x_1^* = 1.25, \quad x_2^* = 1.75, \quad x_3^* = 2.25, \quad \text{and} \quad x_4^* = 2.75.
\]

The resulting midpoint Riemann sum is

\[
\sum_{k=1}^{n} f(x_k^*) \Delta x = \sum_{k=1}^{4} f(x_k^*)(0.5)
\]

\[
= f(1.25)(0.5) + f(1.75)(0.5) + f(2.25)(0.5) + f(2.75)(0.5)
\]

\[
= (-0.5625 - 2.0625 - 4.0625 - 6.5625)(0.5)
\]

\[
= -6.625.
\]

All values of \( f(x_k^*) \) are negative, so the Riemann sum is also negative. Because area is always a nonnegative quantity, this Riemann sum does not approximate the area of the region between the curve and the \( x \)-axis on \([1, 3]\). Notice, however, that the values of \( f(x_k^*) \) are the negative of the heights of the corresponding rectangles (Figure 5.16). Therefore, the Riemann sum approximates the negative of the area of the region bounded by the curve.

b. The length of each subinterval is \( \Delta x = \frac{b - a}{n} = \frac{3 - 0}{6} = 0.5 \), and the grid points are

\[
x_0 = 0, \quad x_1 = 0.5, \quad x_2 = 1, \quad x_3 = 1.5, \quad x_4 = 2, \quad x_5 = 2.5, \quad \text{and} \quad x_6 = 3.
\]

To calculate the left Riemann sum, we set \( x_1^* \), \( x_2^* \), \( x_3^* \), \( x_4^* \), \( x_5^* \), \( x_6^* \) equal to the left endpoints of the subintervals:

\[
x_1^* = 0, \quad x_2^* = 0.5, \quad x_3^* = 1, \quad x_4^* = 1.5, \quad x_5^* = 2, \quad \text{and} \quad x_6^* = 2.5.
\]
The resulting left Riemann sum on $[0, 3]$ is 3.875.
The left Riemann sum on $[0, 1.5]$ is 0.875.
The left Riemann sum on $[1.5, 3]$ is 4.75.
The resulting left Riemann sum is an\[\sum_{k=1}^{6} f(x_k) \Delta x = \sum_{k=1}^{6} f(x_k)(0.5)\]
onnegative contribution negative contribution\[= (f(0) + f(0.5) + f(1) + f(1.5) + f(2) + f(2.5))0.5\]
\[= (1 + 0.75 + 0 - 1.25 - 3 - 5.25)0.5\]
\[= -3.875.\]

In this case, the values of $f(x_k)$ are nonnegative for $k = 1, 2, \text{and } 3, \text{and negative for } k = 4, 5, \text{and } 6$ (Figure 5.17). Where $f$ is positive, we get positive contributions to the Riemann sum, and where $f$ is negative, we get negative contributions to the sum.

**Related Exercises 11–20**

Let’s recap what we learned in Example 1. On intervals where $f(x) < 0$, Riemann sums approximate the negative of the area of the region bounded by the curve (Figure 5.18).

In the more general case that $f$ is positive on only part of $[a, b]$, we get positive contributions to the sum where $f$ is positive and negative contributions to the sum where $f$ is negative. In this case, Riemann sums approximate the area of the regions that lie above the $x$-axis minus the area of the regions that lie below the $x$-axis (Figure 5.19). This difference between the positive and negative contributions is called the net area; it can be positive, negative, or zero.

**Definition** Net Area

Consider the region $R$ bounded by the graph of a continuous function $f$ and the $x$-axis between $x = a$ and $x = b$. The net area of $R$ is the sum of the areas of the parts of $R$ that lie above the $x$-axis minus the sum of the areas of the parts of $R$ that lie below the $x$-axis on $[a, b]$.

**Quick Check** 1 Suppose $f(x) = -5$. What is the net area of the region bounded by the graph of $f$ and the $x$-axis on the interval $[1, 5]$? Make a sketch of the function and the region. ©
The Definite Integral

Riemann sums for \( f \) on \([a, b]\) give approximations to the net area of the region bounded by the graph of \( f \) and the \( x \)-axis between \( x = a \) and \( x = b \), where \( a < b \). How can we make these approximations exact? If \( f \) is continuous on \([a, b]\), it is reasonable to expect the Riemann sum approximations to approach the exact value of the net area as the number of subintervals \( n \to \infty \) and as the length of the subintervals \( \Delta x \to 0 \) (Figure 5.20). In terms of limits, we write

\[
\text{net area} = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x.
\]

Quick Check 2 Sketch a continuous function \( f \) that is positive over the interval \([0, 1]\) and negative over the interval \((1, 2]\), such that the net area of the region bounded by the graph of \( f \) and the \( x \)-axis on \([0, 2]\) is zero.

As the number of subintervals \( n \) increases, the Riemann sums approach the net area of the region between the curve \( y = f(x) \) and the \( x \)-axis on \([a, b]\).

The Riemann sums we have used so far involve regular partitions in which the subintervals have the same length \( \Delta x \). We now introduce partitions of \([a, b]\) in which the lengths of the subintervals are not necessarily equal. A general partition of \([a, b]\) consists of the \( n \) subintervals

\[
[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n],
\]

where \( x_0 = a \) and \( x_n = b \). The length of the \( k \)-th subinterval is \( \Delta x_k = x_k - x_{k-1} \), for \( k = 1, \ldots, n \). We let \( x_k^* \) be any point in the subinterval \([x_{k-1}, x_k]\). This general partition is used to define the general Riemann sum.


**DEFINITION** General Riemann Sum

Suppose \([x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\) are subintervals of \([a, b]\) with

\[
a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.
\]

Let \(\Delta x_k\) be the length of the subinterval \([x_{k-1}, x_k]\) and let \(x^*_k\) be any point in \([x_{k-1}, x_k]\), for \(k = 1, 2, \ldots, n\).

If \(f\) is defined on \([a, b]\), the sum

\[
\sum_{k=1}^{n} f(x^*_k) \Delta x_k = f(x^*_1) \Delta x_1 + f(x^*_2) \Delta x_2 + \cdots + f(x^*_n) \Delta x_n
\]

is called a general Riemann sum for \(f\) on \([a, b]\).

As was the case for regular Riemann sums, if we choose \(x^*_k\) to be the left endpoint of \([x_{k-1}, x_k]\), for \(k = 1, 2, \ldots, n\), then the general Riemann sum is a left Riemann sum. Similarly, if we choose \(x^*_k\) to be the right endpoint \([x_{k-1}, x_k]\), for \(k = 1, 2, \ldots, n\), then the general Riemann sum is a right Riemann sum, and if we choose \(x^*_k\) to be the midpoint of the interval \([x_{k-1}, x_k]\), for \(k = 1, 2, \ldots, n\), then the general Riemann sum is a midpont Riemann sum.

Now consider the limit of \(\sum_{k=1}^{n} f(x^*_k) \Delta x_k\) as \(n \to \infty\) and as all the \(\Delta x_k \to 0\). We let \(\Delta\) denote the largest value of \(\Delta x_k\); that is, \(\Delta = \max \{ \Delta x_1, \Delta x_2, \ldots, \Delta x_n \}\). Observe that if \(\Delta \to 0\), then \(\Delta x_k \to 0\), for \(k = 1, 2, \ldots, n\). For the limit \(\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x^*_k) \Delta x_k\) to exist, it must have the same value over all general partitions of \([a, b]\) and for all choices of \(x^*_k\) on a partition.

**DEFINITION** Definite Integral

A function \(f\) defined on \([a, b]\) is integrable on \([a, b]\) if \(\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x^*_k) \Delta x_k\) exists and is unique over all partitions of \([a, b]\) and all choices of \(x^*_k\) on a partition. This limit is the definite integral of \(f\) from \(a\) to \(b\), which we write

\[
\int_{a}^{b} f(x) \, dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x^*_k) \Delta x_k.
\]

When the limit defining the definite integral of \(f\) exists, it equals the net area of the region bounded by the graph of \(f\) and the \(x\)-axis on \([a, b]\). It is imperative to remember that the indefinite integral \(\int f(x) \, dx\) is a family of functions of \(x\) (the antiderivatives of \(f\)) and that the definite integral \(\int_{a}^{b} f(x) \, dx\) is a real number (the net area of a region).

**Notation** The notation for the definite integral requires some explanation. There is a direct match between the notation on either side of the equation in the definition (Figure 5.21). In the limit as \(\Delta \to 0\), the finite sum, denoted \(\Sigma\), becomes a sum with an infinite number of terms, denoted \(\int f\). The integral sign \(\int\) is an elongated \(S\) for sum. The limits of integration, \(a\) and \(b\), and the limits of summation also match: The lower limit in the sum,
For Leibniz, who introduced this notation in 1675, \( dx \) represented the width of an infinitesimally thin rectangle and \( f(x) \, dx \) represented the area of such a rectangle. He used \( \int_a^b f(x) \, dx \) to denote the sum of these areas from \( a \) to \( b \).

- A function \( f \) is bounded on an interval \( I \) if there is a number \( M \) such that \( |f(x)| \leq M \) for all \( x \) in \( I \).

When \( f \) is continuous on \([a, b]\), we have seen that the definite integral \( \int_a^b f(x) \, dx \) is the net area bounded by the graph of \( f \) and the \( x \)-axis on \([a, b]\). Figure 5.23 illustrates how the idea of net area carries over to piecewise continuous functions (Exercises 76–80).

**Quick Check 3** Graph \( f(x) = x \) and use geometry to evaluate \( \int_{1}^{3} x \, dx \).

**Example 2** Identifying the limit of a sum

Assume that

\[
\lim_{\Delta \to 0} \sum_{k=1}^{n} (3x_k^2 + 2x_k + 1) \Delta x_k
\]

is the limit of a Riemann sum for a function \( f \) on \([1, 3]\). Identify the function \( f \) and express the limit as a definite integral. What does the definite integral represent geometrically?

**Solution** By comparing the sum \( \sum_{k=1}^{n} f(x_k) \Delta x_k \), we see that \( f(x) = 3x^2 + 2x + 1 \). Because \( f \) is a polynomial, it is continuous on \([1, 3]\) and is, therefore, integrable on \([1, 3]\). It follows that

\[
\lim_{\Delta \to 0} \sum_{k=1}^{n} (3x_k^2 + 2x_k + 1) \Delta x_k = \int_{1}^{3} (3x^2 + 2x + 1) \, dx.
\]
Because \( f \) is positive on \([1, 3]\), the definite integral \( \int_1^3 (3x^2 + 2x + 1) \, dx \) is the area of the region bounded by the curve \( y = 3x^2 + 2x + 1 \) and the \( x \)-axis on \([1, 3]\) (Figure 5.24).

\[
\lim_{\Delta x \to 0} \sum_{k=1}^{n} (3x_k^2 + 2x_k + 1) \Delta x_k = \int_1^3 (3x^2 + 2x + 1) \, dx
\]

**Figure 5.24**

**Related Exercises 21–24**

**EXAMPLE 3 Evaluating definite integrals using geometry** Use familiar area formulas to evaluate the following definite integrals.

**a.** \( \int_2^4 (2x + 3) \, dx \)  
**b.** \( \int_1^6 (2x - 6) \, dx \)  
**c.** \( \int_3^4 \sqrt{1 - (x - 3)^2} \, dx \)

**SOLUTION** To evaluate these definite integrals geometrically, a sketch of the corresponding region is essential.

**a.** The definite integral \( \int_2^4 (2x + 3) \, dx \) is the area of the trapezoid bounded by the \( x \)-axis and the line \( y = 2x + 3 \) from \( x = 2 \) to \( x = 4 \) (Figure 5.25). The width of its base is 2 and the lengths of its two parallel sides are \( f(2) = 7 \) and \( f(4) = 11 \). Using the area formula for a trapezoid, we have

\[
\int_2^4 (2x + 3) \, dx = \frac{1}{2} \cdot 2 (11 + 7) = 18.
\]

**b.** A sketch shows that the regions bounded by the line \( y = 2x - 6 \) and the \( x \)-axis are triangles (Figure 5.26). The area of the triangle on the interval \([1, 3]\) is \( \frac{1}{2} \cdot 2 \cdot 4 = 4 \). Similarly, the area of the triangle on \([3, 6]\) is \( \frac{1}{2} \cdot 3 \cdot 6 = 9 \). The definite integral is the net area of the entire region, which is the area of the triangle above the \( x \)-axis minus the area of the triangle below the \( x \)-axis:

\[
\int_1^6 (2x - 6) \, dx = \text{net area} = 9 - 4 = 5.
\]
c. We first let \( y = \sqrt{1 - (x - 3)^2} \) and observe that \( y \geq 0 \) when \( 2 \leq x \leq 4 \). Squaring both sides leads to the equation \( (x - 3)^2 + y^2 = 1 \), whose graph is a circle of radius 1 centered at \((3, 0)\). Because \( y \geq 0 \), the graph of \( y = \sqrt{1 - (x - 3)^2} \) is the upper half of the circle. It follows that the integral \( \int_3^4 \sqrt{1 - (x - 3)^2} \, dx \) is the area of a quarter circle of radius 1 (Figure 5.27). Therefore,

\[
\int_3^4 \sqrt{1 - (x - 3)^2} \, dx = \frac{1}{4} \pi (1)^2 = \frac{\pi}{4}.
\]

**Related Exercises 25–32**

**EXAMPLE 4** Definite integrals from graphs Figure 5.28 shows the graph of a function \( f \) with the areas of the regions bounded by its graph and the \( x \)-axis given. Find the values of the following definite integrals.

\[
\begin{align*}
\text{a. } & \int_a^b f(x) \, dx & \text{b. } & \int_b^c f(x) \, dx & \text{c. } & \int_a^c f(x) \, dx & \text{d. } & \int_b^d f(x) \, dx
\end{align*}
\]

**SOLUTION**

a. Because \( f \) is positive on \([a, b]\), the value of the definite integral is the area of the region between the graph and the \( x \)-axis on \([a, b]\); that is, \( \int_a^b f(x) \, dx = 12 \).

b. Because \( f \) is negative on \([b, c]\), the value of the definite integral is the negative of the area of the corresponding region; that is, \( \int_b^c f(x) \, dx = -10 \).

c. The value of the definite integral is the area of the region on \([a, b]\) (where \( f \) is positive) minus the area of the region on \([b, c]\) (where \( f \) is negative). Therefore, \( \int_a^c f(x) \, dx = 12 - 10 = 2 \).

d. Reasoning as in part (c), we have \( \int_b^d f(x) \, dx = -10 + 8 = -2 \).

**Related Exercises 33–40**

**Properties of Definite Integrals**

Recall that the definite integral \( \int_a^b f(x) \, dx \) was defined assuming that \( a < b \). There are, however, occasions when it is necessary to reverse the limits of integration. If \( f \) is integrable on \([a, b]\), we define

\[
\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx.
\]

In other words, reversing the limits of integration changes the sign of the integral.

Another fundamental property of integrals is that if we integrate from a point to itself, then the length of the interval of integration is zero, which means the definite integral is also zero.

**DEFINITION** Reversing Limits and Identical Limits of Integration

Suppose \( f \) is integrable on \([a, b]\).

1. \( \int_b^a f(x) \, dx = - \int_a^b f(x) \, dx \)

2. \( \int_a^a f(x) \, dx = 0 \)

**QUICK CHECK 5** Evaluate \( \int_a^b f(x) \, dx + \int_b^a f(x) \, dx \) assuming \( f \) is integrable on \([a, b]\).
**Integral of a Sum** Definite integrals possess other properties that often simplify their evaluation. Assume \( f \) and \( g \) are integrable on \([a, b]\). The first property states that their sum \( f + g \) is integrable on \([a, b]\) and the integral of their sum is the sum of their integrals:

\[
\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx.
\]

We prove this property assuming that \( f \) and \( g \) are continuous. In this case, \( f + g \) is continuous and, therefore, integrable. We then have

\[
\int_a^b (f(x) + g(x)) \, dx = \lim_{\Delta \to 0} \sum_{k=1}^n (f(x_k^*) + g(x_k^*)) \Delta x_k
\]

\[
= \lim_{\Delta \to 0} \left( \sum_{k=1}^n f(x_k^*) \Delta x_k + \sum_{k=1}^n g(x_k^*) \Delta x_k \right)
\]

\[
= \lim_{\Delta \to 0} \sum_{k=1}^n f(x_k^*) \Delta x_k + \lim_{\Delta \to 0} \sum_{k=1}^n g(x_k^*) \Delta x_k
\]

\[
= \int_a^b f(x) \, dx + \int_a^b g(x) \, dx.
\]

**Constants in Integrals** Another property of definite integrals is that constants can be factored out of the integral. If \( f \) is integrable on \([a, b]\) and \( c \) is a constant, then \( cf \) is integrable on \([a, b]\) and

\[
\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx.
\]

The justification for this property (Exercise 81) is based on the fact that for finite sums,

\[
\sum_{k=1}^n c f(x_k^*) \Delta x_k = c \sum_{k=1}^n f(x_k^*) \Delta x_k.
\]

**Integrals over Subintervals** If the point \( p \) lies between \( a \) and \( b \), then the integral on \([a, b]\) may be split into two integrals. As shown in Figure 5.29, we have the property

\[
\int_a^b f(x) \, dx = \int_a^p f(x) \, dx + \int_p^b f(x) \, dx.
\]
It is surprising that this property also holds when $p$ lies outside the interval $[a, b]$. For example, if $a < b < p$ and $f$ is integrable on $[a, p]$, then it follows (Figure 5.30) that

$$\int_a^b f(x) \, dx = \int_a^p f(x) \, dx - \int_p^b f(x) \, dx.$$ 

Because $\int_p^b f(x) \, dx = -\int_p^a f(x) \, dx$, we have the original property:

$$\int_a^b f(x) \, dx = \int_a^p f(x) \, dx + \int_p^b f(x) \, dx.$$

**Integrals of Absolute Values** Finally, how do we interpret $\int_a^b |f(x)| \, dx$, the integral of the absolute value of an integrable function? The graphs $f$ and $|f|$ are shown in Figure 5.31. The integral $\int_a^b |f(x)| \, dx$ gives the area of regions $R_1$ and $R_2$. But $R_1$ and $R_1'$ have the same area; therefore, $\int_a^b |f(x)| \, dx$ also gives the area of $R_1$ and $R_2$. The conclusion is that $\int_a^b |f(x)| \, dx$ is the area of the entire region (above and below the $x$-axis) that lies between the graph of $f$ and the $x$-axis on $[a, b]$.

All these properties will be used frequently in upcoming work. It’s worth collecting them in one table (Table 5.4).

**Table 5.4 Properties of definite integrals**

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Definition</td>
<td>$\int_a^a f(x) , dx = 0$</td>
</tr>
<tr>
<td>2. Definition</td>
<td>$\int_a^b f(x) , dx = -\int_b^a f(x) , dx$</td>
</tr>
<tr>
<td>3.</td>
<td>$\int_a^b (f(x) + g(x)) , dx = \int_a^b f(x) , dx + \int_a^b g(x) , dx$</td>
</tr>
<tr>
<td>4. For any constant $c$</td>
<td>$\int_a^b cf(x) , dx = c \int_a^b f(x) , dx$</td>
</tr>
<tr>
<td>5.</td>
<td>$\int_a^b f(x) , dx = \int_a^p f(x) , dx + \int_p^b f(x) , dx$</td>
</tr>
<tr>
<td>6. The function $</td>
<td>f</td>
</tr>
</tbody>
</table>

**EXAMPLE 5 Properties of integrals** Assume that $\int_0^5 f(x) \, dx = 3$ and $\int_0^7 f(x) \, dx = -10$. Evaluate the following integrals, if possible.

<table>
<thead>
<tr>
<th>Integral</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $\int_0^2 2f(x) , dx$</td>
<td>b. $\int_0^2 f(x) , dx$</td>
</tr>
<tr>
<td>c. $\int_0^2 f(x) , dx$</td>
<td>d. $\int_0^2 6f(x) , dx$</td>
</tr>
<tr>
<td>e. $\int_0^2</td>
<td>f(x)</td>
</tr>
</tbody>
</table>
SOLUTION

a. By Property 4 of Table 5.4,
\[ \int_0^7 2f(x) \, dx = 2 \int_0^7 f(x) \, dx = 2 \cdot (-10) = -20. \]

b. By Property 5 of Table 5.4, \( \int_0^7 f(x) \, dx = \int_0^5 f(x) \, dx + \int_5^7 f(x) \, dx \). Therefore,
\[ \int_5^7 f(x) \, dx = \int_0^7 f(x) \, dx - \int_0^5 f(x) \, dx = -10 - 3 = -13. \]

c. By Property 2 of Table 5.4,
\[ \int_5^0 f(x) \, dx = -\int_0^5 f(x) \, dx = -3. \]

d. Using Properties 2 and 4 of Table 5.4, we have
\[ \int_7^0 6f(x) \, dx = -\int_0^7 6f(x) \, dx = -6 \int_0^7 f(x) \, dx = (-6)(-10) = 60. \]

e. This integral cannot be evaluated without knowing the intervals on which \( f \) is positive and negative. It could have any value greater than or equal to 10.

Related Exercises 41–46 🔄

QUICK CHECK 6 Evaluate \( \int_{-1}^2 x \, dx \) and \( \int_{-1}^2 |x| \, dx \) using geometry. 🔄

Evaluating Definite Integrals Using Limits

In Example 3, we used area formulas for trapezoids, triangles, and circles to evaluate definite integrals. Regions bounded by more general functions have curved boundaries for which conventional geometrical methods do not work. At the moment, the only way to handle such integrals is to appeal to the definition of the definite integral and the summation formulas given in Theorem 5.1.

We know that if \( f \) is integrable on \([a, b]\), then
\[ \int_a^b f(x) \, dx = \lim_{\Delta \to 0} \sum_{k=1}^n f(x_k^*) \Delta x_k, \]
for any partition of \([a, b]\) and any points \( x_k^* \). To simplify these calculations, we use equally spaced grid points and right Riemann sums. That is, for each value of \( n \), we let \( \Delta x = \frac{b-a}{n} \) and \( x_k^* = a + k \Delta x \), for \( k = 1, 2, \ldots, n \). Then as \( n \to \infty \) and \( \Delta \to 0 \),
\[ \int_a^b f(x) \, dx = \lim_{\Delta \to 0} \sum_{k=1}^n f(x_k^*) \Delta x_k = \lim_{n \to \infty} \sum_{k=1}^n f(a + k \Delta x) \Delta x. \]

EXAMPLE 6 Evaluating definite integrals Find the value of \( \int_0^2 (x^3 + 1) \, dx \) by evaluating a right Riemann sum and letting \( n \to \infty \).

SOLUTION Based on approximations found in Example 5, Section 5.1, we conjectured that the value of this integral is 6. To verify this conjecture, we now evaluate the integral exactly. The interval \([a, b] = [0, 2]\) is divided into \( n \) subintervals of length \( \Delta x = \frac{b-a}{n} = \frac{2}{n} \), which produces the grid points
\[ x_k^* = a + k \Delta x = 0 + k \cdot \frac{2}{n} = \frac{2k}{n} \quad \text{for} \quad k = 1, 2, \ldots, n. \]
Letting \( f(x) = x^3 + 1 \), the right Riemann sum is

\[
\sum_{k=1}^{n} f(x_k) \Delta x = \sum_{k=1}^{n} \left( \frac{2k}{n} \right)^3 + 1 \frac{2}{n} \]

\[
= \frac{2}{n} \sum_{k=1}^{n} \left( \frac{8k^3}{n^3} + 1 \right)
= \frac{2}{n} \left( \frac{8}{n} \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 1 \right)
= \frac{2}{n} \left( \frac{8}{n^3} \left( n^2(n+1)^2 \right) + n \right)
= \frac{4(n^2 + 2n + 1)}{n^2} + 2.
\]

Now we evaluate \( \int_{0}^{2} (x^3 + 1) \, dx \) by letting \( n \to \infty \) in the Riemann sum:

\[
\int_{0}^{2} (x^3 + 1) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k) \Delta x
= \lim_{n \to \infty} \left( \frac{4(n^2 + 2n + 1)}{n^2} + 2 \right)
= 4 \lim_{n \to \infty} \left( \frac{n^2 + 2n + 1}{n^2} \right) + \lim_{n \to \infty} 2
= 4(1) + 2 = 6.
\]

Therefore, \( \int_{0}^{2} (x^3 + 1) \, dx = 6 \), confirming our conjecture in Example 5, Section 5.1.

**Related Exercises 47–52**

The Riemann sum calculations in Example 6 are tedious even if \( f \) is a simple function. For polynomials of degree 4 and higher, the calculations are more challenging, and for rational and transcendental functions, advanced mathematical results are needed. The next section introduces more efficient methods for evaluating definite integrals.

**SECTION 5.2 EXERCISES**

**Review Questions**

1. What does net area measure?
2. What is the geometric meaning of a definite integral if the integrand changes sign on the interval of integration?
3. Under what conditions does the net area of a region equal the area of a region? When does the net area of a region differ from the area of a region?
4. Suppose that \( f(x) < 0 \) on the interval \([a, b]\). Using Riemann sums, explain why the definite integral \( \int_{a}^{b} f(x) \, dx \) is negative.
5. Use graphs to evaluate \( \int_{0}^{2\pi} \sin x \, dx \) and \( \int_{0}^{2\pi} \cos x \, dx \).
6. Explain how the notation for Riemann sums, \( \sum_{k=1}^{n} f(x_k) \Delta x \), corresponds to the notation for the definite integral, \( \int_{a}^{b} f(x) \, dx \).
7. Give a geometrical explanation of why \( \int_{a}^{b} f(x) \, dx = 0 \).
8. Use Table 5.4 to rewrite \( \int_{0}^{1} (2x^3 - 4x) \, dx \) as the difference of two integrals.
9. Use geometry to find a formula for \( \int_{a}^{b} x \, dx \), in terms of \( a \).
10. If \( f \) is continuous on \([a, b]\) and \( \int_{a}^{b} |f(x)| \, dx = 0 \), what can you conclude about \( f \) ?

**Basic Skills**

11–14. **Approximating net area** The following functions are negative on the given interval.

a. Sketch the function on the given interval.

b. Approximate the net area bounded by the graph of \( f \) and the \( x \)-axis on the interval using a left, right, and midpoint Riemann sum with \( n = 4 \).

11. \( f(x) = -2x - 1 \) on \([0, 4]\)
12. \( f(x) = -4 - x^3 \) on \([3, 7]\)
33–36. Net area from graphs The figure shows the areas of regions bounded by the graph of \( f \) and the \( x \)-axis. Evaluate the following integrals.

\[
\begin{align*}
33. & \quad \int_{0}^{a} f(x) \, dx \\
34. & \quad \int_{0}^{b} f(x) \, dx \\
35. & \quad \int_{c}^{e} f(x) \, dx \\
36. & \quad \int_{a}^{e} f(x) \, dx
\end{align*}
\]

37–40. Net area from graphs The accompanying figure shows four regions bounded by the graph of \( y = x \sin x \): \( R_1 \), \( R_2 \), \( R_3 \), and \( R_4 \), whose areas are \( 1 \), \( \pi - 1 \), \( \pi + 1 \), and \( 2\pi - 1 \), respectively. (We verify these results later in the text.) Use this information to evaluate the following integrals.

\[
\begin{align*}
37. & \quad \int_{0}^{\pi} x \sin x \, dx \\
38. & \quad \int_{0}^{3\pi/2} x \sin x \, dx \\
39. & \quad \int_{0}^{2\pi} x \sin x \, dx \\
40. & \quad \int_{\pi/2}^{2\pi} x \sin x \, dx
\end{align*}
\]

41. Properties of integrals Use only the fact that \( \int_{0}^{4} 3x(4 - x) \, dx = 32 \) and the definitions and properties of integrals to evaluate the following integrals, if possible.

\[
\begin{align*}
a. & \quad \int_{0}^{3} 3x(4 - x) \, dx \\
b. & \quad \int_{0}^{4} x(x - 4) \, dx \\
c. & \quad \int_{4}^{6} 6x(4 - x) \, dx \\
d. & \quad \int_{0}^{3} 3x(4 - x) \, dx
\end{align*}
\]

42. Properties of integrals Suppose \( \int_{1}^{4} f(x) \, dx = 8 \) and \( \int_{4}^{6} f(x) \, dx = 5 \). Evaluate the following integrals.

\[
\begin{align*}
a. & \quad \int_{1}^{4} (-3f(x)) \, dx \\
b. & \quad \int_{1}^{4} 3f(x) \, dx \\
c. & \quad \int_{4}^{6} 12f(x) \, dx \\
d. & \quad \int_{4}^{6} 3f(x) \, dx
\end{align*}
\]
43. Properties of integrals Suppose \( \int_a^b f(x) \, dx = 2 \), \( \int_a^b g(x) \, dx = -5 \), and \( \int_a^b h(x) \, dx = 1 \). Evaluate the following integrals.

a. \( \int_0^3 f(x) \, dx \)

b. \( \int_3^6 (-3g(x)) \, dx \)

c. \( \int_3^6 (3f(x) - g(x)) \, dx \)

d. \( \int_3^6 (f(x) + 2g(x)) \, dx \)

44. Properties of integrals Suppose \( f(x) \equiv 0 \) on \([0, 2]\), \( f(x) \equiv 0 \) on \([2, 5]\). \( \int_0^3 f(x) \, dx = 6 \), and \( \int_2^5 f(x) \, dx = -8 \). Evaluate the following integrals.

a. \( \int_0^5 f(x) \, dx \)

b. \( \int_0^4 |f(x)| \, dx \)

c. \( \int_2^5 4 \, |f(x)| \, dx \)

d. \( \int_3^6 (f(x) + |f(x)|) \, dx \)

45–46. Using properties of integrals Use the value of the first integral \( I \) to evaluate the two given integrals.

45. \( I = \int_0^1 (x^3 - 2x) \, dx = -\frac{1}{4} \)

a. \( \int_0^1 (4x - 2x^3) \, dx \)

b. \( \int_0^1 (2x - x^3) \, dx \)

46. \( \int_0^{\pi/2} (2 \sin \theta - \cos \theta) \, d\theta \)

a. \( \int_0^{\pi/2} (2 \sin \theta - \cos \theta) \, d\theta \)

b. \( \int_0^{\pi/2} (4 \cos \theta - 8 \sin \theta) \, d\theta \)

47–52. Limits of sums Use the definition of the definite integral to evaluate the following definite integrals. Use right Riemann sums and Theorem 5.1.

47. \( \int_0^2 (2x + 1) \, dx \)

48. \( \int_1^4 (1 - x) \, dx \)

49. \( \int_3^7 (4x + 6) \, dx \)

50. \( \int_0^2 (x^2 - 1) \, dx \)

51. \( \int_1^4 (x^2 - 1) \, dx \)

52. \( \int_0^2 4x^3 \, dx \)

Further Explorations

53. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If \( f \) is a constant function on the interval \([a, b]\), then the right and left Riemann sums give the exact value of \( \int_a^b f(x) \, dx \), for any positive integer \( n \).

b. If \( f \) is a linear function on the interval \([a, b]\), then a midpoint Riemann sum gives the exact value of \( \int_a^b f(x) \, dx \), for any positive integer \( n \).

c. \( \int_0^{\pi/2} \sin ax \, dx = \int_0^{\pi/2} \cos ax \, dx = 0 \). (Hint: Graph the functions and use properties of trigonometric functions.)

d. If \( \int_a^b f(x) \, dx = \int_a^b f(x) \, dx \), then \( f \) is a constant function.

e. Property 4 of Table 5.4 implies that \( \int_a^b x f(x) \, dx = x \int_a^b f(x) \, dx \).

54–57. Approximating definite integrals Complete the following steps for the given integral and the given value of \( n \).

a. Sketch the graph of the integrand on the interval of integration.

b. Calculate \( \Delta x \) and the grid points \( x_0, x_1, \ldots, x_n \), assuming a regular partition.

c. Calculate the left and right Riemann sums for the given value of \( n \).

d. Determine which Riemann sum (left or right) underestimates the value of the definite integral and which overestimates the value of the definite integral.

54. \( \int_0^3 (x^2 - 2) \, dx; \ n = 4 \)

55. \( \int_3^6 (1 - 2x) \, dx; \ n = 6 \)

56. \( \int_0^\pi \cos x \, dx; \ n = 4 \)

57. \( \int_1^\pi \frac{1}{x} \, dx; \ n = 6 \)

58–62. Approximating definite integrals with a calculator Consider the following definite integrals.

a. Write the left and right Riemann sums in sigma notation for an arbitrary number of steps for the given integral and the given value of \( n \).

b. Based on your answers to part (a), make a conjecture about the value of the definite integral.

58. \( \int_0^9 3\sqrt{x} \, dx \)

59. \( \int_0^1 (x^2 + 1) \, dx \)

60. \( \int_1^e \ln x \, dx \)

61. \( \int_0^1 \cos^{-1} x \, dx \)

62. \( \int_{-1}^1 \pi \cos \left( \frac{\pi x}{2} \right) \, dx \)

63–66. Midpoint Riemann sums with a calculator Consider the following definite integrals.

a. Write the midpoint Riemann sum in sigma notation for an arbitrary value of \( n \).

b. Evaluate each sum using a calculator with \( n = 20, 50, \) and \( 100 \).

63. \( \int_1^4 2\sqrt{x} \, dx \)

64. \( \int_{-1}^2 \sin \left( \frac{\pi x}{4} \right) \, dx \)

65. \( \int_0^4 (4x - x^2) \, dx \)

66. \( \int_0^{1/2} \sin^{-1} x \, dx \)

67. More properties of integrals Consider two functions \( f \) and \( g \) on \([1, 6]\) such that \( \int_1^6 f(x) \, dx = 10 \), \( \int_1^6 g(x) \, dx = 5 \), \( \int_1^4 f(x) \, dx = 5 \), and \( \int_1^4 g(x) \, dx = 2 \). Evaluate the following integrals.

a. \( \int_1^4 3f(x) \, dx \)

b. \( \int_1^6 (f(x) - g(x)) \, dx \)

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68. The region between the graph of \( y = 4x - 8 \) and the \( x \)-axis, for \(-4 \leq x \leq 0\).

69. The region between the graph of \( y = -3x \) and the \( x \)-axis, for \(-2 \leq x \leq 2\).

70. The region between the graph of \( y = 3x - 6 \) and the \( x \)-axis, for \(0 \leq x \leq 2\).

71. The region between the graph of \( y = 1 - |x| \) and the \( x \)-axis, for \(-2 \leq x \leq 2\).

72–75. Area by geometry Use geometry to evaluate the following integrals.

72. \( \int_{-2}^{2} |x + 1| \, dx \)

73. \( \int_{1}^{4} |2x - 4| \, dx \)

74. \( \int_{2}^{6} (3x - 6) \, dx \)

75. \( \int_{-6}^{4} \sqrt{24 - 2x - x^2} \, dx \)

77–78. Integrating piecewise continuous functions Use geometry and the result of Exercise 76 to evaluate the following integrals.

77. \( \int_{0}^{10} f(x) \, dx \), where \( f(x) = \begin{cases} 2 & \text{if } 0 \leq x \leq 5 \\ 3 & \text{if } 5 < x \leq 10 \end{cases} \)

78. \( \int_{1}^{6} f(x) \, dx \), where \( f(x) = \begin{cases} 2x & \text{if } 1 \leq x < 4 \\ 10 - 2x & \text{if } 4 \leq x \leq 6 \end{cases} \)

79–80. Integrating piecewise continuous functions Recall that the floor function \( \lfloor x \rfloor \) is the greatest integer less than or equal to \( x \) and that the ceiling function \( \lceil x \rceil \) is the least integer greater than or equal to \( x \). Use the result of Exercise 76 and the graphs to evaluate the following integrals.

79. \( \int_{1}^{5} \lfloor x \rfloor \, dx \)

80. \( \int_{0}^{1} \frac{1}{\sqrt{x}} \, dx \)

81. Constants in integrals Use the definition of the definite integral to justify the property \( \int_{a}^{b} c f(x) \, dx = c \int_{a}^{b} f(x) \, dx \), where \( f \) is continuous and \( c \) is a real number.

82. Zero net area If \( 0 < c < d \), then find the value of \( b \) (in terms of \( c \) and \( d \)) for which \( \int_{c}^{d} (x + b) \, dx = 0 \).

83. A nonintegrable function Consider the function defined on \([0, 1]\) such that \( f(x) = 1 \) if \( x \) is a rational number and \( f(x) = 0 \) if \( x \) is irrational. This function has an infinite number of discontinuities, and the integral \( \int_{0}^{1} f(x) \, dx \) does not exist. Show that the right, left, and midpoint Riemann sums on regular partitions with \( n \) subintervals equal 1 for all \( n \). (Hint: Between any two real numbers lie a rational and an irrational number.)

84. Powers of \( x \) by Riemann sums Consider the integral \( \int_{a}^{b} x^n \, dx \), where \( p \) is a positive integer.

a. Write the left Riemann sum for the integral with \( n \) subintervals.

b. It is a fact (proved by the 17th-century mathematicians Fermat and Pascal) that

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left( \frac{k}{n} \right)^p = \frac{1}{p+1} \]

Use this fact to evaluate \( \int_{a}^{b} x^n \, dx \).

85. An exact integration formula Evaluate \( \int_{a}^{b} \frac{dx}{x^2} \), where \( 0 < a < b \), using the definition of the definite integral and the following steps.

a. Assume \( \{x_0, x_1, \ldots, x_n\} \) is a partition of \([a, b]\) with \( \Delta x_k = x_k - x_{k-1} \), for \( k = 1, 2, \ldots, n \). Show that \( x_{k-1} \leq x_k \leq x_{k+1} \) for \( k = 1, 2, \ldots, n \).

b. Show that \( \frac{1}{x_k} - \frac{1}{x_{k+1}} = \frac{\Delta x_k}{x_k x_{k+1}} \) for \( k = 1, 2, \ldots, n \).

c. Simplify the general Riemann sum for \( \int_{a}^{b} \frac{dx}{x^2} \) using \( \sum_{k=1}^{n} \frac{1}{x_k} = \frac{x_{n+1} - x_1}{x_n} \).

d. Conclude that \( \int_{a}^{b} \frac{dx}{x^2} = \frac{1}{a} - \frac{1}{b} \).

(Source: The College Mathematics Journal, 32, 4, Sep 2001)

Quick Check Answers

1. \(-20\)  2. \(f(x) = 1 - x\) is one possibility.  3. 0

4. 10; \(c(b - a)\)  5. 0  6. \(\frac{3}{2}\)  7. \(\frac{3}{2}\)
5.3 Fundamental Theorem of Calculus

Evaluating definite integrals using limits of Riemann sums, as described in Section 5.2, is usually not possible or practical. Fortunately, there is a powerful and practical method for evaluating definite integrals, which is developed in this section. Along the way, we discover the inverse relationship between differentiation and integration, expressed in the most important result of calculus, the Fundamental Theorem of Calculus. The first step in this process is to introduce area functions (first seen in Section 1.2).

Area Functions

The concept of an area function is crucial to the discussion about the connection between derivatives and integrals. We start with a continuous function \( y = f(t) \) defined for \( t \in [a, b] \), where \( a \) is a fixed number. The area function for \( f \) with left endpoint \( a \) is denoted \( A(x) \); it gives the net area of the region bounded by the graph of \( f \) and the \( t \)-axis between \( t = a \) and \( t = x \) (Figure 5.32). The net area of this region is also given by the definite integral

\[
A(x) = \int_a^x f(t) \, dt.
\]

Notice that \( x \) is the upper limit of the integral and the independent variable of the area function: As \( x \) changes, so does the net area under the curve. Because the symbol \( x \) is already in use as the independent variable for \( A \), we must choose another symbol for the variable of integration. Any symbol—except \( x \)—can be used because it is a dummy variable; we have chosen \( t \) as the integration variable.

Figure 5.33 gives a general view of how an area function is generated. Suppose that \( f \) is a continuous function and \( a \) is a fixed number. Now choose a point \( b > a \). The net area

\[
A(x) = \int_a^x f(t) \, dt
\]

is usually not possible or practical. Fortunately, there is a powerful and practical method for evaluating definite integrals, which is developed in this section. Along the way, we discover the inverse relationship between differentiation and integration, expressed in the most important result of calculus, the Fundamental Theorem of Calculus. The first step in this process is to introduce area functions (first seen in Section 1.2).

Area Functions

The concept of an area function is crucial to the discussion about the connection between derivatives and integrals. We start with a continuous function \( y = f(t) \) defined for \( t \in [a, b] \), where \( a \) is a fixed number. The area function for \( f \) with left endpoint \( a \) is denoted \( A(x) \); it gives the net area of the region bounded by the graph of \( f \) and the \( t \)-axis between \( t = a \) and \( t = x \) (Figure 5.32). The net area of this region is also given by the definite integral

\[
A(x) = \int_a^x f(t) \, dt.
\]

Notice that \( x \) is the upper limit of the integral and the independent variable of the area function: As \( x \) changes, so does the net area under the curve. Because the symbol \( x \) is already in use as the independent variable for \( A \), we must choose another symbol for the variable of integration. Any symbol—except \( x \)—can be used because it is a dummy variable; we have chosen \( t \) as the integration variable.

Figure 5.33 gives a general view of how an area function is generated. Suppose that \( f \) is a continuous function and \( a \) is a fixed number. Now choose a point \( b > a \). The net area

\[
A(x) = \int_a^x f(t) \, dt
\]
of the region between the graph of \( f \) and the \( t \)-axis on the interval \([a, b]\) is \( A(b)\). Moving the right endpoint to \((c, 0)\) or \((d, 0)\) produces different regions with net areas \( A(c)\) and \( A(d)\), respectively. In general, if \( x > a \) is a variable point, then \( A(x) = \int_a^x f(t) \, dt \) is the net area of the region between the graph of \( f \) and the \( t \)-axis on the interval \([a, x]\).

Figure 5.33 shows how \( A(x) \) varies with respect to \( x \). Notice that \( A(a) = \int_a^a f(t) \, dt = 0 \). Then for \( x > a \), the net area increases for \( x < c \), at which point \( f(c) = 0 \). For \( x > c \), the function \( f \) is negative, which produces a negative contribution to the area function. As a result, the area function decreases for \( x > c \).

**DEFINITION Area Function**

Let \( f \) be a continuous function, for \( t \geq a \). The **area function for \( f \) with left endpoint \( a \)** is

\[
A(x) = \int_a^x f(t) \, dt,
\]

where \( x \geq a \). The area function gives the net area of the region bounded by the graph of \( f \) and the \( t \)-axis on the interval \([a, x]\).

The following two examples illustrate the idea of area functions.

**EXAMPLE 1** **Comparing area functions** The graph of \( f \) is shown in Figure 5.34 with areas of various regions marked. Let \( A(x) = \int_{a}^{x} f(t) \, dt \) and \( F(x) = \int_{a}^{x} f(t) \, dt \) be two area functions for \( f \) (note the different left endpoints). Evaluate the following area functions.

a. \( A(3) \) and \( F(3) \)  

b. \( A(5) \) and \( F(5) \)  

c. \( A(9) \) and \( F(9) \)

**SOLUTION**

a. The value of \( A(3) = \int_{-1}^{3} f(t) \, dt \) is the net area of the region bounded by the graph of \( f \) and the \( t \)-axis on the interval \([-1, 3]\). Using the graph of \( f \), we see that \( A(3) = -27 \) (because this region has an area of 27 and lies below the \( t \)-axis). On the other hand, \( F(3) = \int_{-1}^{3} f(t) \, dt = 0 \) by Property 1 of Table 5.4. Notice that \( A(3) - F(3) = -27 \).

b. The value of \( A(5) = \int_{-1}^{5} f(t) \, dt \) is found by subtracting the area of the region that lies below the \( t \)-axis on \([-1, 3]\) from the area of the region that lies above the \( t \)-axis on \([3, 5]\). Therefore, \( A(5) = 3 - 27 = -24 \). Similarly, \( F(5) \) is the net area of the region bounded by the graph of \( f \) and the \( t \)-axis on the interval \([3, 5]\); therefore, \( F(5) = 3 \). Notice that \( A(5) - F(5) = -27 \).

c. Reasoning as in parts (a) and (b), we see that \( A(9) = -27 + 3 - 35 = -59 \) and \( F(9) = 3 - 35 = -32 \). As before, observe that \( A(9) - F(9) = -27 \).

**Related Exercises 11–12**

Example 1 illustrates the important fact (to be explained shortly) that two area functions of the same function differ by a constant; in Example 1, the constant is \(-27\).

**QUICK CHECK 1** In Example 1, let \( B(x) \) be the area function for \( f \) with left endpoint 5. Evaluate \( B(5) \) and \( B(9) \).

**EXAMPLE 2** **Area of a trapezoid** Consider the trapezoid bounded by the line \( f(t) = 2t + 3 \) and the \( t \)-axis from \( t = 2 \) to \( t = x \) (Figure 5.35). The area function \( A(x) = \int_{2}^{x} f(t) \, dt \) gives the area of the trapezoid, for \( x \geq 2 \).

a. Evaluate \( A(2) \).

b. Evaluate \( A(5) \).

c. Find and graph the area function \( y = A(x) \), for \( x \geq 2 \).

d. Compare the derivative of \( A \) to \( f \).
SOLUTION

a. By Property 1 of Table 5.4, \( A(2) = \int_2^5 (2t + 3) \, dt = 0. \)

b. Notice that \( A(5) \) is the area of the trapezoid (Figure 5.35) bounded by the line \( y = 2t + 3 \) and the \( t \)-axis on the interval \([2, 5]\). Using the area formula for a trapezoid (Figure 5.36), we find that

\[
A(5) = \int_2^5 (2t + 3) \, dt = \frac{1}{2} \cdot 3(7 + 13) = 30.
\]

c. Now the right endpoint of the base is a variable \( x \geq 2 \) (Figure 5.37). The distance between the parallel sides of the trapezoid is \( x-2 \).

By the area formula for a trapezoid, the area of this trapezoid for any \( x \geq 2 \) is

\[
A(x) = \frac{1}{2} \cdot (x - 2)(7 + 2x + 3) = (x - 2)(x + 5) = x^2 + 3x - 10.
\]

Expressing the area function in terms of an integral with a variable upper limit we have

\[
A(x) = \int_2^x (2t + 3) \, dt = x^2 + 3x - 10.
\]

Because the line \( f(t) = 2t + 3 \) is above the \( t \)-axis, for \( t \geq 2 \), the area function \( A(x) = x^2 + 3x - 10 \) is an increasing function of \( x \) with \( A(2) = 0 \) (Figure 5.38).

d. Differentiating the area function, we find that

\[
A'(x) = \frac{d}{dx}(x^2 + 3x - 10) = 2x + 3 = f(x).
\]

Therefore, \( A'(x) = f(x) \), or equivalently, the area function \( A \) is an antiderivative of \( f \).

We soon show that this relationship is not an accident; it is the first part of the Fundamental Theorem of Calculus.

Related Exercises 13–22
Fundamental Theorem of Calculus

Example 2 suggests that the area function $A$ for a linear function $f$ is an antiderivative of $f$; that is, $A'(x) = f(x)$. Our goal is to show that this conjecture holds for more general functions. Let’s start with an intuitive argument; a formal proof is given at the end of the section.

Assume that $f$ is a continuous function defined on an interval $[a, b]$. As before, $A(x) = \int_a^x f(t) \, dt$ is the area function for $f$ with a left endpoint $a$: it gives the net area of the region bounded by the graph of $f$ and the $t$-axis on the interval $[a, x]$, for $x \geq a$. Figure 5.39 is the key to the argument.

![Figure 5.39](image)

Note that with $h > 0$, $A(x + h)$ is the net area of the region whose base is the interval $[a, x + h]$ and $A(x)$ is the net area of the region whose base is the interval $[a, x]$. So the difference $A(x + h) - A(x)$ is the net area of the region whose base is the interval $[x, x + h]$. If $h$ is small, the region in question is nearly rectangular with a base of length $h$ and a height $f(x)$. Therefore, the net area of this region is

$$A(x + h) - A(x) = hf(x).$$

Dividing by $h$, we have

$$\frac{A(x + h) - A(x)}{h} = f(x).$$

An analogous argument can be made with $h < 0$. Now observe that as $h$ tends to zero, this approximation improves. In the limit as $h \to 0$, we have

$$\lim_{h \to 0} \frac{A(x + h) - A(x)}{h} = \lim_{h \to 0} \frac{A(x)}{f(x)}.$$

We see that indeed $A'(x) = f(x)$. Because $A(x) = \int_a^x f(t) \, dt$, the result can also be written

$$A'(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x),$$

which says that the derivative of the integral of $f$ is $f$. This conclusion is the first part of the Fundamental Theorem of Calculus.

Recall that

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

If the function $f$ is replaced with $A$, then

$$A'(x) = \lim_{h \to 0} \frac{A(x + h) - A(x)}{h}.$$
Chapter 5 • Integration

THEOREM 5.3 (PART 1) Fundamental Theorem of Calculus

If \( f \) is continuous on \([a, b]\), then the area function

\[
A(x) = \int_a^x f(t) \, dt, \quad \text{for} \quad a \leq x \leq b,
\]

is continuous on \([a, b]\) and differentiable on \((a, b)\). The area function satisfies \( A'(x) = f(x) \). Equivalently,

\[
A'(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x),
\]

which means that the area function of \( f \) is an antiderivative of \( f \) on \([a, b]\).

Given that \( A \) is an antiderivative of \( f \) on \([a, b]\), it is one short step to a powerful method for evaluating definite integrals. Remember (Section 4.9) that any two antiderivatives of \( f \) differ by a constant. Assuming that \( F \) is any other antiderivative of \( f \) on \([a, b]\), we have

\[
F(x) = A(x) + C, \text{ for } a \leq x \leq b.
\]

Noting that \( A(a) = 0 \), it follows that

\[
F(b) - F(a) = (A(b) + C) - (A(a) + C) = A(b).
\]

Writing \( A(b) \) in terms of a definite integral leads to the remarkable result

\[
A(b) = \int_a^b f(x) \, dx = F(b) - F(a).
\]

We have shown that to evaluate a definite integral of \( f \), we

- find any antiderivative of \( f \), which we call \( F \); and
- compute \( F(b) - F(a) \), the difference in the values of \( F \) between the upper and lower limits of integration.

This process is the essence of the second part of the Fundamental Theorem of Calculus.

THEOREM 5.3 (PART 2) Fundamental Theorem of Calculus

If \( f \) is continuous on \([a, b]\) and \( F \) is any antiderivative of \( f \) on \([a, b]\), then

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

It is customary and convenient to denote the difference \( F(b) - F(a) \) by \( F(x) \bigg|_a^b \).

Using this shorthand, the Fundamental Theorem is summarized in Figure 5.40.
The Inverse Relationship between Differentiation and Integration  It is worth pausing to observe that the two parts of the Fundamental Theorem express the inverse relationship between differentiation and integration. Part 1 of the Fundamental Theorem says

\[ \frac{d}{dx} \int_a^x f(t) \, dt = f(x), \]

or the derivative of the integral of \( f \) is \( f \) itself.

Noting that \( f \) is an antiderivative of \( f' \), Part 2 of the Fundamental Theorem says

\[ \int_a^b f'(x) \, dx = f(b) - f(a), \]

or the definite integral of the derivative of \( f \) is given in terms of \( f \) evaluated at two points. In other words, the integral “undoes” the derivative.

This last relationship is important because it expresses the integral as an accumulation operation. Suppose we know the rate of change of \( f \) (which is \( f' \)) on an interval \([a, b]\). The Fundamental Theorem says that we can integrate (that is, sum or accumulate) the rate of change over that interval and the result is simply the difference in \( f \) evaluated at the endpoints. You will see this accumulation property used many times in the next chapter. Now let’s use the Fundamental Theorem to evaluate definite integrals.

EXAMPLE 3  Evaluating definite integrals  Evaluate the following definite integrals using the Fundamental Theorem of Calculus, Part 2. Interpret each result geometrically.

a. \( \int_0^{10} (60x - 6x^2) \, dx \)

b. \( \int_0^{2\pi} 3 \sin x \, dx \)

c. \( \int_{1/16}^{\sqrt{t} - 1} \frac{1}{t} \, dt \)

SOLUTION

a. Using the antiderivative rules of Section 4.9, an antiderivative of \( 60x - 6x^2 \) is \( 30x^2 - 2x^3 \). By the Fundamental Theorem, the value of the definite integral is

\[ \int_0^{10} (60x - 6x^2) \, dx = \left( 30x^2 - 2x^3 \right) \bigg|_0^{10} \]

\[ = (30 \cdot 10^2 - 2 \cdot 10^3) - (30 \cdot 0^2 - 2 \cdot 0^3) \]

\[ = (3000 - 2000) - 0 \]

\[ = 1000. \]

Because \( f \) is positive on \([0, 10]\), the definite integral \( \int_0^{10} (60x - 6x^2) \, dx \) is the area of the region between the graph of \( f \) and the \( x \)-axis on the interval \([0, 10]\) (Figure 5.41).

b. As shown in Figure 5.42, the region bounded by the graph of \( f(x) = 3 \sin x \) and the \( x \)-axis on \([0, 2\pi]\) consists of two parts, one above the \( x \)-axis and one below the \( x \)-axis. By the symmetry of \( f \), these two regions have the same area, so the definite integral over \([0, 2\pi]\) is zero. Let’s confirm this fact. An antiderivative of \( f(x) = 3 \sin x \) is \(-3 \cos x\). Therefore, the value of the definite integral is

\[ \int_0^{2\pi} 3 \sin x \, dx = -3 \cos x \bigg|_0^{2\pi} \]

\[ = (-3 \cos (2\pi)) - (-3 \cos (0)) \]

\[ = -3 - (-3) = 0. \]

Simplify.

c. Although the variable of integration is \( t \), rather than \( x \), we proceed as in parts (a) and (b) after simplifying the integrand:

\[ \frac{\sqrt{t} - 1}{t} = \frac{1}{\sqrt{t}} - \frac{1}{t} \]

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Finding antiderivatives with respect to \( t \) and applying the Fundamental Theorem, we have

\[
\int_{1/16}^{3/16} \frac{\sqrt{t} - 1}{t} \, dt = \int_{1/16}^{3/16} \left( t^{1/2} - \frac{1}{t} \right) \, dt
\]

Simplify the integrand.

\[
= \left( 2t^{1/2} - \ln |t| \right) \bigg|_{1/16}^{3/16}
\]

Fundamental Theorem

\[
= \left( 2 \left( \frac{3}{16} \right)^{1/2} - \ln \frac{3}{16} \right) - \left( 2 \left( \frac{1}{16} \right)^{1/2} - \ln \frac{1}{16} \right)
\]

Evaluate.

\[
= 1 - \ln \frac{3}{16} + \frac{1}{2} + \ln 2
\]

Simplify.

\[
= \frac{1}{2} - \ln 4 \approx -0.8863.
\]

The definite integral is negative because the graph of \( f \) lies below the \( t \)-axis.

**Related Exercises 23–50**

**EXAMPLE 4** **Net areas and definite integrals** The graph of \( f(x) = 6x(x + 1)(x - 2) \) is shown in Figure 5.44. The region \( R_1 \) is bounded by the curve and the \( x \)-axis on the interval \([-1, 0]\), and \( R_2 \) is bounded by the curve and the \( x \)-axis on the interval \([0, 2]\).

**a.** Find the net area of the region between the curve and the \( x \)-axis on \([-1, 2]\).

**b.** Find the area of the region between the curve and the \( x \)-axis on \([-1, 2]\).

**SOLUTION**

**a.** The net area of the region is given by a definite integral. The integrand \( f \) is first expanded to find an antiderivative:

\[
\int_{-1}^{2} f(x) \, dx = \int_{-1}^{2} (6x^3 - 6x^2 - 12x) \, dx.
\]

Expand \( f \).

\[
= \left( \frac{3}{2} x^4 - 2x^3 - 6x^2 \right) \bigg|_{-1}^{2}
\]

Fundamental Theorem

\[
= -\frac{27}{2}.
\]

Simplify.

The net area of the region between the curve and the \( x \)-axis on \([-1, 2]\) is \(-\frac{27}{2}\), which is the area of \( R_1 \) minus the area of \( R_2 \) (Figure 5.44). Because \( R_2 \) has a larger area than \( R_1 \), the net area is negative.

**b.** The region \( R_1 \) lies above the \( x \)-axis, so its area is

\[
\int_{-1}^{0} (6x^3 - 6x^2 - 12x) \, dx = \left( \frac{3}{2} x^4 - 2x^3 - 6x^2 \right) \bigg|_{-1}^{0} = \frac{5}{2}.
\]

The region \( R_2 \) lies below the \( x \)-axis, so its net area is negative:

\[
\int_{0}^{2} (6x^3 - 6x^2 - 12x) \, dx = \left( \frac{3}{2} x^4 - 2x^3 - 6x^2 \right) \bigg|_{0}^{2} = -16.
\]

Therefore, the area of \( R_2 \) is \(-(-16) = 16\). The combined area of \( R_1 \) and \( R_2 \) is \(\frac{5}{2} + 16 = \frac{37}{2}\). We could also find the area of this region directly by evaluating

\[
\int_{-1}^{2} |f(x)| \, dx.
\]

**Related Exercises 51–60**

Examples 3 and 4 make use of Part 2 of the Fundamental Theorem, which is the most potent tool for evaluating definite integrals. The remaining examples illustrate the use of the equally important Part 1 of the Fundamental Theorem.

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EXAMPLE 5  Derivatives of integrals  Use Part 1 of the Fundamental Theorem to simplify the following expressions.

a. \[ \frac{d}{dx} \int_1^x \sin^2 t \, dt \]

b. \[ \frac{d}{dx} \int_1^x \sqrt{t^2 + 1} \, dt \]

c. \[ \frac{d}{dx} \int_0^x \cos t^2 \, dt \]

**SOLUTION**

a. Using Part 1 of the Fundamental Theorem, we see that

\[ \frac{d}{dx} \int_1^x \sin^2 t \, dt = \sin^2 x. \]

b. To apply Part 1 of the Fundamental Theorem, the variable must appear in the upper limit. Therefore, we use the fact that \[ \int_a^b f(t) \, dt = -\int_b^a f(t) \, dt \] and then apply the Fundamental Theorem:

\[ \frac{d}{dx} \int_1^x \sqrt{t^2 + 1} \, dt = -\frac{d}{dx} \int_x^1 \sqrt{t^2 + 1} \, dt = -\sqrt{x^2 + 1}. \]

c. The upper limit of the integral is not \( x \), but a function of \( x \). Therefore, the function to be differentiated is a composite function, which requires the Chain Rule. We let \( u = x^2 \) to produce

\[ y = g(u) = \int_0^u \cos t^2 \, dt. \]

By the Chain Rule,

\[ \frac{d}{dx} \int_0^{x^2} \cos t^2 \, dt = \frac{dy}{du} \frac{du}{dx} = \frac{dy}{du} \frac{du}{dx} \]

Substitute for \( g \); note that \( u'(x) = 2x \).

\[ = \left( \frac{d}{du} \int_0^{u^2} \cos t^2 \, dt \right)(2x) \]

\[ = 2x \cos u^2 \]

\[ = 2x \cos x^4. \]

**Related Exercises 61–68**

EXAMPLE 6  Working with area functions  Consider the function \( f \) shown in Figure 5.45 and its area function \( A(x) = \int_0^x f(t) \, dt \), for \( 0 \leq x \leq 17 \). Assume that the four regions \( R_1, R_2, R_3, \) and \( R_4 \) have the same area. Based on the graph of \( f \), do the following.

a. Find the zeros of \( A \) on \([0, 17]\).

b. Find the points on \([0, 17]\) at which \( A \) has local maxima or local minima.

c. Sketch a graph of \( A \), for \( 0 \leq x \leq 17 \).

**SOLUTION**

a. The area function \( A(x) = \int_0^x f(t) \, dt \) gives the net area bounded by the graph of \( f \) and the \( t \)-axis on the interval \([0, x]\) (Figure 5.46). Therefore, \( A(0) = \int_0^0 f(t) \, dt = 0. \)

Because \( R_1 \) and \( R_2 \) have the same area but lie on opposite sides of the \( t \)-axis, it follows that \( A(8) = \int_0^8 f(t) \, dt = 0. \) Similarly, \( A(16) = \int_0^{16} f(t) \, dt = 0. \) Therefore, the zeros of \( A \) are \( x = 0, 8, \) and \( 16. \)

b. Observe that the function \( f \) is positive, for \( 0 < t < 4 \), which implies that \( A(x) \) increases as \( x \) increases from 0 to 4 (Figure 5.46b). Then as \( x \) increases from 4 to 8, \( A(x) \) decreases because \( f \) is negative, for \( 4 < t < 8 \) (Figure 5.46c). Similarly, \( A(x) \) increases as \( x \) increases from \( x = 8 \) to \( x = 12 \) (Figure 5.46d) and decreases from \( x = 12 \) to \( x = 16 \). By the First Derivative Test, \( A \) has local minima at \( x = 8 \) and \( x = 16 \) and local maxima at \( x = 4 \) and \( x = 12 \) (Figure 5.46e).
The sine integral function

Let

\[ g(t) = \begin{cases} 
\sin t & \text{if } t > 0 \\
1 & \text{if } t = 0.
\end{cases} \]

Graph the sine integral function \( S(x) = \int_0^x g(t) \, dt \), for \( x \geq 0 \).

**SOLUTION** Notice that \( S \) is an area function for \( g \). The independent variable of \( S \) is \( x \), and \( t \) has been chosen as the (dummy) variable of integration. A good way to start is by graphing the integrand \( g \) (Figure 5.47a). The function oscillates with a decreasing amplitude with \( g(0) = 1 \). Beginning with \( S(0) = 0 \), the area function \( S \) increases until \( x = \pi \) because \( g \) is positive on \((0, \pi)\). However, on \((\pi, 2\pi)\), \( g \) is negative and the net area decreases. On \((2\pi, 3\pi)\), \( g \) is positive again, so \( S \) again increases. Therefore, the graph of \( S \) has alternating local maxima and minima. Because the amplitude of \( g \) decreases, each maximum of \( S \) is less than the previous maximum and each minimum of \( S \) is greater than the previous minimum (Figure 5.47b). Determining the exact value of \( S \) at these maxima and minima is difficult.

**EXAMPLE 7** The sine integral function

\[ S(x) = \int_0^x g(t) \, dt, \text{ for } x \geq 0. \]

**Related Exercises** 69–80

![Figure 5.46](image1)

![Figure 5.47](image2)
Appealing to Part 1 of the Fundamental Theorem, we find that

\[ S'(x) = \frac{d}{dx} \int_0^x g(t) \, dt = \frac{\sin x}{x}, \text{ for } x > 0. \]

As anticipated, the derivative of \( S \) changes sign at integer multiples of \( \pi \).
Specifically, \( S' \) is positive and \( S \) increases on the intervals \((0, \pi), \left(2\pi, 3\pi\right), \ldots, (2n\pi, (2n + 1)\pi), \ldots \), while \( S' \) is negative and \( S \) decreases on the remaining intervals. Clearly, \( S \) has local maxima at \( x = \pi, 3\pi, 5\pi, \ldots \), and it has local minima at \( x = 2\pi, 4\pi, 6\pi, \ldots \).

One more observation is helpful. It can be shown that although \( S \) oscillates for increasing \( x \), its graph gradually flattens out and approaches a horizontal asymptote. (Finding the exact value of this horizontal asymptote is challenging; see Exercise 111.) Assembling all these observations, the graph of the sine integral function emerges (Figure 5.47b).

Related Exercises 81–84

We conclude this section with a formal proof of the Fundamental Theorem of Calculus.

**Proof of the Fundamental Theorem:** Let \( f \) be continuous on \([a, b]\) and let \( A \) be the area function for \( f \) with left endpoint \( a \). The first step is to prove that \( A \) is differentiable on \((a, b)\) and \( A'(x) = f(x) \), which is Part 1 of the Fundamental Theorem. The proof of Part 2 then follows.

**Step 1.** We assume that \( a < x < b \) and use the definition of the derivative,

\[ A'(x) = \lim_{h \to 0} \frac{A(x + h) - A(x)}{h}. \]

First assume that \( h > 0 \). Using Figure 5.48 and Property 5 of Table 5.4, we have

\[ A(x + h) - A(x) = \int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt = \int_x^{x+h} f(t) \, dt. \]

That is, \( A(x + h) - A(x) \) is the net area of the region bounded by the curve on the interval \([x, x + h]\).

![Figure 5.48](image)

Let \( m \) and \( M \) be the minimum and maximum values of \( f \) on \([x, x + h]\), respectively, which exist by the continuity of \( f \). Suppose \( x = t_0 < t_1 < t_2 < \ldots < t_n = x + h \) is a general partition of \([x, x + h]\) and let \( \sum_{k=1}^n f(t_k) \Delta t_k \) be a corresponding general Riemann sum, where \( \Delta t_k = t_k - t_{k-1} \). Because \( m \leq f(t) \leq M \) on \([x, x + h]\), it follows that

\[ \sum_{k=1}^n m \Delta t_k \leq \sum_{k=1}^n f(t_k) \Delta t_k \leq \sum_{k=1}^n M \Delta t_k. \]
or

\[ mh \leq \sum_{k=1}^{n} f(t_k^*) \Delta t_k \leq Mh. \]

We have used the facts that \( \sum_{k=1}^{n} m \Delta t_k = m \sum_{k=1}^{n} \Delta t_k = mh \) and similarly, \( \sum_{k=1}^{n} M \Delta t_k = Mh \).

Notice that these inequalities hold for every Riemann sum for \( f \) on \([x, x+h]\); that is, for all partitions and for all \( n \). Therefore, we are justified in taking the limit as \( n \to \infty \) across these inequalities to obtain

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} f(t_k^*) \Delta t_k}{\int_{x}^{x+h} f(t) \, dt} \leq \lim_{n \to \infty} \frac{Mh}{\int_{x}^{x+h} f(t) \, dt}.
\]

Evaluating each of these three limits results in

\[ mh \leq \int_{x}^{x+h} f(t) \, dt \leq Mh. \]

Substituting for the integral, we find that

\[ mh \leq A(x+h) - A(x) \leq Mh. \]

Dividing these inequalities by \( h > 0 \), we have

\[ m \leq \frac{A(x+h) - A(x)}{h} \leq M. \]

The case \( h < 0 \) is handled similarly and leads to the same conclusion.

We now take the limit as \( h \to 0 \) across these inequalities. As \( h \to 0 \), \( m \) and \( M \) approach \( f(x) \), because \( f \) is continuous at \( x \). At the same time, as \( h \to 0 \), the quotient that is sandwiched between \( m \) and \( M \) approaches \( A'(x) \):

\[
\lim_{h \to 0} m = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} \frac{Mh}{h}.
\]

By the Squeeze Theorem (Theorem 2.5), we conclude that \( A'(x) \) exists and \( A \) is differentiable for \( a < x < b \). Furthermore, \( A'(x) = f(x) \). Finally, because \( A \) is differentiable on \((a, b)\), \( A \) is continuous on \((a, b)\) by Theorem 3.1. Exercise 116 shows that \( A \) is also right- and left-continuous at the endpoints \( a \) and \( b \), respectively.

**Step 2.** Having established that the area function \( A \) is an antiderivative of \( f \), we know that \( F(x) = A(x) + C \), where \( F \) is any antiderivative of \( f \) and \( C \) is a constant. Noting that \( A(a) = 0 \), it follows that

\[ F(b) - F(a) = (A(b) + C) - (A(a) + C) = A(b). \]

Writing \( A(b) \) in terms of a definite integral, we have

\[ A(b) = \int_{a}^{b} f(x) \, dx = F(b) - F(a), \]

which is Part 2 of the Fundamental Theorem.
SECTION 5.3 EXERCISES

Review Questions
1. Suppose \( A \) is an area function of \( f \). What is the relationship between \( f \) and \( A \)?
2. Suppose \( F \) is an antiderivative of \( f \) and \( A \) is an area function of \( f \). What is the relationship between \( F \) and \( A \)?
3. Explain in words and write mathematically how the Fundamental Theorem of Calculus is used to evaluate definite integrals.
4. Let \( f(x) = c \), where \( c \) is a positive constant. Explain why an area function of \( f \) is an increasing function.
5. The linear function \( f(x) = 3 - x \) is decreasing on the interval \([0, 3]\). Is the area function for \( f \) (with left endpoint 0) increasing or decreasing on the interval \([0, 3]\)? Draw a picture and explain.
6. Evaluate \( \int_0^2 3x^2 \, dx \) and \( \int_0^2 3x^3 \, dx \).
7. Explain in words and express mathematically the inverse relationship between differentiation and integration as given by Part 1 of the Fundamental Theorem of Calculus.
8. Why can the constant of integration be omitted from the antiderivative when evaluating a definite integral?
9. Evaluate \( \frac{d}{dx} \int_a^b f(t) \, dt \) and \( \frac{d}{dx} \int_a^b f(t) \, dt \), where \( a \) and \( b \) are constants.
10. Explain why \( \int_a^b f'(x) \, dx = f(b) - f(a) \).

Basic Skills
11. Area functions The graph of \( f \) is shown in the figure. Let \( A(x) = \int_0^x f(t) \, dt \) and \( F(x) = \int_0^x f(t) \, dt \) be two area functions for \( f \). Evaluate the following area functions.
   a. \( A(-2) \)  b. \( F(8) \)  c. \( A(4) \)  d. \( F(4) \)  e. \( A(8) \)

12. Area functions The graph of \( f \) is shown in the figure. Let \( A(x) = \int_0^x f(t) \, dt \) and \( F(x) = \int_0^x f(t) \, dt \) be two area functions for \( f \). Evaluate the following area functions.
   a. \( A(2) \)  b. \( F(5) \)  c. \( A(0) \)  d. \( F(8) \)  e. \( A(8) \)  f. \( A(5) \)  g. \( F(2) \)

13–16. Area functions for constant functions Consider the following functions \( f \) and real numbers \( a \) (see figure).
   a. Find and graph the area function \( A(x) = \int_a^x f(t) \, dt \) for \( f \).
   b. Verify that \( A'(x) = f(x) \).

13. \( f(t) = 5, \ a = 0 \)  14. \( f(t) = 10, \ a = 4 \)
15. \( f(t) = 5, \ a = -5 \)  16. \( f(t) = 2, \ a = -3 \)

17. Area functions for the same linear function Let \( f(t) = t \) and consider the two area functions \( A(x) = \int_0^x f(t) \, dt \) and \( F(x) = \int_0^x f(t) \, dt \).
   a. Evaluate \( A(2) \) and \( A(3) \). Then use geometry to find an expression for \( A(x) \), for \( x \geq 0 \).
   b. Evaluate \( F(4) \) and \( F(6) \). Then use geometry to find an expression for \( F(x) \), for \( x \geq 2 \).
   c. Show that \( A(x) = F(x) \) is a constant and that \( A'(x) = F'(x) = f(x) \).

18. Area functions for the same linear function Let \( f(t) = 2t - 2 \) and consider the two area functions \( A(x) = \int_0^x f(t) \, dt \) and \( F(x) = \int_0^x f(t) \, dt \).
   a. Evaluate \( A(2) \) and \( A(3) \). Then use geometry to find an expression for \( A(x) \), for \( x \geq 1 \).
   b. Evaluate \( F(5) \) and \( F(6) \). Then use geometry to find an expression for \( F(x) \), for \( x \geq 4 \).
   c. Show that \( A(x) = F(x) \) is a constant and that \( A'(x) = F'(x) = f(x) \).

19–22. Area functions for linear functions Consider the following functions \( f \) and real numbers \( a \) (see figure).
   a. Find and graph the area function \( A(x) = \int_a^x f(t) \, dt \).
   b. Verify that \( A'(x) = f(x) \).

19. \( f(t) = t + 5, \ a = -5 \)  20. \( f(t) = 2t + 5, \ a = 0 \)
21. \( f(t) = 3t + 1, \ a = 2 \)  22. \( f(t) = 4t + 2, \ a = 0 \)
23–24. Definite integrals Evaluate the following integrals using the Fundamental Theorem of Calculus. Explain why your result is consistent with the figure.

23. \[ \int_{0}^{1} (x^2 - 2x + 3) \, dx \]
24. \[ \int_{\pi/4}^{7\pi/4} (\sin x + \cos x) \, dx \]

25–28. Definite integrals Evaluate the following integrals using the Fundamental Theorem of Calculus. Sketch the graph of the integrand and shade the region whose net area you have found.

25. \[ \int_{-2}^{3} (x^2 - x - 6) \, dx \]
26. \[ \int_{0}^{1} (x - \sqrt{x}) \, dx \]
27. \[ \int_{0}^{5} (x^2 - 9) \, dx \]
28. \[ \int_{1/2}^{2} \left( 1 - \frac{1}{x} \right) \, dx \]

29–50. Definite integrals Evaluate the following integrals using the Fundamental Theorem of Calculus.

29. \[ \int_{0}^{2} 4x^3 \, dx \]
30. \[ \int_{0}^{2} (3x^2 + 2x) \, dx \]
31. \[ \int_{0}^{1} (x + \sqrt{x}) \, dx \]
32. \[ \int_{\pi/4}^{\pi/2} 2 \cos x \, dx \]
33. \[ \int_{1}^{e} \frac{2}{\sqrt{x}} \, dx \]
34. \[ \int_{0}^{\pi/4} \frac{2}{x} \, dx \]
35. \[ \int_{-2}^{2} (x^2 - 4) \, dx \]
36. \[ \int_{0}^{\ln 2} e^x \, dx \]
37. \[ \int_{1/2}^{1} (x^3 - 8) \, dx \]
38. \[ \int_{0}^{4} x(x - 2)(x - 4) \, dx \]
39. \[ \int_{\pi/4}^{\pi/2} \sec^2 \theta \, d\theta \]
40. \[ \int_{0}^{1/2} \frac{dx}{\sqrt{1 - x^2}} \]
41. \[ \int_{-2}^{1} x^3 \, dx \]
42. \[ \int_{0}^{\pi} (1 - \sin x) \, dx \]
43. \[ \int_{1}^{4} (1 - x)(x - 4) \, dx \]
44. \[ \int_{\pi/2}^{\pi} \cos x - 1 \, dx \]
45. \[ \int_{-\pi/2}^{\pi/2} \frac{\sqrt{t}}{t} \, dt \]
46. \[ \int_{1}^{3} \frac{1}{x^3 - 4} \, dx \]
47. \[ \int_{0}^{\pi/8} \cos 2x \, dx \]
48. \[ \int_{0}^{1} 10e^{2x} \, dx \]
49. \[ \int_{1}^{\sqrt{7}} \frac{dx}{1 + x^2} \]
50. \[ \int_{\pi/16}^{\pi/8} \frac{8 \csc^2 2x \, dx}{x} \]

51–54. Areas Find (i) the net area and (ii) the area of the following regions. Graph the function and indicate the region in question.

51. The region bounded by \( y = x^{1/2} \) and the \( x \)-axis between \( x = 1 \) and \( x = 4 \)

52. The region above the \( x \)-axis bounded by \( y = 4 - x^2 \)
53. The region below the \( x \)-axis bounded by \( y = x^4 - 16 \)
54. The region bounded by \( y = 6 \cos x \) and the \( x \)-axis between \( x = -\pi/2 \) and \( x = \pi \)

55–60. Areas of regions Find the area of the region bounded by the graph of \( f \) and the \( x \)-axis on the given interval.

55. \( f(x) = x^2 - 25 \) on \([2, 4]\)
56. \( f(x) = x^3 - 1 \) on \([-1, 2]\)
57. \( f(x) = \frac{1}{x} \) on \([-2, -1]\)
58. \( f(x) = x(x + 1)(x - 2) \) on \([-1, 2]\)
59. \( f(x) = \sin x \) on \([-\pi/4, 3\pi/4]\)
60. \( f(x) = \cos x \) on \([\pi/2, \pi]\)

61–68. Derivatives of integrals Simplify the following expressions.

61. \[ \frac{d}{dx} \left( \int_{0}^{1} (t^2 + 1) \, dt \right) \]
62. \[ \frac{d}{dx} \left( \int_{0}^{t} e^t \, dt \right) \]
63. \[ \frac{d}{dx} \left( \int_{2}^{1} \frac{dp}{p^2} \right) \]
64. \[ \frac{d}{dx} \left( \int_{-\frac{9}{2}}^{0} \frac{dz}{z^2 + 1} \right) \]
65. \[ \frac{d}{dx} \left( \int_{-\sqrt{1}}^{1} \sqrt{1 + t^2} \, dt \right) \]
66. \[ \frac{d}{dx} \left( \int_{-\ln 2}^{0} \frac{dp}{p^2 + 1} \right) \]
67. \[ \frac{d}{dx} \left( \int_{-\frac{9}{2}}^{0} \sqrt{1 + t^2} \, dt \right) \]
68. \[ \frac{d}{dx} \left( \int_{-\ln 2}^{0} \ln t^2 \, dt \right) \]

69. Matching functions with area functions Match the functions \( f \), whose graphs are given in a–d, with the area functions \( A(x) = \int_{0}^{x} f(t) \, dt \), whose graphs are given in A–D.
### 70–73. Working with area functions

Consider the function \( f \) and its graph.

a. Estimate the zeros of the area function \( A(x) = \int_0^x f(t) \, dt \) for \( 0 \leq x \leq 10 \).

b. Estimate the points (if any) at which \( A \) has a local maximum or minimum.

c. Sketch a graph of \( A \), for \( 0 \leq x \leq 10 \), without a scale on the \( y \)-axis.

### 74. Area functions from graphs

The graph of \( f \) is given in the figure. Let \( A(x) = \int_0^x f(t) \, dt \) and evaluate \( A(1) \), \( A(2) \), \( A(4) \), and \( A(6) \).

### 75. Area functions from graphs

The graph of \( f \) is given in the figure. Let \( A(x) = \int_0^x f(t) \, dt \) and evaluate \( A(2) \), \( A(5) \), \( A(8) \), and \( A(12) \).

### 76–80. Working with area functions

Consider the function \( f \) and the points \( a, b, \) and \( c \).

a. Find the area function \( A(x) = \int_0^x f(t) \, dt \) using the Fundamental Theorem.

b. Graph \( f \) and \( A \).

c. Evaluate \( A(b) \) and \( A(c) \). Interpret the results using the graphs of part (b).

| 76. \( f(x) = \sin x; a = 0, b = \pi/2, c = \pi \) |
| 77. \( f(x) = e^x; a = 0, b = \ln 2, c = \ln 4 \) |
| 78. \( f(x) = -12x(x-1)(x-2); a = 0, b = 1, c = 2 \) |
| 79. \( f(x) = \cos \pi x; a = 0, b = 1, c = 1 \) |
| 80. \( f(x) = \frac{1}{x}; a = 1, b = 4, c = 6 \) |

### 81–84. Functions defined by integrals

Consider the function \( g \), which is given in terms of a definite integral with a variable upper limit.

a. Graph the integrand.

b. Calculate \( g'(x) \).

c. Graph \( g \), showing all your work and reasoning.

| 81. \( g(x) = \int_0^x \sin^2 t \, dt \) |
| 82. \( g(x) = \int_0^x (t^2 + 1) \, dt \) |
| 83. \( g(x) = \int_0^x \sin (\pi t^2) \, dt \) (a Fresnel integral) |
| 84. \( g(x) = \int_0^x \cos (\pi \sqrt{t}) \, dt \) |

### Further Explorations

85. Explain why or why not

Determine whether the following statements are true and give an explanation or counterexample.

a. Suppose that \( f \) is a positive decreasing function, for \( x > 0 \). Then the area function \( A(x) = \int_0^x f(t) \, dt \) is an increasing function of \( x \).

b. Suppose that \( f \) is a negative increasing function, for \( x > 0 \). Then the area function \( A(x) = \int_0^x f(t) \, dt \) is a decreasing function of \( x \).

c. The functions \( p(x) = \sin 3x \) and \( q(x) = 4 \sin 3x \) are antiderivatives of the same function.

d. If \( A(x) = 3x^3 - x - 3 \) is an area function for \( f \), then \( B(x) = 3x^3 - x \) is also an area function for \( f \).

e. \( \frac{d}{dx} \int_a^b f(t) \, dt = 0 \).
86–94. **Definite integrals** Evaluate the following definite integrals using the Fundamental Theorem of Calculus.

86. \( \int_0^{\ln 2} e^x \, dx \)
87. \( \int_1^x \frac{x - 2}{\sqrt{x}} \, dx \)
88. \( \int_1^x \frac{2}{\sqrt{x}} \, dx \)
89. \( \int_0^{\pi/3} \sec x \, dx \)
90. \( \int_{\pi/4}^{\pi/2} \tan x \, dx \)
91. \( \int_0^\pi x \, dy \)
92. \( \int_1^2 \frac{dx}{x \sqrt{x^2 - 1}} \)
93. \( \int_1^2 \frac{z^2 + 4}{z} \, dz \)
94. \( \int_0^\pi \frac{3 \, dx}{9 + x^2} \)

**Areas of regions** Find the area of the region \( R \) bounded by the graph of \( f \) and the \( x \)-axis on the given interval. Graph \( f \) and show the region \( R \).

95. \( f(x) = 2 - |x| \) on \([-2, 4]\)
96. \( f(x) = (1 - x^2)^{-1/2} \) on \([-1/2, \sqrt{3}/2]\)
97. \( f(x) = x^4 - 4 \) on \([1, 4]\)
98. \( f(x) = x^2(x - 2) \) on \([-1, 3]\)

**Derivatives and integrals** Simplify the given expressions.

99. \( \int_3^8 f(t) \, dt \), where \( f(t) \) is continuous on \([3, 8]\)
100. \( \frac{d}{dx} \int_0^x e^t \, dt \)
101. \( \frac{d}{dx} \int_0^x t^3 \, dt \)
102. \( \frac{d}{dx} \left( \int_0^x \left( \frac{3}{x} \right) \, dx \right) \)
103. \( \frac{d}{dx} \left( \int_0^x \left( \frac{1}{x^2 + 1} + \frac{1}{x^2 + 1} \right) \, dx \right) \)

**Additional Exercises**

105. **Zero net area** Consider the function \( f(x) = x^2 - 4x \).
   a. Graph \( f \) on the interval \( x \geq 0 \).
   b. For what value of \( b > 0 \) is \( \int_0^b f(x) \, dx = 0 \)?
   c. In general, for the function \( f(x) = x^2 - ax \), where \( a > 0 \), for what value of \( b > 0 \) (as a function of \( a \)) is \( \int_0^b f(x) \, dx = 0 \)?

106. **Cubic zero area** Consider the graph of the cubic \( y = x(x - a)(x - b) \), where \( 0 < a < b \). Verify that the graph bounds a region above the \( x \)-axis, for \( 0 < x < a \), and bounds a region below the \( x \)-axis, for \( a < x < b \). What is the relationship between \( a \) and \( b \) if the areas of these two regions are equal?

107. **Maximum net area** What value of \( b > -1 \) maximizes the integral
   \( \int_{-1}^b x^2 (3 - x) \, dx \)?

108. **Maximum net area** Graph the function \( f(x) = 8 + 2x - x^2 \) and determine the values of \( a \) and \( b \) that maximize the value of the integral
   \( \int_a^b (8 + 2x - x^2) \, dx \).

109. **An integral equation** Use the Fundamental Theorem of Calculus, Part 1, to find the function \( f \) that satisfies the equation
   \( \int_0^t f(t) \, dt = 2 \cos x + 3x - 2 \).

Verify the result by substitution into the equation.

**Max/min of area functions** Suppose \( f \) is continuous on \([0, \infty)\) and \( A(x) \) is the net area of the region bounded by the graph of \( f \) and the \( t \)-axis on \([0, x]\). Show that the local maxima and minima of \( A \) occur at the zeros of \( f \). Verify this fact with the function \( f(x) = x^2 - 10x \).

**Asymptote of sine integral** Use a calculator to approximate
   \( \lim_{x \to \infty} S(x) = \lim_{x \to \infty} \int_0^x \frac{\sin t}{t} \, dt \),
where \( S \) is the sine integral function (see Example 7). Explain your reasoning.

112. **Sine integral** Show that the sine integral \( S(x) = \int_0^x \frac{\sin t}{t} \, dt \) satisfies the (differential) equation \( xS''(x) + 2S'(x) + xS(x) = 0 \).

113. **Fresnel integral** Show that the Fresnel integral \( S(x) = \int_0^x \sin t^2 \, dt \) satisfies the (differential) equation
   \( (S'(x))^2 + \left( \frac{S(x)}{2x} \right)^2 = 1 \).

114. **Variable integration limits** Evaluate \( \frac{d}{dx} \int_a^x (t^2 + t) \, dt \).
   (Hint: Separate the integral into two pieces.)

115. **Discrete version of the Fundamental Theorem** In this exercise, we work with a discrete problem and show why the relationship \( \sum f(x) \, dx = f(b) - f(a) \) makes sense. Suppose we have a set of equally spaced grid points
   \( \{a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\} \),
   where the distance between any two grid points is \( \Delta x \). Suppose also that at each grid point \( x_k \), a function value \( f(x_k) \) is defined, for \( k = 0, \ldots, n \).
   a. We now replace the integral with a sum and replace the derivative with a difference quotient. Explain why \( \sum f(x) \, dx \) is analogous to \( \frac{\sum f(x_k) - f(x_{k-1})}{\Delta x} \).
   b. Simplify the sum in part (a) and show that it is equal to \( f(b) - f(a) \).
   c. Explain the correspondence between the integral relationship and the summation relationship.

116. **Continuity at the endpoints** Assume that \( f \) is continuous on \([a, b]\) and let \( A \) be the area function for \( f \) with left endpoint \( a \). Let \( M^a \) and \( M^b \) be the absolute minimum and maximum values of \( f \) on \([a, b]\), respectively.
   a. Prove that \( M^a(x - a) = A(x) = M^b(x - a) \) for all \( x \) in \([a, b]\). Use this result and the Squeeze Theorem to show that \( A \) is continuous from the right at \( x = a \).
   b. Prove that \( M^a(b - x) = A(b) - A(x) = M^b(b - x) \) for all \( x \) in \([a, b]\). Use this result to show that \( A \) is continuous from the left at \( x = b \).

**Quick Check Answers**
1. 0, 35
2. 44; 120
3. \( \frac{5}{7} \)
4. If \( f \) is differentiable, we get \( f' \). Therefore, \( f \) is an antiderivative of \( f' \).

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5.4 Working with Integrals

With the Fundamental Theorem of Calculus in hand, we may begin an investigation of integration and its applications. In this section, we discuss the role of symmetry in integrals, we use the slice-and-sum strategy to define the average value of a function, and we explore a theoretical result called the Mean Value Theorem for Integrals.

Integrating Even and Odd Functions

Symmetry appears throughout mathematics in many different forms, and its use often leads to insights and efficiencies. Here we use the symmetry of a function to simplify integral calculations.

Section 1.1 introduced the symmetry of even and odd functions. An even function satisfies the property \( f(-x) = f(x) \), which means that its graph is symmetric about the y-axis (Figure 5.49a). Examples of even functions are \( f(x) = \cos x \) and \( f(x) = x^n \), where \( n \) is an even integer. An odd function satisfies the property \( f(-x) = -f(x) \), which means that its graph is symmetric about the origin (Figure 5.49b). Examples of odd functions are \( f(x) = \sin x \) and \( f(x) = x^n \), where \( n \) is an odd integer.

Special things happen when we integrate even and odd functions on intervals centered at the origin. First suppose \( f \) is an even function and consider \( \int_{-a}^{a} f(x) \, dx \). From Figure 5.49a, we see that the integral of \( f \) on \([-a, 0]\) equals the integral of \( f \) on \([0, a]\). Therefore, the integral on \([-a, a]\) is twice the integral on \([0, a]\), or

\[
\int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx.
\]

On the other hand, suppose \( f \) is an odd function and consider \( \int_{-a}^{a} f(x) \, dx \). As shown in Figure 5.49b, the integral on the interval \([-a, 0]\) is the negative of the integral on \([0, a]\). Therefore, the integral on \([-a, a]\) is zero, or

\[
\int_{-a}^{a} f(x) \, dx = 0.
\]

We summarize these results in the following theorem.

**THEOREM 5.4** Integrals of Even and Odd Functions

Let \( a \) be a positive real number and let \( f \) be an integrable function on the interval \([-a, a]\). If \( f \) is even, \( \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx \). If \( f \) is odd, \( \int_{-a}^{a} f(x) \, dx = 0 \).

**QUICK CHECK** 1 If \( f \) and \( g \) are both even functions, is the product \( fg \) even or odd? Use the facts that \( f(-x) = f(x) \) and \( g(-x) = g(x) \).

The following example shows how symmetry can simplify integration.

**EXAMPLE 1** Integrating symmetric functions  Evaluate the following integrals using symmetry arguments.

a. \( \int_{-2}^{2} (x^4 - 3x^3) \, dx \)  

b. \( \int_{-\pi/2}^{\pi/2} (\cos x - 4 \sin^3 x) \, dx \)
Not for Sale

SOLUTION

a. Note that \( x^4 - 3x^3 \) is neither odd nor even so Theorem 5.4 cannot be applied directly. However, we can split the integral and then use symmetry:

\[
\int_{-2}^{2} (x^4 - 3x^3) \, dx = \int_{-2}^{2} x^4 \, dx - 3 \int_{-2}^{2} x^3 \, dx
\]

Properties 3 and 4 of Table 5.4

\[
= 2 \int_{0}^{2} x^4 \, dx - 0 \quad \text{\( x^4 \) is even; \( x^3 \) is odd.}
\]

\[
= 2 \left( \frac{x^5}{5} \right) \bigg|_{0}^{2} \quad \text{Fundamental Theorem}
\]

\[
= 2 \left( \frac{32}{5} \right) = \frac{64}{5}. \quad \text{Simplify.}
\]

Notice how the odd-powered term of the integrand is eliminated by symmetry. Integration of the even-powered term is simplified because the lower limit is zero.

b. The \( \cos x \) term is an even function, so it can be integrated on the interval \([0, \pi/2]\).

What about \( \sin^3 x \)? It is an odd function raised to an odd power, which results in an odd function; its integral on \([-\pi/2, \pi/2]\) is zero. Therefore,

\[
\int_{-\pi/2}^{\pi/2} (\cos x - 4 \sin^3 x) \, dx = 2 \int_{0}^{\pi/2} \cos x \, dx - 0 \quad \text{Symmetry}
\]

\[
= 2 \sin x \bigg|_{0}^{\pi/2} \quad \text{Fundamental Theorem}
\]

\[
= 2(1 - 0) = 2. \quad \text{Simplify.}
\]

Related Exercises 7–20

Average Value of a Function

If five people weigh 155, 143, 180, 105, and 123 lb, their average (mean) weight is

\[
\frac{155 + 143 + 180 + 105 + 123}{5} = 141.2 \text{ lb.}
\]

This idea generalizes quite naturally to functions. Consider a function \( f \) that is continuous on \([a, b]\). Using a regular partition \( x_0 = a, x_1, x_2, \ldots, x_n = b \) with \( \Delta x = \frac{b - a}{n} \), we select a point \( x_k^* \) in each subinterval and compute \( f(x_k^*) \), for \( k = 1, \ldots, n \). The values of \( f(x_k^*) \) may be viewed as a sampling of \( f \) on \([a, b]\). The average of these function values is

\[
\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}.
\]

Noting that \( n = \frac{b - a}{\Delta x} \), we write the average of the \( n \) sample values as the Riemann sum

\[
\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{(b - a)/\Delta x} = \frac{1}{b - a} \sum_{k=1}^{n} f(x_k^*) \Delta x.
\]

Now suppose we increase \( n \), taking more and more samples of \( f \), while \( \Delta x \) decreases to zero. The limit of this sum is a definite integral that gives the average value \( \bar{f} \) on \([a, b]\):

\[
\bar{f} = \frac{1}{b - a} \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x
\]

\[
= \frac{1}{b - a} \int_{a}^{b} f(x) \, dx.
\]

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This definition of the average value of a function is analogous to the definition of the average of a finite set of numbers.

**DEFINITION** Average Value of a Function

The average value of an integrable function \( f \) on the interval \([a, b]\) is

\[
\overline{f} = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

The average value of a function \( f \) on an interval \([a, b]\) has a clear geometrical interpretation. Multiplying both sides of the definition of average value by \( \frac{1}{b-a} \), we have

\[
\frac{1}{b-a} \int_a^b f(x) \, dx = \frac{1}{b-a} \left( \int_a^b f(x) \, dx \right).
\]

We see that \( \overline{f} \) is the height of a rectangle with base \([a, b]\), and that rectangle has the same net area as the region bounded by the graph of \( f \) on the interval \([a, b]\) (Figure 5.50). Note that \( \overline{f} \) may be zero or negative.

**Quick Check 2** What is the average value of a constant function on an interval? What is the average value of an odd function on an interval \([-a, a]\)?

**Example 2** Average elevation A hiking trail has an elevation given by

\[
f(x) = 60x^3 - 650x^2 + 1200x + 4500,
\]

where \( f \) is measured in feet above sea level and \( x \) represents horizontal distance along the trail in miles, with \( 0 \leq x \leq 5 \). What is the average elevation of the trail?

**Solution** The trail ranges between elevations of about 2000 and 5000 ft (Figure 5.51). If we let the endpoints of the trail correspond to the horizontal distances \( a = 0 \) and \( b = 5 \), the average elevation of the trail in feet is

\[
\overline{f} = \frac{1}{5} \int_0^5 (60x^3 - 650x^2 + 1200x + 4500) \, dx
\]

\[
= \frac{1}{5} \left[ 60 \frac{x^4}{4} - 650 \frac{x^3}{3} + 1200 \frac{x^2}{2} + 4500x \right]_0^5
\]

Simplify.

The average elevation of the trail is slightly less than 3960 ft.

**Mean Value Theorem for Integrals**

The average value of a function brings us close to an important theoretical result. The Mean Value Theorem for Integrals says that if \( f \) is continuous on \([a, b]\), then there is at least one point \( c \) in the interval \((a, b)\) such that \( f(c) \) equals the average value of \( f \) on the interval \([a, b]\).
In other words, the horizontal line \( y = f \) intersects the graph of \( f \) for some point \( c \) in \((a, b)\) (Figure 5.52). If \( f \) were not continuous, such a point might not exist.

**THEOREM 5.5** Mean Value Theorem for Integrals
Let \( f \) be continuous on the interval \([a, b]\). There exists a point \( c \) in \((a, b)\) such that

\[
f(c) = \bar{f} = \frac{1}{b-a} \int_a^b f(t) \, dt.
\]

**Proof:** We begin by letting \( F(x) = \int_a^x f(t) \, dt \) and noting that \( F \) is continuous on \([a, b]\) and differentiable on \((a, b)\) (by Theorem 5.3, Part 1). We now apply the Mean Value Theorem for derivatives (Theorem 4.9) to \( F \) and conclude that there exists at least one point \( c \) in \((a, b)\) such that

\[
F'(c) = \frac{F(b) - F(a)}{b - a} = \frac{f(c)}{b - a}.
\]

By Theorem 5.3, Part 1, we know that \( F'(c) = f(c) \), and by Theorem 5.3, Part 2, we know that

\[
F(b) - F(a) = \int_a^b f(t) \, dt.
\]

Combining these observations, we have

\[
f(c) = \frac{1}{b - a} \int_a^b f(t) \, dt,
\]

where \( c \) is a point in \((a, b)\).

**Quick Check 3** Explain why \( f(x) = 0 \) for at least one point of \((a, b)\) if \( f \) is continuous and \( \int_a^b f(x) \, dx = 0 \).

**Example 3** Average value equals function value Find the point(s) on the interval \((0, 1)\) at which \( f(x) = 2x(1 - x) \) equals its average value on \([0, 1]\).

**Solution** The average value of \( f \) on \([0, 1]\) is

\[
\bar{f} = \frac{1}{1 - 0} \int_0^1 2x(1 - x) \, dx = \left. \left( x^2 - \frac{2}{3} x^3 \right) \right|_0^1 = \frac{1}{3}.
\]
We must find the points on \((0, 1)\) at which \(f(x) = \frac{1}{2}\) (Figure 5.53). Using the quadratic formula, the two solutions of \(f(x) = 2x(1 - x) = \frac{1}{2}\) are
\[
\frac{1 - \sqrt{1/3}}{2} \approx 0.211 \quad \text{and} \quad \frac{1 + \sqrt{1/3}}{2} \approx 0.789.
\]
These two points are located symmetrically on either side of \(x = \frac{1}{2}\): The two solutions, 0.211 and 0.789, are the same for \(f(x) = ax(1 - x)\) for any nonzero value of \(a\) (Exercise 57).

**Related Exercises 35–40**
35–40. **Mean Value Theorem for Integrals** Find or approximate all points at which the given function equals its average value on the given interval.

35. \( f(x) = 8 - 2x \) on \([0, 4]\)
36. \( f(x) = e^x \) on \([0, 2]\)
37. \( f(x) = 1 - x^2/a^2 \) on \([0, a]\), where \( a \) is a positive real number
38. \( f(x) = \frac{\pi}{4} \sin x \) on \([0, \pi]\)
39. \( f(x) = 1 - |x| \) on \([-1, 1]\)
40. \( f(x) = 1/x \) on \([1, 4]\)

**Further Explorations**

41. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.
   a. If \( f \) is symmetric about the line \( x = 2 \), then \( \int_{a}^{b} f(x) \, dx = 2 \int_{0}^{2} f(x) \, dx \).
   b. If \( f \) has the property \( f(a + x) = -f(a - x) \), for all \( x \), where \( a \) is a constant, then \( \int_{-a}^{a} f(x) \, dx = 0 \).
   c. The average value of a linear function on an interval \([a, b]\) is the function value at the midpoint of \([a, b]\).
   d. Consider the function \( f(x) = x(a - x) \) on the interval \([0, a]\), for \( a > 0 \). Its average value on \([0, a]\) is \( \frac{a}{4} \) of its maximum value.

42–45. **Symmetry in integrals** Use symmetry to evaluate the following integrals.

42. \( \int_{-\pi/4}^{\pi/4} \tan x \, dx \)
43. \( \int_{-\pi/4}^{\pi/4} \sec^2 x \, dx \)
44. \( \int_{-2}^{2} (1 - |x|^3) \, dx \)
45. \( \int_{-2}^{2} \frac{x^3 - 4x}{x^3 + 1} \, dx \)

**Applications**

46. **Root mean square** The root mean square (or RMS) is another measure of average value, often used with oscillating functions (for example, sine and cosine functions that describe the current, voltage, or power in an alternating circuit). The RMS of a function \( f \) on the interval \([0, T]\) is

\[
\mathcal{F}_{\text{RMS}} = \sqrt{\frac{1}{T} \int_{0}^{T} [f(t)]^2 \, dt}.
\]

Compute the RMS of \( f(t) = A \sin (\omega t) \), where \( A \) and \( \omega \) are positive constants and \( T \) is any integer multiple of the period of \( f \), which is \( 2\pi/\omega \).

47. **Gateway Arch** The Gateway Arch in St. Louis is 630 ft high and has a 630-ft base. Its shape can be modeled by the parabola

\[
y = 630 \left( 1 - \left( \frac{x}{315} \right)^2 \right).
\]

Find the average height of the arch above the ground.

48. **Another Gateway Arch** Another description of the Gateway Arch is

\[
y = 1260 - 315(e^{-0.00418t} + e^{0.00418t}),
\]

where the base of the arch is \([-315, 315]\) and \( x \) and \( y \) are measured in feet. Find the average height of the arch above the ground.

49. **Planetary orbits** The planets orbit the Sun in elliptical orbits with the Sun at one focus (see Section 10.4 for more on ellipses). The equation of an ellipse whose dimensions are \( 2a \) in the \( x \)-direction and \( 2b \) in the \( y \)-direction is \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).
   a. Let \( d^2 \) denote the square of the distance from a planet to the center of the ellipse at \((0, 0)\). Integrate over the interval \([-a, a]\) to show that the average value of \( d^2 \) is \( (a^2 + 2b^2)/3 \).
   b. Show that in the case of a circle \((a = b = R)\), the average value in part (a) is \( R^2 \).
   c. Assuming \( 0 < b < a \), the coordinates of the Sun are \((\sqrt{a^2 - b^2}, 0)\). Let \( D^2 \) denote the square of the distance from the planet to the Sun. Integrate over the interval \([-a, a]\) to show that the average value of \( D^2 \) is \( (4a^2 - b^2)/3 \).
5.4 Working with Integrals

Additional Exercises

50. Comparing a sine and a quadratic function Consider the functions \( f(x) = \sin x \) and \( g(x) = \frac{4}{\pi^2}x(\pi - x) \).
   a. Carefully graph \( f \) and \( g \) on the same set of axes. Verify that both functions have a single local maximum on the interval \([0, \pi]\) and that they have the same maximum value on \([0, \pi]\).
   b. On the interval \([0, \pi]\), which is true: \( f(x) \geq g(x) \), \( g(x) \geq f(x) \), or neither?
   c. Compute and compare the average values of \( f \) and \( g \) on \([0, \pi]\).

51. Using symmetry Suppose \( f \) is an even function and \( \int_{-a}^{a} f(x) \, dx = 18 \).
   a. Evaluate \( \int_{-8}^{8} f(x) \, dx \)
   b. Evaluate \( \int_{-8}^{8} xf(x) \, dx \)

52. Using symmetry Suppose \( f \) is an odd function, \( \int_{0}^{a} f(x) \, dx = 3 \), and \( \int_{-a}^{a} f(x) \, dx = 9 \).
   a. Evaluate \( \int_{-a}^{a} f(x) \, dx \)
   b. Evaluate \( \int_{-a}^{a} f(x) \, dx \)

53–56. Symmetry of composite functions Prove that the integrand is either even or odd. Then give the value of the integral or show how it can be simplified. Assume that \( f \) and \( g \) are even functions and \( p \) and \( q \) are odd functions.

53. \( \int_{-a}^{a} f(g(x)) \, dx \)
54. \( \int_{-a}^{a} f(p(x)) \, dx \)
55. \( \int_{-a}^{a} p(g(x)) \, dx \)
56. \( \int_{-a}^{a} p(q(x)) \, dx \)

57. Average value with a parameter Consider the function \( f(x) = ax(1 - x) \) on the interval \([0, 1]\), where \( a \) is a positive real number.
   a. Find the average value of \( f \) as a function of \( a \).
   b. Find the points at which the value of \( f \) equals its average value and prove that they are independent of \( a \).

58. Square of the average For what polynomials \( f \) is it true that the square of the average value of \( f \) equals the average value of the square of \( f \) over all intervals \([a, b]\)?

59. Problems of antiquity Several calculus problems were solved by Greek mathematicians long before the discovery of calculus. The following problems were solved by Archimedes using methods that predated calculus by 2000 years.
   a. Show that the area of a segment of a parabola is \( \frac{1}{3} \) that of its inscribed triangle of greatest area. In other words, the area bounded by the parabola \( y = a^2 - x^2 \) and the \( x \)-axis is \( \frac{1}{3} \) the area of the triangle with vertices \(( \pm a, 0)\) and \((0, a^2)\). Assume that \( a > 0 \) but is unspecified.
   b. Show that the area bounded by the parabola \( y = a^2 - x^2 \) and the \( x \)-axis is \( \frac{1}{2} \) the area of the rectangle with vertices \(( \pm a, 0)\) and \(( \pm a, a^2)\). Assume that \( a > 0 \) but is unspecified.

60. Unit area sine curve Find the value of \( c \) such that the region bounded by \( y = c \sin x \) and the \( x \)-axis on the interval \([0, \pi]\) has area 1.

61. Unit area cubic Find the value of \( c > 0 \) such that the region bounded by the cubic \( y = x(x - c)^2 \) and the \( x \)-axis on the interval \([0, c]\) has area 1.

62. Unit area
   a. Consider the curve \( y = 1/x \), for \( x \approx 1 \). For what value of \( b > 0 \) does the region bounded by this curve and the \( x \)-axis on the interval \([1, b]\) have an area of 1?
   b. Consider the curve \( y = 1/x^p \), where \( x \approx 1 \), and \( p < 2 \) with \( p \neq 1 \). For what value of \( b \) (as a function of \( p \)) does the region bounded by this curve and the \( x \)-axis on the interval \([1, b]\) have unit area?
   c. Is \( b(p) \) in part (b) an increasing or decreasing function of \( p \)? Explain.

63. A sine integral by Riemann sums Consider the integral \( I = \int_{0}^{\pi/2} \sin x \, dx \).
   a. Write the left Riemann sum for \( I \) with \( n \) subintervals.
   b. Show that \( \lim_{\varepsilon \to 0} \left( \frac{\cos \varepsilon + \sin \varepsilon - 1}{2(1 - \cos \varepsilon)} \right) = 1 \).
   c. It is a fact that \( \sum_{k=0}^{n-1} \sin \left( \frac{\pi k}{2n} \right) = \frac{\cos \left( \frac{\pi}{2n} \right) + \sin \left( \frac{\pi}{2n} \right) - 1}{2(1 - \cos \left( \frac{\pi}{2n} \right))} \)

Use this fact and part (b) to evaluate \( I \) by taking the limit of the Riemann sum as \( n \to \infty \).

64. Alternate definitions of means Consider the function \( f(t) = \frac{\int_{a}^{b} x^{r+1} \, dx}{\int_{a}^{b} x^{r} \, dx} \).
   Show that the following means can be defined in terms of \( f \).
   a. Arithmetic mean: \( f(0) = \frac{a + b}{2} \)
   b. Geometric mean: \( f\left( \frac{3}{2} \right) = \sqrt{ab} \)
   c. Harmonic mean: \( f(-3) = \frac{2ab}{a + b} \)
   d. Logarithmic mean: \( f(-1) = \frac{b - a}{\ln b - \ln a} \)

(Source: Mathematics Magazine 78, 5, Dec 2005)

65. Symmetry of powers Fill in the following table with either even or odd, and prove each result. Assume \( n \) is a nonnegative integer and \( f^n \) means the \( n \)th power of \( f \).

<table>
<thead>
<tr>
<th>( f ) is even</th>
<th>( f ) is odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n ) is even</td>
<td>( f^n ) is ____</td>
</tr>
<tr>
<td>( n ) is odd</td>
<td>( f^n ) is ____</td>
</tr>
</tbody>
</table>
66. **Average value of the derivative** Suppose that \( f' \) is a continuous function for all real numbers. Show that the average value of the derivative on an interval \([a, b]\) is \( f' = \frac{f(b) - f(a)}{b - a} \). Interpret this result in terms of secant lines.

67. **Symmetry about a point** A function \( f \) is symmetric about a point \((c, d)\) if whenever \((c - x, d - y)\) is on the graph, then so is \((c + x, d + y)\). Functions that are symmetric about a point \((c, d)\) are easily integrated on an interval with midpoint \( c \).

a. Show that if \( f \) is symmetric about \((c, d)\) and \( a > 0 \), then \( \int_{-a}^{a} f(x) \, dx = 2a f(c) = 2ad \).

b. Graph the function \( f(x) = \sin^2 x \) on the interval \([0, \pi/2]\) and show that the function is symmetric about the point \((\frac{\pi}{4}, \frac{1}{2})\).

c. Using only the graph of \( f \) (and no integration), show that \( \int_{0}^{\pi/2} \sin^2 x \, dx = \frac{\pi}{4} \) (See the Guided Project Symmetry in Integrals.)

68. **Bounds on an integral** Suppose \( f \) is continuous on \([a, b]\) with \( f''(x) > 0 \) on the interval. It can be shown that

\[
(b - a) f \left( \frac{a + b}{2} \right) \leq \int_{a}^{b} f(x) \, dx \leq (b - a) \frac{f(a) + f(b)}{2}.
\]

a. Assuming \( f \) is nonnegative on \([a, b]\), draw a figure to illustrate the geometric meaning of these inequalities. Discuss your conclusions.

b. Divide these inequalities by \((b - a)\) and interpret the resulting inequalities in terms of the average value of \( f \) on \([a, b]\).

69. **Generalizing the Mean Value Theorem for Integrals** Suppose \( f \) and \( g \) are continuous on \([a, b]\) and let

\[
h(x) = (x - b) \int_{a}^{f(t)} dt + (x - a) \int_{c}^{g(t)} dt.
\]

a. Use Rolle’s theorem to show that there is a number \( c \) in \((a, b)\) such that

\[
\int_{a}^{c} f(t) \, dt + \int_{c}^{b} g(t) \, dt = f(c)(b - c) + g(c)(c - a),
\]

which is a generalization of the Mean Value Theorem for Integrals.

b. Show that there is a number \( c \) in \((a, b)\) such that

\[
\int_{a}^{b} f(t) \, dt = f(c)(b - c).
\]

c. Use a sketch to interpret part (b) geometrically.

d. Use the result of part (a) to give an alternate proof of the Mean Value Theorem for Integrals.


**QUICK CHECK ANSWERS**

1. \( f(-x)g(-x) = f(x)g(x) \); therefore, \( fg \) is even.
2. The average value is the constant; the average value is 0.
3. The average value is zero on the interval; by the Mean Value Theorem for Integrals, \( f(x) = 0 \) at some point on the interval.

### 5.5 Substitution Rule

Given just about any differentiable function, with enough know-how and persistence, you can compute its derivative. But the same cannot be said of antiderivatives. Many functions, even relatively simple ones, do not have antiderivatives that can be expressed in terms of familiar functions. Examples are \( \sin x^2 \), \( (\sin x)/x \), and \( x^x \). The immediate goal of this section is to enlarge the family of functions for which we can find antiderivatives. This campaign resumes in Chapter 7, where additional integration methods are developed.

**Indefinite Integrals**

One way to find new antiderivative rules is to start with familiar derivative rules and work backward. When applied to the Chain Rule, this strategy leads to the Substitution Rule. A few examples illustrate the technique.

**EXAMPLE 1** Antiderivatives by trial and error Find \( \int \cos 2x \, dx \).

**SOLUTION** The closest familiar indefinite integral related to this problem is

\[
\int \cos x \, dx = \sin x + C,
\]

which is true because

\[
\frac{d}{dx}(\sin x + C) = \cos x.
\]

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Therefore, we might incorrectly conclude that the indefinite integral of \(\cos 2x\) is 
\[\sin 2x + C.\]
However, by the Chain Rule,
\[
\frac{d}{dx}(\sin 2x + C) = 2 \cos 2x \neq \cos 2x.
\]

Note that \(\sin 2x\) fails to be an antiderivative of \(\cos 2x\) by a multiplicative factor of 2. A small adjustment corrects this problem. Let’s try \(\frac{1}{2} \sin 2x:\)

\[
\frac{d}{dx}\left(\frac{1}{2} \sin 2x\right) = \frac{1}{2} \cdot 2 \cos 2x = \cos 2x.
\]

It works! So we have
\[
\int \cos 2x \, dx = \frac{1}{2} \sin 2x + C.
\]

The trial-and-error approach of Example 1 is impractical for complicated integrals. To develop a systematic method, consider a composite function \(F(g(x))\), where \(F\) is an antiderivative of \(f\); that is, \(F' = f\). Using the Chain Rule to differentiate the composite function \(F(g(x))\), we find that

\[
\frac{d}{dx}\left(F(g(x))\right) = F'(g(x))g'(x) = f(g(x))g'(x).
\]

This equation says that \(F(g(x))\) is an antiderivative of \(f(g(x))g'(x)\), which is written

\[
\int f(g(x))g'(x) \, dx = F(g(x)) + C, \quad (1)
\]

where \(F\) is any antiderivative of \(f\).

Why is this approach called the **Substitution Rule (or Change of Variables Rule)**?

In the composite function \(f(g(x))\) in equation (1), we identify the “inner function” as \(u = g(x)\), which implies that \(du = g'(x) \, dx\). Making this identification, the integral in equation (1) is written

\[
\int f(g(x))g'(x) \, dx = \int f(u) \, du = F(u) + C.
\]

We see that the integral \(\int f(g(x))g'(x) \, dx\) with respect to \(x\) is replaced with a new integral \(\int f(u) \, du\) with respect to the new variable \(u\). In other words, we have substituted the new variable \(u\) for the old variable \(x\). Of course, if the new integral with respect to \(u\) is no easier to find than the original integral, then the change of variables has not helped. The Substitution Rule requires plenty of practice until certain patterns become familiar.

**THEOREM 5.6** **Substitution Rule for Indefinite Integrals**

Let \(u = g(x)\), where \(g'\) is continuous on an interval, and let \(f\) be continuous on the corresponding range of \(g\). On that interval,

\[
\int f(g(x))g'(x) \, dx = \int f(u) \, du.
\]

In practice, Theorem 5.6 is applied using the following procedure.
Chapter 5 • Integration

### Example 2  Perfect Substitutions

Use the Substitution Rule to find the following indefinite integrals. Check your work by differentiating.

#### a.

\[ \int 2(2x + 1)^3 \, dx \]

**Solution**

We identify \( u = 2x + 1 \) as the inner function of the composite function \((2x + 1)^3\).

Therefore, we choose the new variable \( u = 2x + 1 \), which implies that \( du = 2 \, dx \). Notice that \( du = 2 \, dx \) appears as a factor in the integrand. The change of variables looks like this:

\[
\int (2x + 1)^3 \cdot 2 \, dx = \int u^3 \, du \quad \text{Substitute } u = 2x + 1, \, du = 2 \, dx.
\]

\[
= \frac{u^4}{4} + C \quad \text{Antiderivative}
\]

\[
= \frac{(2x + 1)^4}{4} + C. \quad \text{Replace } u \text{ with } 2x + 1.
\]

Notice that the final step uses \( u = 2x + 1 \) to return to the original variable.

#### b.

\[ \int 10e^{10x} \, dx \]

The composite function \( e^{10x} \) has the inner function \( u = 10x \), which implies that \( du = 10 \, dx \). The change of variables appears as

\[
\int e^{10x} \cdot 10 \, dx = \int e^{u} \, du \quad \text{Substitute } u = 10x, \, du = 10 \, dx.
\]

\[
= e^u + C \quad \text{Antiderivative}
\]

\[
= e^{10x} + C. \quad \text{Replace } u \text{ with } 10x.
\]

In checking, we see that \( \frac{d}{dx}(e^{10x} + C) = e^{10x} \cdot 10 = 10e^{10x} \).

**Related Exercises 13–16**

#### Quick Check

Find a new variable \( u \) so that

\[ \int 4x^3(x^4 + 5)^{10} \, dx = \int u^{10} \, du. \]

Most substitutions are not perfect. The remaining examples show more typical situations that require introducing a constant factor.

### Example 3  Introducing a Constant

Find the following indefinite integrals.

#### a.

\[ \int x^4(x^5 + 6)^9 \, dx \]

#### b.

\[ \int \cos^3 x \sin x \, dx \]
SOLUTION

a. The inner function of the composite function \((x^5 + 6)^9\) is \(x^5 + 6\) and its derivative \(5x^4\) also appears in the integrand (up to a multiplicative factor). Therefore, we use the substitution \(u = x^5 + 6\), which implies that \(du = 5x^4\,dx\), or \(x^4\,dx = \frac{1}{5}\,du\). By the Substitution Rule,

\[
\int (x^5 + 6)^9 x^4 \,dx = \int \frac{1}{5} \,du
\]

Substitute \(u = x^5 + 6\).

\[
\frac{1}{5} \int ud\!u = \frac{1}{5} \int f(x) \,dx = \frac{1}{5} \int f(x) \,dx
\]

\[
\frac{1}{5} \int ud\!u = \frac{1}{5} \int ud\!u = \frac{1}{5} \int ud\!u
\]

Antiderivative

\[
= \frac{1}{50} (x^5 + 6)^{10} + C
\]

Replace \(u\) with \(x^5 + 6\).

b. The integrand can be written as \((-\cos x)^3 \sin x\). The inner function in the composition \((-\cos x)^3\) is \(-\cos x\), which suggests the substitution \(u = -\cos x\). Note that \(du = -\sin x\,dx\) or \(\sin x\,dx = -du\). The change of variables appears as

\[
\int \frac{\cos^3 x \sin x}{\cos x} \,dx = \int \frac{\cos^3 x}{\cos x} \,du = \int \frac{\cos^4 x}{\cos x} \,du
\]

Substitute \(u = -\cos x\), \(du = -\sin x\,dx\).

\[
= \frac{-u^4}{4} + C
\]

Antiderivative

\[
= \frac{-\cos^5 x}{4} + C
\]

Replace \(u\) with \(-\cos x\).

Related Exercises 17–32

Quick Check 2 In Example 3a, explain why the same substitution would not work as well for the integral \(\int x^3(x^5 + 6)^7\,dx\).

Sometimes the choice for a \(u\)-substitution is not so obvious or more than one \(u\)-substitution works. The following example illustrates both of these points.

Example 4 Variations on the substitution method Find \(\int \frac{x}{\sqrt{x+1}} \,dx\).

SOLUTION

Substitution 1 The composite function \(\sqrt{x+1}\) suggests the new variable \(u = x + 1\). You might doubt whether this choice will work because \(du = dx\), which leaves the \(x\) in the numerator of the integrand unaccounted for. But let’s proceed. Letting \(u = x + 1\), we have \(x = u - 1\), \(du = dx\), and

\[
\int \frac{x}{\sqrt{x+1}} \,dx = \int \frac{u - 1}{\sqrt{u}} \,du
\]

Substitute \(u = x + 1\), \(du = dx\).

\[
= \int \left( \sqrt{u} - \frac{1}{\sqrt{u}} \right) \,du
\]

Rewrite integrand.

\[
= \int \left( u^{1/2} - u^{-1/2} \right) \,du
\]

Fractional powers

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We integrate each term individually and then return to the original variable $x$:

$$
\int \left( u^{1/2} - u^{-1/2} \right) du = \frac{2}{3} u^{3/2} - 2u^{1/2} + C \quad \text{Antiderivatives}
$$

$$
= \frac{2}{3} (x + 1)^{3/2} - 2(x + 1)^{1/2} + C \quad \text{Replace } u \text{ with } x + 1.
$$

$$
= \frac{2}{3} (x + 1)^{1/2} (x - 2) + C. \quad \text{Factor out } (x + 1)^{1/2} \text{ and simplify.}
$$

> In Substitution 2, you could also use the fact that

$$
u'(x) = \frac{1}{2\sqrt{x + 1}},
$$

which implies

$$
\frac{du}{dx} = \frac{1}{2\sqrt{x + 1}} \quad \text{or } du = \frac{1}{2\sqrt{x + 1}} \, dx.
$$

**Substitution 2**  Another possible substitution is $u = \sqrt{x + 1}$. Now $u^2 = x + 1$, $x = u^2 - 1$, and $dx = 2u \, du$. Making these substitutions leads to

$$
\int \frac{x}{\sqrt{x + 1}} \, dx = \int \frac{u^2 - 1}{u} \, 2u \, du
$$

Substitute $u = \sqrt{x + 1}$, $x = u^2 - 1$.

$$
= 2 \int \left( u^2 - 1 \right) \, du
$$

Simplify the integrand.

$$
= 2 \left( \frac{u^3}{3} - u \right) + C \quad \text{Antiderivatives}
$$

$$
= \frac{2}{3} (x + 1)^{3/2} - 2(x + 1)^{1/2} + C \quad \text{Replace } u \text{ with } \sqrt{x + 1}.
$$

$$
= \frac{2}{3} (x + 1)^{1/2} (x - 2) + C. \quad \text{Factor out } (x + 1)^{1/2} \text{ and simplify.}
$$

Observe that the same indefinite integral is found using either substitution.

*Related Exercises 33–38*

**Definite Integrals**

The Substitution Rule is also used for definite integrals; in fact, there are two ways to proceed.

- You may use the Substitution Rule to find an antiderivative $F$ and then use the Fundamental Theorem to evaluate $F(b) - F(a)$.

- Alternatively, once you have changed variables from $x$ to $u$, you also may change the limits of integration and complete the integration with respect to $u$. Specifically, if $u = g(x)$, the lower limit $x = a$ is replaced with $u = g(a)$ and the upper limit $x = b$ is replaced with $u = g(b)$.

The second option tends to be more efficient, and we use it whenever possible. This approach is summarized in the following theorem, which is then applied to several definite integrals.

**Theorem 5.7**  Substitution Rule for Definite Integrals

Let $u = g(x)$, where $g'$ is continuous on $[a, b]$, and let $f$ be continuous on the range of $g$. Then

$$
\int_a^b f(g(x)) g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.
$$

**Example 5**  Definite integrals  Evaluate the following integrals.

a. $\int_0^2 \frac{dx}{(x + 3)^3}$

b. $\int_0^4 \frac{x}{x^2 + 1} \, dx$

c. $\int_0^{\pi/2} \sin^4 x \cos x \, dx$
5.5 Substitution Rule

**SOLUTION**

a. Let the new variable be \( u = x + 3 \) and then \( du = dx \). Because we have changed the variable of integration from \( x \) to \( u \), the limits of integration must also be expressed in terms of \( u \). In this case,

\[
\begin{align*}
&x = 0 \text{ implies } u = 0 + 3 = 3, \quad \text{Lower limit} \\
&x = 2 \text{ implies } u = 2 + 3 = 5. \quad \text{Upper limit}
\end{align*}
\]

The entire integration is carried out as follows:

\[
\int_{0}^{2} \frac{dx}{(x + 3)^3} = \int_{3}^{6} u^{-3} \, du \quad \text{Substitute } u = x + 3, \, du = dx.
\]

\[
= \frac{u^{-2}}{2} \Bigg|_{3}^{5} \quad \text{Fundamental Theorem}
\]

\[
= \frac{1}{2} \left( 5^{-2} - 3^{-2} \right) = \frac{8}{225} \quad \text{Simplify.}
\]

b. Notice that a multiple of the derivative of the denominator appears in the numerator; therefore, we let \( u = x^2 + 1 \), which implies that \( du = 2x \, dx \), or \( x \, dx = \frac{1}{2} du \). Changing limits of integration,

\[
\begin{align*}
&x = 0 \text{ implies } u = 0 + 1 = 1, \quad \text{Lower limit} \\
&x = 4 \text{ implies } u = 4^2 + 1 = 17. \quad \text{Upper limit}
\end{align*}
\]

Changing variables, we have

\[
\int_{0}^{4} \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \int_{1}^{17} u^{-1} \, du \quad \text{Substitute } u = x^2 + 1, \, du = 2x \, dx.
\]

\[
= \frac{1}{2} \ln |u| \Bigg|_{1}^{17} \quad \text{Fundamental Theorem}
\]

\[
= \frac{1}{2} \left( \ln 17 - \ln 1 \right) \quad \text{Simplify.}
\]

\[
= \frac{1}{2} \ln 17 \approx 1.417. \quad \ln 1 = 0
\]

\[
\begin{align*}
&x = \sin x, \text{ which implies that } du = \cos x \, dx. \text{ The lower limit of integration becomes } u = 0 \text{ and the upper limit becomes } u = 1. \text{ Changing variables, we have}
\end{align*}
\]

\[
\int_{0}^{\pi/2} \sin^4 x \cos x \, dx = \int_{0}^{1} u \, du \quad u = \sin x, \, du = \cos x \, dx
\]

\[
= \left( \frac{u^5}{5} \right) \bigg|_{0}^{1} = \frac{1}{5} \quad \text{Fundamental Theorem}
\]

\[
\begin{align*}
\text{Related Exercises 39–52} \quad \Rightarrow
\end{align*}
\]

The Substitution Rule enables us to find two standard integrals that appear frequently in practice, \( \int \sin^2 x \, dx \) and \( \int \cos^2 x \, dx \). These integrals are handled using the identities

\[
\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}.
\]
EXAMPLE 6  Integral of \( \cos^2 \theta \)  Evaluate \( \int_{0}^{\pi/2} \cos^2 \theta \, d\theta \).

SOLUTION  Working with the indefinite integral first, we use the identity for \( \cos^2 \theta \):

\[
\int \cos^2 \theta \, d\theta = \int \left( \frac{1 + \cos 2\theta}{2} \right) d\theta = \frac{1}{2} \int d\theta + \frac{1}{2} \int \cos 2\theta \, d\theta.
\]

The change of variables \( u = 2\theta \) (or Table 4.9) is now used for the second integral, and we have

\[
\int \cos^2 \theta \, d\theta = \frac{1}{2} \int d\theta + \frac{1}{2} \int \cos 2\theta \, d\theta = \frac{1}{2} \int d\theta + \frac{1}{2} \cdot \frac{1}{2} \int \cos u \, du = \frac{\theta}{2} + \frac{1}{4} \sin 2\theta + C.
\]

Evaluating integrals; \( u = 2\theta \).

Using the Fundamental Theorem of Calculus, the value of the definite integral is

\[
\int_{0}^{\pi/2} \cos^2 \theta \, d\theta = \left[ \frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right]_{0}^{\pi/2} = \frac{\pi}{4}.
\]

\[\text{Related Exercises 53–60} \]

Geometry of Substitution

The Substitution Rule has a geometric interpretation. To keep matters simple, consider the integral \( \int_{0}^{\pi/2} 2(2x + 1) \, dx \). The graph of the integrand \( y = 2(2x + 1) \) on the interval \([0, 2]\) is shown in Figure 5.54a, along with the region \( R \) whose area is given by the integral. The change of variables \( u = 2x + 1, \, du = 2 \, dx \), \( u(0) = 1, \, u(2) = 5 \) leads to the new integral

\[
\int_{0}^{5} 2(2x + 1) \, dx = \int_{1}^{5} u \, du.
\]

Figure 5.54b also shows the graph of the new integrand \( y = u \) on the interval \([1, 5]\) and the region \( R' \) whose area is given by the new integral. You can check that the areas of \( R \) and \( R' \) are equal. An analogous interpretation may be given to more complicated integrands and substitutions.

QUICK CHECK 3  Changes of variables occur frequently in mathematics. For example, suppose you want to solve the equation \( x^4 - 13x^2 + 36 = 0 \). If you use the substitution \( u = x^2 \), what is the new equation that must be solved for \( u \)? What are the roots of the original equation?

SECTION 5.5 EXERCISES

Review Questions

1. On which derivative rule is the Substitution Rule based?
2. Why is the Substitution Rule referred to as a change of variables?
3. The composite function \( f(g(x)) \) consists of an inner function \( g \) and an outer function \( f \). If an integrand includes \( f(g(x)) \), which function is often a likely choice for a new variable \( u \)?
4. Find a suitable substitution for evaluating \( \int \tan x \sec^2 x \, dx \) and explain your choice.

5. When using a change of variables \( u = g(x) \) to evaluate the definite integral \( \int_{a}^{b} f(g(x)) \, g'(x) \, dx \), how are the limits of integration transformed?
6. If the change of variables \( u = x^2 - 4 \) is used to evaluate the definite integral \( \int_{5}^{7} f(x) \, dx \), what are the new limits of integration?
7. Find \( \int \cos^2 x \, dx \).
8. What identity is needed to find \( \int \sin^2 x \, dx \)?
Basic Skills

9–12. Trial and error Find an antiderivative of the following functions by trial and error. Check your answer by differentiating.

9. \( f(x) = (x + 1)^{12} \)  
10. \( f(x) = e^{3x+1} \)  
11. \( f(x) = \sqrt{2x+1} \)  
12. \( f(x) = \cos(2x+5) \)

13–16. Substitution given Use the given substitution to find the following indefinite integrals. Check your answer by differentiating.

13. \( \int 2x(x^2 + 1)^4 \, dx, \quad u = x^2 + 1 \)  
14. \( \int 8x \cos(4x^2 + 3) \, dx, \quad u = 4x^2 + 3 \)  
15. \( \int \sin^3 x \cos x \, dx, \quad u = \sin x \)  
16. \( \int (6x + 1) \sqrt{3x^2 + x} \, dx, \quad u = 3x^2 + x \)

17–32. Indefinite integrals Use a change of variables to find the following indefinite integrals. Check your work by differentiating.

17. \( \int 2x(x^2 - 1)^9 \, dx \)  
18. \( \int xe^2 \, dx \)  
19. \( \int \frac{2x^3}{\sqrt{1 - 4x^4}} \, dx \)  
20. \( \int \left( \sqrt{x} + 1 \right)^4 \, dx \)  
21. \( \int (x^2 + x)^{10} (2x + 1) \, dx \)  
22. \( \int \frac{1}{10x - 3} \, dx \)  
23. \( \int x^5(x^2 + 16)^6 \, dx \)  
24. \( \int \sin^{10} \theta \cos \theta \, d\theta \)  
25. \( \int \frac{dx}{\sqrt{1 - 9x^2}} \)  
26. \( \int x^9 \sin x^{10} \, dx \)  
27. \( \int (x^6 - 3x^4)^2 (x^5 - x) \, dx \)  
28. \( \int \frac{x}{x - 2} \, dx (\text{Hint: Let } u = x - 2) \)  
29. \( \int \frac{dx}{1 + 4x^2} \)  
30. \( \int \frac{3}{1 + 25y^2} \, dy \)  
31. \( \int \frac{2}{x\sqrt{4x^2 - 1}} \, dx, \quad x > \frac{1}{2} \)  
32. \( \int \frac{8x + 6}{2x^2 + 3x} \, dx \)

33–38. Variations on the substitution method Find the following integrals.

33. \( \int \frac{x}{\sqrt{4 - 4x^2}} \, dx \)  
34. \( \int \frac{y^2}{(y + 1)^4} \, dy \)  
35. \( \int \frac{x}{\sqrt{x + 4}} \, dx \)  
36. \( \int e^x - e^{-x} \, dx \)  
37. \( \int \sqrt{2x + 1} \, dx \)  
38. \( \int (z + 1) \sqrt{3z + 2} \, dz \)

39–52. Definite integrals Use a change of variables to evaluate the following definite integrals.

39. \( \int_0^1 2x(4 - x^2) \, dx \)  
40. \( \int_0^2 \frac{2x}{(x^2 + 1)^2} \, dx \)  
41. \( \int_{\sin^2 \theta}^0 \sin^2 \theta \cos \theta \, d\theta \)  
42. \( \int_0^{\sin^2 \theta} \sin x \, dx \)  
43. \( \int_{-1}^2 x^2e^{x^4} \, dx \)  
44. \( \int_0^3 p \sqrt{9 + p^2} \, dp \)  
45. \( \int_{\sin^2 \theta}^{\sin \theta} \cos x \, dx \)  
46. \( \int_0^{\sin^2 \theta} \sin \theta \cos \theta \, d\theta \)  
47. \( \int_2^{1/\sqrt{5}} \frac{dx}{x\sqrt{25x^2 - 1}} \)  
48. \( \int_0^3 \frac{v^2 + 1}{\sqrt{v^3 + 3v^2 + 4}} \, dv \)  
49. \( \int_0^1 \frac{x}{x^4 + 1} \, dx \)  
50. \( \int_0^{1/\sqrt{5}} \frac{x}{\sqrt{1 - 16x^2}} \, dx \)  
51. \( \int_{1/3}^{4/9} \frac{dx}{x^2 + 1} \)  
52. \( \int_0^{e} \frac{e^x}{3 + 2e^x} \, dx \)

53–60. Integrals with \( \sin^2 x \) and \( \cos^2 x \) Evaluate the following integrals.

53. \( \int_{-\pi}^{\pi} \cos^2 x \, dx \)  
54. \( \int_{0}^{\pi} \sin^2 x \, dx \)  
55. \( \int_{-\pi}^{\pi} \sin^2 \left( \theta + \frac{\pi}{6} \right) \, d\theta \)  
56. \( \int_{0}^{\pi/2} \cos^2 8\theta \, d\theta \)  
57. \( \int_{-\pi/2}^{\pi/2} \sin^2 2\theta \, d\theta \)  
58. \( \int_{0}^{\pi/2} x \cos^2 (x^2) \, dx \)  
59. \( \int_{0}^{\pi/6} \sin^2 y \, dy \)  
60. \( \int_{0}^{\pi/2} \sin^4 \theta \, d\theta \)

Further Explorations

61. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample. Assume that \( f, f', \) and \( f'' \) are continuous functions for all real numbers.

a. \( \int f(x)f'(x) \, dx = \frac{1}{2} (f(x))^2 + C. \)

b. \( \int (f(x))^{n} f'(x) \, dx = \frac{1}{n+1} (f(x))^{n+1} + C, \quad n \neq -1. \)

c. \( \int \sin 2x \, dx = 2 \int \sin x \, dx. \)

d. \( \int (x^2 + 1)^{10} \, dx = \frac{(x^2 + 1)^{10}}{10} + C. \)

e. \( \int_a^b f'(x)f''(x) \, dx = f'(b) - f'(a). \)
62–78. Additional integrals Use a change of variables to evaluate the following integrals.

62. \[ \int \sec 4w \tan 4w \, dw \]
63. \[ \int \sec^2 10x \, dx \]

64. \[ \int (\sin^5 x + 3 \sin^3 x - \sin x) \cos x \, dx \]

65. \[ \int \frac{\csc^2 x}{\cot^2 x} \, dx \]
66. \[ \int (x^{3/2} + 8)^{3/2} \sqrt{x} \, dx \]

67. \[ \int \sin x \sec^8 x \, dx \]
68. \[ \int \frac{e^{2x}}{e^{2x} + 1} \, dx \]

69. \[ \int \frac{x^2}{\sqrt{x^2 - 1}} \, dx \]
70. \[ \int \frac{2^x}{2^x} \, dx \]

71. \[ \int \frac{x}{\sqrt{x^2 - 1}} \, dx \]
72. \[ \int \frac{6^x}{25x^2 + 36} \, dx \]

73. \[ \int \frac{x^2}{\sqrt{x^2 - 1}} \, dx \]
74. \[ \int (x - 1)(x^2 - 2x)^7 \, dx \]

75. \[ \int \frac{\sin x}{x^2 + \cos x} \, dx \]
76. \[ \int (v + 1)(v + 2) \, dv \]

77. \[ \int \frac{4}{9x^2 + 6x + 1} \, dx \]
78. \[ \int e^{\sin x} \sin 2x \, dx \]

79–82. Areas of regions Find the area of the following regions.

79. The region bounded by the graph of \( f(x) = x \sin x^2 \) and the \( x \)-axis between \( x = 0 \) and \( x = \sqrt{\pi} \).

80. The region bounded by the graph of \( f(\theta) = \cos \theta \sin \theta \) and the \( \theta \)-axis between \( \theta = 0 \) and \( \theta = \pi/2 \).

81. The region bounded by the graph of \( f(x) = (x - 4)^4 \) and the \( x \)-axis between \( x = 2 \) and \( x = 6 \).

82. The region bounded by the graph of \( f(x) = \frac{x}{\sqrt{x^2 - 9}} \) and the \( x \)-axis between \( x = 4 \) and \( x = 5 \).

83. Morphing parabolas The family of parabolas \( y = \left(\frac{1}{a}\right) - x^2 / a^3 \), where \( a > 0 \), has the property that for \( x = 0 \), the \( x \)-intercept is \((a, 0)\) and the \( y \)-intercept is \((0, 1/a)\). Let \( A(a) \) be the area of the region in the first quadrant bounded by the parabola and the \( x \)-axis. Find \( A(a) \) and determine whether it is an increasing, decreasing, or constant function of \( a \).

84. Substitutions Suppose that \( f \) is an even function with \( F^b_a f(x) \, dx = 9 \). Evaluate each integral.

a. \[ \int_{-1}^1 x f(x^2) \, dx \]

b. \[ \int_{-2}^2 x^2 f(x^3) \, dx \]

85. Substitutions Suppose that \( p \) is a nonzero real number and \( f \) is an odd integrable function with \( F^b_a f(x) \, dx = \pi \). Evaluate each integral.

a. \[ \int_0^{\pi/(2p)} \cos px \, f(\sin px) \, dx \]

b. \[ \int_{-\pi/2}^{\pi/2} \cos x \, f(\sin x) \, dx \]

Applications

86. Periodic motion An object moves along a line with a velocity in m/s given by \( v(t) = 8 \cos (\pi t/6) \). Its initial position is \( s(0) = 0 \).

a. Graph the velocity function.

b. As discussed in Chapter 6, the position of the object is given by \( s(t) = \int_0^t v(y) \, dy \), for \( t \geq 0 \). Find the position function, for \( t \geq 0 \).

c. What is the period of the motion—that is, starting at any point, how long does it take the object to return to that position?

87. Population models The population of a culture of bacteria has a growth rate given by \( p'(t) = \frac{200}{(t + 1)^2} \) bacteria per hour, for \( t \geq 0 \), where \( r > 1 \) is a real number. In Chapter 6 it is shown that the increase in the population over the time interval \([0, t]\) is given by \( \int_0^t p'(x) \, dx \). (Note that the growth rate decreases in time, reflecting competition for space and food.)

a. Using the population model with \( r = 2 \), what is the increase in the population over the time interval \( 0 \leq t \leq 4? \)

b. Using the population model with \( r = 3 \), what is the increase in the population over the time interval \( 0 \leq t \leq 6? \)

c. Let \( \Delta P \) be the increase in the population over a fixed time interval \([0, T]\). For fixed \( T \), does \( \Delta P \) increase or decrease with the parameter \( r \)? Explain.

d. A lab technician measures an increase in the population of 350 bacteria over the 10-hr period \([0, 10]\). Estimate the value of \( r \) that best fits this data point.

e. Looking ahead: Use the population model in part (b) to find the increase in population over the time interval \([0, T]\), for any \( T > 0 \). If the culture is allowed to grow indefinitely \((T \to \infty)\), does the bacteria population increase without bound? Or does it approach a finite limit?

88. Average distance on a triangle Consider the right triangle with vertices \((0, 0), (0, b), \) and \((a, 0)\), where \( a > 0 \) and \( b > 0 \). Show that the average vertical distance from points on the \( x \)-axis to the hypotenuse is \( b/2 \), for all \( a > 0 \).

89. Average value of sine functions Use a graphing utility to verify that the functions \( f(x) = \sin kx \) have a period of \( 2\pi/k \), where \( k = 1, 2, 3, \ldots \). Equivalently, the first “hump” of \( f(x) = \sin kx \) occurs on the interval \([0, \pi/k]\). Verify that the average value of the first hump of \( f(x) = \sin kx \) is independent of \( k \). What is the average value?

Additional Exercises

90. Looking ahead: Integrals of \( \tan x \) and \( \cot x \) Use a change of variables to verify each integral.

a. \[ \int \tan x \, dx = -\ln |\cos x| + C = \ln |\sec x| + C \]

b. \[ \int \cot x \, dx = \ln |\sin x| + C \]

91. Looking ahead: Integrals of \( \sec x \) and \( \csc x \)

a. Multiply the numerator and denominator of \( \sec x \) by \( \sec x + \tan x \); then use a change of variables to show that \[ \int \sec x \, dx = \ln |\sec x + \tan x| + C. \]

b. Use a change of variables to show that \[ \int \csc x \, dx = -\ln |\csc x + \cot x| + C. \]
92. **Equal areas** The area of the shaded region under the curve 
\[ y = 2 \sin 2x \] 
in (a) equals the area of the shaded region under the curve 
\[ y = \sin x \] 
in (b). Explain why this is true without computing areas.

![Graph of y = 2 sin 2x and y = sin x](image)

93. **Equal areas** The area of the shaded region under the curve 
\[ y = \frac{(\sqrt{x} - 1)^2}{2\sqrt{x}} \] 
on the interval [4, 9] in (a) equals the area of the shaded region under the curve 
\[ y = x^2 \] 
on the interval [1, 2] in (b). Without computing areas, explain why.

![Graph of y = (sqrt(x) - 1)^2 / 2sqrt(x) and y = x^2](image)

### 94–98. General results
Evaluate the following integrals in which the function \( f \) is unspecified. Note that \( f^{(p)} \) is the \( p \)th derivative of \( f \) and \( f^n \) is the \( n \)th power of \( f \). Assume \( f \) and its derivatives are continuous for all real numbers.

94. \[ \int (5f'(x) + 7f^2(x) + f(x)) f'(x) \, dx \]
95. \[ \int_1^2 (5f'(x) + 7f^2(x) + f(x)) f'(x) \, dx \], where \( f(1) = 4 \), \( f(2) = 5 \)
96. \[ \int_0^1 f'(x)f''(x) \, dx \], where \( f'(0) = 3 \) and \( f'(1) = 2 \)
97. \[ \int (f^{(p)}(x))^{p+1} f(x) \, dx \], where \( p \) is a positive integer, \( n \neq -1 \)
98. \[ \int_0^1 2(f''(x) + 2f(x)) f(x) f'(x) \, dx \]

99–101. **More than one way** Occasionally, two different substitutions do the job. Use each substitution to evaluate the following integrals.

99. \[ \int_0^1 x\sqrt{x} + a \, dx \], \( a > 0 \) \( (u = \sqrt{x} + a \) and \( u = x + a) \)
100. \[ \int_0^1 \sqrt{x} + a \, dx \], \( a > 0 \) \( (u = \sqrt{x} + a \) and \( u = x + a) \)
101. \[ \int \sec^3 \theta \tan \theta \, d\theta \] \( (u = \cos \theta \) and \( u = \sec \theta) \)

102. **\( \sin^2 ax \) and \( \cos^2 ax \) integrals** Use the Substitution Rule to prove that
\[
\int \sin^2 ax \, dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} + C \quad \text{and} \\
\int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin(2ax)}{4a} + C.
\]

103. **Integral of \( \sin^2 x \cos^2 x \)** Consider the integral 
\[ I = \int \sin^2 x \cos^2 x \, dx. \]

a. Find \( I \) using the identity \( \sin 2x = 2 \sin x \cos x. \)

b. Find \( I \) using the identity \( \cos^2 x = 1 - \sin^2 x. \)

104. **Substitution: shift** Perhaps the simplest change of variables is the shift or translation given by \( u = x + c \), where \( c \) is a real number.

a. Prove that shifting a function does not change the net area under the curve, in the sense that
\[ \int_a^b f(x + c) \, dx = \int_{a+c}^{b+c} f(u) \, du. \]

b. Draw a picture to illustrate this change of variables in the case that \( f(x) = \sin x, a = 0, b = \pi, \) and \( c = \pi/2. \)

105. **Substitution: scaling** Another change of variables that can be interpreted geometrically is the scaling \( u = cx \), where \( c \) is a real number. Prove and interpret the fact that
\[ \int_a^b f(cx) \, dx = \frac{1}{c} \int_{ac}^{bc} f(u) \, du. \]

Draw a picture to illustrate this change of variables in the case that \( f(x) = \sin x, a = 0, b = \pi, \) and \( c = \frac{1}{2}. \)

106–109. **Multiple substitutions** If necessary, use two or more substitutions to find the following integrals.

106. \[ \int x \sin^2 x \cos x \, dx \] (Hint: Begin with \( u = x^2 \), then use \( v = \sin u) \)
107. \[ \int \frac{dx}{\sqrt{1 + x^2}} \] (Hint: Begin with \( u = \sqrt{1 + x} \))
108. \[ \int_0^1 x\sqrt{1 - \sqrt{x}} \, dx \]
109. \[ \int_0^1 \sqrt{x - x\sqrt{x}} \, dx \]
110. \[ \int \tan^{10} 4x \sec^2 4x \, dx \] (Hint: Begin with \( u = 4x \))
111. \[ \int_0^{\pi/2} \frac{-\cos \theta \sin \theta}{\sqrt{\cos^2 \theta + 16}} \, d\theta \] (Hint: Begin with \( u = \cos \theta \), \( u = x + 5 \)  
2. With \( u = x^5 + 6 \), we have \( du = 5x^4 \), and \( x^5 \) does not appear in the integrand.  
3. New equation: \( u^2 - 13u + 36 = 0; \) roots: \( x = \pm 2, \pm 3 \).

### Quick Check Answers
1. \( u = x^3 + 5 \)
2. With \( u = x^5 + 6 \), we have \( du = 5x^4 \), and \( x^5 \) does not appear in the integrand.  
3. New equation: \( u^2 - 13u + 36 = 0; \) roots: \( x = \pm 2, \pm 3 \) &lt;
1. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample. Assume $f$ and $f'$ are continuous functions for all real numbers.
   a. If $A(x) = \int_a^b f(t) \, dt$ and $f(t) = 2t - 3$, then $A$ is a quadratic function.
   b. Given an area function $A(x) = \int_a^x f(t) \, dt$ and an antiderivative $F$ of $f$, it follows that $A'(x) = F(x)$.
   c. $\int_a^b f'(x) \, dx = F(b) - F(a)$.
   d. If $f$ is continuous on $[a, b]$ and $\int_a^b |f(x)| \, dx = 0$, then $f(x) = 0$ on $[a, b]$.
   e. If the average value of $f$ on $[a, b]$ is zero, then $f(x) = 0$ on $[a, b]$.
   f. $\int_a^b (2f(x) - \sqrt{3g(x)}) \, dx = 2 \int_a^b f(x) \, dx + 3 \int_a^b g(x) \, dx$.
   g. $\int_a^b f'(g(x))g'(x) \, dx = f(g(x)) + C$.

2. **Velocity to displacement** An object travels on the $x$-axis with a velocity given by $v(t) = 2t + 5$, for $0 \leq t \leq 4$.
   a. How far does the object travel, for $0 \leq t \leq 4$?
   b. What is the average value of $v$ on the interval $[0, 4]$?
   c. True or false: The object would travel as far in part (a) if it traveled at its average velocity (a constant), for $0 \leq t \leq 4$.

3. **Area by geometry** Use geometry to evaluate the following definite integrals, where the graph of $f$ is given in the figure.
   a. $\int_0^4 f(x) \, dx$
   b. $\int_6^8 f(x) \, dx$
   c. $\int_3^7 f(x) \, dx$
   d. $\int_0^7 f(x) \, dx$

4. **Displacement by geometry** Use geometry to find the displacement of an object moving along a line for the time intervals (i) $0 \leq t \leq 5$, (ii) $3 \leq t \leq 7$, and (iii) $0 \leq t \leq 8$, where the graph of its velocity $v = g(t)$ is given in the figure.

5. **Area by geometry** Use geometry to evaluate $\int_0^4 \sqrt{8x - x^2} \, dx$.
   (Hint: Complete the square.)

6. **Bagel output** The manager of a bagel bakery collects the following production rate data (in bagels per minute) at seven different times during the morning. Estimate the total number of bagels produced between 6:00 and 7:30 A.M., using a left and right Riemann sum.

<table>
<thead>
<tr>
<th>Time of day (A.M.)</th>
<th>Production rate (bagels/min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6:00</td>
<td>45</td>
</tr>
<tr>
<td>6:15</td>
<td>60</td>
</tr>
<tr>
<td>6:30</td>
<td>75</td>
</tr>
<tr>
<td>6:45</td>
<td>60</td>
</tr>
<tr>
<td>7:00</td>
<td>50</td>
</tr>
<tr>
<td>7:15</td>
<td>40</td>
</tr>
<tr>
<td>7:30</td>
<td>30</td>
</tr>
</tbody>
</table>

7. **Integration by Riemann sums** Consider the integral $\int_1^2 (3x - 2) \, dx$.
   a. Evaluate the right Riemann sum for the integral with $n = 3$.
   b. Use summation notation to express the right Riemann sum in terms of a positive integer $n$.
   c. Evaluate the definite integral by taking the limit as $n \to \infty$ of the Riemann sum of part (b).
   d. Confirm the result of part (c) by graphing $y = 3x - 2$ and using geometry to evaluate the integral. Then evaluate $\int_1^2 (3x - 2) \, dx$ with the Fundamental Theorem of Calculus.

8–11. **Limit definition of the definite integral** Use the limit definition of the definite integral with right Riemann sums and a regular partition to evaluate the following definite integrals. Use the Fundamental Theorem of Calculus to check your answer.
   8. $\int_0^1 (4x - 2) \, dx$
   9. $\int_0^2 (x^2 - 4) \, dx$
   10. $\int_1^2 (3x^2 + x) \, dx$
   11. $\int_0^4 (x^3 - x) \, dx$

12. **Evaluating Riemann sums** Consider the function $f(x) = 3x + 4$ on the interval $[3, 7]$. Show that the midpoint Riemann sum with $n = 4$ gives the exact area of the region bounded by the graph.

13. **Sum to integral** Evaluate the following limit by identifying the integral that it represents:
   $$\lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{4k}{n} \right)^5 + 1 \left( \frac{4}{n} \right)$$

14. **Area function by geometry** Use geometry to find the area $A(x)$ that is bounded by the graph of $f(t) = 2t - 4$ and the $t$-axis between the point $(2, 0)$ and the variable point $(x, 0)$, where $x \geq 2$. Verify that $A'(x) = f(x)$. 
15–30. Evaluating integrals Evaluate the following integrals.

15. \[ \int_{-2}^{2} (3x^4 - 2x + 1) \, dx \]
16. \[ \int \cos 3x \, dx \]
17. \[ \int_{0}^{1} (x + 1)^3 \, dx \]
18. \[ \int_{0}^{1} (4x^2 - 2x^4 + 1) \, dx \]
19. \[ \int_{0}^{1} \sqrt{x} (\sqrt{x} + 1) \, dx \]
20. \[ \int_{-2}^{2} e^{4x} \, dx \]
21. \[ \int_{0}^{1} \frac{x}{\sqrt{4 - x^2}} \, dx \]
22. \[ \int \frac{y^2}{y^2 + 27} \, dy \]
23. \[ \int_{0}^{25} \frac{x}{\sqrt{25 - x^2}} \, dx \]
24. \[ \int y^2 (3y^3 + 1)^4 \, dy \]
25. \[ \int_{0}^{\pi} \sin^2 50 \, d\theta \]
26. \[ \int_{0}^{\infty} (1 - \cos^2 3\theta) \, d\theta \]
27. \[ \int_{0}^{2} x^3 + 3x^2 - 6x \, dx \]
28. \[ \int_{0}^{10} e^x \, dx \]
29. \[ \int_{1}^{4} (x^2 + 3x + 1) \, dx \]
30. \[ \int_{1}^{4} (3x^3 - 2g(x)) \, dx \]

31–34. Area of regions Compute the area of the region bounded by the graph of \( f \) and the \( x \)-axis on the given interval. You may find it useful to sketch the region.

31. \( f(x) = 16 - x^2 \) on \([-4, 4]\)
32. \( f(x) = x^3 - x \) on \([-1, 0]\)
33. \( f(x) = 2 \sin(\pi/4) \) on \([0, 2\pi]\)
34. \( f(x) = 1/(x^2 + 1) \) on \([-1, \sqrt{3}]\)

35–36. Area versus net area Find (i) the net area and (ii) the area of the region bounded by the graph of \( f \) and the \( x \)-axis on the given interval. You may find it useful to sketch the region.

35. \( f(x) = x^4 - x^2 \) on \([-1, 1]\)
36. \( f(x) = x^2 - x \) on \([0, 3]\)

37. Symmetry properties Suppose that \( \int_{-a}^{a} f(x) \, dx = 10 \) and \( \int_{-b}^{b} g(x) \, dx = 20 \). Furthermore, suppose that \( f \) is an even function and \( g \) is an odd function. Evaluate the following integrals.

a. \( \int_{-4}^{4} f(x) \, dx \)  
   b. \( \int_{-4}^{4} 3g(x) \, dx \)

c. \( \int_{-4}^{4} (4f(x) - 3g(x)) \, dx \)  
   d. \( \int_{0}^{1} 8x f(4x^3) \, dx \)

e. \( \int_{-2}^{2} 3x f(x) \, dx \)

38. Properties of integrals The figure shows the areas of regions bounded by the graph of \( f \) and the \( x \)-axis. Evaluate the following integrals.

a. \( \int_{a}^{b} f(x) \, dx \)  
   b. \( \int_{c}^{d} f(x) \, dx \)  
   c. \( \int_{e}^{f} 2f(x) \, dx \)

d. \( \int_{a}^{d} 4f(x) \, dx \)  
   e. \( \int_{a}^{b} 3f(x) \, dx \)  
   f. \( \int_{b}^{c} 2f(x) \, dx \)

39–44. Properties of integrals Suppose that \( \int_{1}^{2} f(x) \, dx = 6 \), \( \int_{1}^{2} g(x) \, dx = 4 \), and \( \int_{1}^{2} f(x) \, dx = 2 \). Evaluate the following integrals or state that there is not enough information.

39. \( \int_{1}^{4} 3f(x) \, dx \)  
40. \( \int_{4}^{1} 2f(x) \, dx \)

41. \( \int_{1}^{4} (3f(x) - 2g(x)) \, dx \)  
42. \( \int_{1}^{4} f(x)g(x) \, dx \)

43. \( \int_{1}^{3} f(x) \, dx \)  
44. \( \int_{1}^{3} (f(x) - g(x)) \, dx \)

45. Displacement from velocity A particle moves along a line with a velocity given by \( v(t) = 5 \sin \pi t \) starting with an initial position \( s(0) = 0 \). Find the displacement of the particle between \( t = 0 \) and \( t = 2 \), which is given by \( s(t) = \int_{0}^{2} v(t) \, dt \). Find the distance traveled by the particle during this interval, which is \( \int_{0}^{2} |v(t)| \, dt \).

46. Average height A baseball is launched into the outfield on a parabolic trajectory given by \( y = 0.01x(200 - x) \). Find the average height of the baseball over the horizontal extent of its flight.

47. Average values Integration is not needed.

a. Find the average value of \( f \) shown in the figure on the interval \([1, 6]\) and then find the point(s) \( c \) in \((1, 6)\) guaranteed to exist by the Mean Value Theorem for Integrals.
b. Find the average value of \( f \) shown in the figure on the interval \([2, 6]\) and then find the point(s) \( c \) in \((2, 6)\) guaranteed to exist by the Mean Value Theorem for Integrals.

\[ y = f(x) \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad x \]

48. An unknown function The function \( f \) satisfies the equation
\[ 3x^4 - 48 = \int_2^4 f(t) \, dt. \]
Find \( f \) and check your answer by substitution.

49. An unknown function Assume \( f' \) is continuous on \([2, 4]\),
\[ \int_2^4 f'(2x) \, dx = 10, \text{ and } f(4) = 4. \]
Evaluate \( f(4) \).

50. Function defined by an integral Let
\[ H(x) = \int_0^x \sqrt{4 - t^2} \, dt, \]
for \(-2 \leq x \leq 2\).

a. Evaluate \( H(0) \).
b. Evaluate \( H'(1) \).
c. Evaluate \( H'(2) \).
d. Use geometry to evaluate \( H(2) \).
e. Find the value of \( s \) such that \( H(x) = sH(-x) \).

51. Function defined by an integral Make a graph of the function
\[ f(x) = \int_1^x \frac{dt}{t} \]
for \( x \geq 1 \). Be sure to include all of the evidence you used to arrive at the graph.

52. Identifying functions Match the graphs \( A, B, \) and \( C \) in the figure with the functions \( f(x), f'(x), \) and \( \int_0^x f(t) \, dt \).

\[ y \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad x \]

53. Ascent rate of a scuba diver Divers who ascend too quickly in the water risk decompression illness. A common recommendation for a maximum rate of ascent is 30 feet/minute with a 5-minute safety stop 15 feet below the surface of the water. Suppose that a diver ascends to the surface in 8 minutes according to the velocity function
\[ v(t) = \begin{cases} 
30 & \text{if } 0 \leq t \leq 2 \\
0 & \text{if } 2 < t \leq 7 \\
15 & \text{if } 7 < t \leq 8.
\end{cases} \]
a. Graph the velocity function \( v \).
b. Compute the area under the velocity curve.

c. Interpret the physical meaning of the area under the velocity curve.

54. Area functions Consider the graph of the continuous function \( f \) in the figure and let
\[ F(x) = \int_0^x f(t) \, dt \]
and \( G(x) = \int_1^x f(t) \, dt \).
Assume the graph consists of a line segment from \((0, -2)\) to \((2, 2)\) and two quarter circles of radius 2.

\[ y = f(t) \]

\[ -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 1 \quad 2 \quad 3 \quad 4 \]

a. Evaluate \( F(2), F(-2), \) and \( F(4) \).
b. Evaluate \( G(-2), G(0), \) and \( G(4) \).
c. Explain why there is a constant \( C \) such that \( F(x) = G(x) + C \), for \(-2 \leq x \leq 4\). Fill in the blank with a number: \( F(x) = G(x) + \_ \), for \(-2 \leq x \leq 4\).

55–56. Area functions and the Fundamental Theorem Consider the function
\[ f(t) = \begin{cases} 
t & \text{if } -2 \leq t < 0 \\
t^2 & \text{if } 0 \leq t \leq 2
\end{cases} \]
and its graph shown below.

\[ y = f(t) \]

\[ -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 1 \quad 2 \quad 3 \quad 4 \]

Let \( F(x) = \int_{-2}^x f(t) \, dt \) and \( G(x) = \int_{-2}^1 f(t) \, dt \).

a. Evaluate \( F(-2) \) and \( F(2) \).
b. Use the Fundamental Theorem to find an expression for \( F'(x) \), for \(-2 \leq x < 0\).
c. Use the Fundamental Theorem to find an expression for \( F'(x) \), for \( 0 \leq x \leq 2\).
d. Evaluate \( F'(-1) \) and \( F'(1) \). Interpret these values.
e. Evaluate \( F''(-1) \) and \( F''(1) \).
f. Find a constant \( C \) such that \( F(x) = G(x) + C \).
56. a. Evaluate \( G(-1) \) and \( G(1) \).
   b. Use the Fundamental Theorem to find an expression for \( G'(x) \), for \(-2 \leq x < 0 \).
   c. Use the Fundamental Theorem to find an expression for \( G'(x) \), for \( 0 \leq x \leq 2 \).
   d. Evaluate \( G'(0) \) and \( G'(1) \). Interpret these values.
   e. Find a constant \( C \) such that \( F(x) = G(x) + C \).

57–58. Limits with integrals Evaluate the following limits.

57. \[ \lim_{x \to 2} \frac{\int x e^x \, dt}{x - 2} \]
58. \[ \lim_{x \to 1} \frac{\int x^2 e^x \, dt}{x - 1} \]

59. Geometry of integrals Without evaluating the integrals, explain why the following statement is true for positive integers \( n \):
   \[ \int_0^1 x^n \, dx + \int_0^1 \sqrt{2} \, dx = 1. \]

60. Change of variables Use the change of variables \( u^3 = x^2 - 1 \) to evaluate the integral \( \int_0^1 x^2 \sqrt{x^2 - 1} \, dx \).

61. Inverse tangent integral Prove that for nonzero constants \( a \) and \( b \),
   \[ \int \frac{dx}{a x^2 + b} = \frac{1}{ab} \tan^{-1} \left( \frac{ax}{b} \right) + C. \]

62–67. Additional integrals Evaluate the following integrals.

62. \[ \int \frac{\sin 2x}{1 + \cos 2x} \, dx \] \( \text{[Hint: } \sin 2x = 2 \sin x \cos x.] \)
63. \[ \int \frac{1}{x^2 \sin \frac{1}{x}} \, dx \]
64. \[ \int \frac{\tan^{-1} x}{1 + x^2} \, dx \]
65. \[ \int \frac{dx}{(\tan^{-1} x)(1 + x^2)} \]
66. \[ \int \frac{\sin^{-1} x}{\sqrt{1 - x^2}} \, dx \]
67. \[ \int e^x - e^{-x} \, dx \]

68. Area with a parameter Let \( a > 0 \) be a real number and consider the family of functions \( f(x) = \sin ax \) on the interval \([0, \pi/a] \).

   a. Graph \( f \), for \( a = 1, 2, 3 \).
   b. Let \( g(a) \) be the area of the region bounded by the graph of \( f \) and the \( x \)-axis on the interval \([0, \pi/a] \). Graph \( g \) for \( 0 < a < \infty \). Is \( g \) an increasing function, a decreasing function, or neither?

69. Equivalent equations Explain why if a function \( u \) satisfies the equation \( u(x) + 2 \int_0^x u(t) \, dt = 10 \), then it also satisfies the equation \( u'(x) + 2u(x) = 0 \). Is it true that if \( u \) satisfies the second equation, then it satisfies the first equation?

70. Area function properties Consider the function \( f(t) = t^2 - 5t + 4 \) and the area function \( A(x) = \int_0^x f(t) \, dt \).

   a. Graph \( f \) on the interval \([0, 6] \).
   b. Compute and graph \( A \) on the interval \([0, 6] \).
   c. Show that the local extrema of \( A \) occur at the zeros of \( f \).
   d. Give a geometrical and analytical explanation for the observation in part (c).
   e. Find the approximate zeros of \( A \), other than 0, and call them \( x_1 \) and \( x_2 \), where \( x_1 < x_2 \).
   f. Find \( b \) such that the area bounded by the graph of \( f \) and the \( x \)-axis on the interval \([0, t_1] \) equals the area bounded by the graph of \( f \) and the \( x \)-axis on the interval \([t_1, b] \).
   g. If \( f \) is an integrable function and \( A(x) = \int_0^x f(t) \, dt \), is it always true that the local extrema of \( A \) occur at the zeros of \( f \)? Explain.

71. Function defined by an integral Let \( f(x) = \int_0^x (t - 1)(t - 2) \, dt \).

   a. Find the intervals on which \( f \) is increasing and the intervals on which \( f \) is decreasing.
   b. Find the intervals on which \( f \) is concave up and the intervals on which \( f \) is concave down.
   c. For what values of \( x \) does \( f \) have local minima? Local maxima?
   d. Where are the inflection points of \( f \)?

72. Exponential inequalities Sketch a graph of \( f(t) = e^t \) on an arbitrary interval \([a, b] \). Use the graph and compare areas of regions to prove that
   \[ e^{(a+b)/2} \leq \frac{e^b - e^a}{b - a} \leq \frac{e^a + e^b}{2} \]
   \( \text{(Source: Mathematics Magazine 81, 5, Dec 2008)} \)